TECHNOLOGY AND DISTRIBUTION: HISTORICAL TRAJECTORIES A LA MARX

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RÉSUMÉ

TECHNOLOGIE ET RÉPARTITION:
TRAJECTOIRES HISTORIQUES A LA MARX

Au livre III du *Capital*, Marx introduit un système de tendances historiques des grandes variables décrivant la technique et la distribution: productivités, composition du capital, part des profits et taux de profit. La principale tendance est la fameuse baisse du taux de profit, dont le dernier siècle d'évolution du capitalisme confirme la pertinence factuelle. Ces trajectoires *à la Marx* sont différentes des sentiers usuels de croissance homothétique équilibrée des modèles de croissance, avec taux de profit constant. Cette étude propose une formalisation de ces tendances *à la Marx*. Les théories du changement technique et du salaire y jouent chacune un rôle central. On montre que l'intuition principale de Marx est celle d'une *difficulté d'innnover*: il n'est pas aisé de trouver de nouveaux procédés autorisant la croissance de la productivité du travail sans trop diminuer celle du capital. Mais le salaire est également en cause. Sa croissance est stimulée par celle de l'emploi, malgré les détentes résultant des crises récurrentes. Pourtant, la stabilisation de la part des salaires (ou du taux de la plus-value) le long de trajectoires *à la Marx* interdit de voir dans la baisse du taux de profit l'effet d'une croissance excessive des salaires.

ABSTRACT

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HISTORICAL TRAJECTORIES A LA MARX

In *Capital*, Marx introduces a system of historical tendencies accounting for the trends of the major technological and distributional variables (notably the profit rate), quite distinct from the usual steady states. Marx's main insight is that of a *difficulty to innovate*: It is difficult to find new processes allowing for the growth of labor productivity while not considerably reducing the productivity of capital. The growth of the wage rate is stimulated by the rise of employment, but the stabilization of the share of wages forbids a direct imputation of the decline of the profit rate to the excessive growth of the wage rate.

MOTS CLEFS : Changement technique, productivité du travail, productivité du capital, coût salarial, taux de profit, tendances historiques, modèle évolutionniste.

KEYWORDS : Technical change, labor productivity, productivity of capital, labor cost, profit rate, historical tendencies, evolutionary model.

J.E.L. Nomenclature: 040.

Introduction

While standard growth models in economic theory account for steady states with a constant growth rate and a constant situation of distribution, one can locate in Marx's work a description of quite specific trajectories, called "historical tendencies," that we denote as *trajectories à la Marx*. Along such long-term paths, the growth rates of capital, output, and employment are gradually reduced, the share of wages in total income is constant or diminishing, and the profit rate declines.

After nearly a century and a half, it is interesting to notice that such patterns of evolution are observable during specific phases — several decades long — of the evolution of capitalism. In other works, we have shown that such trajectories *à la Marx* are well in line with the trends of the major technological and distributional variables in the US economy during approximately the second half of the 19th century and the second half of the 20th century (DUMÉNIL G., LÉVY D. 1993, 1996, 2000).

This paper provides an analytical treatment of Marx's analysis of historical tendencies, in particular the famous *tendency for the profit rate to fall*. The purpose of the exercise is to make Marx's assumptions explicit, discuss their consistency, and derive a number of additional properties. As we will show, Marx's analysis of historical tendencies cannot be separated from other important components of his work, notably the theories of technical change, wages, and accumulation. In the interpretation of Marx's work, the most difficult issue is the determination of wages, and it is impossible to summarize unambiguously Marx's view in this respect. The theory of technical change is also a crucial element; we use the *classical-Marxian evolutionary model of technical change* that we presented a few years ago (DUMÉNIL G., LÉVY D. 1995, 1998 available on our web site). This paper does not discuss the relationship between the falling rate of profit and crises.[1]

We show that dynamic models can be built in which trajectories *à la Marx* correspond to stable equilibria.[2] This means that, if the economy tends to deviate from such paths, there are mechanisms that push the variables back to their trajectories. This result is well in line with the notion of *tendency* or *law*. Both the conditions of innovation and the dynamics of wages are at issue in the formation of trajectories *à la Marx*:

1. The basic insight is that innovation is difficult, in general and particularly within capitalism, in the sense that is not easy to find innovations which increase simultaneously the productivities of labor and capital. This may or may not be due to an *a priori* labor-saving capital-consuming bias in the innovation process.

2. If the conditions of innovation are assumed constant, the profit rate declines exponentially. As a result of the rapidity of this fall, the rise of the capital stock is bounded above, and both output and employment diminish. Thus, difficult conditions of innovation are not sufficient to account for all features of trajectories à la Marx, in particular Marx's assumption of continuing growth (though at a gradually reduced pace). Such trajectories actually require a *gradual erosion* of the conditions of innovation diminishing the pace of technical change: a form of "productivity slowdown," analogous to what would be now

1. A large litterature has been devoted to these problems since Henryk Grossmann and Maurice Dobb (GROSSMANN H. 1970; DOBB M. 1967). These issues are treated in our books listed above.
2. In this paper, *equilibrium* refers to the fixed point of a dynamic model, a mathematical notion, susceptible to many applications in economics (appendix A.2).

labeled the "exhaustion of a socio-technical paradigm" or "diminishing returns" of R&D activities. The assumption of a continuous deterioration of the conditions of innovation as a power function of time allows for growth trajectories with a declining profit rate (also as a power function of time).

3. In spite of recurrent crises, capitalists cannot stop the tendential rise of real wages as employment grows. However, the notion of *wage squeeze* does not account for trajectories *à la Marx*, since the rate of surplus-value (or the share of wages) stabilizes while the profit rate continues its slide downward.

Obviously, the parameters which define both innovation and the dynamics of wages in the models are susceptible to historical variation. As shown in other works, a declining profit rate will lead at some point to a structural crisis, and "something" will happen with respect to technical change.[3] The historical investigation of the US economy since the Civil War reveals that the transformation of what we denote as the *conditions of innovation*, or the *difficulty to innovate*, seems to have been the crucial change that accounted for the successive phases of rise and decline in the profit rate. These conditions of innovation refer to complex historical settings in which technology in the narrow sense, relations of production, class patterns and class struggle, and a whole range of institutions are involved. Thus, a very compact formulation of Marx's insight that adequately matches historical observation is the intrinsic difficulty of capitalism to maintain a steady pace of efficient innovation (concerning production technology).

Section 1 is devoted to Marx's analysis in *Capital*. A formalization is then performed in a step by step fashion. Section 2 introduces the notation and framework of analysis and discusses alternative analytical representations of Marx's historical tendencies. Section 3 deals with the modeling of technical change and wages. Finally, section 4 is devoted to the study of two models and their economic interpretation.

1 - Marx's analysis of historical tendencies and related mechanisms

Marx's analysis of historical tendencies in Volume III of *Capital* is not always easy to interpret. The concept refers to a rather complex set of tendencies, quite explicit and convincing in some respects, although sometimes obscure. Moreover, this analysis can only be understood as a component of a broader theoretical framework in which the theories of technical change, distribution, and accumulation are also implied. This first section recalls the main aspects of Marx's analysis as a preliminary to a formal treatment in the following sections.

3. DUMÉNIL G., LÉVY D. 1993; 2001. This interpretation is in line with that found in MANDEL E. 1975.

1.1 Historical tendencies

Marx's famous "law of the tendential fall in the profit rate" is well known, at least in its simplest formulation at the beginning of chapter 13 of Volume III of *Capital* (MARX K. 1981). In that chapter, Marx addresses the issue of the falling profit rate under the assumption of a constant rate of surplus-value[4]: "a gradual fall in the general rate of profit, given that the rate of surplus-value, or the level of exploitation of labour by capital, remains the same" (MARX K. 1981, p. 318). How can the profit rate decline while the rate of surplus-value remains constant? Marx's answer is straightforward: It is the effect of the rising composition of capital, the fact that more and more constant capital is required compared to variable capital.[5] The assumption of a constant rate of surplus-value is used by Marx to contend that the fall of the profit rate is not due to excessive wages, but to a specific feature of technical change. This analysis sharply contrasts with Ricardo who locates the declining profitability of capital in the rise of the relative price of corn and, thus, in a rising nominal wage and wage share. In a modern formulation, Marx contends that the downward trend of the profit rate should not be understood as a *profit squeeze* by wages.

1.1.1 A set of tendencies and countertendencies

A careful reading of *Capital* shows, however, that Marx's analysis actually refers to a broad set of variables, notably:

1. *Labor productivity*. A first law expresses the fact that, with the progress of productive forces, the production of a given commodity requires less and less labor. The tendency for the profit rate to fall is, thus, an "expression" of the progress of labor productivity:

> Thus the same development in the social productivity of labour is expressed, with the advance of the capitalist mode of production, ... in a progressive tendency for the profit rate to fall (MARX K. 1981, p. 329).

Marx stresses in this statement the *a priori* paradoxical association of the growth of labor productivity and declining profitability.

2. *The value-composition of capital* (the ratio of the value of the means of production, or constant capital, to the value of labor-power, or variable capital). The rise of the value-composition of capital is a crucial feature of technical change[6]:

> Moreover, it has been shown to be a law of the capitalist mode of production that its development does in fact involve a relative decline in the relation of variable capital to constant .. (MARX K. 1981, p. 318).

The value-composition of capital reflects, in value terms, the technical composition of capital, a physical index. Marx often uses the notion of *organic composition of capital* in

4. A constant rate of surplus-value or rate of exploitation is equivalent to a constant profit share.
5. Constant capital is, loosely, the capital stock. Variable capital is the payment to labor. The composition of capital is the ratio of these two.
6. *Value of a product* means the amount of labor embodied in it in the sense of the sum of labor directly used to produce it plus the labor used to produce the inputs. Little is lost if one thinks here of value as a cost in the ordinary sense.

place of *value-composition*.[7]

3. *Accumulation and growth.* Marx's description of historical tendencies also includes a statement concerning the path of accumulation: profits accumulated, the growth rates of capital, employment, profits, and output. The central idea is that the falling profit rate is associated with a tendentially accelerating accumulation:

> A fall in the profit rate, and accelerating accumulation, are simply different expressions of the same process, in so far as both express the development of productivity ... there is an acceleration of accumulation, as far as its mass is concerned, even though the rate of this accumulation falls together with the rate of profit (MARX K. 1981, p. 349).

