Reprinted from JOURNAL OF ECONOMIC THEORY Vol. 53, No. 2, April 1991

Micro Adjustment toward Long-Term Equilibrium*

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Received March 15, 1988; revised June 4, 1990

A dynamic model is constructed in which the behavior of economic agents is based on adjustment. Decisions are made in disequilibrium: prices respond to differences between supply and demand (inventories), capital is moved according to profitability differentials, and money is issued by a bank which reacts to the general price level. A proof of local stability of long-term equilibrium with production prices is demonstrated. We further distinguish between two aspects of the stability problem: concerning the allocation of capital and relative prices, "stability in proportions," and with respect to the general levels of activity and prices, "stability in dimension." *Journal of Economic Literature* Classification Number: 020. © 1991 Academic Press, Inc.

INTRODUCTION

The purpose of this article is to demonstrate that the decentralized adjustment to disequilibrium by economic agents can insure the stability of long-term equilibrium. By adjustment we mean the reaction to the observation of past disequilibrium (unequal rates of return or involuntary inventories). Long-term equilibrium denotes a situation in which profit rates are equalized between sectors and supply equals demand. Beginning with a situation in which profit rates are not uniform and supply differs from demand in the various activities, the aim must be to show that the reactions of investors and producers to the evidence of these disequilibria create centripetal forces capable of achieving a convergence toward long-term equilibrium. The Walrasian tradition has been unable to achieve this aim in its study of long-term equilibrium with an explicit treatment of capital

* We thank an anonymous referee for helpful suggestions and Mark Glick for his aid in the translation of this text into English.

0022-0531/91 $3.00

mobility. (An important exception is the work by Hugo Sonnenschein [32].)

Equilibrium. Embedded in the works of the classics, Smith and Ricardo, as well as in Marx, is a conception of long-term equilibrium [31, Chap. 7; 29, Chap. 4; 26, Chap. 10] accompained by a notion of natural or production prices, i.e., prices which insure a uniform profit rate throughout the economy. In Walras' *Élements* [35, Sect. V], a long-term equilibrium is also described with a uniform rate of return (*taux de revenu net*). Later work has assimilated only the Walrasian notion of short-term equilibrium [35, Sect. III and IV]. Long-term equilibrium has only survived within the framework of growth theory or in von Neumann's model, in spite of its emperical relevance [20].

Disequilibrium. Many studies have been devoted to the project of modeling the formation of a short-term or long-term equilibrium through decentralized procedures. In our opinion, however, these attempts have been largely unsuccessful for various reasons.

A first deficiency in these models is that the treatment of disequilibrium is not extended to the commodity market. Long-term equilibrium is often studied as the outcome of a sequence of short-term equilibria, rather than the outcome of a sequence of disequilibria. Sonnenschein, for example, sets supply equal to demand in each period, as a result of the adjustment of prices by the auctioneer. Thus, disequilibrium is manifested only in the fact that profits are unequal. Most models of classical inspiration adopt this temporary equilibrium point of view.

A second deficiency concerns the usual treatment of disequilibrium. Models of *non-tâtonnement* [21, 34] should more adequately be denoted as *models of tâtonnement with trading out of equilibrium* [2]. This label very clearly accounts for the mechanism described and the limit of this approach. An auctioneer announces prices to the real agents who react *as if equilibrium would prevail on the market.* The difference from a *tâtonnement* is that transactions do take place at such prices (but no consumption is permitted before equilibrium prevails). The auctioneer then adjusts prices and the same process is repeated. Fisher, in his study, defines an interesting disequilibrium framework [17]. Agents are price-makers. They ignore what will happen on the markets that immediately follow the decisions (the so-called "present markets"). Production and consumption are allowed at each period, i.e., in disequilibrium. However, Fisher does not take advantage of the centuries of economic thinking on this matter, and fails to incorporate in his model any of the three basic laws of adjustment which played a central role in the development of economic thinking: law of supply and demand, law of the mobility of capital, and direct adjustment of outputs to demand signals. The decision making, therefore, remains fully arbitrary. No relevant results can be obtained in such a general perspective. Fisher derives his results from the assumption of "no favorable surprises" borrowed from Hahn and Negishi. This assumption, translated into this more general framework, appears purely ad hoc.

Interest in the dynamics associated with classical long-term equilibrium only developed in the recent years.[1]

Adjustment. In spite of the prominent role played by adjustment in the history of economic thought, it is certainly not the dominant paradigm in modern economics. In the work of the classics, the laws of "supply and demand" and "mobility of capital" were presented in terms of adjustment. Excess supply induces enterprises to diminish prices. Large profit rates induce capitalists to invest more capital in specific fields. Walras described similar procedures but, as is well known, he finally dissolved the treatment of disequilibrium and adjustment into the fictitious world of the *tâtonnement.* Adjustment behavior has survived in Keynesian models of the very short run [33; 25, Chap. 8.2], as well as in the Hungarian school [23, 24]. In such models, the evidence of disequilibrium corresponds to the existence of involuntary inventories, and the reaction involves the decision to produce. (Concerning adjustment in modern literature, see also [10].)

Indeed, in this study we adopt the point of view of adjustment, but we will not discuss its rationality (its relation to optimization). Aspects of such a discussion can be found in other recent works (cf. [4–6, 15, 37 and 38]). The framework of analysis in these studies is in some respects quite sophisticated, but in other respects it is also "simple," since the perspective adopted is that of short-term partial equilibrium with strong assumptions concerning information (knowledge of the probability distribution of stochastic variables).

The Framework of Analysis. The emphasis in this analysis is placed on the two classical disequilibria: profitability differential and maladjustment between supply and demand, i.e., involuntary inventories. (We will abstract from the direct adjustment of output to the difference between supply and demand.) These two disequilibria induce capitalists to modify the allocation of their capital, and enterprises to adjust their prices.

For simplicity, we assume that only one capitalist exists who controls all enterprises. However, a difficulty in this analysis concerns the treatment of money. Very simple monetary mechanisms are modeled, in which a bank

[1] We presented a first simple model in 1983 [13]. Other examples of models of classical inspiration are [1, 8, 9, 11, 12, 14, 18, 22, 28].

issues a monetary asset. This behavior is also modeled in terms of adjustment, in which the bank reacts to a third disequilibrium, the variation of the general level of prices.

We call this generalized adjustment in the modeling of the behavior of economic agents "Disequilibrium Microeconomics." In this framework, the services of the auctioneer are not required. No agent computes any equilibrium prices prior to the occurence of the market, and no auctioneer announces prices on the market.

Outline. The study divides into two parts. Part I is devoted to the presentation of the model, and Part II to its resolution. In this second part, we present the existence conditions and properties of equilibrium (cf. Proposition 1), and the conditions to which the local stability of long-term equilibrium is subject (cf. Proposition 2). An appendix is devoted to the definition of a specific type of matrices which we call H-matrices, and to the analysis of some of their properties.

I. The Model

The agents and the main mechanisms are presented in Section A. Section B deals with the modeling of monetary transactions. Section C is devoted to final consumption. Decisions made by enterprises concerning prices are described in Section D. Last, reduced variables and the recursion are introduced in Section E.

A. *Agents and Basic Mechanisms*

Four agents, or groups of agents, are considered in the model: enterprises which are in charge of production and which determine prices (Subsection 1), wage earners (Subsection 2), the capitalist (Subsection 3), and the bank which creates money by making loans to the capitalist (Subsection 4). The total amount of money is controlled by the bank, and the final destination of funds is controlled by the capitalist.