This path of accumulation combines a continued growth of the mass of profits accumulated, hence of investment, and a declining growth rate of the stock of capital (as implied in the last statement). The view that the declining profit rate should be reflected in a declining growth rate of fixed capital is implicitly based on the simple assumption that a constant share of profits is accumulated at each period. Concerning the growth of employment and profits, one can consider the following summary statement:

> The same development of the productivity of social labour, the same laws that are evident in the relative fall in variable capital as a proportion of the total capital and the accelerated accumulation that follows from this ... — this same development is expressed, leaving aside temporary fluctuations, in the progressive increase in the total labour-power applied and in the progressive growth in the absolute mass of surplus-value and therefore in profit (MARX K. 1981, p. 326).

The growth of output is obviously part of this profile, with both employment and labor productivity rising. One must distinguish the historical trajectories, described by Marx as "tendencies" from more transitory developments such as structural crises or recessions. During recessions, profits, output, and employment decline. Marx's analysis of tendencies only considers historical trends, averaging over such shorter-term patterns of events.

In the same chapters Marx points to two evolutions that actually have a *counter-tendential* effect:

1. *The rising rate of surplus-value.* Marx is not content with his original assumption of a constant rate of surplus-value. The rise of the rate of exploitation is the first counteracting factor presented in chapter 14:

> The rise in the rate of surplus-value ... is a factor which contributes to the determination of the mass of surplus-value and hence also the rate of profit. It does not annul the general law. But it has the effect that this law operates more as a tendency, i.e. as a law whose absolute realization is held up, delayed and weakened by counteracting factors (MARX K. 1981, p. 341).

At the end of the chapter, one can find the following:

7. "I call the value-composition of capital, in so far as it is determined by its technical composition and mirrors the change in the latter, the organic composition of capital" (MARX K. 1977, p. 762). This statement implies that the value-composition is not exclusively determined by the technical composition. The difference may be due to variations of the value of labor-power or of the relative values of commodities. Thus, the organic composition of capital can be understood as an expression of the value-composition, "abstracting" from these variations. The use of the notion of organic composition in lieu of value-composition is of the nature of a simplifying assumption.

> The tendential fall in the profit rate is linked with a tendential rise in the rate of surplus-value (MARX K. 1981, p. 347).

2. *The rising share of profits accumulated.* The share of profits devoted to accumulation tends to rise. This is implicit in Marx's insistence on the "acceleration of accumulation." In combination with the rise of the rate of surplus-value, this mechanism has a counter-tendential effect vis-à-vis the decline of the growth rate of the capital stock. It adds to the rise of profits accumulated.

1.1.2 Loose ends

Marx's analysis of the historical tendencies of capitalism, while fascinating, is deficient in several important respects, three beeing noteworthy:

1. *Real wages.* The variable that Marx uses to account for distribution is the rate of surplus-value. Although he considers real wages in other parts of his work[8], Marx did not refer explicitly to the movement of the real wage in the analysis of historical tendencies. He was, however, quite conscious of the fact that his own simultaneous assumptions of a constant rate of surplus-value and a rising labor productivity *implied that real wages must rise*[9], hence the statement:

> We entirely leave aside here the fact that the same amount of value represents a progressively rising mass of use-values and satisfactions, with the progress of capitalist production (MARX K. 1981, p. 325).

Thus, Marx's original assumption concerning a constant rate of surplus-value does not imply that rising real wages play no role in the decline of the profit rate. But the constancy, or possible rise, of the rate of surplus-value is explicitly used by Marx to refute the view that the responsibility of the decline of the profit rate should be placed on the *excessive* rise of wages. After pointing to the possible rise of the rate of surplus-value, Marx comments:

> Nothing is more absurd, then, to explain the fall in the rate of profit in terms of a rise in wage rates (MARX K. 1981, p. 347).

2. *Mechanization.* Although Marx insists on the tendency for the composition of capital to rise, he is not very clear concerning the origin of this tendency. Is mechanization a feature of technical change in general? Does mechanization always require the rise of the composition of capital? In other parts of his work, Marx analyzes the impact of the cost of labor-power on technological change as a factor contributing to the rise of the composition of capital (section 1.2.2), but this effect is not discussed in the chapters devoted to historical tendencies.

3. *Competition.* It is also necessary to recall that Marx's description of the mechanisms leading to a diminished average profit rate is problematic. Marx's mechanism is well known: (1) Individual producers may introduce a new technique on account of the incremental profit that it yields prior to its diffusion; (2) Once it is generalized to all producers and a uniform profit rate is reestablished, the average profit rate is diminished. However, in order to draw a conclusion concerning the comparison between the profit rate prevailing

8. See, for example, the quotations used in section 1.2.2, or the chapter 25 of Volume I of *Capital*.
9. Under the assumption of a rising labor productivity, only a rate of surplus-value rising toward infinity or a wage share declining toward zero would be compatible with a constant real wage.

before the introduction of the new process of production and the eventual profit rate after its diffusion, one additional assumption must be made concerning distribution. Nobuo Okishio (1961) has shown, in his famous theorem, that the profit rate must rise if the real wage is maintained, that is, if the capitalists take the entire advantage of the new improved conditions of production. The profit rate can decline only if the workers benefit from at least a portion of the progress accomplished, that is, if the real wage increases to an extent. *It is therefore not possible to establish a falling profit rate under the assumption of a constant real wage rate, unless capitalists do not select techniques yielding the maximum profit rate.*[10]

1.2 The theories of technical change, wages, and accumulation

The various components of Marx's analysis of historical tendencies are obviously interrelated, and a more careful examination shows that major components of economic theory are involved, in particular the theories of wages and technical change.

1.2.1 A system of variables

The major relationships between the variables at issue are illustrated in diagram 1:

1. When technology and wages are given, it is possible to *compute* the profit rate, r (arrows [1]).

2. The comparative profit rate is a crucial variable in the choice of a technology (arrow [2]). Thus, the rate of wages, among other variables, also affects technology, though indirectly. Here, the *theory of technical change* is at issue.

3. The increase of employment influences the determination of the real wage (arrow [3]). A second theoretical framework is at issue, *the determination of wages*.

4. The profit rate impacts accumulation (arrow [4]). We will use the simple model, implicit in Marx's analysis, and assume that the growth rate of the stock of fixed capital follows in a straightforward manner from the profit rate, under the assumption that a given fraction of profits is accumulated.

5. and 6. The accumulation of capital conditions the growth of output and employment (arrow [5]). But the degree to which accumulation translates into output and employment depends on technology (arrow [6]). Like [1], these relations are only computational and represented by dotted lines.

1.2.2 Technical change and wages

Two aspects of Marx's analysis of technical change are relevant to the present study: the role of the profit rate and the indirect impact of the wage rate, *via* the profit rate.

That capitalists choose technologies which maximize their profit rate is quite explicit in *Capital*. For example, one can find the following statement in Volume III:

> No capitalist voluntarily applies a new method of production, no matter how much more productive it may be or how much it might rise the rate of surplus-value, if it reduces the rate of profit (MARX K. 1981, p. 373).

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10. For example, one can assume that capitalists choose, for competitive reasons, techniques which do not maximize the profit rate but profit margins (SHAIKH A. 1980).

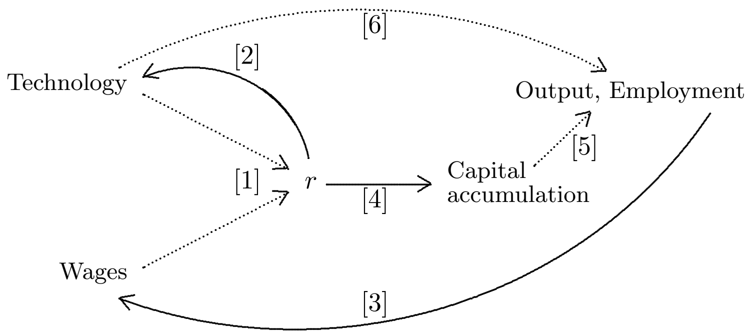

Diagram 1

When comparing various technologies, savings on wages by mechanizing is a crucial criterion in the capitalist's choice. It is explicit in Marx's analysis that innovations are implemented depending on a comparison between the cost of the equipment and the cost of labor saved. In the following extract from the chapter of *Capital* entitled *Machinery and Large-Scale Industry*, the discussion is couched in terms of values or prices proportional to values (or gravitating around such values), in conformity with the perspective of Volume I. Marx compares the labor time embodied in the machine (which will be transferred to the product) to the value of labor-power saved as a result of the use of the machine:

> The use of machinery for the exclusive purpose of cheapening the product is limited by the requirement that less labor must be expended in producing the machinery than is displaced by the employment of that machinery. For the capitalist, however, there is a further limit on its use. Instead of paying for the labor, he pays only for the value of the labor-power employed; the limit to his using a machine is therefore fixed by the difference between the value of the machine and the value of the labor-power replaced by it. Since the division of the day's work into necessary labour and surplus labour differs in different countries, and even in the same country at different periods, or in different branches of industry; and further, since the actual wage of the worker sometimes sinks below the value of his labor power, and sometimes rises above it, it is possible for the difference between the price of the machinery and the price of the labour-power replaced by that machinery to undergo great variations, while the difference between the quantity of labour needed to produce the machine and the total quantity of labour replaced by it remains constant. But it is only the former difference [*between the price of the machinery and the price of the labour-power replaced*] that determines the cost to the capitalist producing a commodity, and influences his actions through the pressure of competition (MARX K. 1977, p. 515-516).

Marx contends in this passage that it is not the minimization of labor that matters, but the economy of labor *paid*, the actual wage (which may, in addition, diverge from the value of labor-power). Although this discussion relies on a comparison of costs, instead of profit rates, since the variable *profit over the stock of capital* is not introduced until Volume III, it is clear that Marx considers the cost of labor as a crucial factor in the choice whether to implement a new technique.[11]

11. Marx sometimes contends that capitalists push the use of machinery even beyond purely direct

1.2.3 Technical change, output, and employment

As already stated in section 1.1.1, output and employment grow along a trajectory *à la Marx*. Arrows [5] and [6] in diagram 1 depict the fact that output and employment are the combined effects of accumulation and technical change. Accumulation contributes to the rise of the stock of fixed capital, but the decline of the productivity of capital and the rise of the technical composition of capital can result in declining output and employment if the evolution of technology is rapid in comparison to accumulation.

The contradictory effects of accumulation and technical change are quite explicit in chapter 25 of Volume I of *Capital*, entitled *The general law of capitalist accumulation*. Marx combines the effects of technical change (the changes in the composition of capital) with the rhythms of accumulation. This is clearly stated in the French edition, in which Marx considerably revised this chapter.[12] There, Marx examines three cases depending on the relative impact of accumulation and technical change on the rise of the number of workers and variable capital. As in Volume III, he concludes that, in the long run, the number of workers must rise:

> Considering a period of several years, for example, a decade, we will generally find that the number of exploited workers increased with the progress of social accumulation, although each particular year, taken separately, contributed very inequally to this outcome or, sometimes, did not contribute at all . . . (MARX K. 1967, p. 72).

1.2.4 The determination of wages

In the mechanisms considered in this section, the real wage is the relevant variable. The rate of surplus-value, or the wage or profit shares, is an important analytical variable but it does not impact directly on the behavior of economic agents. Its influence is indirect. The reference to firm behavior is not alien to Marx's analysis, where one must distinguish between "immanent laws" and "motives" (the behaviors of economic agents).[13] Independently of working conditions, workers fight for higher wages; capitalists strive for greater profitability, as measured by profit rates, not shares of profits.

In the consideration of wages, it is important to distinguish between two types of issues: (1) the analysis of the concept of wage as the price of a given *commodity*[14], with a use-value and an exchange value; and (2) the mechanisms at work in the quantitative determination of wages and specific theses concerning the variation of the real wage in the short and long runs. Only the second issue is considered in this study.

The determination of wages and their historical trend is one of the most difficult and controversial issues in Marx's work, and his opinion varied considerably over his lifetime from the manuscripts of 1844 to *Capital* (LAPIDES K. 1998). A constant aspect of

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economical incentives in order to control the workers (BRAVERMAN H. 1974).

12. MARX K. 1967.

13. Marx claims "it is not our intention here to consider the way in which the immanent laws of capitalist production manifest themselves in the external movements of individual capitals, assert themselves as the coercive laws of competition, and therefore enter into the consciousness of the individual capitalist as the motives which drive him forward" (MARX K. 1977, p. 433).

14. Incidentally, the analysis of labor-power as a commodity largely justifies the reference to a *labor market*. Such a reference does not imply, however, a neoclassical treatment of the mechanisms prevailing on this market, in particular immediate market clearing by prices.

Marx's analysis remained the role of the struggle of workers. His early views were rather pessimistic, but gradually changed. The situation of employment in comparison to the available labor force in the short and long runs was a crucial component of his analysis. The growth of employment is favorable to the rise of wages:

> For since in each year more labourers are employed than in its predecessor, sooner or later a point must be reached, at which the requirements of accumulation begin to outgrow the customary supply of labour, and, therefore, a rise of wages takes place (MARX K. 1977, p. 763).

It is certainly not coincidental that Marx returned to the issue of real wages — in section III of chapter 15, devoted to the *internal contradictions* of the tendency for the profit rate to fall — in his theory of *overaccumulation*[15]:

> Thus as soon as capital has grown in such proportion as to the working population that neither the absolute labour-time that this working population supplies nor its relative surplus labor-time can be extended (the latter would not be possible in any case in *a situation where the demand for labour was so strong, and there was thus a tendency for wages to rise* [*our emphasis*]) ... there would even be a sharper and more sudden fall in the general rate of profit, but this time on account of a change in the composition of capital which would not be due to a development in productivity, but rather to a rise in the money value of variable capital on account of higher wages (MARX K. 1981, p. 360).

Because of this dependency of wages on the demand for labor, the conclusion in chapter 25 of Volume I of *Capital* (confirmed in Volume III) that employment tends to rise in the long run implies that this rise actually poses a threat on the profitability of capital. The main argument in chapter 25 is, however, that, because of recurrent crises, capitalism constantly recreates a reserve army which allows capitalists to reduce the growth of wages (or to diminish wages) to levels compatible with the functioning of the system:

> The rise of wages [*caused by rapid accumulation and, hence, employment*] is therefore confined within limits that not only leave intact the foundations of the capitalist system, but also secure its reproduction over an increasing scale (MARX K. 1977, p. 771).

Thus, both historical trends and fluctuations are involved in Marx's sophisticated analysis of the relationship between accumulation, employment, and wages. In a recurrent manner, accumulation pushes employment to the limits of the available labor force, creating a tension which induces a rise of wages. The profit rate declines, and this fall sparks the contraction of output and the devaluation of capital. During the ensuing crisis the reserve army is recreated, and the increase in employment is temporarily curtailed, diminishing the pressure that accumulation put on wages. But a new phase of accumulation is initiated pushing employment to new levels in spite of the rise in the composition of capital.

Unfortunately, Marx never organized these various aspects of his analysis into a single coherent framework, thus leaving room for endless controversies. One can, however, contend that various mechanisms are rather clearly set out: (1) the impact of wages on

15. Marx often uses the term "overproduction," borrowed from Ricardo: "Overproduction of capital and not of individual commodities — though this overproduction of capital always involves overproduction of commodities — is nothing more than overaccumulation of capital" (MARX K. 1981, p. 359).

technical change stimulating the rise of the composition of capital if wages rise, (2) the twofold impact of accumulation and technical change on the growth of employment, and the tendential increase of employment, and (3) the positive effect of the growth of employment on real wages, moderated by the rise of the composition of capital and recurrent crises. Only the overall outcome concerning *the historical trend of real wages* remained somewhat ambiguous.

1.3 Historical tendencies and related mechanisms: Major components

This study does not purport to provide a coherent and complete formal treatment of all aspects of Marx's analysis as discussed above, but only its major components. The investigation in the remainder of this study elaborates on the following elements:

1. Historical "tendencies" are the focus. This means that shorter fluctuations, such as business-cycle fluctuations are set aside. Although the tendency for the profit rate to fall manifests in periodical crises, such developments are not considered; only the trends are.

2. The trajectories under investigation are growth trajectories, with the continued accumulation of a fraction of profits. Fixed capital, output, employment, investment, and profits grow, though possibly at diminished rates.

3. The other main tendencies are the rise of the composition of capital, a constant rate of exploitation, and a falling profit rate.

4. Despite the constancy of the rate of exploitation, that Marx uses to deny the responsibility of wages, the tendency for the profit rate to fall requires a rise in real wages. This is implicit in Marx's assumption of a constant rate of exploitation and rising labor productivity, a discussion explicitly set aside by Marx. In addition, this necessity has been demonstrated in OKISHIO N. 1961.

5. The rise of the composition of capital follows from: (a) specific features of technical change, and (b) the influence of rising wages due to the maximizing of the profit rate.

6. The variable directly at issue in the conflict over the distribution between workers and capitalists is the real wage rate. (The reference to rate of exploitation has other analytical purposes.) The wage is determined by (a) socio-political conditions ("class struggle" and other institutional determinations), and (b) competition on the labor market. Here, both business-cycle and long-term determinants are involved. The rapidity of accumulation is the crucial variable in these mechanisms, despite the recurrent breaks following from the also recurrent crises.

2 - A formalism for the description of historical tendencies

Section 2.1 defines the variables and expresses the main tendencies in this formalism. Section 2.2 is devoted to the modeling of historical tendencies: Is any formal description (such as exponential functions) of the decline of the profit rate compatible with the various aspects of Marx's analysis? Important conclusions can already be derived from this preliminary approach. In particular, the growth of output is not compatible with the exponential decline of the profit rate. Power decrease must be favored instead of exponential functions.

2.1 General framework of analysis

The formalism in this paper is simple in many respects. We adopt the same technical assumptions as Marx does in his analysis of historical tendencies. We do not distinguish between productive and unproductive labor, and labor is assumed to be homogeneous. Only one good exists[16], making the distinction between relative prices or relative values irrelevant. A constant currency unit is used. The division of profits into various components is not at issue.

In chapter 13 of Volume III of *Capital*, the tendency for the profit rate to fall is presented within the framework used in Volume I to account for the theory of surplus-value: Capital is the sum of two flows, variable and constant capital. This framework abstracts from the circulation of capital introduced (later in Marx's life) in Volume II, where capital is a stock, the sum of the three components, productive, commodity, and money capitals. The definition of the profit rate in this paper conforms to that of Volume II — capital is a stock — but, for simplicity, we abstract from commodity and money capital and, thus, restrict the measure of capital to fixed capital.

The main definitions and notation are as follows:

1. *Production.* Y denotes output, net of all productive consumption including the depreciation of fixed capital. K denotes fixed capital (the total capital stock). H denotes the number of workers or, equivalently, the total number of hours worked (under the assumption of a constant number of hours per period).

2. *Technology.* We call *productivity of labor*, the ratio $P_L = Y/H$ and, by analogy, *productivity of capital*, the ratio $P_K = Y/K$.[17] The technical composition of capital is K/H.

3. *Incomes.* We use w to denote the unit wage, $W = Hw$, total wages (total compensation), and Π, total profits. After division by Y of the identity $Y = W + \Pi$, one obtains

$$1 = \omega + \pi \tag{1}$$

in which ω and π respectively denote the share of wages and the share of profits. We also define the rate of surplus-value:

$$\tau = \frac{\Pi}{W} = \frac{1-\omega}{\omega}. \tag{2}$$

The three variables ω, π, and τ provide alternative measures of distribution, and the assumption that one of these variables is constant implies that the two others are also constant. While ω declines (rises), π and τ rise (fall) in tandem. The following relation is useful:

$$\omega = \frac{w}{P_L}. \tag{3}$$

---

16. Marx's analysis of historical tendencies always considers output globally, except in the description of the introduction of new techniques within competition.

17. Note that the use of the term "productivity" in relation to capital is totally alien to the fact that, in Marx's labor theory of value, only the productive labor force produces value.

Main variables and parameters:

α, c	Parameters in the exponential (or power) trajectories of P_K and r
β, c'	Parameters in the exponential (or power) trajectories of P_L
C	Parameter in the trajectories of K ($C = s\pi c$)
γ	Composition of capital
H	Employment
$K, \Delta K$	Stock of fixed capital, Variation of this stock
ω	Share of wages
P_L, P_K	Productivity of labor, Productivity of capital
π, Π	Share of profits, Profits
r	Profit rate
s	Share of profits devoted to accumulation
$t, \theta = ln\, t$	Time, Logarithm of time
τ	Rate of surplus-value
w, W	Rate of wages, Wages
Y	Output

Parameters of the models of sections 3 and 4:

A, L	Capital and labor inputs required for the production of one unit of output
a, l	Rates of saving on these inputs at each period
$\overline{a}(\omega), \overline{l}(\omega)$	Average values of a, l
N, n	Population, Growth rate of population
ρ_w, δ	Parameters in the wage equations

Notation:

$\dot{x} = dx/dt$	Derivative of x
$\rho(x) = \dot{x}/x$	Growth rate of x

4. *The profit rate and the composition of capital.* The profit rate is

$$r = \frac{\Pi}{K} = P_K \pi = \frac{\tau}{\gamma} \tag{4}$$

with γ denoting the composition of capital:

$$\gamma = \frac{K}{W} = \frac{1}{P_K \omega}. \tag{5}$$

The major tendencies can be easily formulated with the above definitions and notation:

1. *Mechanization.* The use of increasing amounts, K, of fixed capital can be assessed in relation to output, Y, labor, H, or total wages, Hw. The relevant variables are, respectively, (1) the productivity of capital, $P_K = Y/K$, (2) the technical composition of capital, K/H, and (3) the composition of capital, $\gamma = K/Hw$. Increased mechanization refers to a diminished productivity of capital and a larger composition of capital. However, these various notions are not equivalent. The technical composition may rise while the composition of capital in equation (5) declines.