1. *Enterprises*

The number of enterprises is equal to n. Each enterprise produces a single product. Where two enterprises produce the same physical good, we consider their products to be two different commodities. Thus, it is possible

TABLE I

List of the Main Symbols	
u_n	Vector in $\mathbb{R}^n$ with all components equal to 1.
$\hat{V}$	Diagonal matrix in which the ith element of the diagonal is equal to the ith component of vector V. Depending on the context, $\hat{V}$ can be an $(n-1)\times(n-1)$ or $n\times n$ matrix.
I_n	Identity matrix of order n ($I_n = \hat{u}_n$).
$\mathbb{R}_+$, $\mathbb{R}_{++}$	Set of positive reals, set of strictly positive reals.
$\mathbb{C}_-$	Set of complex numbers with non-positive real parts.
Row vectors	Quantities, levels of activity.
Column vectors	Prices, labor inputs.
i, n	Superscript of enterprises, number of enterprises.
A, L	Good inputs, labor inputs.
p, x, w	Prices, relative prices ($x = p/p^n$), wage rate.
Y, y	Outputs, relative outputs ($y = Y/Y^n$).
S, s	Inventories, ratio of inventories ($s^i = S^i/Y^i$).
π, r	Profits, profit rate.
β, γ, ω	Coefficients of reaction to disequilibrium
α	Propensity to consume
ρ	Growth rate of capital

to use the same superscript, i, to index enterprises as well as commodities.[2]

Enterprises use techniques of production with constant returns to scale. There is no fixed capital and no joint products. The production process of commodity i is described by a set of fixed coefficients $(A^i, L^i) \in \mathbb{R}^n_+ \times \mathbb{R}_{++}$ which denote the physical and labor inputs required for the production of one unit of good i. We denote by Y_i, the level of production of commodity i and by p^i its price. Since the markets are not necessarily in equilibrium, a certain number of unsold commodities can exist. These commodities form a stock of inventories, S^i, held by enterprises during the period of production, and transmitted without alteration to the next market.

A square matrix, A, of order $n \times n$, represents the complete set of techniques of production. Its ith row is vector A^i. The column vector L represents the set of labor inputs. Its ith element is L^i. We also make the usual following assumption:

[2] The purpose of this assumption is to simplify the formalism (in particular, with regard to prices and the functioning of the market). The alternative assumption, "Each physical good corresponds to a unique commodity, independently of the producer," would allow the existence of different prices for the same commodity (as many prices as producers, since enterprises are price-makers). This latter assumption would require a "price dependent rationing scheme." Such an approach was adopted in [11, Sect. IV.G].

Assumption 1. The technical matrix A is irreducible.[3]

2. *Wage Earners*

There is only one type of wage earner. Sufficient labor always exists, and no labor market is considered in which the level of wages is determined in order to equalize the supply and demand for labor. On the contrary, we assume that the purchasing power of wage earners is guaranteed,

$$w_t = dp_t \tag{1}$$

where $d \in \mathbb{R}^n_+$ denotes a constant bundle of commodities which indexes the nominal wage rate, w_t, to prices p_t. The wage earners are not supposed to buy d rather than any other bundle. It is assumed that the wage earners do not save, but immediately spend their income to buy final consumption goods.

It will be necessary to make certain assumptions concerning the level of wages, in order to ensure that the equilibrium profit rate is positive (cf. Assumption 3 in Section A of Part II).

3. *The Capitalist*

"Capital" is used here in the sense of a fund of purchasing power, rather than in the sense of a physical asset. Enterprises are subject to a *capital constraint*,[4] since the size of their output is determined by the availability of funds.

We refer to the agents which have control over capital, decide on accumulation and distribution for consumption, and allocate capital among the various enterprises, as the "capitalists." In traditional microeconomics, "intertemporal consumers" play a role similar to that of our capitalists. For simplicity, only one such capitalist is considered (or, what is the equivalent, we assume that all capitalists are identical).[5]

In actual ecomomies, these agents are not exclusively final consumers, or "households." They also exist in a number of institutional forms: holding companies, investment banks, enterprises themselves.... Holding companies allocate resources between subsidiaries. Enterprises allocate their funds to the production of different products.

[3] For a definition and a presentation of the properties of irreducible matrices, consult [19, Vol. 2, Chap. 13].

[4] The notion of "Credit Constraint" refers to the same phenomenon [7].

[5] Models in which several capitalists exist can be found in [11], for two capitalists, and in [12] for any number of capitalists.

4. *The Bank and Monetary Flows*

Only the capitalist has access to bank credit, and all money is issued through new credit lines to the capitalist. This amount of credit can vary at each period. The equilibrium growth rate of the money stock is not known *ex ante*. The bank can adjust the rate at which money is issued, in reaction to the disequilibrium in the general price level. The behavior of the bank in the model is that of a central bank or Federal Reserve which controls the quantity of money created.

The total money made available can have two destinations, either final consumption or allocation to enterprises.

It is possible to decompose the sequence of operations into a succession of two types of periods: (1) Production periods during which no exchange and no financial transactions occur (money returns to the bank before production occurs), and (2) Periods of "circulation" during which financial transactions take place and the market is held. Money is issued at the beginning of the circulation period, and it flows back to the bank when transactions are terminated. During the circulation period, the sequence is as follows:

1. Determination of the amount of money in the bank, made available to the capitalist.

2. Distribution of income destined for final consumption, and allocation of capital to enterprises.

3. Payment of wages.

4. Market transactions. (All funds made available are spent and, when transactions have occurred, the whole amount of money is held by enterprises.)

5. Money flows back from enterprises to the capitalist and eventually to the bank.[6]

At each period, decisions concerning each of the five steps above are limited by the availability of liquidity. This constraint differs from a budget constraint in which these five steps are considered globally.

B. *Monetary Transactions*

In this section, we introduce the treatment of the monetary transactions described above. We first consider the division between consumption and accumulation (Subsection 1). The decision made by the bank actually determines the new total amount of capital to be allocated (Subsection 2).

[6] This procedure is equivalent to that in which the capitalist conserves this money until the next transaction period and the bank is only involved in the variation of this amount.

Last, the model for the allocation of capital is described (Subsection 3). Recall that no stock of money is transferred from one period to the next. The whole stock of money. M_t, is created by the bank and made available to the capitalist.

1. *Consumption and Accumulation*

The capitalist allocates M_t into two fractions corresponding to consumption and accumulation. The exact form of the procedure is not crucial to the argument made in this study. We assume that a given fraction, α, of *net* income (or profits), π_t, of capitalist is devoted to consumption.[7] With $C = A + ld$, and using Eq. (1), one obtains

$$\pi_t = \sum_{i=1}^{n} Y_t^i (p_t^i - A^i p_t - l^i w_t) = Y_t(I - C) p_t.$$

In this computation, the inputs have been evaluated at replacement cost in order to avoid the bias resulting from the variation of the price level.

The remainder of the stock of money, $\mathscr{K}_{t+1} = M_t - \alpha\pi_t$, finances accumulation. Note that the decision to issue money affects the total quantity of funds that can be allocated to enterprises by the capitalist, and not the sums destined for consumption.