Instead of referring, as Marx does, to the rate of surplus-value and the composition of capital, τ and γ, we usually consider the share of wages and productivity of capital: ω and P_K. (Equations (2) and (5) show that these sets of variables are equivalent.) If the rate of surplus-value, τ (or ω or π) is constant, the decline of the productivity of capital is equivalent to the rise of the composition of capital (equation (5)), and Marx's statements concerning the rise of the composition of capital can be translated into *a declining productivity of capital*. Note that with $Y = KP_K$ and P_K declining, nothing ensures that Y rises. (The same is true of H.)

2. *Accumulation and growth.* Following Marx, we assume that a given share of profit is accumulated. Denoting this share as s, profits accumulated as ΔK, and the growth rate of the capital stock as $\rho(K)$, one has

$$\Delta K = s\Pi \quad or \quad \rho(K) = \frac{\Delta K}{K} = sr. \tag{6}$$

From $Y = KP_K$ and $H = KP_K/P_L$ (and equations (6) and (4)), one can derive the following relationships:

$$\rho(Y) = s(1-\omega)P_K + \rho(P_K), \quad and \tag{7}$$

$$\rho(H) = s(1-\omega)P_K + \rho(P_K) - \rho(P_L). \tag{8}$$

3. *Rising wages.* A final remark concerns the statement in section 1.1.2 that the simultaneous assumption of a constant rate of surplus-value (or constant ω) and a rising productivity of labor implies a rising real wage. This property follows directly from equation (3), which can be written as

$$\rho(\omega) = \rho(w) - \rho(P_L) \quad or \quad \rho(w) = \rho(\omega) + \rho(P_L). \tag{9}$$

If ω is constant ($\rho(\omega) = 0$), the growth rates of w and P_L are equal.

2.2 Exponential or power trajectories?

This section shows that the choice of a particular analytical form for trajectories has important implications concerning growth. Continuing growth of output is only compatible with trajectories in which the rate of decline of the profit rate gradually diminishes over time (as in a slowdown).

Along trajectories in which the profit share (or wage share, or rate of surplus-value) is constant, the rates of decline of the profit rate and of the productivity of capital are equal, and it is equivalent to consider either one of these variables. Below, trajectories *à la Marx* are studied for two alternative profiles of the *decline* of the productivity of capital and of the *rise* of the productivity of labor: (1) exponential or (2) power functions of time. (Appendix A.1 provides additional results and proofs.) As a preliminary to this investigation, section 2.2.1 defines a *trajectory à la Marx* and contrasts such a trajectory with a steady state (as in traditional growth models *à la Solow*).

2.2.1 Trajectories à la Marx and steady states

We call *trajectory à la Marx* a trajectory with the following properties:

1. *Technical change.* The rise of labor productivity is paralleled by a comparative rise of fixed capital, signaling a strong mechanization of production:

$$\rho(P_L) > 0 \qquad \rho(K/H) > 0 \qquad \rho(P_K) < 0 \qquad \rho(\gamma) > 0. \tag{10}$$

2. *Distribution.* Real wages rise while the share of profits and the rate of suplus-value are constant and the profit rate declines:

$$\rho(w) > 0 \qquad \rho(\omega) = \rho(\pi) = \rho(\tau) = 0 \qquad \rho(r) < 0. \tag{11}$$

3. *Growth.* Growth is observed at a positive but declining rate:

$$\begin{array}{lll} \rho(H) > 0 & \rho(K) > 0 & \rho(Y) > 0 \\ \dot{\rho}(H) < 0 & \dot{\rho}(K) < 0 & \dot{\rho}(Y) < 0. \end{array} \tag{12}$$

A more general definition could include a rising share of profits and rate of surplus-value. In this paper, the investigation is limited to trajectories in which a constant rate of surplus-value is obtained.

Such trajectories *à la Marx* differ from traditional steady states in the three following respects:

1. *Technical change.* The productivity of capital and composition of capital are not constant:

$$\rho(P_K) < 0 \qquad \rho(\gamma) > 0 \quad \text{ instead of } \quad \rho(P_K) = \rho(\gamma) = 0. \tag{13}$$

2. *Distribution.* The profit rate is not constant but declines:

$$\rho(r) < 0 \quad \text{ instead of } \quad \rho(r) = 0. \tag{14}$$

3. *Growth.* Growth rates are not constant but decline:

$$\dot{\rho}(H) < 0,\ \dot{\rho}(K) < 0,\ \dot{\rho}(Y) < 0 \quad \text{ instead of } \quad \dot{\rho}(H) = \dot{\rho}(K) = \dot{\rho}(Y) = 0. \tag{15}$$

2.2.2 Exponential trajectories

Consider the first assumption (that the productivity of capital and, hence, the profit rate decline exponentially) or, what is equivalent, that the growth rates of these variables are constant and negative:

$$P_K = ce^{-\alpha t}, \quad r = \pi c e^{-\alpha t} \quad \text{or} \quad \rho(P_K) = \rho(r) = -\alpha. \tag{16}$$

The productivity of labor follows a similar trajectory, but rises exponentially over time: $P_L = c' e^{-\beta t}$ or $\rho(P_L) = \beta$. From $\rho(K) = sr$, one can derive the capital stock (equation (33)). When t tends to infinity, the capital stock, K, rises but remains below an upper bound (equation (34)). It follows that capital accumulated, $\Delta K = K\rho(K)$, diminishes. Since $Y = KP_K$, the decline of the productivity of capital results in a similar decrease of output. Since $H = KP_K/P_L$, the decline of the productivity of capital and the rise of labor productivity provoke the decline of employment.[18] (These properties can be generalized and are independent from the assumption of constant π and s. If π and s are functions of time, one can still prove that the capital stock is bounded.)

These results show that, along a trajectory à la Marx, the profit rate must not decline too rapidly: An exponential decline of the profit rate results in a bounded rise of the stock of capital, as well as an asymptotic decline of output and employment. At some point, output and employment will decline, a feature which is not in line with Marx's analysis of historical tendencies as growth trajectories. Such trajectories are not *trajectories à la Marx*.

2.2.3 Power trajectories

We now consider the second assumption, that of a productivity of capital and profit rate proportional to a power function of time. (This type of trajectory is only adequate when the number of periods tends to infinity, but not when it is close to 0.) We assume

$$P_K = \frac{c}{t^\alpha}, \quad r = \frac{\pi c}{t^\alpha}, \quad \textit{or} \quad \rho(P_K) = \rho(r) = -\frac{\alpha}{t} \quad \textit{with} \quad \alpha > 0. \tag{17}$$

The productivity of labor follows a similar trajectory, but rises over time: $P_L = c' t^\beta$ or $\rho(P_L) = \beta/t$. With this model, the absolute values of the rates of variation of the two productivities decline with time. Thus, the economic meaning of the choice of this alternative framework is the assumption of a *slowdown* in the rates of variation of the two productivities.

The capital stock can be derived (equation (35)). If α is larger than 1, K is bounded (equation (36)) as in the previous section. Therefore, α must be larger than 0 and smaller or equal to 1 ($0 < \alpha \leq 1$) along a trajectory *à la Marx*. For α belonging to this interval[19], where a productivity of capital and a profit rate decline not too rapidly, K is not bounded. The growth rate of output can also be determined (equation (37)). With α smaller than 1, $\rho(Y)$ is positive when t tends to infinity. Thus, output Y, profits accumulated ΔK, and profits Π all increase. Using the assumption made on labor productivity, the growth rate of employment can also be determined (equation (38)). As in the case of $\rho(Y)$, this rate is positive when t tends to infinity. From the asymptotic growth rates of Y and H, one can derive their dominant components along an asymptotic trajectory (equation (39)).

Overall, the rise of labor productivity and the decline of the productivity of capital at sufficiently low rates as power functions of time and the assumption of a constant share of wages are consistent with all other features specific of trajectories *à la Marx*: (1) a declining profit rate; (2) rising capital, profits, profits accumulated, output, and employment, but at a declining growth rate.

---

18. Duncan Foley and T. Michl recognize this problem (1999). See their section 7.1 (in particular p. 120). Their consumption function, $C = (1-s)(K+\Pi)$, is not the same as ours, $C = (1-s)\Pi$. For this reason, the formal conditions differ, but the basic idea is identical.
19. The limit case $\alpha = 1$ requires a specific discussion (appendix A.1).

3 - Modeling technical change and wages

As shown in diagram 1, the modeling of historical trajectories à la Marx requires the joint treatment of accumulation, technical change, and wages. We make the following assumptions:

1. A given fraction of profits is invested.

2. Concerning technical change we will use our earlier model[20] (section 3.1). Innovation is treated as a random process. New techniques are selected depending on their profitability. Distribution influences the path of technical change because of this profitability criterion.

3. The wage rate grows with employment (section 3.2). (We do not assume a constant wage share.)

3.1 Modeling technical change

Below, we present the simplest possible form of the modeling of technical change. Only one good exists, and it is produced by a representative firm. At a given point in time, the production of one unit of this commodity requires a certain amount of itself, A, used as fixed capital, and a quantity of labor (also assumed homogeneous), L.[21] Thus, a technique is denoted (A, L). The ratio of output to either one of the inputs is the productivity of this input. The productivity of capital is $P_K = 1/A$, and labor productivity is $P_L = 1/L$.

A new technique, an "innovation," (A_+, L_+), appears in each period. It can be compared to the existing technique by the rates a and l of saving on each input:

$$A_+ = A/(1+a) \quad \textit{and} \quad L_+ = L/(1+l). \tag{18}$$

If the new technique is adopted, a and l are also the growth rates of the two productivities:

$$\rho(P_K) = a \quad \textit{and} \quad \rho(P_L) = l. \tag{19}$$

Technical change can be decomposed into two distinct steps:

1. Innovations result from R&D activities. We make the following assumptions: (1) the outcome of R&D is to a large extent unpredictable, and (2) new techniques are devised on the basis of the existing technology, which is modified only gradually (innovation is local). Innovation is conveniently described by the variables a and l. Thus, innovation is modeled as a random process, which follows a *probability distribution*, $\mu(a, l)$, whose support is bounded and denoted as the *innovation set* (diagram 2). Maintaining the existing technique is always a possibility, therefore the origin belongs to the innovation set.