2. *The Decision of the Bank to Issue Money*

The funds, $\mathscr{K}$, are destined for the purchase of inputs for the next period:

$$\mathscr{K}_{t+1} = Y_{t+1} A p_t + Y_{t+1} l w_t = Y_{t+1} C p_t.$$

Consider the growth rate of total capital between periods t and $t+1$ (capital is again evaluated at replacement cost):

$$\rho_{t+1} = \frac{Y_{t+1} C p_t}{Y_t C p_t} - 1.$$

The decision to issue money by the bank can be modeled as a decision on ρ_{t+1}. Rather than replicate this growth rate, as in the previous period, the bank adjusts it, in response to variations in the general level of prices:

$$\rho_{t+1} = \rho_t - \omega j_t \qquad (2)$$

In this first behavioral equation, ω is a reaction coefficient and j_t is the rate of variation of the general price level ($1 + j_t = Y_t p_t / Y_t p_{t-1}$). Inflation leads

[7] Another possible option would be to consider *gross* income $Y_t p_t$.

to a reduction of the growth rate of money injected in the economy. The converse holds in the case of deflation.[8] Although the bank plays a crucial role in the achievement of equilibrium, the equilibrium growth rate does not depend on its behavior (cf. Proposition 1).

3. *The Allocation of Capital*

A simple rule for the capitalist would be to apply the same growth rate ρ_{t+1} for the allocation of capital to enterprises. However, he/she is sensitive to disequilibria concerning profitability, and his/her behavior is also a function of the relative values of profit rates in the various enterprises.

The profit rate in enterprise i is defined as the ratio of profits to the stock of productive capital $Y_t^i C^i p_t$ plus the actual inventories $S_t^i p_t^i$:

$$r_t^i = \frac{Y_t^i (p_t^i - C^i p_t)}{Y_t^i C^i p_t + S_t^i p_t^i} = \frac{(I-C)^i p_t}{(C+\hat{s}_t)^i p_t}. \tag{3}$$

In this equation $s^i = S^i/Y^i$. The average profit rate is

$$\bar{r}_t = \frac{Y_t(I-C)\,p_t}{Y_t(C+\hat{s}_t)\,p_t}$$

As new funds are allowed, the capitalist will favor enterprises on the basis of their profit rate r_t^i, in comparison to his/her average profit rate $\bar{r}_t$:

$$Y_{t+1}^i = Y_t^i(1+\rho'_{t+1})(1+\gamma(r_t^i - \bar{r}_t)). \tag{4}$$

It is irrelevant that Eq. (4) has substituted output for capital, since all funds allocated will be used for production. The reaction coefficient γ measures the capitalist's sensitivity to profitability differentials $r_t^i - \bar{r}_t$. Parameter ρ' is determined in such a way that the total amount of funds allocated is equal to the available: $Y_{t+1}Cp_t = (1+\rho_{t+1})\,Y_tCp_t$. If $\gamma = 0$ or $r_t^i - \bar{r}_t = 0$ $\forall i$, then $\rho'_{t+1} = \rho_{t+1}$. If this is not the case, one has

$$1+\rho'_{t+1} = (1+\rho_{t+1})\left(1+\frac{\gamma}{Y_tCp_t}\sum_{i=1}^{n} Y_t^i C^i p_t (r_t^i - \bar{r}_t)\right).$$

(The summation over i does not yield exactly 0, since the weights $Y_t^i C^i p_t$ are not identical to those used in the definition of $\bar{r}_t$.

[8] This behavior is sensible, in this model, since the variation of prices reflects only demand fluctuation, and demand in turn is related to the amount of money in a straightforward manner.

C. *Final Consumption*

Each final consumer faces a separable utility function: $U^{(c)}$ for the capitalist, $U^{(w)}$ for the wage earners. We assume that utility functions are continuous functions from $\mathbb{R}^n_+$ in $\mathbb{R}$, and homogeneous of degree 1. Below we analyze a growth model in which equilibrium is defined as a situation of homothetic growth with constant prices; such a framework of analysis requires the definition of an assumption concerning homogeneity. It guarantees that, at equilibrium, the optimal consumptions will also grow homothetically.

On the basis of these functions, demand is proportional to income: $W_t = Y_{t+1} l w_t$ for wage earners, and $\alpha\pi_t$ for the capitalist. Considering the demand functions $d^{(c)}(p)$ and $d^{(w)}(p)$ which are homogeneous of degree -1 with regard to prices, and normalized such that their price is equal to 1, $d^{(c)}(p)p = d^{(w)}(p)p = 1$, it is easy to determine final demand (on market t)

$$D^{\mathrm{f}}_t = W_t d^{(w)}(p_t) + \alpha\pi_t d^{(c)}(p_t) = Y_{t+1} D^{(w)}(p_t) + Y_t D^{(c)}(p_t),$$

in which $D^{(w)}$ and $D^{(c)}$ are two matrices, homogeneous of degree 0 with respect to prices, and defined as

$$D^{(w)}(p) = dp\, l d^{(w)}(p) \qquad \text{and} \qquad D^{(c)}(p) = \alpha(I - C) p d^{(c)}(p).$$

In the study of the stability of equilibrium, the following assumption will be necessary:

Assumption 2. The demand functions $d^{(w)}$ and $d^{(c)}$ are continuously differentiable.

D. *Decisions Made by Enterprises*

Enterprises use all funds allocated by the capitalist for production. Therefore, their output is determined by this allocation (cf. Eq. 4). They must decide, however, on prices. When exchanges occur on market t, the prices, p_t, remain constant. The new prices (to prevail on market $t+1$), p_{t+1}, are decided when market t is over. These prices are determined according to the outcome of the market—the difference, Supply − Demand. This behavior is the expression of what has tradionally been called the "law of supply and demand." The difference between supply and demand is measured by the level of inventories:[9]

$$S_{t+1} = (S_t + Y_t) - (Y_{t+1} A + Y_{t+1} D^{(w)}(p_t) + Y_t D^{(c)}(p_t)).$$

[9] In what follows we limit our study to equilibrium and local stability. For this reason, and because of the existence of positive normal inventories in the model, we can abstract from situations such as a rationing of buyers corresponding to "negative" inventories.

Supply on market t is the sum of the new output Y_t and the stock of inventories S_t inherited from the previous market, and held during production t. Demand is also the sum of two components: inputs for the next production period, $Y_{t+1}A$, and final consumption, $Y_{t+1}D^{(w)}(p_t)+Y_tD^{(c)}(p_t)$.

In Subsection 3 of Section B above, the notation s was introduced for the ratio of inventories to output. Enterprises tend to maintain this ratio at a certain optimal value $\bar{s}$ (for example, about three weeks of sales for inventories of finished goods in U.S. manufacturing) which we assume strictly positive. If inventories are larger than this target value ($s>\bar{s}$), they will diminish the price. The converse occurs in cases of low inventories ($s<\bar{s}$):

$$p^i_{t+1}=p^i_t(1-\beta^i(s^i_{t+1}-\bar{s}^i)) \tag{5}$$

A priori, the reaction coefficients β^i are positive and differ for each enterprise.

E. *Reduced Variables and the Recursion*

The above equations define a relation of recursion for the variables p^i, Y^i, S^i (for $i=1, ..., n$), and ρ. All functions which play a role in this recursion (such as r, $D^{(w)}$, $D^{(c)}$ and ρ') are homogeneous, of degree 0 with regard to their arguments. Therefore, one can define a recursion on the *reduced variables* x^i, y^i, s^i, and ρ. Variables s^i and ρ have already been introduced, and x^i and y^i are defined as

$$x^i=p^i/p^n \qquad \text{and} \qquad y^i=Y^i/Y^n.$$

The recursion can be written:

$$x^i_{t+1}=x^i_t\frac{1-\beta^i(s^i_{t+1}-\bar{s}^i)}{1-\beta^n(s^n_{t+1}-\bar{s}^n)} \tag{6}$$

$$y^i_{t+1}=y^i_t\frac{1+\gamma(r^i_t-\bar{r}_t)}{1+\gamma(r^n_t-\bar{r}_t)} \tag{7}$$

$$s^i_{t+1}=\frac{1+s^i_t-y_tD^{(c)}(x_t)^i/y^i_t}{(1+\rho'_{t+1})(1+\gamma(r^i_t-\bar{r}_t))}-\frac{y_{t+1}(A+D^{(w)}(x_t))^i}{y^i_{t+1}} \tag{8}$$

$$\rho_{t+1}=\rho_t-\omega j_t \tag{9}$$

with

$$r_t^i = \frac{(I-C)^i x_t}{(C+\hat{s}_t)^i x_t}$$

$$\bar{r}_t = \frac{y_t(I-C)\, x_t}{y_t(C+\hat{s}_t)\, x_t}$$

$$1+j_t = \frac{y_t x_t}{y_t x_{t-1}} (1-\beta^n(s_t^n - \bar{s}^n))$$

$$1+\rho'_{t+1} = (1+\rho_{t+1})\left(1+\frac{\gamma}{y_t C x_t}\sum_{i=1}^{n} y_t^i\, C^i x_t (r_t^i - \bar{r}_t)\right).$$

II. Equilibrium and Stability

This part analyzes two properties of the above model: its equilibrium and the local stability of this equilibrium. Both existence and stability are conditional. The existence conditions are easy to formulate. Stability represents the difficult part of the demonstration: We define a *sufficient condition* on the value of reaction coefficients which guarantees the local stability of long-term equilibrium. In the proof of this stability condition, we distinguish (1) the ability of the system to allocate capital and to define relative prices, or *stability in proportions*, and (2) its capability to converge toward equilibrium with respect to the general levels of activity and prices, or *stability in dimension*.

Section A is devoted to the definition and existence of equilibrium. The sufficient condition for stability is presented in Section B, and the general guidelines governing its analytical proof are formulated. The Jacobian is determined in Section C and its eigenvalues studied using the perturbation method. The proof of stability is provided in section D.

A. *Definition and Existence of Equilibrium*

We define an equilibrium in the following way:

Definition 1. *Equilibrium.* An equilibrium is a state of homothetic growth with constant prices, positive profit rates, and normal ratios of inventories ($s_i = \bar{s}_i$).

The existence of an equilibrium growth path with constant prices and normal inventories is guaranteed in the model. One assumption must be made, however, to insure that the profit rate is positive. We define the matrix

$$F = (I + \hat{\bar{s}})^{-1} (C + \hat{\bar{s}})$$

and make an assumption on the dominant eigenvalue of F (which is larger than 0, since $F \geqslant 0$):

Assumption 3. The dominant eigenvalue $\bar{\lambda}$ of matrix F is smaller than 1.

This assumption is equivalent to the statement that the bundle of commodities d which defines the purchasing power of workers must be bounded.

Thus, it is possible to state the following proposition concerning the condition of existence of equilibrium and its properties:

PROPOSITION 1. *Under Assumptions* 1 *and* 3, *and if the reaction coefficients* γ, β^i, *and* ω *are strictly positive, the relation of recursion defined by Eqs.* (6) *to* (9) *has a fixed point, which is unique, is an equilibrium* (*cf. Definition* 1), *and has the following properties*:

(i) *The profit rates are uniform* ($r^i = \bar{r}$).

(ii) *Prices are strictly positive and satisfy*

$$(I + \hat{\bar{s}})\, \bar{x} = (1 + \bar{r})(C + \hat{\bar{s}})\, \bar{x}. \tag{10}$$

(*This equation defines production prices, including normal inventories in the capital stock.*)

(iii) *The growth rate is stricktly positive and equal to*

$$\bar{\rho} = (1 - \alpha)\, \bar{r}$$

(*i.e., the well-known "Cambridge" equation*).

(iv) *The levels of activity are positive and satisfy*

$$\bar{y}(I + \hat{\bar{s}}) = (1 + \bar{\rho})\, \bar{y}(A + D^{(w)}(\bar{x}) + \hat{\bar{s}}) + \bar{y}D^{(c)}(\bar{x}). \tag{11}$$

(*This equation defines homothetic growth with demand functions and inventories.*)

Proof. A fixed point of the recursion satisfies Eqs. (6) to (9) in which the time subscripts have been deleted:

From Eq. (6), it follows that prices vary at the same rate.

From Eq. (9), it follows that the rate of variation of prices is null. Thus, $s^i = \bar{s}^i\ \forall i$.

From Eq. (7), it follows that profit rates are equal. Therefore prices satisfy Eq. (10) (cf. Eq. (3)). This equation can be written

$$F\bar{x} = \frac{1}{1+\bar{r}}\,\bar{x}.$$

Under Assumption 3, F is non-negative, irreducible, and its dominant eigenvalue is smaller than 1. Thus, $\bar{r} = (1-\bar{\lambda})/\bar{\lambda} > 0$, and the corresponding eigenvector is strictly positive.

From Eq. (8), it follows that vector $\bar{y}$ satisfies Eq. (11). Multiplying this equation by $\bar{x}$, one obtains $\bar{\rho} = (1-\alpha)\,\bar{r}$. Equation (11) can be expressed as $\bar{y}F' = \bar{y}$, with $F' = (I+\hat{s})^{-1}\,((1+\bar{\rho})(A+D^{(w)}(\bar{x})) + D^{(c)}(\bar{x}))$. Vector $\bar{x}$ is the right eigenvector of F' associated with the eigenvalue 1. Since $\bar{x}$ is strictly positive and F' is irreducible, 1 is the dominant eigenvalue of F' (cf. [19, Chap. 13]. Thus, $\bar{y}$ is the strictly positive left eigenvalue which corresponds to this dominant eigenvalue.

If one of the reaction coefficients γ, β^i, or ω is equal to zero, then the recursion still has a fixed point which is generally not an equilibrium in the sense of Definition 1. ▌

B. *A Condition for Stability*

This section divides into two subsections. In Subsection 1 we state a sufficient condition for local stability. The general guidelines governing the proof of this condition are introduced in Subsection 2. The proof itself will be presented in the following sections.

1. *A Sufficient Condition for Stability*

We demonstrate the following proposition which constitutes the main result of this study.

PROPOSITION 2. *Consider the recursion defined by Eqs.* (6) *to* (9) *in Part I and satisfying Assumptions* 1 *to* 3. *A set of values of reaction coefficients β^i, γ, and ω exists for which the local stability of the fixed point is insured. More specifically, for any very small value of coefficient γ, then $\bar{\beta}$ and $\bar{\omega}(\beta)$ exist such that local stability is insured if β^i and ω are choosen such that*

(i) $\beta^i = \beta z^i \gamma$ *for any* $\beta \in\,]0, \bar{\beta}[$ *and* z^i *as in Eq.* (13).

(ii) $\omega \in\,]0, \bar{\omega}(\beta)[$.

The reaction to profit rate differentials, measured by coefficient γ, must be small. The reaction on prices responding to disequilibrium between supply and demand, measured by β^i, must not be too large in comparison to γ: this is expressed by the condition $\beta < \bar{\beta}$. In a similar manner, the sensitivity of the bank to the variation of the general level of prices measured by coefficient ω must be limited in relation to the reaction of enterprises. This is expressed by the condition $\omega < \bar{\omega}(\beta)$.

2. *Introduction to the Proof of the Stability Condition*

The remainder of this part is devoted to the proof of Proposition 2. In order to study the local stability of the recursion, Eqs. (6) to (9) must be linearized in the vicinity of equilibrium. (This is possible since $\bar{x}$ and $\bar{y}$ are strictly positive (cf. Proposition 1) and because of Assumption 2). This calculation leads to the determination of the Jacobian matrix of the recursion. The dimension of this matrix is equal to $3n-1$, the number of variables. The analysis of the $3n-1$ eigenvalues of the Jacobian matrix allows for the formulation of the sufficient condition for the stability of the fixed point. One must show that the moduli of all the eigenvalues of the matrix are smaller than 1.