2. The criterion used in the decision to adopt a new technique is its ability to yield a larger profit rate at prevailing prices (including the wage rate). The *viability frontier* separates the adopted ($r_+ > r$) from the rejected techniques ($r_+ < r$). As shown in diagram 2, it is a downward sloping line crossing the origin. We denote Π as the *profitable innovation set*,

20. DUMÉNIL G., LÉVY D. 1995; 1998.
21. Returns are constant: $K = AY$ and $H = LY$.

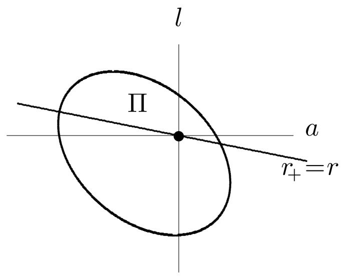

Diagram 2

the subset of the innovation set which lies above this frontier. Only innovations falling in this region are selected.

A single relative price is required in this model in which only one good is considered: the unit wage deflated by the price of the good ("the wage" for short), w. The profit rates of the existing and new techniques are

$$r = \frac{1 - Lw}{A} \quad \textit{and} \quad r_+ = \frac{1 - L_+ w}{A_+}. \tag{20}$$

If the innovation set is small, the profit rate, r_+, of the new technique can be developed linearly in the vicinity of the prevailing profit rate r, and the equation for the viability frontier, $r_+ = r$, is

$$\tau a + l = 0 \quad \textit{or} \quad (1-\omega)a + \omega l = 0. \tag{21}$$

This framework defines a dynamic model that determines the technique in any period from the technique prevailing in the previous period using the two stochastic variables a and l (equation (18)). Assuming a given probability distribution (thus, a given innovation set), the wage w, or the share of wages ω, is the only exogenous variable. More generally, beginning with a technique (A_0, L_0), one can derive a sequence of techniques A_t, L_t (with $t = 1, 2, \ldots$) from a given sequence of wages w_t or share of wages ω_t (with $t = 0, 1, 2, \ldots$). We denote such a sequence as a *technical trajectory*. From the point of view of the formalism, a *stochastic dynamic model* is obtained[22].

A deterministic approximation of this model can be defined by substituting the average value of innovations, $(\overline{a}, \overline{l})$, for their stochastic values (a, l) in equation (18)[23]. The "average" features of the trajectories of the stochastic model can be easily studied using this *deterministic approximation*.[24]

22. This model can be alternatively expressed in discrete or continuous time.
23. One has

$$\overline{a} = \iint_{\Pi} a \, d\mu(a,l) \quad \textit{and} \quad \overline{l} = \iint_{\Pi} l \, d\mu(a,l).$$

in which the integrals are limited to selected innovations (to the profitable innovation set Π).
24. Although the basic framework is different, there are significant formal convergences between the *deterministic approximation* of our model and the *innovation possibility frontier* (IPF) of KENNEDY C. 1964; DRANDAKIS E.M., PHELPS E.S. 1966; and Von WEIZSACKER C.C. 1966. The IPF is similar to our set of centers of gravity $(\overline{a}, \overline{l})$, for $0 \leq \omega \leq 1$. There are also a number of common aspects concerning selection criteria. In Kennedy's model, an entrepreneur chooses "an improvement that reduces his total unit cost in the greatest proportion" [p. 543]. This is equivalent to the selection of the innovation providing the best possible profit rate. There are also

For a given innovation set, $\overline{a}$ and $\overline{l}$ only depend on the slope of the viability frontier or, what is equivalent, on ω: $\overline{a} = \overline{a}(\omega)$ and $\overline{l} = \overline{l}(\omega)$. An increase of ω rotates the viability frontier in a counterclockwise fashion. Consequently, $\overline{a}$ is a decreasing function of ω, and $\overline{l}$, an increasing function of ω. (Other properties of $\overline{a}$ and $\overline{l}$ are presented at the beginning of appendix A.3.) The growth rates of the two productivities (equation (19)) can be expressed as functions of the wage share:

$$\rho(P_K) = \overline{a}(\omega) \quad \textit{and} \quad \rho(P_L) = \overline{l}(\omega). \tag{22}$$

3.2 The determination of the wage

As recalled in section 1.2.4, the effect of labor market conditions on the wage is a crucial component of Marx's analysis (in combination with other mechanisms which are not treated explicitly in the present analysis). This dependency is a rather common view within economics. However, the many suggested models are not equivalent. Before presenting our own framework, we briefly recall three such alternative models used in traditional growth theories:

1. In neoclassical growth models, as in all Walrasian models, wages, just as all prices, are determined to clear markets *ex ante*.

2. In a growth model *à la Kaldor*[25], without technical change, the situation of distribution (the wage rate) is determined in order to equalize the growth rate of employment to an exogenous growth rate of the labor force. (Accumulation is proportional to profits and makes the link between distribution and growth.)

3. Goodwin's model[26] is closer to Marx's analytical framework. A variable, v, is introduced to denote the ratio between employment H and the exogenous available labor force N: $v = H/N$. (The rate of unemployment is $1 - v$.) The wage equation is

$$\rho(w) = \delta_0 + \delta_1 v. \tag{23}$$

We will not discuss here the theoretical foundations of these three models. The following technical comments must, however, be made. The assumption of an exogenous growth rate n of population N is typical of such models: $N = N_0\, exp(nt)$. The trajectories of neoclassical growth models and of Kaldor's model are standard *exponential* growth trajectories, *with a constant profit rate and constant growth rates* (a steady state). The same is true of the Goodwin model with the exception that the variables fluctuate around such trajectories. The modeling of the impact of employment on the wage in these models cannot be generalized to models accounting for trajectories *à la Marx* with a declining profit rate and declining growth rates. It is impossible to combine a population growing exponentially with a growth trajectory in which employment can only grow at a rate declining over time as along a power trajectory (equations (38) and (39), or equation (40)).

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important divergences. The purpose of the model of Kennedy and that of Drandakis and Phelps is not to build an *alternative* framework to the production function but a model of technical change *compatible* with the production function. The structure of one of Foley's models, the "neoclassical model" with endogenous technical change, is close to that of Drandakis and Phelps (FOLEY D. 2000). We thank Howard Petith for drawing our attention to these models.

25. KALDOR N. 1957.

26. GOODWIN R.M. 1967.

We do not make the assumption of an exogenous labor force, but use a model in which *the growth of the wage is a function of the growth of employment*. As in the above models, such a model does not provide a detailed analysis of the labor market — and does not make explicit the combination between market forces and class struggle — but it confirms rather satisfactorily our views of the relationship between the labor force and employment in both the short and long runs:

1. The labor force is *not* exogenous in the long run. The proletarianization of small producers and small capitalists, immigration, the involvement of children and women in the labor force, and the variations of the duration of labor are, in particular, crucial endogenous mechanisms in the slow adjustment of the labor force to the requirements of accumulation in the long run.
2. In the short run (along business-cycle fluctuations), employment periodically reaches the limits of the available labor force (Marx's overaccumulation), and the reserve army is periodically reproduced during crises, as also analyzed by Marx (Marx's law of accumulation). It is only in the analysis of these recurrent shortages of labor that the reference to a limited and given labor force is relevant.

Thus, wages tend to rise during periods of overaccumulation and to decline or stagnate during crises. Long-term adjustments of the labor force contribute to the limitation of these recurrent impacts of employment on the wage.

Our modeling of wages assumes that, despite these short and long-term relaxations, *the growth of employment contributes to the growth of the wage* in the long run. This dependency will be modeled in the wage equation by a term $\delta\rho(H)$, where parameter δ accounts for the reactiveness of the wage to the situation of the labor market.[27]

4 - Modeling trajectories à la Marx

We build two models (sections 4.1 and 4.2, and appendices A.2, A.3, and A.4). In the first model, the conditions of innovation are constant. The trajectories either display a constant or declining profit rate. In this latter case, the profit rate declines exponentially, and growth is not ensured. In the second model, we assume a gradual deterioration of the conditions of innovation (as a power function of time). This "productivity slowdown" allows for *growth* trajectories à la Marx (growth at a declining rate) with a declining profit rate. These models suggest an interpretation of Marx's insight concerning the rise of the composition of capital (section 4.3). The main results are summarized in section 4.4.

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27. In the German edition of Volume I of *Capital*, one can find the following statement: "To use a mathematical language, the size of accumulation [*which governs employment*] is the independent variable, the level of wages being the dependent variable, and not the inverse," MARX K. 1969, p. 648. An example of such a model of the determination of wages in relation to accumulation can be found in THOMPSON F. 1998.

4.1 First model: Given conditions of innovation

We first assume that the probability distribution of innovation is given, using the following wage equation:

$$\rho(w) = \rho_w + \delta\rho(H). \tag{24}$$

Under certain conditions, this model allows for a number of features of trajectories à la Marx, notably a declining profit rate, but these trajectories are not *growth trajectories*, since output and employment diminish asymtotically.

From equations (8) and (22), it follows that

$$\rho(H) = s(1-\omega)P_K + \overline{a}(\omega) - \overline{l}(\omega). \tag{25}$$

Using equation (9), one finally obtains a dynamic system with two variables, ω and P_K:

$$\begin{cases} \rho(\omega) = \delta s(1-\omega)P_K - f(\omega) + \rho_w \\ \rho(P_K) = \overline{a}(\omega) \end{cases} \tag{26}$$

with $f(\omega) = (1+\delta)\overline{l}(\omega) - \delta\overline{a}(\omega)$. As stated in section 2.1, the two variables, ω and P_K, stand in a straightforward manner for Marx's rate of surplus-value and composition of capital.

Equilibria and their stability are studied in appendix A.3. Under condition (47), the model displays two types of stable equilibria depending on the value of the two parameters, ρ_w and δ, of the wage equation (equation (24)). It is first possible to obtain steady states with a constant profit rate. A second possibility is a trajectory with a profit rate declining exponentially.[28]

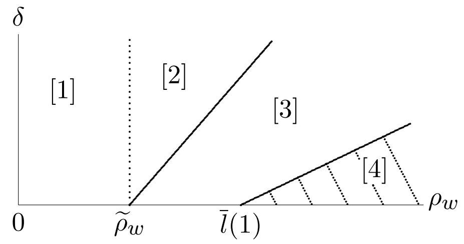

Diagram 3

The discussion is illustrated in diagram 3. (Parameter $\widetilde{\rho}_w$ is defined in equation (48).) Four zones are distinguished in this diagram:

1. For values of ρ_w and δ corresponding to zones [1] and [2], trajectories with a constant profit rate are obtained. The equilibrium wage share, $\omega^\star$, is equal to $\widetilde{\omega}$ (equation (46)) and, thus, is independent from ρ_w and δ. Conversely, the constant equilibrium profit rate depends on these parameters:

$$r^\star = \frac{\widetilde{\rho}_w(1+\delta) - \rho_w}{\delta s}. \tag{27}$$

28. Foley presents a model similar to that used in this section (2000). The modeling of technical change is the same, and the wage is assumed to depend on unemployment. Foley only obtains steady states with a constant profit rate.