The method adopted is based on the *perturbation calculation* of eigenvalues [36, Chap. 2]. If we nullify the reaction coefficients β^i and γ, and compute the complete set of eigenvalues, we find $\lambda = 1$ with a multiplicity $2n-1$, and $\lambda = 1/(1+\bar{\rho}) < 1$ with a multiplicity n ($\bar{\rho}$ denotes the positive equilibrium value of the growth rate). One can then develop the equation in the vicinity of this point. The dimension of the problem is, thus, reduced. With $\mathbb{C}_-$ denoting the set of complex numbers whose real component is smaller or equal to zero and $Q(\mu)$ a polynomial of order $2n-1$, one must then show that $Q(\mu)$ is different from 0 if $\mu \in \mathbb{C}_-$, i.e., that all zeros of $Q(\mu)$ have strictly positive real parts. Polynomial $Q(\mu)$ can be written as $Q(\mu) = \mu Q_1(\mu) + \omega Q_2(\mu)$. We will prove that (1) A set of reaction coefficients exists such that $Q_1(\mu) \neq 0$ if $\mu \in \mathbb{C}_-$ and $Q_1(0) < 0$, (2) $Q_2(0)$ is larger than 0, and (3) A set of values of ω exists such that $Q(\mu) \neq 0$, if $\mu \in \mathbb{C}_-$. In order to complete the last stage of this proof, we will use the properties of the H-matrices whose presentation has been left for the appendix.

In the development of this proof, we will first study what has been called *stability in proportions*, and then turn to the analysis of *stability in dimension*.

C. *The Jacobian and the Perturbation Method*

The Jacobian is presented in Subsection 1. The perturbation method is addressed in Subsection 2. Subsection 3 is devoted to the study of a polynomial denoted $Q(\mu)$ which will be defined below.

1. *The Jacobian*

The order of the variables is x^i for $i = 1, ..., n-1$, y^i for $i = 1, ..., n-1$, s^i for $i = 1, ..., n$, and ρ. Thus, the Jacobian can be determined as

$$P(\lambda)=\begin{vmatrix}(1-\lambda)\,I_{n-1} & 0 & -\lambda Z_4\hat{\beta} & 0\\ \gamma Z_1 & (1-\lambda)\,I_{n-1} & -\gamma Z_5 & 0\\ Z_2+\gamma Z_2' & Z_3'+\lambda Z_3''+\gamma Z_3''' & \left(\frac{1}{1+\bar{\rho}}-\lambda\right) I_n+\gamma Z_6 & Z_7\\ \omega(\lambda-1)\frac{\bar{y}}{\bar{y}\bar{x}} & 0 & \beta^n\omega Z_8\lambda & 1-\lambda\end{vmatrix},$$

in which Z_i for $i=1, ..., 8$ denotes a set of matrices independent from the reaction coefficients and λ. For example:

$$(Z_1)^{ij}=\bar{y}^i\left(\frac{\partial r^i}{\partial x^j}-\frac{\partial r^n}{\partial x^j}\right) \tag{12}$$

$$(Z_3')^{ij}=-\frac{1}{1+\bar{\rho}}\frac{1}{\bar{y}^i}\frac{\partial yD^{(c)}(\bar{x})^i/y^i}{\partial y^j}$$

$$(Z_3'')^{ij}=-\frac{1}{\bar{y}^i}\frac{\partial y(A+D^{(w)}(\bar{x}))^i/y^i}{\partial y^j}$$

$$Z_4=\hat{\bar{x}}\,U_{n-1} \qquad \text{with } U_{n-1}=(I_{n-1}, -u_{n-1})$$

$$Z_5=\hat{\bar{y}}\,U_{n-1}\hat{z} \qquad \text{with } z^i=\bar{r}\frac{\bar{x}^i}{C^i\bar{x}+\bar{s}^i\bar{x}^i} \tag{13}$$

$$(Z_7)^i=-\frac{1}{1+\bar{\rho}}\left(\bar{s}^i+\frac{\bar{y}(A+D^{(w)}(\bar{x}))^i}{\bar{y}^i}\right) \tag{14}$$

$$(Z_8)^i=\begin{cases}0 & \text{if } i\leqslant n-1\\ 1 & \text{if } i=n.\end{cases}$$

This Jacobian is a function of reaction coefficients β^i ($i=1, ..., n$), γ, and ω.

2. *The Perturbation Method*

By setting the value of coefficients β^i and γ to 0, it is possible to calculate all the eigenvalues. One obtains

$$\lambda=1 \qquad 2n-1 \text{ times}$$

$$\lambda=\frac{1}{1+\bar{\rho}} \qquad n \text{ times.}$$

If these coefficients are very small, n eigenvalues remain in the vicinity of $1/(1+\bar{\rho})$ which is smaller than 1 since $\bar{\rho}$ is larger than 0. Therefore, no difficulty is raised concerning convergence. The $2n-1$ other eigenvalues remain in the vicinity of 1, and can be studied using the perturbation method.

Assuming that the reaction coefficients β^i and γ are infinitely small and of the same order (cf. a), we can develop the eigenvalues in the vicinity of 1, as functions of these coefficients (cf. b). In this way, the problem originally posed is transformed and rendered manageable (cf. c).

a. *Infinitely Small Values of β^i and γ.* Our computation will be made by focusing on γ. Since the β^i's and γ are assumed to be infinitely small and of the same order, we will choose β^i's proportional to γ, with the model

$$\beta^i = \beta z^i \gamma, \tag{15}$$

in which β is a new coefficient and z^i is as defined in Eq. (13). The β^i's are chosen to be unequal in order to simplify the formalism, and this choice has a straightforward economic interpretation. Equation (5) which models the behavior of enterprises with respect to prices, can be written as $p^i_{t+1} = p^i_t(1 - \beta' \sigma^i (s^i_{t+1} - \bar{s}^i)/\bar{s}^i)$, in which $\sigma^i = \bar{s}^i \bar{x}^i/(C^i \bar{x} + \bar{s}^i \bar{x}^i)$ denotes the share of inventories in the total balance-sheet of enterprises (at equilibrium), and $\beta' = \beta \bar{r} \gamma$ is independent from i. Thus, Eq. (15) can be interpreted as follows: The greater the share of inventories in the total balance-sheet, the greater the reaction of enterprises to the disequilibrium on inventories in their price behavior.

b. *Development of the Eigenvalues.* With the above model for the β^i's, we are left with three coefficients: γ, assumed to be infinitely small, β, and ω. The Jacobian and the eigenvalues are, thus, functions of these three coefficients: $P(\lambda) = P(\lambda; \gamma, \beta, \omega)$ and $\lambda = \lambda(\gamma, \beta, \omega)$. We assume that we can develop $\lambda(\gamma, \beta, \omega)$ in the vicinity of $\gamma = 0$, as a series in γ,

$$\lambda(\gamma, \beta, \omega) = 1 - (\mu\gamma + \mu_2\gamma^2 + \mu_3\gamma^3 + \cdots), \tag{16}$$

in which $\mu, \mu_2, \mu_3, \ldots$ are functions of β and ω. Substituting this expression of λ for its value in $P(\lambda; \gamma, \beta, \omega)$, and nullifying all the coefficients in the series in γ thus obtained, a set of equations is determined which allows for the recursive calculation of $\mu, \mu_2, \mu_3, \ldots$. The coefficient of the term of lowest degree in γ in the development of $P(\lambda(\gamma, \beta, \omega); \gamma, \beta, \omega)$ depends only on μ, and not on the μ_i's for $i \geqslant 2$. We denote it $Q(\mu)$.

c. *The New Form of the Problem.* In order to show that values of β^i, γ, and ω exist for which all moduli of the eigenvalues are smaller than 1, it is sufficient to show, as suggested in Fig. 1, that values of β and ω exist for which the real parts of all the roots of $Q(\mu)$ are strictly positive, and to choose a sufficiently small value of γ.

The original problem, whether the moduli of the roots of polynomial

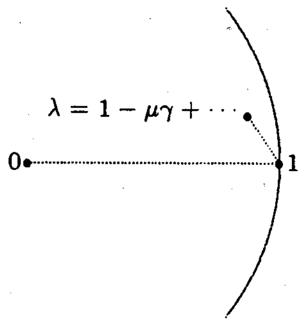

FIG. 1. From $\operatorname{Re}\mu > 0$ to $|\lambda| < 1$.

$P(\lambda)$ are smaller than 1, has been transformed above into the investigation of *whether all the roots of $Q(\mu)$ have a positive real part.*

3. *Polynomial $Q(\mu)$*

The term of lowest degree in γ in the development of $P(\lambda(\gamma, \beta, \omega); \gamma, \beta, \omega)$ can be written $(-1)^{n+1}Q(\mu)\gamma^{2n-1}$ with

$$Q(\mu) = \begin{vmatrix} \mu I_{n-1} & 0 & \beta\hat{\bar{x}}U_{n-1}\hat{z} & 0 \\ Z_1 & \mu I_{n-1} & \hat{\bar{y}}U_{n-1}\hat{z} & 0 \\ Z_2 & Z_3 & \dfrac{\bar{\rho}}{1+\bar{\rho}} I_n & Z_7 \\ \dfrac{\omega\mu}{\bar{y}\bar{x}}\bar{y} & 0 & (0, ..., 0, \beta\omega z^n) & -\mu \end{vmatrix},$$

in which $Z_3 = Z_3' + Z_3''$:

$$(Z_3)^{ij} = -\frac{1}{1+\bar{\rho}} \frac{\partial y E^i / y^i}{\partial y^j} \qquad \text{with} \quad E = D^{(c)}(\bar{x}) + (1+\bar{\rho})(A + D^{(w)}(\bar{x})). \quad (17)$$

$Q(\mu)$ is a polynomial, which depends on coefficients β and ω, and of degree $(2n-1)$. This degree is equal to the multiplicity of the eigenvalue $\lambda = 1$ if $\gamma = 0$. This justifies the assumption made above (Eq. (16)) concerning the development of $\lambda(\gamma, \beta, \omega)$ for $\gamma = 0$ [36, Chap. 2, Theorem 2, p. 65], which is satisfied with $m^i = 1\ \forall i$.

D. *The Proof of Stability*

The purpose of this section is to prove that values of β and γ exist such that $Q(\mu) \neq 0\ \forall \mu \in \mathbb{C}_-$. This step will conclude the proof of Proposition 2.

In this proof, we will use a decomposition which makes explicit the dependency on ω,

$$Q(\mu) = \mu Q_1(\mu) + \omega Q_2(\mu), \tag{18}$$

in which Q_1 and Q_2 are two polynomials in μ of degree $2(n-1)$, which depend only on coefficient β. We will establish two preliminary results: (1) An interval, $]0, \bar{\beta}[$, in β exists such that $Q_1(\mu) \neq 0 \ \forall \mu \in \mathbb{C}_-$, and $Q_1(0) < 0$ (cf. Subsection 1), and (2) For any value of β, one has $Q_2(0) > 0$ (cf. Subsection 2). The end of the proof will be presented in Subsection 3.

Using the distinction between proportions and dimension, it is possible to state that Subsection 1 deals with the first aspect of the stability problem, *stability in proportions*, and Subsection 3 with the second, *stability in dimension.*

1. *Capital Allocation and Relative Prices: Stability in Proportions*

The analysis in this subsection focuses on polynomial Q_1. This polynomial can be derived directly from the Jacobian $P(\lambda)$ restricted to its $3n-2$ first lines and columns, which correspond to the set of variables involved in the analysis of stability in proportions: relative prices x, relative outputs y, and ratios of inventories s. We will prove the following lemma:

LEMMA 1. *If $Q_1(\mu)$ is the polynomial defined in Eq.* 18, *then*

(i) *$\bar{\beta}$ can be determined such that for all β satisfying $0 < \beta < \bar{\beta}$, $Q_1(\mu) \neq 0$ if $\mu \in \mathbb{C}_-$.*

(ii) $Q_1(0) < 0$.

Proof. One has

$$Q_1(\mu) = -\begin{vmatrix} \mu I_{n-1} & 0 & \beta \hat{\bar{x}} U_{n-1} \hat{z} \\ Z_1 & \mu I_{n-1} & \hat{\bar{y}} U_{n-1} \hat{z} \\ Z_2 & Z_3 & \dfrac{\bar{\rho}}{1+\bar{\rho}} I_n \end{vmatrix}.$$

We pre-multiply the determinant which defines $Q_1(\mu)$ by

$$\begin{pmatrix} -\hat{\bar{x}}^{-1} & \beta \hat{\bar{y}}^{-1} & 0 \\ 0 & -\beta^{1/2} \hat{\bar{y}}^{-1} & \beta^{1/2} \dfrac{1+\bar{\rho}}{\bar{\rho}} U_{n-1} \hat{z} \\ 0 & 0 & I_n \end{pmatrix}$$

(which is actually defined, since $\bar{\rho}$, $\bar{x}$, and $\bar{y}$ are strictly positive, cf. Proposition 1) and post-multiply it by

$$\begin{pmatrix} \hat{\hat{x}} & 0 & 0 \\ 0 & \beta^{-1/2}\hat{\hat{y}} & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

After suppressing n lines and columns, one obtains a determinant of order $2(n-1)\times 2(n-1)$: $Q_1(\mu) = -\det M(\mu)$, with,

$$M(\mu) = \begin{pmatrix} \beta\hat{\hat{y}}^{-1}Z_1\hat{\hat{x}} - \mu I_{n-1} & \beta^{1/2}\mu I_{n-1} \\ -\beta^{1/2}\hat{\hat{y}}^{-1}Z_1\hat{\hat{x}} + \beta^{1/2}\tilde{Z}_2\hat{\hat{x}} & \tilde{Z}_3\hat{\hat{y}} - \mu I_{n-1} \end{pmatrix},$$

in which $\tilde{Z}_i$ is defined by

$$\tilde{Z}_i = \frac{1+\bar{\rho}}{\bar{\rho}} U_{n-1}\hat{z}Z_i. \tag{19}$$

We will now use the notion of H-matrix defined in the appendix. Lemma 6 in the appendix can be applied to $M(\mu)$ with $N_1 = \hat{\hat{y}}^{-1}Z_1\hat{\hat{x}}$, $N_2 = \tilde{Z}_3\hat{\hat{y}}$, two H-matrices (cf. Lemma 7, also in the appendix), and $N_3 = -\hat{\hat{y}}^{-1}Z_1\hat{\hat{x}} + Z_2\hat{\hat{x}}$. Then $\bar{\beta}$ exists such that, if $0 < \beta \leqslant \bar{\beta}$, the matrix $M(\mu)$ is not singular when $\operatorname{Re}\mu \leqslant 0$, and all the zeros of $Q_1(\mu) = -\det M(\mu)$ have a strictly positive real part. This proves the first point in Lemma 1.