In zone [1], employment is increasing. In zone [2], it diminishes over time.

2. For values of ρ_w and δ corresponding to zone [3], the equilibrium value of the wage share is defined by $f(\omega^\star) = \rho_w$. The growth rate of the profit rate is negative: $\rho(r) = \rho(P_K) = \overline{a}(\omega^\star) < 0$. Employment declines.

3. For values of ρ_w and δ corresponding to zone [4], there is no equilibrium.

In all instances, larger values of ρ_w have a negative impact on the profit rate. In zones [1] and [2], larger values of ρ_w are associated with smaller (constant) equilibrium profit rates $r^\star$, and in zone [3], with smaller rates of growth of r (larger rates of decline of r). Larger values of δ have a negative impact on the profit rate if the growth rate of employment is positive, as in zone [1]. In cases in which employment declines (in zones [2] and [3]), larger values of δ have a positive impact on r. Either, the equilibrium profit rate is higher, as in zone [2], or the profit rate declines less rapidly, as in zone [3].

Equilibrium, when it exists, is always stable. This property is rather intuitive as can be understood from the first of equations (26):

1. $\delta s(1-\omega)P_K$. A large wage share diminishes the profit rate and, consequently, accumulation. A slower accumulation diminishes the growth rate of employment, thus of the wage and, finally, the wage share. This effect is reminiscent of Goodwin's model.

2. $-f(\omega)$, with $f(\omega) = \overline{l}(\omega) + \delta(\overline{l}(\omega) - \overline{a}(\omega))$. This term accounts for mechanisms deriving from technical change. The profit share impacts on technical change that, in turn, impacts on the profit share. Two such mechanisms can be identified:

- $-\overline{l}(\omega)$. A large wage share stimulates the rise of labor productivity that, in turn, diminishes the wage share (equation (9)).
- $-\delta(\overline{l}(\omega) - \overline{a}(\omega))$, in which $\overline{l}(\omega) - \overline{a}(\omega)$ is the growth rate of the capital-labor ratio. A large wage share induces the substitution of capital for labor, or rather a decline of employment for a given capital stock, and diminishes the growth rate of employment, with consequences similar to those described for $\delta s(1-\omega)P_K$.

Independently of the cross dynamic $\omega \to P_K \to \omega$, these mechanisms would be sufficient to prove stability.

4.2 Second model: The deterioration of the conditions of innovation

Growth trajectories à la Marx are obtained under the assumption of the variation of the profit rate and productivities as power functions of time, *a productivity slowdown*, or what is equivalent (section 2.2):

$$\rho(P_K) = \rho(r) = -\frac{\alpha}{t} \quad \text{and} \quad \rho(P_L) = \frac{\beta}{t}. \tag{28}$$

This section shows that such trajectories follow from the gradual reduction over time of the innovation set: a homothetical reduction centered in the origin at a constant rate $1/t$ of all parameters defining the innovation set.[29] Within the deterministic approximation of the model, the growth rates of the two productivities decline at a rate $1/t$:

$$\rho(P_K) = \frac{\overline{a}(\omega)}{t} \quad \text{and} \quad \rho(P_L) = \frac{\overline{l}(\omega)}{t}. \tag{29}$$

29. As stated in section 2.2, this type of trajectory is only adapted when the number of periods tends to infinity, but not when it is close to 0.

For wages, we use the following equation[30]:

$$\rho(w) = \delta\rho(H). \tag{30}$$

The growth rate of employment can be derived from equations (8) and (29):

$$\rho(H) = s(1-\omega)P_K + \frac{\overline{a}(\omega) - \overline{l}(\omega)}{t}. \tag{31}$$

Using equations (9) and (30), and again with $f(\omega) = (1+\delta)\overline{l}(\omega) - \delta\overline{a}(\omega)$, a dynamic system with two variables, ω and P_K, is obtained:

$$\begin{cases} \rho(\omega) = \delta s(1-\omega)P_K - \dfrac{f(\omega)}{t} \\ \rho(P_K) = \dfrac{\overline{a}(\omega)}{t}. \end{cases} \tag{32}$$

Equilibria and their stability are studied in appendix A.4. The main results can be summarized as follows:

1. Growth trajectories *à la Marx* are obtained, including the growth of the capital stock, profits accumulated, employment, profits, and output.

2. These trajectories *à la Marx* are stable, that is, if as a result of a shock, the variables deviate from such a trajectory, they tend to return to it.

3. Neither the existence of equilibrium nor its stability depend on parameter δ in the wage equation (30), provided that δ is strictly positive. The equilibrium share of wages $\omega^\star$ only depends on the innovation set, not on parameter δ. Since employment rises, larger values of δ are associated with lower trajectories of the profit rate.

4.3 The roots of the tendency for the profit rate to fall

The two previous models account to different extents for basic features of trajectories à la Marx, in particular the tendency for the profit rate to fall. These models may contribute to the interpretation of Marx's emphasis, in his analysis of the historical tendencies of capitalism, on the rise of the composition of capital.

At the beginning of chapter 13 of Volume III of *Capital*, Marx presents the rise of the composition of capital in combination with a constant rate of surplus-value as the *cause* of the tendency for the profit rate to fall. As stated in section 1.1.2, Marx is, however, not very clear concerning the origin of the rise of the composition of capital. We interpret Marx's analysis of the tendency for the profit rate to fall as a statement concerning the features of innovation. According to Marx, innovation displays certain features such that the profit rate will tend to decline, even if the rate of exploitation is constant.

We must therefore confront two questions: (1) What are these features? (2) Why do such features prevail, in particular within capitalism? Formally, the problem is the sign

30. Using equation (24) with $\rho_w \neq 0$, the model would have no equilibrium. As could be expected, trajectories where all growth rates tend toward zero are not compatible with the inclusion of a constant in the equation for the growth rate of the wage. An alternative, more complex, model of the wage would be

$$\rho(w) = \delta'\rho(w)_{t-1} + \delta\rho(H) \quad \textit{with} \quad 0 \leq \delta' < 1.$$

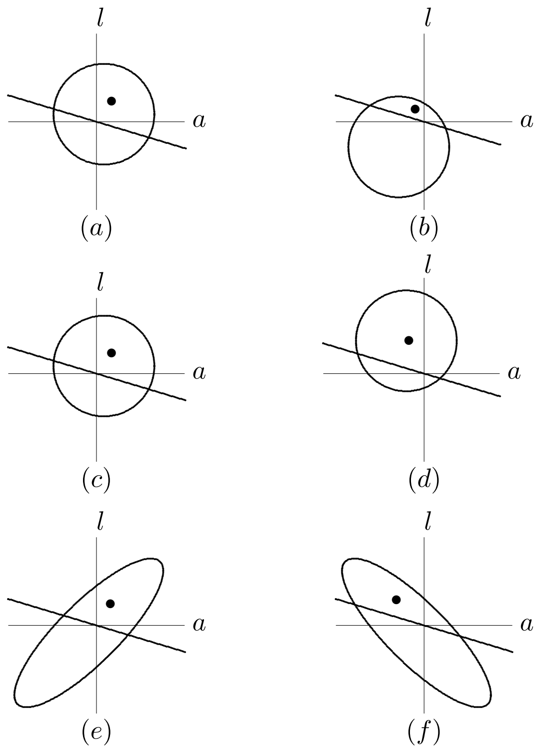

Diagram 4

of $\overline{a}(\omega^\star)$, *which must be strictly negative.* (In the first model, the profit rate declines if $\overline{a}(\omega^\star) < 0$; in the second model, one has: $\overline{a}(\omega^\star) = -1$, *cf.* equation (60).) The crucial issue is, in our opinion, the *difficulty to innovate.*

Diagram 4 illustrates three types of properties of innovation. We first abstract from the viability frontier and the position of the center of gravity of the profitable innovation set. Thus, the following comments can be made concerning the position and shape of the innovation set:

1. Panels (a) and (b) show how the *difficulty* to innovate can be expressed in the model. In these two diagrams the radius of the circle is the same, and the two centers are located on the first bisector. In panel (b), finding profitable innovations is more difficult in comparison to the situation in panel (a). It is the location of the center, its distance from the origin, that accounts for the difficulty to innovate.

2. Panels (c) and (d) elaborate on the notion of *bias* often underlying Marx's analysis. In panel (c), the circle is centered on the first bisector, and innovations economizing on each input are equally likely. There is, therefore, no *a priori* bias. The converse is true of panel (d), where the circle has been shifted toward the upper left-hand side. Consequently, the probability of finding labor-saving capital-consuming innovations ($l > 0$ and $a < 0$) is larger.

3. Panels (e) and (f) describe two distinct patterns of saving on the two inputs when

innovations occur. The circle has been replaced by an ellipse. In panel (e), the use of the two inputs tends to vary in the same direction, both increased or reduced. In panel (f), the use of one input tends to increase while the use of the other tends to diminish.[31] The pattern in panel (e) matches, for example, the complementary features of structures and labor (like an office environment), while panel (f) may correspond to the case of equipment and labor. In this latter case, it is difficult to save simultaneously on the two inputs (to diminish simultaneously $\overline{a}$ and $\overline{l}$), and this pattern is conducive to declining profit rates.

Consider now the viability frontier and the center of gravity ($\bullet$) of the innovation set. It is easy to identify visually on these panels the cases corresponding to a falling profit rate. The element $\overline{a}(\omega^\star)$ is proportional to the coordinate of the center of gravity on the horizontal axis. Whenever it is negative (when the center of gravity is located to the left of the vertical axis) the profit rate falls along an asymptotic trajectory. It is clear that this configuration is observed for each panel in the right column.

The configuration described in panel (d) is quite reflective of Marx's insight concerning the composition of capital. Innovations can be found which diminish the productivity of capital (signaling heavy mechanization); other cases are possible, but rare. The configuration in panel (b) is interesting, since it signals that this propensity of innovations to display characteristics *à la Marx* can be the expression of the *difficulty* to find profitable innovations in general, independently of any *a priori* bias.[32]

4.4 A summing up

The two models presented in this section elaborate on how to model technical change and wages in section 3. The basic difference between these two models concerns the assumptions made on the conditions of innovation. These conditions are assumed constant in the first model. In the second model, the conditions are assumed to "degenerate" over time, as in a productivity slowdown. This difference implies that the wage equation must be adjusted. In both models, however, wages respond positively to the growth of employment.