If $\mu = 0$, Q_1 is equal to

$$Q_1(0) = -\det M(0) = \begin{vmatrix} \beta\hat{\hat{y}}^{-1}Z_1\hat{\hat{x}} & 0 \\ \beta^{1/2}N_3 & \tilde{Z}_3\hat{\hat{y}} \end{vmatrix} = -\beta^{n-1}\det N_1 \det N_2.$$

Since the determinant of an H-matrix is always strictly positive (cf. Lemma 5 in the appendix), $Q_1(0)$ is strictly negative, and this proves the second point. ∎

2. *The Positiveness of* $Q_2(0)$

In this subsection, the analysis is devoted to the sign of $Q_2(0)$ (cf. Eq. (18)). We will first establish a technical lemma:

LEMMA 2. *Consider M an $n \times n$ matrix such as*

$$M = \begin{pmatrix} & N & & b_1 \\ & & & \vdots \\ & & & b_{n-1} \\ a_1 & \cdots & a_{n-1} & c \end{pmatrix}$$

in which a_i and b_i are negative, c is strictly negative, and N is an $(n-1)\times(n-1)$ matrix with a positive strictly dominant diagonal. Then $\det M < 0$.

Proof. This lemma can be proved by recursion. The lemma is true for $n=2$. We develop $\det M$ along the last line:

$$\det M = -\sum_{i=1}^{n-1} a_i \det M_i + c \det N.$$

Matrices M_i are of order $(n-1)\times(n-1)$, and they satisfy the conditions posed in the lemma. Thus, their determinants are strictly negative. Matrix N has a positive strictly dominant diagonal, and its determinant is, therefore, strictly larger than 0 [27]. Consequently, $\det M < 0$. ▮

We now turn to the lemma concerning the positiveness of $Q_2(0)$:

LEMMA 3. *If $Q_2(\mu)$ is the polynomial defined in Eq.* (18), *then* $Q_2(0) > 0$.

Proof. One has

$$Q_2(0) = -\prod_{i=1}^{n} (\beta \bar{x}^i z^i) \det Z_1 \det (Z_3, Z_7).$$

The sign of this expression is determined by $-\det(Z_3, Z_7)$, since the determinant of Z_1 has the same sign as that of $N_1 = \hat{\bar{y}}^{-1} Z_1 \hat{\bar{x}}$, which is positive, since N_1 is an H-matrix (cf. Lemma 5). The sign of $\det(Z_3, Z_7)$ can be derived from Lemma 2, whose assumptions are satisfied: For the $n-1$ first columns this follows from the proof of Lemma 7, and for the last column, from the definition of Z_7 (cf. Eq. (14)). Thus, $\det(Z_3, Z_7) < 0$ and $Q_2(0) > 0$. ▮

3. *The General Level of Activity*: *Stability in Dimension*

This subsection concludes the proof of Proposition 2: We show that values of the reaction coefficients exist such that $Q(\mu)$ has no roots for $\mu \in \mathbb{C}_-$. (Recall that this is equivalent to showing that all roots of $P(\lambda)$ have a modulus shaller than 1 if γ is small enough.)

This last stage of the demonstration corresponds to the stability in dimension of the economy. Coefficient ω plays a crucial role in this stability. If $\omega = 0$, the roots of $Q(\mu)$ are (1) the $2(n-1)$ zeros of $Q_1(\mu)$ which have strictly positive real parts (cf. Lemma 1), and (2) $\mu = 0$. This latter value corresponds to $\lambda = 1$ and mirrors the fact that the general dimension of the economy, or rather its growth rate, has been pegged to a given constant determined by initial conditions. The problem in this subsection is to show that positive values of ω exist such that the $2n-1$ roots of $Q(\mu)$ have strictly positive real parts. In addition to the proof of stability in proportions, which corresponds to the $2(n-1)$ first eigenvalues, the proof of stability in dimension, which corresponds to the last eigenvalue, will thus be completed.

LEMMA 4. *Consider $Q_1(\mu)$ and $Q_2(\mu)$, two polynomials, dependent on parameter β, of the same degree in μ satisfying*:

(i) $Q_1(0) < 0$ *and* $Q_1(\mu) \neq 0 \ \forall \mu \in \mathbb{C}_-$.

(ii) $Q_2(0) > 0$.

Then, $\bar{\omega}(\beta)$ exists such that, for all strictly positive values of ω smaller than $\bar{\omega}(\beta)$, $Q(\mu)$ defined by Eq. (18) *satisfies*

$$Q(\mu) \neq 0 \qquad \forall \mu \in \mathbb{C}_-.$$

Proof. Again we can use the perturbation method, which in this simple framework is reduced to a problem of continuity of the roots of a polynomial, as functions of the coefficients of this polynomial.

If $\omega = 0$, $2(n-1)$ roots exist with strictly positive real parts together with a root equal to zero (cf. Eq. (18) and Assumption *i*)). We study the roots of $Q(\mu)$ in the vicinity of $\omega = 0$. By continuity, the $2(n-1)$ first roots conserve strictly positive real parts, provided that ω remains sufficiently small. Only the last remaining root can be the problem. This root can be developed in ω: $\mu = v\omega + v_2\omega^2 + \cdots$. One can replace μ by this expression in $Q(\mu)$. Nullifying the term of lowest degree in ω, one obtains the value of v:

$$v = -\frac{Q_2(0)}{Q_1(0)}.$$

Under the assumptions made, v is strictly positive. Thus, a maximum value, $\bar{\omega}(\beta)$, of ω exists such that for all strictly positive ω and smaller than $\bar{\omega}(\beta)$, the $2n-1$ roots have strictly positive real parts. ∎

From Lemmas 1 and 3, it follows that assumptions (i) and (ii) are satisfied.

APPENDIX: *H*-MATRICES

In this appendix we define a specific type of matrix called "*H*-matrix," study some of its properties, and verify that two matrices are *H*-matrices.

DEFINITION 2. *H*-Matrix. A square matrix, N, or order $(n-1) \times (n-1)$, is an *H*-matrix if a square matrix M, of order $n \times n$, exists such that:

(i) Matrices N and M are related by

$$N=(I_{n-1},\ -u_{n-1})\,M\begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}; \tag{20}$$

(ii) Diagonal elements in M are strictly positive and the off-diagonal elements are negative or zero;

(iii) The sum of the elements in each row is equal to zero,

$$Mu_n=0; \tag{21}$$

(iv) Matrix M is irreducible.

LEMMA 5. (i) *Every H-matrix N is similar to a matrix such as*

$$D=\begin{pmatrix} \lambda_1 & \omega_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \omega_{n-2} \\ 0 & \cdots & \cdots & 0 & \lambda_{n-1} \end{pmatrix} \tag{22}$$

in which the λ^i*s and* ω^i*s satisfy*

- $\forall i=1, ..., n-1$, $\mathrm{Re}\lambda_i>0$;
- $\forall i=1, ..., n-2$, $\omega_i \in R_+$ *and either* $\omega_i=0$, *or* $\omega_i=v$, *with* $0<v<\min_i \mathrm{Re}\lambda_i$.

(ii) *The determinant of an H-matrix is strictly positive.*

Proof. Let M be the matrix of order $n\times n$ which verifies the conditions stated in Definition 2. We consider a, a positive scalar smaller than the inverse of the greatest element of the diagonal of M. Matrix $I-aM$ is an irreducible matrix with positive elements, of which u_n is the eigenvector associated with 1 as the eigenvalue. Thus, 1 is the Frobenius eigenvalue of $I-aM$; the moduli of the $n-1$ other eigenvalues are smaller than 1.