Both models show the possibility of the reproduction of trajectories with a declining profit rate, in relation to specific features of innovation that we denote as the "difficulty to innovate." On average not many innovations allow for the simultaneous rise of labor productivity and the productivity of capital. Put differently, the rise of labor productivity is, on average, obtained at the cost of a decline of the productivity of capital. Another result is that the ensuing decline of the profit rate is associated with a constant share of profits.

The results described above depend on the model in two respects:

1. In both models, the growth of employment stimulates the rise of wages. In the first model, the decline in the profit rate requires a certain degree of variation of wages (as in zone [3] in diagram 3), whereas in the second model, a positive response is sufficient. Thus, in the second model, the condition placed on wages dynamics is very weak.

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31. All techniques in this model are represented by fixed coefficients. The patterns of variation described in panels (e) and (f) are, however, evocative of the notions of complementary and substituable factors.

32. Note that what is at issue concerning the falling profit rate is *process innovation*, not *product innovation*. Product innovation is not a countertendency to the falling profit rate. *A priori* a new product can be produced by any kind of technique, with a low or large composition of capital.

2. Only the second model allows for growth trajectory *à la Marx*, asymptotically combining a positive accumulation at a diminishing rate and a declining profit rate.

In our opinion, the second model represents an appropriate formal synthesis of Marx's analysis of historical tendencies. The primary issue is the difficulty to innovate: (1) it is a general feature of technical change within capitalism, and (2) the effectiveness of innovation tends to decline with time.[33] In addition: (1) the falling profit rate is compatible with growth, (2) it is associated with a constant share of wages (so that the falling profit rate cannot be interpreted as a wages squeeze), and (3) the dynamics of wages are implied in a very weak manner. Last, the stability of the trajectories provides precise meaning to the notions of *tendency* or *law*.

Appendices

A.1 Capital, output, and employment along various trajectories

We begin with exponential trajectories. From $\rho(K) = s\pi ce^{-\alpha t}$, one can derive

$$K = K_0\, exp\left(-\frac{C}{\alpha}(exp(-\alpha t) - 1)\right) \quad with \quad C = s\pi c. \tag{33}$$

When t tends to infinity, the capital stock, K, rises but remains below an upper bound:

$$K < K_0\, exp\left(\frac{C}{\alpha}\right). \tag{34}$$

If π and s are functions of time, the growth rate of the capital stock is smaller than $ce^{-\alpha t}$, and the capital stock is bounded: $K < K_0\, exp\left(\frac{c}{\alpha}\right)$.

Under the assumption of π and s constant, we now consider a decline of the productivity of capital proportionally to a power of time. The capital stock can be easily derived from $\rho(K) = C/t^\alpha$:

$$K = K_0\, exp\left(C\frac{t^{1-\alpha} - t_0^{1-\alpha}}{1-\alpha}\right). \tag{35}$$

If α is larger than 1, K is bounded:

$$K < K_0\, exp\left(\frac{C}{(\alpha - 1)t_0^{\alpha-1}}\right). \tag{36}$$

For $0 < \alpha < 1$, K is not bounded. The growth rates of output and employment can be determined using equations (7) and (8) (assuming that P_L also grows as a power of time,

33. This analysis introduces the notion of possible transformations of relations of production allowing for the recurrent restorations of these conditions, along the phases of capitalism (DUMÉNIL G., LÉVY D. 2001).

$P_L = c' t^\beta$):

$$\rho(Y) = \frac{C}{t^\alpha} - \frac{\alpha}{t} \tag{37}$$

$$\rho(H) = \frac{C}{t^\alpha} - \frac{\alpha}{t} - \frac{\beta}{t}. \tag{38}$$

With α smaller than 1, the first term is dominant when t tends to infinity and, therefore, $\rho(Y)$ and $\rho(H)$ are positive. From the asymptotic growth rates of Y and H, one can derive the dominant component (obtained from the dominant term of the growth rate) of Y and H along an asymptotic trajectory. For example, for H,

$$H = H_0 \, exp \left(C \frac{t^{1-\alpha} - t_0^{1-\alpha}}{1-\alpha} \right). \tag{39}$$

Y and H rise, and are not bounded. (The same is true of $\Delta K = s\pi Y$.)

In the limit case $\alpha = 1$, the expressions of K, Y, and H are different:

$$K = K_0 t^C, \quad Y = Y_0 t^{C-1}, \quad \textit{and} \quad H = H_0 t^{C-1-\beta}. \tag{40}$$

The capital stock, K, is never bounded, output, Y, rises if $C > 1$, and employment, H, rises if $C > 1 + \beta$.

A.2 Trajectories à la Marx and stable equilibria of a dynamic system

In this paper, trajectories à la Marx have been interpreted as stable equilibria of dynamic systems. This analytic framework appears very elegant, since it allows the presentation of a historical tendency as the outcome of a stable dynamic process. If the existence and stability of the equilibrium can be established, it means that forces are at work, pushing the economy toward such trajectories. It therefore is very relevant in such models to refer to *tendencies* or *laws*. The following remarks can be made:

1. *Asymptotic trajectories.* We interpret trajectories à la Marx as *equilibria* entirely defined by the parameters of the model, and not as preasymptotic trajectories (convergence trajectories) which also depend on the initial values of the variables. In particular we do not build a model in which the profit rate would converge toward a constant value, beginning arbitrarily with a profit rate larger than its equilibrium value.

2. *Fixed points of a dynamic model.* In this paper, equilibrium refers to the fixed point of a dynamic model, a mathematical notion, susceptible to many applications in economics. It is common in growth models to call *equilibrium* a situation in which a variable changes while its growth rate is constant. More generally, if it is possible to find a transformation of the variables and the time scale such that the dynamic system that results from this transformation has a fixed point, then the path corresponding to the inverse of the transformation of the fixed point can be called equilibrium.

3. Since historical tendencies are at issues, these equilibria could be defined as *long-term equilibria*. Two such equilibria are considered in this paper, steady states and trajectories à la Marx[34]:

34. The distinction between a traditional steady state and what we call a trajectory *à la Marx* is also discussed in FOLEY D., MICHL T. 1999: "The key difference is that with biased technical change, the economy never reaches a steady state because the rate of profit changes over time, generating changes in the rate of capital accumulation and growth" [p. 117-118].

- By *steady state*, we mean an equilibrium in which the profit rate, the wage share, the *growth rates* of capital, output, and employment are constant. (The variables, capital, output..., grow exponentially.)
- By *trajectory à la Marx*, we mean an equilibrium in which the profit rate and the *growth rates* of capital, output, and labor diminish. Concerning the wage share, this paper only considers equilibria in which it is constant. It is also possible to build models in which the wage share tends toward zero (the rate of surplus-value tends toward infinity).[35]

4. *The convergence of growth rates.* This paper shows that the growth rates of the productivity of capital P_K and the profit rate r converge toward constant values. In the points below, (a) explains how this demonstration is technically made, (b) makes the terminology more explicit, and (c) signals a technical difficulty that is not treated in the paper.

(a) Consider the first model. The dynamic system in equations (26), with two variables ω and P_K has the following general form:

$$\begin{cases} \dot{\omega} = f(\omega, P_K) \\ \dot{P}_K = g(\omega, P_K). \end{cases} \tag{41}$$

We first study the equilibrium $(\omega^\star, P_K^\star)$ of this system, and then its stability. To these two variables, one can add the growth rate of the productivity of capital: $y = \rho(P_K)$. This third variable is a function of the wage share (second of equations (26)):

$$y = h(\omega). \tag{42}$$

The function $h(\omega)$ being continuous, the convergence of y toward $y^\star = h(\omega^\star)$ follows from that of ω toward $\omega^\star$. The same procedure is used for the second model.

(b) The equilibrium of the model possesses the following properties:

$$P_K^\star = 0 \quad \textit{and} \quad \rho(P_K)^\star = y^\star = h(\omega^\star) < 0. \tag{43}$$

The most appropriate expression to describe this equilibrium is "the equilibrium value of the growth rate of P_K is negative." We sometimes use other formulations, economically more straightforward, such as: "asymptotically, P_K diminishes exponentially toward 0," "P_K converges toward a declining exponential trajectory," or even "at equilibrium, P_K declines exponentially." These formulations always mean: $\rho(P_K)^\star < 0$.

(c) The two formulations "P_K converges toward an exponential trajectory" and "the growth rate of P_K converges toward a constant" are not strictly equivalent. Consider the following example:

$$P_K = P_{K0} e^{\alpha t + 2\beta\sqrt{t}} \quad \text{and} \quad \rho(P_K) = \alpha + \frac{\beta}{\sqrt{t}}. \tag{44}$$

35. Michl (1999) obtains trajectories with a declining profit rate and a rising rate of surplus-value. In his model, the growth rates of labor productivity and capital productivity, that we denote $\overline{l}$ and $\overline{a}$, are assumed constant, and positive and negative respectively. Thus, they do not respond to the variations of wages. In our model, $\overline{l}$ and $\overline{a}$ are functions of the wage share, and the tendency for the rate of surplus-value to rise (the decline of the share of wages toward 0) results in a vertical viability frontier. In this situation, $\overline{a}$ (with $\overline{a} = \rho(P_K) = \rho(r)$) is positive, and the profit rate necessarily rises asymptotically.

If $t \to \infty$, $\rho(P_K) \to \alpha$: the growth rate of P_K tends toward a constant. But P_K *does not* converge toward an exponential: $P_K/e^{\alpha t}$ does not converge toward a constant. This paper proves the convergence of $\rho(P_K)$ toward a constant $h(\omega^\star)$ but not the convergence of $z = P_K/\,exp(h(\omega^\star)t)$ toward a constant γ. It would be interesting to compare the speed of the convergence of z toward γ (the convergence of P_K toward an exponentially declining trajectory) and the speed of the convergence of $exp\,(h(\omega^\star)t)$ toward 0 (the convergence of this exponential trajectory toward 0).

5. By *stability*, we always mean local stability. Independently of the difficulty of the mathematical treatment of global stability, we consider that capitalist economies are not globally stable (only limited shocks can be corrected autonomously).

A.3 First model: Equilibrium and stability

The following properties of $\overline{a}(\omega)$ and $\overline{l}(\omega)$ can be easily derived geometrically:

1. $\overline{a}(\omega)$ and $\overline{l}(\omega)$ are respectively declining and increasing functions of ω. Thus, $f(\omega) = (1+\delta)\overline{l}(\omega) - \delta\overline{a}(\omega)$ is an increasing function.

2. $\overline{a}(0)$ and $\overline{l}(1)$ are positive.

3. For a same value of ω, $\overline{a}(\omega)$ and $\overline{l}(\omega)$ cannot be simultaneously negative. Consequently, if $\overline{a}(\omega)$ is negative, $\overline{a}(\omega) - \overline{l}(\omega)$ is also negative.

With the notation $\dot{x}$ for the derivative of x, the dynamic system of equations (26) can be written as follows in order to study the equilibria and their stability:

$$\begin{cases} \dot{\omega} = \omega(\delta s(1-\omega)P_K - f(\omega) + \rho_w) \\ \dot{P}_K = P_K\overline{a}(\omega). \end{cases} \tag{45}$$

We define two parameters $\widetilde{\omega}$ and $\widetilde{\rho}_w$, $\widetilde{\omega}$ being defined by:

$$\overline{a}(\widetilde{\omega}) = 0. \tag{46}$$

Since $\overline{a}(0) > 0$ and $\overline{a}(\omega)$ is a decreasing function, a necessary and sufficient condition for the existence of $\widetilde{\omega}$ is[36]

$$\overline{a}(1) < 0. \tag{47}$$

This is an assumption concerning the features of innovation which can be interpreted in the framework of the discussion in section 4.3. When the share of wages is equal to 1, the viability frontier is horizontal. The profitable innovation set corresponds to the fraction of the innovation set located above the horizontal axis. Thus, condition (47) means that the center of gravity of this zone must be located to the left of the vertical axis. This configuration is consistent with the general notion of the *difficulty to innovate*. Obviously, it is observed in the three cases displayed on the right column of diagram 4.

We assume that this condition holds, and $\widetilde{\rho}_w$ is defined by

$$\widetilde{\rho}_w = \overline{l}(\widetilde{\omega}). \tag{48}$$

36. When condition (47) is not satisfied, equilibrium trajectories, if they exist, correspond to a constant profit rate, a share of wages equal to 1, and a productivity of capital tending toward infinity. The mathematical study of these trajectories is difficult, and they appear economically less relevant.

Since $\bar{l}(\omega)$ is an increasing function, and using the third property above, one has

$$0 < \widetilde{\rho}_w < \bar{l}(1). \tag{49}$$

Two types of equilibria are distinguished:

1. *With constant profit rate and productivity of capital.*

The equilibrium value of the wage share, $\omega^\star$, is given by $\rho(P_K) = \overline{a}(\omega^\star) = 0$. Thus, $\omega^\star$ is equal to $\widetilde{\omega}$ (equation (46)). The equilibrium value of the productivity of capital can be derived from $\rho(\omega) = 0$. One obtains

$$P_K^\star = \frac{f(\widetilde{\omega}) - \rho_w}{\delta s(1-\widetilde{\omega})} = \frac{(1+\delta)\widetilde{\rho}_w - \rho_w}{\delta s(1-\widetilde{\omega})} \quad \text{and} \quad r^\star = (1-\widetilde{\omega})P_K^\star. \tag{50}$$

The equilibrium productivity of capital $P_K^\star$ must be positive:

$$\rho_w < (1+\delta)\widetilde{\rho}_w. \tag{51}$$

In an equilibrium, one has $\rho(H) = sr^\star - \widetilde{\rho}_w = \frac{\widetilde{\rho}_w - \rho_w}{\delta}$. Thus, the growth or decline of employment depends on the comparative values of ρ_w and $\widetilde{\rho}_w$. Employment grows if ρ_w is smaller than $\widetilde{\rho}_w$, and declines if ρ_w is larger than $\widetilde{\rho}_w$.

2. *With a profit rate declining exponentially.*

In such an equilibrium, the productivity of capital is equal to zero. (This is equivalent to saying that, if equilibrium is stable, the productivity of capital tends asymptotically toward zero.) The condition $\rho(\omega) = 0$ allows for the determination of $\omega^\star$ from the equation:

$$f(\omega^\star) = \rho_w. \tag{52}$$

Thus, one has $\rho(P_K) = \overline{a}(\omega^\star)$. Equilibrium exists if: (1) $\overline{a}(\omega^\star) < 0$ (the productivity of capital actually declines exponentially), that is if $\widetilde{\omega} < \omega^\star$, and (2) $\omega^\star < 1$ (the equilibrium share of profits is positive). Since $f(\omega)$ is an increasing function of its argument, the double condition $\widetilde{\omega} < \omega^\star < 1$ is satisfied under the condition

$$f(\widetilde{\omega}) < \rho_w < f(1) \quad \text{or} \quad (1+\delta)\widetilde{\rho}_w < \rho_w < (1+\delta)\bar{l}(1) - \delta\overline{a}(1). \tag{53}$$

This defines the boundaries of zone [3]. One has $\rho(H) = \overline{a}(\omega^\star) - \bar{l}(\omega^\star) < 0$ (employment declines exponentially).

The results can be summarized as follows:

- For $0 \le \rho_w < \widetilde{\rho}_w$, in zone [1], $r^\star$ is constant and H rises.
- For $\widetilde{\rho}_w < \rho_w < f(\widetilde{\omega})$, in zone [2], $r^\star$ is constant and H declines.
- For $f(\widetilde{\omega}) < \rho_w < f(1)$, in zone [3], both the profit rate and employment decline exponentially.
- For $f(1) < \rho_w$, in zone [4], no equilibrium exists.

In diagram 3, the continuity is not always ensured when δ tends toward zero. There is no problem in zone [3], including the frontier $\delta = 0$, where the profit rate tends toward zero. The same is true in zone [4] and its frontier, where no equilibrium exists. Conversely, in zone [1], the general outcome of horizontal trajectories of the profit rate (with $\omega = \widetilde{\omega}$) does not hold for $\delta = 0$. In this case, for ρ_w larger than $\bar{l}(0)$, the profit rate follows a rising exponential trajectory with a constant share of profits, defined by $\bar{l}(\omega^\star) = \rho_w$.

We now study the stability of these two types of equilibria:

1. For the equilibrium with a constant profit rate $r^\star$ (zones [1] and [2] in diagram 3), the Jacobian matrix is

$$\begin{pmatrix} -\widetilde{\omega}(\delta s P_K^\star + f'(\widetilde{\omega})) & \delta s\widetilde{\omega}(1-\widetilde{\omega}) \\ \overline{a}'(\widetilde{\omega})P_K^\star & 0 \end{pmatrix}. \tag{54}$$

Thus, the signs of the elements of the Jacobian matrix are, schematically:

$$\begin{pmatrix} <0 & >0 \\ <0 & =0 \end{pmatrix} \tag{55}$$

and the equilibrium is stable.

2. For the equilibrium with a declining profit rate (zone [3]), the Jacobian matrix is:

$$\begin{pmatrix} -\omega^\star f'(\omega^\star) & \delta s\omega^\star(1-\omega^\star) \\ 0 & \overline{a}(\omega^\star) \end{pmatrix}. \tag{56}$$

The signs of the elements of the Jacobian matrix are:

$$\begin{pmatrix} <0 & >0 \\ =0 & <0 \end{pmatrix} \tag{57}$$

and the equilibrium is also stable (ω and P_K converge toward $\omega^\star$ and 0). From this, it follows that $\rho(P_K) = \overline{a}(\omega)$ and $\rho(r) = \rho(\pi) + \rho(P_K)$ converge toward $\overline{a}(\omega^\star) < 0$ or, equivalently, P_K and r converge toward exponentially decreasing trajectories (appendix A.2, 3).

A.4 The equilibrium of the model of equations (32) and its stability

A problem in the study of the dynamic model of equations (32) is that these equations depend explicitly on time. In order to eliminate this dependency, two successive transformations can be made:

1. We substitute $\kappa = tP_K$ for P_K. Equations (32) can be written:

$$\begin{cases} \dot{\omega} = \dfrac{1}{t} g(\omega, \kappa) \\ \dot{\kappa} = \dfrac{1}{t} h(\omega, \kappa) \end{cases} \tag{58}$$

$$\begin{aligned} \textit{with} \quad g(\omega,\kappa) &= \omega\left(\delta s(1-\omega)\kappa - f(\omega)\right) \\ h(\omega,\kappa) &= \kappa(1+\overline{a}(\omega)). \end{aligned}$$

2. We substitute variable $\theta = ln\, t$ for t. The derivatives (denoted $\mathring{x}$) are now computed vis-à-vis this new variable. One has: $\mathring{x} = \dfrac{dx}{d\theta} = \dfrac{dx}{dt/t} = t\dot{x}$. It follows that

$$\begin{cases} \mathring{\omega} = g(\omega, \kappa) \\ \mathring{\kappa} = h(\omega, \kappa). \end{cases} \tag{62}$$

The equilibrium of this system is defined by $\mathring{\omega} = \mathring{\kappa} = 0$. From $h(\omega^\star, \kappa^\star) = 0$, or:

$$\overline{a}(\omega^\star) = -1 \tag{60}$$

one can determine $\omega^\star$. We assume that this equation has a solution. Then, $g(\omega^\star, \kappa^\star) = 0$ allows for the determination of $\kappa^\star$:

$$\kappa^\star = \left(\frac{\bar{l}(\omega^\star)}{\delta} + \bar{l}(\omega^\star) + 1 \right) \frac{1}{s(1-\omega^\star)}. \tag{63}$$

The equilibrium values (asymptotic trajectories) of the other variables are

$$\begin{aligned} &P_K^\star = \frac{\kappa^\star}{t}, \;\; r^\star = \frac{(1-\omega^\star)\kappa^\star}{t}, \;\; P_L^\star = c' t^{\bar{l}(\omega^\star)}, \\ &K = K_0\, t^{\frac{\bar{l}(\omega^\star)}{\delta} + \bar{l}(\omega^\star) + 1}, \quad Y = Y_0\, t^{\frac{\bar{l}(\omega^\star)}{\delta} + \bar{l}(\omega^\star)}, \quad \textit{and} \quad H = H_0\, t^{\frac{\bar{l}(\omega^\star)}{\delta}}. \end{aligned} \tag{62}$$

This defines a power trajectory with $\alpha = 1$ and $C = s(1-\omega^\star)\kappa^\star$ (equation (40)).

The Jacobian matrix is:

$$J = \begin{pmatrix} -\omega^\star \left(\delta s \kappa^\star + f'(\omega^\star) \right) & \omega^\star \delta s (1-\omega^\star) \\ \kappa^\star \overline{a}'(\omega^\star) & 0 \end{pmatrix}. \tag{63}$$

Since $\overline{a}(\omega)$ declines and $f(\omega)$ rises, the signs of the elements of the Jacobian matrix are easily determined:

$$\begin{pmatrix} < 0 & > 0 \\ < 0 & = 0 \end{pmatrix} \tag{64}$$

and the equilibrium is stable.

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