If λ' is an eigenvalue of $I-aM$, $\lambda=(1-\lambda')/a$ is an eigenvalue of M. Therefore, M has an eigenvalue equal to 0, and the $n-1$ other eigenvalues, λ_i ($i=1, ..., n-1$), have a strictly positive real part, $\mathrm{Re}\lambda_i>0$.

We denote by S the matrix of change of basis which allows the transformation of M into its Jordan's reduced form. We define a matrix $T=S\hat{\alpha}$, with $\alpha_i=(v)^{i-1}$. Since u_n is an eigenvector of M, it can be choosen as the last column of the T matrix. This matrix can, thus, be written

$$T=\begin{pmatrix} \bar{T} & u_{n-1} \\ t & 1 \end{pmatrix}, \tag{23}$$

where $\bar{T}$ is a matrix of order $(n-1)\times(n-1)$, and t is a matrix of order $1\times(n-1)$. T^{-1} can be decomposed in the same manner,

$$T^{-1}=\begin{pmatrix}\bar{T}' & t''\\ t' & t'''\end{pmatrix}, \tag{24}$$

where $\bar{T}'$, t', t'', and t''' are matrices of order $(n-1)\times(n-1)$, $1\times(n-1)$, $(n-1)\times 1$, and 1×1, respectively. From the identity $T^{-1}T=I_n$, it follows that

$$(\bar{T}')^{-1}=\bar{T}-u_{n-1}t. \tag{25}$$

Thus

$$M=TM'T^{-1} \qquad \text{with } M'=\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix}, \tag{26}$$

where D is a matrix of order $(n-1)\times(n-1)$, such as matrix D in Eq. (22). From Eqs. (20) to (26), it follows that

$$N=(\bar{T}')^{-1}D\bar{T}',$$

i.e., N is similar to D. If we choose $v<\operatorname{Re}\lambda^i\ \forall i$, matrix D satisfies all the conditions stated in the lemma.

The second part of the lemma is a straightforward consequence of the first part. Two similar matrices have the same determinant, and the determinant of D is strictly positive, since it is the product of strictly positive numbers (the real eigenvalues and the squares of the moduli of the complex eigenvalues). ▮

LEMMA 6. *We consider two H-matrices N_1 and N_2, and any matrix N_3. A strictly positive real number $\bar{\beta}$ exists, such that, for all β satisfying $0<\beta<\bar{\beta}$ and for all complex μ with a non-positive real part ($\mu\in\mathbb{C}_-$), matrix M defined by*

$$M=\begin{pmatrix}\beta N_1-\mu I_{n-1} & \beta^{1/2}\mu I_{n-1}\\ \beta^{1/2}N_3 & N_2-\mu I_{n-1}\end{pmatrix}$$

is not singular ($\det M\neq 0$).

Proof. From Lemma 5 it follows that two matrices of change of basis, T_1 and T_2, exist, such that $D_i=T_iN_iT_i^{-1}$ (for $i=1,2$) are matrices which satisfy properties (i) and (ii) in Lemma 5. We define

$$T=\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix} \qquad \text{and} \qquad D=TMT^{-1}=\begin{pmatrix}\beta D_1-\mu I_{n-1} & \beta^{1/2}\mu T_1T_2^{-1}\\ b^{1/2}T_2N_3T_1^{-1} & D_2-\mu I_{n-1}\end{pmatrix}.$$

We will show that matrix D has a dominant diagonal. Consider, successively, the $(n-1)$ first rows of D and the $(n-1)$ last rows:

1. On account of the specific form of D_1, the issue is to show that

$$\|\beta\lambda_i - \mu\| > \beta\omega_i + \beta^{1/2}\|\mu\| f_i \qquad \text{with} \quad f_i = \sum_{j=1}^{n} \|(T_1 T_2^{-1})^{ij}\|.$$

We write $\mu = -(a+ib)$, with $a > 0$, and $\lambda_i = x_i + iy_i$, with $x_i > 0$. Then we square the inequality under considerations:

$$(\beta x_i + a)^2 + (\beta y_i + b)^2 > \beta^2\omega^i + 2\beta^{3/2}\omega^i\|\mu\| f_i + \beta(a^2+b^2) f_i^2.$$

Since $\omega_1 \leqslant v$ (cf. Lemma 5), $\|\mu\| < \alpha + |\beta|$, $(y_i + \beta\alpha)^2 \geqslant (|y_i| - |\beta v|\, a)^2$, it is sufficient to show that

$$b^2(1-\beta f_i^2) - 2|b|\, \beta(|y_i| + \beta^{1/2} v f_i)$$
$$+ \beta^2(\|\lambda_i^2\| - v^2) + a^2(1-\beta f_i^2) + 2\beta a(x_i - \beta^{1/2} v f_i) > 0.$$

This polynomial of the second degree in $|b|$ is greater than its minimum. Under the assumptions made, and defining a strictly positive coefficient $\bar{\beta}^i$ as

$$\bar{\beta}^i = \frac{1}{f_i^2} \min\left(1, \left(\frac{x_i^2 - v^2}{2|y_i|\, v + \beta^{1/2} f_i \|\lambda_i\|^2}\right)^2\right),$$

this minimum is larger than 0, if $\beta < \bar{\beta}^i$.

2. Since D_2 has a dominant diagonal, the same property holds for the last $(n-1)$ rows of D for β sufficiently small. Therefore, $\bar{\beta}^0$ exists, such that, if $0 \leqslant \beta < \bar{\beta}^0$, the diagonal in the last $n-1$ rows is dominant.

3. We choose $\bar{\beta} = \inf \bar{\beta}^i$ for $i = 0, 1, ..., n-1$. If $\mu \in \mathbb{C}_-$, and $\beta \leqslant \bar{\beta}$, D has a strictly dominant diagonal and, thus, is not singular [27, Theorem 1]. The same is true of M which is similar to D.

LEMMA 7. (i) $\hat{\bar{y}}^{-1} Z_1 \hat{\bar{x}}$ *is an H-matrix, with M defined by*

$$M^{ij} = \bar{x}^j \frac{\partial r^i}{\partial x^j};$$

(ii) $\tilde{Z}_3 \hat{\bar{y}}$ *is an H-matrix, with M defined by*

$$M^{ij} \doteq -\frac{z^i \bar{y}^i}{\bar{p}} \frac{\partial y E^i / y^i}{\partial y^j}.$$

Proof. The matrix Z_1 is defined by Eq. (12). Condition (i) in the definition of an H-matrix follows from this equation in a straightforward manner.

It is easy to verify that (cf. Eq. (3))

$$\frac{\partial r^i}{\partial x^j} = -\frac{1+\bar{r}}{(C+\hat{\bar{s}})^i \bar{x}} C^{ij} < 0 \qquad \text{if} \quad i \neq j;$$

$$\sum_{j=1}^{n} \bar{x}^j \frac{\partial r^i}{\partial x^j} = 0 \qquad \text{since } r^i(x) \text{ is homogeneous of degree zero in } x.$$

From these two results, it follows that:

(i) The matrix whose general element is $\bar{x}^j \, \partial r^i/\partial x^j$ is irreducible, since C is irreducible (and since only the off-diagonal elements are relevant in the definition of irreducibility);

(ii) $\bar{x}^i \, \partial r^i/\partial x^i$ is strictly positive.

Thus, conditions (ii), (iii), and (iv) in the definition of an H-matrix are also satisfied.

The matrix $\tilde{Z}_3$ is defined by Eqs. (17) and (19). Therefore, condition (i) is satisfied. The three other conditions can be proved as above. ▮

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium