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Demographic transition, intergenerational transfers and the increase in public and national debts

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Transition démographique, transferts intergénérationnels et dettes publique et nationale

Résumé: Ce papier analyse les politiques et réformes dynamiquement cohérentes d'un système public de transferts intergénérationnels. Si l'Etat accorde un poids suffisamment modéré aux personnes âgées vivantes, les Gouvernements successifs mettront en oeuvre des politiques donnant des résultats équitables pour les différentes générations, alors même que leurs fonctions de bien-être social ne sont pas équitables à l'égard des générations non encore nées. Le rapport de la dette publique au PIB ne changera pas au cours du temps et les consommations des générations successives croîtront au taux naturel de l'économie. Cependant, si le Gouvernement donne un poids plus élevé aux personnes âgées, le rapport de la dette publique au PIB augmentera au cours du temps. Alors, les générations futures paieront des impôts de plus en plus élevés et consommeront de moins en moins. La transition démographique n'interfère pas avec ces résultats, bien qu'elle rende tous les consommateurs plus pauvres. Cependant, il y a la possibilité que le poids des générations âgées dans les préférences de l'Etat ait augmenté récemment et que certains pays industrialises soient entrés dans un processus d'endettement public croissant et d'appauvrissement des générations futures.

Mots clefs : Transferts intergénérationnels, système de retraites par répartition, modèle à générations imbriquées, politiques dynamiquement cohérentes.
Classification JEL : E62, F43, H55.

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Abstract: This paper investigates time consistent policies and reforms of intergenerational transfers. If the weight the Government gives to the living elderly is low enough, successive Governments will implement policies with equitable results across generations, even if their social welfare function is not equitable with the unborn. The ratio of Government public debt to GDP will not change over time, and the consumption flows of successive generations will grow at the natural rate of the economy. However, if the Government gives a higher weight to the elderly, the ratio of public debt to GDP will increase over time. Then, future generations will have to pay higher and higher taxes and consume less and less. Demographic transition does not interfere with these results although it makes every consumer poorer. However, there is the possibility that the weight of the elderly in Government preferences has increased recently, and that some Western democracies are entering a process of increasing public indebtedness and immiserisation of future generations.

Keywords: Intergenerational transfers, pay-as-you-go pension system, overlapping generation model, time consistent policies.
JEL Classification: E62, F43, H55.

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Introduction

In his excellent survey of retirement Cairncross (2004) writes: "A larger generation of old folk than ever before will need support for longer than ever before from a population of working age that is shrinking continuously in absolute size for the first time since the Black Death. Moreover, if things look bad in America and worse in continental Europe, they will one day look calamitous in some parts of the developing world". There are two causes of this demographic transition: a widespread fall in the fertility rate and an increase in life expectancy. This demographic evolution creates a financing problem for the pay-as-you-go systems existing in many industrialised countries. To maintain the level of pensions unchanged, contributions to pension funds must be repeatedly increased. Moreover, the gap between the market interest rate and the implicit rate of return of the system becomes wider. This causes a decline in the economic condition of the young. The most natural reform to reduce the burden on the young is to decrease the level of pensions, which will lower the increase in contributions and free some income of the young generation. This can be invested in the financial markets at better rates. This policy will increase the welfare of young and unborn, but it will decrease the income of the pensioners and the expected income of the people who are planning to retire early.

We can think of adding an ingredient to this reform. The loss of income of pensioners can be compensated by a transfer from the Government. This transfer can be financed by public borrowing. Economists, who advocate this idea, argue that the implicit public debt will just be made more visible, and that the increase in public debt will be financed by the income of the young, freed by the decrease or the slower increase of their contribution. This point of view is of course debatable. The consequence of this policy will be to increase the taxation of future generations. So, it will transfer a sacrifice by the elderly currently alive to people who are not yet born. This proposal is often canvassed in current political discussions. For instance, we can refer to a very good article in *The Economist* ("From slogan to legacy", 13 November 2004), on the state of the debate inside the American administration and the Republican party or to the Economic Focus published by this magazine on 11 December 2004. Feldstein (2005) gives a stimulating discussion of these issues.

Pension funds must balance their budget, and so can manage only intergenerational transfers between people who are currently alive. However, the Government, which can run a deficit, can achieve much more. It can organise any intergenerational transfer between living and still unborn people, under the only constraint of intertertemporal solvency. However, de La Croix and Michel (2002, p. 137 and 188) raise doubt about the justification of using indebtedness to add flexibility to intergenerational transfers. They prove that any Pareto-optimal allocation of consumption between generations and in each age of their life can be obtained by a system of transfers, which is balanced in each period. So, a system of pay-as-you-go compulsory pension funds, which balance their budgets in each period, is sufficient, and the Government has no reason for using public borrowing. However, this result rests on two assumptions. First, the relative weight the Government gives to successive generations in its social welfare function does not change over time. Secondly, the Government can immediately set the levels of transfers for all periods, present and future. Both assumptions are unconvincing. A Government gives more weight to the living than to the unborn. Moreover, if it can set transfers and taxes for the current period, it cannot commit itself to future taxes, which will be decided by other Governments with different objectives. However, the current Government can pressure the next Government by setting the level of its debt, which will be transmitted to its successor.

A quotation by Miles and Cerny (2001) will make our reticence clearer and help to understand why a Government can use public borrowing as an element of its intergenerational transfer policy: « The result that a large proportion of those alive now would be worse off if the unfunded state scheme is phased out – even though every future generation is better off – illustrates the nature of the transition problem rather clearly. Democratically elected governments facing voters who focus on the direct implications to them (and not to all future generations) of changes to state pension systems would find it hard to get support for this kind of transition plan. Table 2 suggests that once a transition from an unfunded to a funded scheme is complete welfare for all subsequent generations will be higher, but without relying on deficit financing the transition will cause certain generations to be worse off, and those generations could form a majority of voters thus permanently blocking any change.”

This paper will adopt the following specification. The Government only takes into account the welfare of the living elderly, the living young and the first unborn generation. The respective weight it gives to each of these generations are 1, A and B. Thus, the relative weight given to a specific generation will change over time. The Government can implement only its current decisions and cannot commit itself to decisions, which will be implemented in 25 or 50 years. Thus, we will look for time consistent policies: in a given period, the Government, which is in power, makes its current decisions by maximising its social welfare function, under the constraint that it perfectly anticipates the future decisions, which will be taken by future Governments. We will especially examine if a democratic Government will favour the elderly to a greater extent, and will finance the cost of its policy by borrowing too much and thereby lowering the welfare of the unborn. We will also consider if this bias against future generations becomes more serious when demographic transition occurs.

The existence and the uniqueness of a series of time consistent policies require two conditions. The first is that the weight given by the Government to the living young relatively to the next generation, A/B, is sufficiently high. This condition allows for sufficient contribution, by future generations, to the financing of public debt and prevents this debt from diverging. The second condition is that the weight the Government gives to the living young must be smaller than or equal to a precise bound $A \leq \overline{A}$. Otherwise, the Government would transfer too much money from the living elderly to future generations. It would do that by running a budgetary surplus, accumulating an increasing level of assets and driving the economy on an inefficient path.

If $A = \overline{A}$, the series of time consistent policies give results, which are equitable for all generations. The deficit of the public sector (pension funds included) and public debts represent constant shares of GDP. The consumption flows of each generation increase at the natural growth rate. However, if $A < \overline{A}$, that is if the weight of the elderly in the social welfare function is high, we get less favourable results. Governments will run higher deficits than before, and the ratio of public debt to GDP will increase over time and converge to a high limit. The successive generations will pay higher and higher taxes to prevent public debt from diverging. Their consumption flows will decrease over time and finally converge to zero. In both cases, demographic transition will make all generations poorer. However, it has no effect on the structure of intergenerational transfers, or on the dynamic path followed by the economy.

Thus, demographic transition should introduce no new difficulty to the problem of intergenerational transfers and of equity with future generations. However, the reality could be less favourable. Cairncross (2004) considers the possibility that the power of the elderly has increased recently in Western countries. This evolution could result from demographic transition and could explain the increases in public deficits and debts observed in many of these countries and their plans to use borrowing to finance a part of pension costs. This change would have dramatic consequences. Heller (2003) gives a fascinating account of the dangers for a society, which puts off costs that should be paid in the current time and which increases the burden of future generations.

The analysis of a pay-as-you-go pension system is usually made by using an overlapping generation model. However, even the simplest versions of this model are complex under the assumption of a closed economy (Azariadis, Demange and Laroque (2000)). The main reason for this complexity comes from the feedback from saving to investment. Hence, in this paper we will use an overlapping generation model of a small open economy. In this case Fisher's separation theorem separates saving from investment decisions as agents can borrow or lend at a fixed interest rate. Then, we will be able to solve the model fully and to answer most of our questions in an analytical manner. Simulations will just be used to illustrate the answers. A consequence of this specification is that it will establish a clear and strong link between the Government deficit and public debt on one hand and the balance of payments and the accumulation of foreign debt on the other. To simplify our presentation we will limit our investigation of demographic transition, to a decrease in the rate of births.

The first section presents the model, shows that it can be solved recursively, and describes the balanced growth path of the economy. It also analyses the consequences of demographic transition when policies are unchanged. The second section computes the time consistent intergenerational transfers and the dynamic path followed by the economy.

1. An overlapping generations model and its properties

The model is constructed for a small open economy. Time is assumed to be discrete, 0 denotes the current period and agents hold perfect expectations. All economic variables are expressed in real terms. $N(t)$ consumers are born at the beginning of period t. They will die at the end of period $t+1$. They work for the first period of their life. Then they retire. Each worker receives wages $w(t)$ and pays the contribution $d(t)$ to a public pension fund in period t. In the same period, each pensioner receives a retirement pension $p(t)$, set by the Government. The number of births increases over time at rate $1+n$. Thus, we have $N(t) = N(1+n)^{t+1}$, for $t \geq -1$, where $N = N(-1)$ denotes the number of people entering the second period of their life at the beginning of period 0.

1.1. Firms

Employment in period t equals $N(t)$. Domestic output in this period $Y(t)$ is given by a Cobb-Douglas function

(1) $Y(t) = P(t)K(t)^{\alpha} N(t)^{1-\alpha}$

$P(t)$, which is the global productivity of factors, grows at the constant rate ρ: $P(t) = P(1+\rho)^t$. Thus, the natural growth rate of the economy is $g = (1+n)(1+\rho)^{1/(1-\alpha)} - 1$. $K(t)$ denotes the amount of capital used in period t. This is available at the beginning of this period. Its accumulation satisfies the accounting identity

(2) $K(t+1) = (1-\delta)K(t) + I(t+1)$,

$I(t+1)$ denotes the investment made at the end of period t, δ is the depreciation rate of capital and $i(t)$ the interest rate in the period. We have the two following marginal conditions

(3) $w(t) = (1-\alpha)Y(t) / N(t)$

(4) $i(t)+\delta=\alpha Y(t)/K(t)$

1.2. The public sector

The public sector has two components. The first is the public pension fund system. It balances its budget in each period, which gives the following condition in period t

(5) $d(t)=p(t)/(1+n)$

The second component of the public sector is Government. In period t, Government consumes $G(2+n)(1+g)^t$ and it makes the lump sums transfers $s_1(t)$ and $s_2(t)$, respectively to each worker and to each pensioner. Transfers are net of taxes and Government's consumption is proportional to population. Let us introduce two new variables

(6) $s_1^{'}(t)=s_1(t)-p(t)/(1+n)$
(7) $s_2^{'}(t)=p(t)+s_2(t)$

$s_1^{'}(t)$ and $s_2^{'}(t)$ denote net public transfers made by the Government *and* pension funds, respectively to each young person and to each old person, in period t.

Government's consumption and net transfers are exogenous and increase at the natural growth rate of the economy. The public debt at the beginning of period t is $B(t-1)$. If we assume that the interest rate is higher than the natural growth rate, then the ratio of public debt to domestic output will diverge. To prevent this from happening, we introduce a tax, indexed on the amount of public debt, of rate $\nu(t)$ and only paid by the young. The budgetary equilibrium of the Government is

(8) $B(t)=G(2+n)(1+g)^t+N(1+n)^{t+1}\left[s_1^{'}(t)+s_2^{'}(t)/(1+n)\right]+(1+i^*-\nu(t))B(t-1)$

1.3 Accounting and equilibrium equations

Domestic and foreign goods are perfect substitute. Then, the surplus of the trade balance is equal to the excess of production over absorption

(9) $BT(t) = Y(t) - C_1(t) - C_2(t) - I(t+1) - G(2+n)(1+g)^t$

$C_1(t)$ represents the consumption in period t of the generation born at the beginning of periods t and $C_2(t)$ is the consumption in the same period of the generation born at the beginning of period $t-1$.

$F(t)$ denotes the stock of foreign assets held at the beginning of period t. It generates the interest income $i(t)F(t)$. Thus, the balance of payments identity in period t is

(10) $BT(t) = F(t+1) - (1+i(t))F(t)$

We assume that the domestic interest rate is equal to the interest rate in the rest of the world, i^*, which is exogenous and constant over time

(11) $i(t) = i^*$

We make the following assumption:

Assumption 1. The interest rate in the rest of the world is higher than the domestic natural growth rate $i^* > g$.

This assumption will imply that the equilibrium of the model will be dynamically efficient. Our results will be very sensitive to this condition, which is considered as realistic by most macroeconomists (for a discussion see Obstfeld and Rogoff (1996), pp. 191-195).

1.4. Plan of a cohort of consumers born at the beginning of period $t \geq 0$

We will assume that consumers receive no endowment at their birth and leave no bequest when they die. The non capital incomes of the cohort born at the beginning of period t, in the two periods of its life are

(12) $\omega_1(t) = N(1+n)w(1+g)^t + N(1+n)^{t+1}s_1^{'}(t) - \nu(t)B(t-1)$

(13) $\omega_2(t+1) = N(1+n)^{t+1}s_2^{'}(t+1)$

The wealth at birth of this cohort is

(14) $W(t) \equiv \omega_1(t) + \omega_2(t+1)/(1+i^*)$

This cohort consumes $C_1(t)$ and $C_2(t+1)$ in the two periods of its life. These consumption flows must satisfy the budgetary constraint

(15) $C_1(t)+C_2(t+1)/(1+i^*)=W(t)$

The wealth of this cohort at the end of period t, $A(t)$, which is identical to national wealth, is

(16) $A(t)=\omega_1(t)-C_1(t)$

This wealth is equal to the sum of domestic capital, foreign assets and public debt

(17) $A(t)=K(t+1)+F(t+1)+B(t)$

We have to be more precise with the definition of the equilibrium of the model in period t. At the end of period $t-1$, consumers' wealth, public debt, foreign assets and productive capital, $A(t-1)$, $B(t-1)$, $F(t-1)$ and $(1-\delta)K(t-1)$, are inherited from the past. The interest rate is set by the rest of the world and employment is exogenous. These two variables, the production function (1) and the marginal condition (4) determine the amount of productive capital used in the period $K(t)$. Thus, we must have at the beginning of period t an investment equal to $I(t)=K(t)-(1-\delta)K(t-1)$, which instantaneously drives the quantity of capital to the level that is required by the conditions in the period. The difference between the consumers' wealth, which was inherited from the past, and the sum of the required amount of productive capital and public debt, sets the quantity of foreign assets held at the beginning of period t: $F(t)=A(t-1)-K(t)-B(t-1)$. The balance of trade in period t is equal to the difference between the output in the period $Y(t)$ and the sum of private consumption, Government consumption and the investment that will be instantaneously made at the beginning of period $t+1$ (equation (9)). The surplus of the balance of trade plus the interest income on foreign assets, $i^*F(t)$, is equal to the increase in foreign assets between the beginning of period t and the beginning of period $t+1$: $F(t+1)-F(t)$.

We assume that consumers take their lifetime decisions at the time of their birth. They perfectly forecast their future incomes. Their discount rate is $\beta>0$ and their preference for consumption is logarithmic. Thus, the utility at birth of the cohort of consumers born at the beginning of period t is

(18) $U(t) = N(t)\left[\ln\left[C_1(t)/N(t)\right] + \ln\left[C_2(t+1)/N(t)\right]/(1+\beta)\right]$

The maximisation of this utility under the budgetary constraint (15) gives the consumption of this cohort in the two periods of its life

(19) $C_1(t) = W(t)(1+\beta)/(2+\beta)$

(20) $C_2(t+1) = W(t)(1+i^*)/(2+\beta)$

1.5. Plan of the cohort of consumers born at the beginning of period -1

This cohort enters the second part of its life at the beginning of period 0. Then, its wealth inherited from the past is $A(-1)$. Moreover, this cohort receives the non-wealth income

(21) $\omega_2(0) = Ns_2^{'}(0)$

The consumption of this cohort in period 0 is equal to the sum of its wealth, the interest income earned on this wealth and its non-wealth income

(22) $C_2(0) = (1+i^*)A(-1) + \omega_2(0)$

The utility of this cohort at the beginning of period 0 is

(23) $U(-1) = N\ln\left[C_2(0)\right]$

We make the following assumption, which implies that the total wealth of each cohort is non negative.

Assumption 2. The Government must satisfy the constraints $s_2^{'}(0) + (1+i^*)A(-1)/N \geq 0$ and

$w(1+n)(1+\rho)^t + (1+n)\left[s_1^{'}(t) + s_2^{'}(t+1)/(1+i^*)\right] - \nu(t)B(t-1)/\left[N(1+n)^t\right] \geq 0$, for $t \geq 0$

1.6. Solution of the model

The model can be solved, analytically and recursively. Equations (11) and (5) give the expressions of the interest rate and the contribution to the public pension fund $i(t)$ and $d(t)$. Equations (1), (3) and (4) give the expression of the wage rate, domestic output and capital

(24) $w(t) = w(1+\rho)^t$, $w = \alpha^{\alpha/(1-\alpha)}(1-\alpha)(i^*+\delta)^{-\alpha/(1-\alpha)}P^{1/(1-\alpha)}$

(25) $Y(t) = Y(1+g)^t$, $Y = P^{1/(1-\alpha)}[\alpha/(i^*+\delta)]^{\alpha/(1-\alpha)} N(1+n)$

(26) $K(t) = K(1+g)^t$, $K = [\alpha P/(i^*+\delta)]^{1/(1-\alpha)} N(1+n)$

Equation (2) determine investment

(27) $I(t) = I(1+g)^{t-1}$, $I = K(g+\delta)$

Equation (8) determines the dynamics of public debt $B(t)$ (the initial level of public debt $B(-1)$ is given). Equations (12), (13) and (14) give the expressions of the incomes and wealth at birth of the cohort born at the beginning of period t, $\omega_1(t)$, $\omega_2(t+1)$ and $W(t)$. Equations (18), (19) and (20) give the expressions of the consumption of this cohort over its lifetime and of its utility at birth, $C_1(t)$, $C_2(t+1)$ and $U(t)$. Finally, equations (21), (22) and (23) give the expressions of the income, the consumption and the utility of the cohort of people who are old in period 0, $\omega_2(0)$, $C_2(0)$ and $U(-1)$ (the initial wealth of this cohort, $A(-1)$, is given).

Equation (9) determines the balance of trade in the same period $BT(t)$. Equation (16) gives national wealth at the end of period t, $A(t)$. Finally, we deduce from equation (17) the amount of foreign assets held at the beginning of period t $F(t) = A(t-1) - K(t) - B(t-1)$. Then, we can easily prove that equation (10) is satisfied, which means that this equation is redundant in the model. This redundancy is equivalent to the Walras' identity.

1.7. Balanced growth paths

On this path, the rate of taxation based on debt is set to the constant value ν and the other policy variables follow the geometric trends $p(t) = p(1+\rho)^t$, $s_1^{'}(t) = s_1^{'}(1+\rho)^t$, $s_2^{'}(t) = s_2^{'}(1+\rho)^t$, for $t \geq 0$. Then, the contribution to the pension fund $d(t)$ also follows a geometric trend, which can be deduced from equation (5). The path of public debt is deduced from equation (8) and equal to

(28) $B = [G(2+n) + s_1^{'}N(1+n) + s_2^{'}N]/(g+\nu-i^*)$, $B(t) = B(1+g)^{t+1}$, for $t \geq -1$

If we exclude the debt stabilisation tax from the definition of primary deficit, this deficit in period 0 is equal to $G(2+n)+s_1^{'}N(1+n)+s_2^{'}N$. The following assumption will establish that a positive (negative) primary deficit of the Government is associated with a positive (negative) amount of public debt.

Assumption 3. On balanced growth paths, the rate of taxation on public debt satisfies the condition $g+\nu-i^*>0$ *and also the constraint that* $\nu \le 1+i^*$. *Moreover, the total public deficits and the Government deficits, generated by each generation are positive* $G+s_1^{'}N, G+s_2^{'}N, G+s_1 N, G+s_2 N>0$.

The incomes flows and the wealth at birth of the cohort born at the beginning of period $t \ge 0$, can be deduced from equations (12), (13) and (14) and are equal to

(29) $\omega_1(t)=\omega_1(1+g)^t$, $\omega_1 = N(w-d+s_1^{'})(1+n)-\nu B$

(30) $\omega_2(t+1)=\omega_2(1+g)^{t+1}$, $\omega_2 = N(p+s_2^{'})$

(31) $W(t)=W(1+g)^t$, $W=\omega_1+\omega_2(1+g)/(1+i^*)$

The consumption flows of this cohort are deduced from equations (19) and (20) and are equal to

(32) $C_1(t)=C_1(1+g)^t$, $C_1 = W(1+\beta)/(2+\beta)$

(33) $C_2(t+1)=C_2(1+g)^{t+1}$, $C_2 = W(1+i^*)/[(1+g)(2+\beta)]$

Finally, equations (9), (16) and (17) show that the trade balance, private wealth at the end of period t and foreign assets at the beginning of period t are respectively equal to

(34) $BT(t)=BT(1+g)^t$, $BT=Y-C_1-C_2-I-G(2+n)$

(35) $A(t)=A(1+g)^{t+1}$, $(1+g)A=\omega_1-C_1$, for $t \ge -1$

(36) $F(t)=F(1+g)^t$, $F=(\omega_1-C_1)/(1+g)-K-B$

1.8. The effects of a permanent decrease in the growth rate of births

We will assume that the number of people entering the second period of their life in period 0, $N(-1) = N$, is given. The number of births in period $t \geq 0$ is $N(t) = N(1+n)^{t+1}$. We interpret demographic transition as a permanent decrease in the growth rate of births n by $\Delta n < 0$. As the growth rate of the global productivity of factors, ρ, does not change, the natural growth rate of the economy decreases by $\Delta g/(1+g) = \Delta n/(1+n)$. We will assume that the levels of pensions $p(t)$ and of, either total transfers to each generation $s_1^{'}(t)$ and $s_2^{'}(t)$, or the levels of the transfers made by the Government $s_1(t)$ and $s_2(t)$, are set by the Government and exogenous. Equations (1), (2), (3) and (4) show that the paths of domestic output, wages, capital and investment remain unchanged. We have the following proposition:

Proposition 1. A decrease in the growth rate of births starting at time 0, leads

a) to an increase in the contributions to the public pension fund by people born in period 0 and afterward,

b) to an increase in public debt per head,

c) has no consequence for the elderly living in period 0. The consumption flows of a consumer born in period $t \geq 0$ decrease, as the wealth of this consumer at the end of period t.

These results stay valid if we assume, either that total public transfers, or that Government's transfers to each generation are kept constant.

Proof. See Appendix.

The demographic transition has two negative consequences for people born at the time when the transition takes place or afterward. First, as pensions are maintained at previous levels, contributions by each worker to public pension funds will have to increase. The implicit rate of return of the pension system becomes even more unfavourable in comparison to the market interest rate. Secondly, public debt per head will increase and converge to a higher public debt ratio, because the difference between the interest rate and the growth rate of the economy increases. So the stabilisation of this higher level of debt will be more costly to the tax payers.

Production and investment decrease because the number of workers declines. Private consumption decreases progressively and public consumption, which is indexed on the total population, decreases. So, the effect of demographic transition on domestic saving and on the balance of trade is undetermined. However, we can note that this effect is more unfavourable in the short than in the long run. With a much richer model, Fehr, Jokisch and Kotlikoff (2003, p. 24) obtain for the European Union some deterioration of the balance of trade in the short-run and an improvement in the long run.

2. Time consistent policies

2.1. The problem

We can easily prove that the sum of the discounted wealth at birth of all generations, computed in period 0, is independent of intergenerational transfers. More precisely we have the relation

$$(37)\quad \begin{aligned} &\sum_{t=0}^{\infty}\left[C_1(t)+C_2(t+1)/(1+i^*)\right]/(1+i^*)^t + C_2(0) = \\ &\left[N(1+n)w - G(2+n)\right](1+i^*)/(i^*-g) + (1+i^*)\left[B(-1)-A(-1)\right] \end{aligned}$$

This result shows that a reform of a pay-as-you-go pension system or more generally of the public transfer policy to the successive generations will improve the welfare of some generations and lead to a decline in the utility of other generations. For instance, we can easily prove that a permanent decrease in retirement pensions will increase the consumption in each period of the living young and of future generations. However, the consumption of the elderly will decrease. We can also prove that an increase in Government’s transfers to a cohort of pensioners, financed by public borrowing, will increase the consumption of this cohort. However, it will also increase taxes paid by future generations and will reduce their consumption. This efficiency result is well known (see for instance Azariadis, 1993, or Feldstein and Liebman, 2002). Its consequence is that the system of pensions and of intergenerational transfers, which prevails in an economy, results from political compromise and arbitrage between the various generations or the people who care for them. We proved that demographic transition changes the relative welfare of the generations. Thus, we can expect that it will induce a political reaction, which will change the system of pensions and of intergenerational transfers.

We will deal with this problem by introducing a social welfare function of the Government, with the following specification

(38) $\Omega = U(-1) + AU(0) + BU(1)$, $A > B > 0$

This function gives different weights to the utilities $U(-1)$ of the elderly alive in period 0, $U(0)$ of the cohort of the youth who are also currently alive, and $U(1)$ of the cohort of people who will be born in the next period. The Government does not care for generations, which will be born in later periods. Other social welfare functions could be considered. However, this specification is simple and sufficient to establish our results.

At time 0, the Government will have to consider the levels of transfers and taxes, $s_1^{'}(t)$, $s_2^{'}(t)$ and $v(t)$ for every period $t \geq 0$. However, the optimal plan it could compute for all these periods, by maximising function Ω, would be time inconsistent. The reason is that the relative weight of the various cohorts in the social welfare function changes over time. Thus, we'd better look for time consistent policies, which assume that only decisions taken for the current period are implemented. The Government cannot commit itself to decisions, which will be implemented by future Governments, which will then use a different social welfare function.

2.2. Characteristic of time consistent policies

In period $t \geq 0$, the Government sets the transfers to the generations living in this period, $s_1^{'}(t)$ and $s_2^{'}(t)$ and the tax rate $v(t)$, by maximising a social welfare function, for given anticipated levels of the policy variables for future periods, and for a predetermined level of the wealth of the old generation and of public debt at the beginning of the period. We have the following lemma

Lemma 1. The time consistent policy, which is implemented in period $t \geq 0$ *:* $s_1^{'}(t)$*,* $s_2^{'}(t)$ *and* $\nu(t)$*, is determined by the following system of equations*

$$(39)\quad \begin{aligned}&[B\nu(t+1)(1+n)/A]\{Nw(1+n)(1+g)^t + N(1+n)^{t+1}[s_1^{'}(t)+s_2^{'}(t+1)/(1+i^*)] - \nu(t)B(t-1)\} = \\ &Nw(1+n)(1+g)^{t+1} + N(1+n)^{t+2}[s_1^{'}(t+1)+s_2^{'}(t+2)/(1+i^*)] - \nu(t+1)B(t)\end{aligned}$$

$$(40)\quad \begin{aligned}&B\nu(t+1)(1+n)^2\{(1+i^*)A(t-1)+N(1+n)^t s_2^{'}(t)\}(2+\beta)/(1+\beta) = \\ &Nw(1+n)(1+g)^{t+1} + N(1+n)^{t+2}[s_1^{'}(t+1)+s_2^{'}(t+2)/(1+i^*)] - \nu(t+1)B(t)\end{aligned}$$

Proof. See Appendix.

The series of time consistent policies, and the paths followed by public indebtedness $B(t)$ and the wealth of the old generation $A(t)$, for $t \geq 0$, are given by the system composed of these two equations, of equation (8) and of the following equation (deduced from equation (16))

$$(41)\quad A(t) = \frac{N(1+n)^{t+1}}{2+\beta}\left[s_1^{'}(t) - \frac{1+\beta}{1+i^*}s_2^{'}(t+1)\right] - \nu(t)B(t-1)/(2+\beta) + \frac{N(1+n)w(1+g)^t}{2+\beta}$$

The initial values of the predetermined variables of this system, $A(-1)$ and $B(-1)$, are given.

Lemma 2. The series of time consistent policies must satisfy the condition

(42) $\nu(t) = \nu = (A^2/B)(1+i^*)/(1+\beta)$, for $t \geq 1$

Proof. See Appendix.

This lemma leaves $\nu(0)$ undetermined. However, if the Government increases $\nu(0)$ by $\Delta\nu$ and $s_1^{'}(0)$ by $\Delta\nu B(-1)/[N(1+n)]$, nothing else changes in the economy, especially the current consumption and saving of the youth. So, to raise this undetermination, we will adopt the following assumption.

Assumption 4. The Government sets the tax rate on public debt in period 0, to the same level ν *as the following Governments.*

To solve the dynamic system, we must, first, rewrite it in terms of reduced variables. These variables are defined as $s_1^{'*}(t) = s_1^{'}(t)/(1+\rho)^t$, $s_2^{'*}(t) = s_2^{'}(t)/(1+\rho)^t$, $B^*(t) = B(t)/(1+g)^{t+1}$, $A^*(t) = A(t)/(1+g)^{t+1}$, $W^*(t) = W(t)/(1+g)^t$ and $C_1^*(t) = C_1(t)/(1+g)^t$, $C_2^*(t) = C_2(t)/(1+g)^t$. Then, we get the following lemma:

Lemma 3. The sequence of time consistent policies is determined by the following system of equations, which includes two predetermined variables $A^*(t)$ and $B^*(t)$ (which appear with a lag), one anticipated policy variable $s_2^{'*}(t)$ (which appears with a lead) and a static variable $s_1^{'*}(t)$, for $t \geq 0$

(43) $A\left[s_2^{'*}(t) + (1+i^*)A^*(t-1)/N\right] = \left[s_2^{'*}(t+1) + (1+i^*)A^*(t)/N\right](1+\beta)(1+\rho)/(1+i^*)$

(44) $$\begin{aligned}&(2+\beta)(1+g)A^*(t) = \\ &\left[(1+g)B^*(t) - (1+i^*)B^*(t-1) - G - Ns_2^{'*}(t) - Ns_2^{'*}(t+1)(1+\beta)(1+g)/(1+i^*) + (1+n)Nw\right]\end{aligned}$$

(45) $(1+g)B^*(t) = G(2+n) + N(1+n)\left[s_1^{'*}(t) + s_2^{'*}(t)/(1+n)\right] + (1+i^*-\nu)B^*(t-1)$

with $A^*(-1) = A(-1)$ and $B^*(-1) = B(-1)$, given.

Proof. See Appendix.

The next lemma will help to determine the solution of the dynamic system and the path followed by the economy. We have:

Lemma 4. a) Time consistent policies can be computed by

1. *Setting $s_1^{'*}(t)$, for $t \geq 1$, to arbitrary levels.*
2. *Solving the system of equations (43), (44) and (45) in $s_2^{'*}(t)$, $A^*(t)$ and $B^*(t)$, for $t \geq 0$ and in $s_1^{'*}(0)$, for $A^*(-1)$, $B^*(-1)$ given.*

b) This system has three non zero eigenvalues, which are respectively equal to $A(1+i^*)/(1+\beta)(1+\rho)$, $(1+i^*)/(1+g)$ and $-\nu/(1+g)$. The necessary and sufficient condition *for the existence and the uniqueness of a consistent policy is that two of these eigenvalues are larger than 1 and one is less or equal to 1.*

c) The path of the economy, which results from a time consistent policy, satisfies the following properties

1. *The wealth at birth and the consumption of the successive generations grow at constant rate*

(46) $W^*(t+1)/W^*(t) = C_2^*(t+1)/C_2^*(t) = C_1^*(t+1)/C_1^*(t) = A(1+i^*)/[(1+\beta)(1+\rho)],\ t \geq 0$

2. *The ratio between the consumption of the elderly and the consumption of the youth in a given period, is given by*

(47) $C_2^*(t)/C_1^*(t) = 1/[A(1+n)]$

d) A change in $s_1^{'*}(t)$ *for* $t \geq 1$ *leaves the consumption flows and the wealth at birth of each generation unchanged, but implies modifications in public debt* $B^*(t-1)$*, in national wealth* $A^*(t-1)$ *and in* $s_2^{'*}(t)$ *. However, if the change takes place for* $t=1$*,* $s_1^{'*}(0)$ *changes but not* $s_2^{'*}(0)$.

The Government of period $t \geq 1$ can set $s_1^{'*}(t)$ and $s_2^{'*}(t)$. If in this period it wants to increase the wealth of the young generation to a level higher than the one the Government of period $t+1$ will consider optimal, the latter Government will destroy this attempt by reducing $s_2^{'*}(t+1)$. So, the Government can set $s_1^{'*}(t)$ at any level, without affecting the welfare and the consumption choices of the generation born in period t. This is a kind of Ricardian equivalence, which is valid in the overlapping generation model when the modulation of transfers and taxes take place inside the same generation. To remove the ambiguity in the values of the $s_1^{'*}(t)$ anticipated in period 0 for periods $t \geq 1$, we will assume that these anticipations are constant, and at a level such that they are equal to the current decision $s_1^{'*}(0)$. However, the Government of period $t \geq 1$ can set $s_2^{'*}(t)$ and the welfare of the old generation in period t to the level it wants because no future Government can correct this decision. Assumption 1 implies that one of the eigenvalues is larger than 1. If

$A(1+i^*)/(1+\beta)(1+\rho) \neq 1$, the system has a unique steady state such that $C_2^* = C_1^* = W^* = 0$. We want to keep available the situation of hysteresis (see Laffargue (2004)) when $A(1+i^*)/(1+\beta)(1+\rho) = 1$ and when the economy does not converge to this degenerated state. However, the existence and uniqueness of a solution under this condition require that the two other eigenvalues are larger than 1. Thus, we make the following assumption.

Assumption 5. In period 0, the Government anticipates that the transfers to the youth, which will be decided by the successive Governments will be the same as its current decision $s_1^{'*}(0) = s_1^{'*}(t) = s_1^{'*}$, for $t \geq 1$. *Moreover, we assume that* $\nu > 1+g$.

Then, a necessary and sufficient condition for the existence and the uniqueness of a time consistent policy is that $A(1+i^*)/(1+\beta)(1+\rho) \leq 1$. A sufficient condition for the validity of the second part of Assumption 5 is that $A/B > 1+n$. This means that the Government gives more weight, in its social welfare function, to a member of the cohort of the youth currently alive than to a member of the next cohort. We must consider successively the two cases when $A(1+i^*)/[(1+\beta)(1+\rho)]$ is equal to 1 and when it is smaller than 1.

2.3. The case of a Government equitable with future generations $A(1+i^*)/[(1+\beta)(1+\rho)] = 1$

The dynamic system has an eigenvalue equal to 1 and two eigenvalues larger than 1. Then, the economy immediately adjusts to the unique balanced growth path consistent with the initial conditions inherited from the past for private financial wealth and public debt, $A(-1)$ and $B(-1)$. More precisely, we have for $t \geq 0$ $A^*(t) = A^*$, $B^*(t) = B^*$, $s_2^{'*}(t+1) = s_2^{'*}$. So, we must compute these three long run values, plus the two values of policies in period 0 $s_1^{'*}$ and $s_2^{'*}(0)$. We have the proposition:

Proposition 2. When policies are time consistent and when the Government is equitable with future generations, the path of the economy is described by the following

a) *Public indebtedness* $B^*(t)$ *and the wealth of the old generation* $A^*(t)$ *stay equal to their initial values, inherited from the past* $B(-1)$ *and* $A(-1)$. *Moreover, the transfers to the elderly stay constant over time* $s_2^{'*}(0) = s_2^{'*}$.

b) *The transfers to the elderly* $s_2^{'*}$ *is determined by the equation*

(48) $[1+(1+g)(1+\beta)/(1+i^*)]Ns_2^{'*} = -(2+\beta)(1+g)A(-1) + (g-i^*)B(-1) - G(2+n) + (1+n)Nw$

c) *The consumption of the youth and the elderly in period* t *is given by the equation*

(49) $C_1^*(1+i^*)/(1+\beta)/(1+g) = C_2^* = Ns_2^{'*} + (1+i^*)A(-1)$

d) *The transfers to the youth* $s_1^{'*}$ *is given by the equation*

(50) $(1+n)Ns_1^{'*} = (g+\nu-i^*)B(-1) - (2+n)G - Ns_2^{'*}$

e) In the case of demographic transition, when n *decreases, then the transfers to the elderly and the consumption of each consumer in each period of its life decreases.*

The model was calibrated on French data. The period was set to 25 years. The expansion rate of the young generations went from 1+n=1.137 to 1+n=0.920, because of demographic transition. The model was simulated and its eigenvalues computed with the software Dynare, which was run under Matlab (Dynare was developed by Michel Juillard, and can be unloaded from the website: http://www.cepremap.cnrs.fr/dynare). We get the following results.

Table 1

	1+n=1.137	1+n=0.920
$s_1^{'*}$	-0.1736	-0.1693
$s_2^{'*}$	-0.1736	-0.1894

$A*$	0.5144	0.5144
$B*$	0.5294	0.5294
C_1^*	7.3827	5.7799
$C_1^*/(1+n)$	6.4931	6.2825
C_2^*	7.0474	6.8187
Y	22.0565	19.4195
BT	1.4352	1.4323
F	-0.5761	-0.5090
S	2.2429	1.9840

The most important result obtained is that demographic transition has no effect on public debt. So, the intergenerational transfers by the Government must respect the same budget constraint as pay-as-you-go systems when the environment changes. Intergenerational transfers are constrained to take place between the livings, which we interpret as a rule of equity with future generations

The implicit return on pension funds decreases when population growth decreases. Then, it is fair for the Government to reduce the contribution rate of the young. So, the elderly suffer at a double extent: there are less young people to finance their pension, and the contributions of these people decreases. The table shows that the contribution of the young decreases, but the transfers to the elderly decrease by more. The consumption of the young decreases much, because there are less of them. However, their consumption per head does decrease by less and in the same proportion as the consumption of the elderly.

Domestic output decreases because there are less young people in the population. Government's consumption decreases because it is indexed on the total population. Finally national saving decreases. As investment follows output and decreases, the balance of trade and net foreign assets hardly move.

We still have to explain why we consider that the Government is equitable with future generations in this case. After all, the condition $A/B > 1+n$ shows that the Government puts more weight in its social welfare function on living young consumers than on consumers who will be born in the next period. It puts no weight on the utility of future generations. These features look hardly consistent with intergenerational equity. However, our results show that reduced public debt, that is the ratio of public debt at the end of a period to domestic output in the period, does not change over time. This means that the Government does not transfer the cost of its transfers to the living generations toward unborn generations. In the same way, the reduced wealth of each consumer at the beginning of the second period of its life remains unchanged. When, the consumers' discount rate is equal to the international interest rate, equation (49) shows that the ratio between the consumption of the youth and the consumption of the elderly in the same period is equal to $1+g$. This is consistent with the fact that each generation has the same consumption in each period of its life, and that consumption flows grow from generation to generation at rate $1+g$. All these results are equitable.

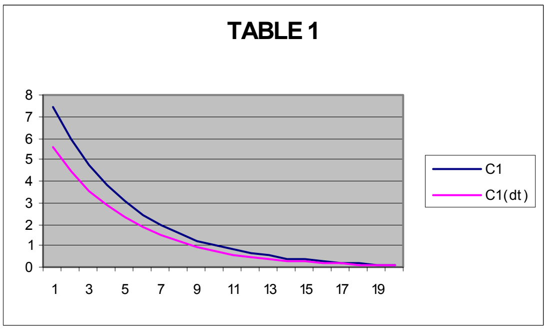
2.4. The case of a Government immiserising future generations $A(1+i^*)/\left[(1+\beta)(1+\rho)\right] < 1$

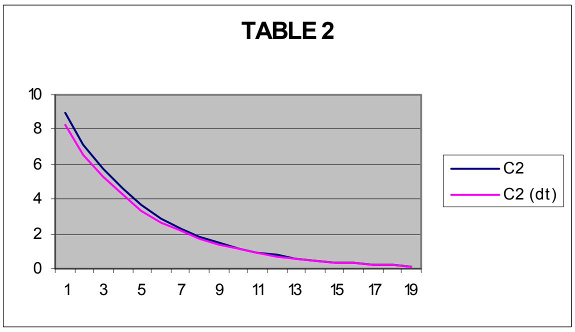
We decrease now the preference of the Government for the young generation relatively to the old generation A, and of the next generation relatively to the young generation B/A, and keep the tax rate on public debt ν unchanged. The value of A is set to 80% of its value in the previous paragraph. So, we get an eigenvalue of 0.8 instead of 1. The two other eigenvalues do not change.

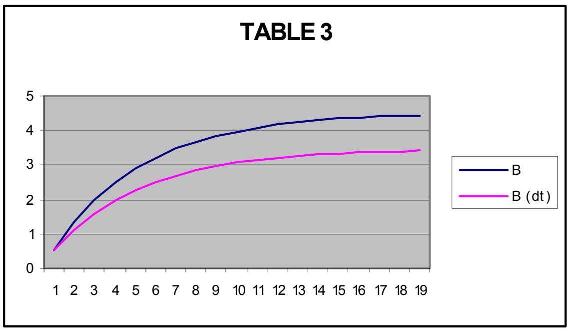
The reduced wealth at birth and the reduced consumption flows of the successive cohorts decrease at a geometrical rate and tend to zero. The reason is that the Government gives too generous transfers to the elderly in period 0. This increases public debt and reduces the disposable income of the following generations. In each period, the Government will continue to be too generous with its elderly and to increase its public debt. Finally, the payments on the debt endured by consumers will become so high that they will have no money left for their own consumption.

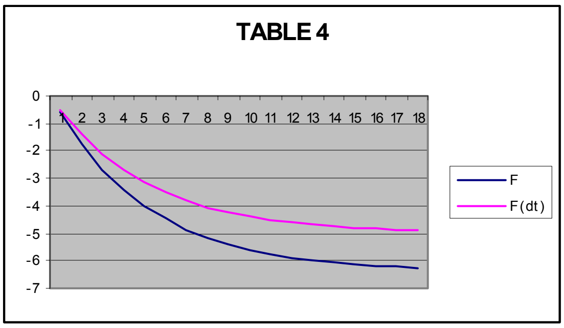
The results of the simulation without demographic transition can be compared to the second column of Table 1. The net transfers to each young person is kept constant over time, but at a lower level than in previous paragraph $s_1^{'*} = -0.2053$. The net transfer to each old person starts at -0.0448 and decreases progressively to -0.1375. It is always higher than in the last paragraph. Public indebtedness increases to a long run level of 4.512. The wealth of the elderly at the beginning of the second part of their life decreases to a long run level of -1.2957. Finally, foreign assets decrease to a long value of -6.3689. The initial values of these three variables are the same as in the previous paragraph.

The following graphs represent the consumption of each generation, Government debts and foreign assets, in both cases without and with demographic transition. The second case is identified by the addition of (dt).









The initial values of the consumption of the youth respectively are 7.4726 and 5.5611. This consumption is slightly higher than in the previous paragraph in the case without demographic transition, and it is lower in the other case. The consumption per young consumer ($C_1^* /(1+n)$) under demographic transition is lower than in the last paragraph. The initial values of the consumption of the elderly respectively are 8.9164 and 8.2376. They are significantly higher than in the previous paragraph. So, decreasing parameter A increases the welfare of the old people who are currently alive and decreases the welfare of the unborn generations. The welfare of the young people, who are currently alive, increases in the case without demographic transition and decreases under demographic transition. Public debt increases over time and foreign assets decrease, with and without demographic transition.

An interesting result is that demographic transition has no effect on the rate of decay of the welfare of successive generations. So, financing the cost of the transfers to living people by borrowing and so by taxing future generations, a political idea considered in the beginning of the paper, will depend on the preference of the Government, but not on demographic transition. Thus, there is no justification for the Government to weaken the costs of the transition from a pay-as-you go system to a capitalisation system by increasing its public debt.

3. Conclusion

This paper uses an overlapping generations model of an open economy to analyse the setting of time consistent intergenerational transfers policies. Governments make their decisions without putting much weight on the welfare of future generations. However, if the weight they put on the welfare of the living elderly is not too high, the results of their policy will give equitable results across generations. If this weight becomes higher, Governments will increase their transfers to the elderly, and finance these higher costs by borrowing more. Public indebtedness will increase, the successive generations will have to pay higher and higher taxes, and the consumption of these generations will become lower and lower. Demographic transition does not change these results. It only makes all consumers poorer. However, there is the possibility that a higher number of pensioners have increased the weight of the elderly in Governments' preferences, and that some Western countries are forsaking their equitable handling of future generations for a policy of increasing indebtedness and immiserisation of the unborn.

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APPENDIX. Proofs

Proof of Proposition 1

a) Equation (5) shows that the contribution paid by a young person born in period $t \geq 0$, increases by $\Delta d(t) = -p(t)\Delta n/(1+n)^2 > 0$.

b) We make the following change of variable $b(t) = B(t)/(1+g)^{t+1}$. Then equation (8) becomes

$(1+g)b(t) = G(2+n) + [s_1^{'}(1+n) + s_2^{'}]N + (1+i^* - \nu)b(t-1)$, $t \geq 0$, with $b(-1) = B(-1)$ given

The steady state solution of this equation is $b = \{G(2+n) + [s_1^{'}(1+n) + s_2^{'}]N\}/(g + \nu - i^*)$. We assume that the initial value $b(-1)$ is equal to the steady state solution of equation (8). Then, the solution of equation (8) is $b(t) - b = \left(\dfrac{1+i^*-\nu}{1+g}\right)^{t+1}[b(-1) - b]$. When n changes by Δn, b and $b(t)$ change by Δb and $\Delta b(t)$:

$$\Delta b = [G + s_1^{'}N - b(1+\rho)]\Delta n/(g+\nu-i^*) = -\frac{(1+i^*-\nu)(G+s_1^{'}N) + (1+\rho)(G+s_2^{'}N)}{(g+\nu-i^*)^2}\Delta n > 0$$

$$\Delta b(t) = \{1 - [(1+i^*-\nu)/(1+g)]^{t+1}\}\Delta b > 0$$

If we assume that s_1 and s_2, instead of s_1' and $s_2^{'}$ are kept unchanged, we have to substitute the two first variables to the two last variables in the expression of Δb and nothing else is changed.

c) Equations (21) and (22) show that the income and the consumption of each old consumer living in period 0 are unchanged. Equations (12), (13) and (14) give the changes in the incomes and wealth of each consumer born in period $t \geq 0$

$$\Delta(W(t)/N(t)) = \Delta(\omega_1(t)/N(t)) = -\nu\Delta(B(t-1)/N(t)) < 0, \ \Delta(\omega_2(t+1)/N(t)) = 0$$

We just have to add the term $p(1+\rho)^t\Delta n/(1+n)^2 < 0$ to the right-hand side of the first equation, if we assume that s_1 and s_2, instead of s_1' and $s_2^{'}$ are kept constant. Equations (19) and (20) show that $\Delta[C_1(t)/N(t)]$ and $\Delta[C_2(t)/N(t)]$ are negative. The change in the wealth at the end of period t of this consumer are given by equation (16) $\Delta(A(t)/N(t)) = \Delta(\omega_1(t)/N(t)) - \Delta(C_1(t)/N(t)) < 0$ □□

Proof of Lemma 1

The program, which determines the time consistent policy of the Government implemented at time $t \geq 0$ is

$$\underset{s_1^{'}(t), s_2^{'}(t), \nu(t)}{Max} \left[U(t-1) + AU(t) + BU(t+1)\right]$$

$$U(t-1) = N(1+n)^t \ln\left\{(1+i^*)A(t-1) + N(1+n)^t s_2^{'}(t)\right\}$$

$$U(t) = \frac{2+\beta}{1+\beta} N(1+n)^{t+1} \ln\left[N(1+n)w(1+g)^t + N(1+n)^{t+1}\left[s_1^{'}(t) + s_2^{'}(t+1)/(1+i^*)\right] - \nu(t)B(t-1)\right]$$

$$U(t+1) = \left[(2+\beta)/(1+\beta)\right]N(1+n)^{t+2}$$
$$\ln\left[N(1+n)w(1+g)^{t+1} + N(1+n)^{t+2}\left[s_1^{'}(t+1) + s_2^{'}(t+2)/(1+i^*)\right] - \nu(t+1)B(t)\right]$$

$$B(t) = G(2+n)(1+g)^t + N(1+n)^{t+1}\left[s_1^{'}(t) + s_2^{'}(t)/(1+n)\right] + (1+i^* - \nu(t))B(t-1)$$

with w, $A(t-1)$, $B(t-1)$, $s_1^{'}(t+1)$, $s_2^{'}(t+1)$, $s_2^{'}(t+2)$, $\nu(t+1)$ given. Equations (39) and (40) are the first order conditions of the program. It is easy to show that the second orders conditions are satisfied. □

Proof of Lemma 2

Equation (39) and (40) can be rewritten $B\nu(t+1)(1+n)W(t) = AW(t+1)$ and $B\nu(t+1)(2+\beta)(1+n)^2 C_2(t)/(1+\beta) = W(t+1)$. For $t \geq 1$, equation (20) implies $C_2(t) = (1+i^*)W(t-1)/(2+\beta)$. So, we get equation (42). □

Proof of Lemma 3

Equations (39) and (40) are equivalent to $A(1+i^*)(1+n)W(t)/(1+\beta) = W(t+1)$, and $A(1+i^*)(1+n)C_2(t)/(1+\beta) = C_2(t+1)$, for $t \geq 0$. The second equation and equation (20) imply the first equation, which is redundant. Equation (44) comes from equation (41) after the elimination of $s_1^*(t)$ thanks to equation (8).□

Proof of Lemma 4

a) We consider the system of equations (43), (44) and (45), with equation (45) rewritten with a lead of one period $(1+g)B^{*}(t+1) = G(2+n) + N(1+n)\left[s_1^{'*}(t+1) + s_2^{'*}(t+1)/(1+n)\right] + (1+i^{*}-\nu)B^{*}(t)$, for $t \geq 0$. Let us set the sequence $s_1^{'*}(t+1)$ to arbitrary values. The system includes two predetermined variables $A^{*}(t)$ and $B^{*}(t)$ and one anticipated variable $s_2^{'*}(t)$. The initial conditions $A^{*}(-1) = A(-1)$ and $B^{*}(-1) = B(-1)$ are given. Let us assume that the system has a unique solution. Then, we can compute $s_1^{'*}(0)$ with the equation (45) written for period 0 $(1+g)B^{*}(0) = G(2+n) + N(1+n)\left[s_1^{'*}(0) + s_2^{'*}(0)/(1+n)\right] + (1+i^{*}-\nu)B^{*}(-1)$.

b) Equations (21) and (22) show that equations (43) can be written $C_2^{*}(t+1) = \dfrac{1+i^{*}}{(1+\beta)(1+g)} C_2^{*}(t)$.

We get the first eigenvalue. Let us define $z(t) = s_2^{'*}(t+1) + (1+i^{*})B^{*}(t)/N$. Equation (44) can be written $(2+\beta)C_2^{*}(t+1)/N = z(t) - \dfrac{1+i^{*}}{1+g} z(t-1) + \dfrac{1+i^{*}}{1+\rho} w - \dfrac{1+i^{*}}{1+g} G(2+n)/N$. We get the second eigenvalue. Equation (45) can be written $(1+g)B^{*}(t+1) = -\nu B^{*}(t) + G(2+n) + N(1+n)s_1^{'*}(t+1) + Nz(t)$. We get the third eigenvalue. The system, so rewritten has two anticipated variables ($C_2^{*}(t+1)$ and $B_2^{*}(t+1)$) and one predetermined variable $z(t-1)$. So, the necessary and sufficient condition for the existence and the uniqueness of a solution is that one of the eigenvalues is smaller than or equal to 1, and the two others are larger than 1 (see Laffargue (2004)).

c1) We use equations (43), (19) and (20).

c2) Equation (19) and (20) show that each generation sets the following ratio between its consumption in the two periods of its life $C_2^{*}(t+1)/C_1^{*}(t) = (1+i^{*})/\left[(1+\beta)(1+g)\right]$. This equation and equation (46) give equation (47).

d) Let us consider a solution obtained for an arbitrary path set for $s_1^{'*}(t)$, with $t \geq 1$. Let us increase $s_1^{'*}(t+1)$ by an arbitrary amount $\Delta s_1^{'*}$ for a period $t+1 \geq 1$ given. Then, a new solution to the system differs from the previous one only by $\Delta B^*(t) = \Delta A^*(t) = N(1+n)\Delta s_1^{'*} / \nu$ and $\Delta s_2^{'*}(t+1) = -(1+n)(1+i^*)\Delta s_1^{'*} / \nu$. That means that after these changes, the new path satisfies all the equation. We can notice that $\Delta s_2^{'*}(t+2) + (1+i^*)\Delta A^*(t+1)/N = 0$. So, $C_2^*(t+2)$ does not change.□

Proof of Proposition 2

We can derive from the equations of the system of Lemma 3 the following 5 equations with 5 unknown variables

$$(g+\nu-i^*)B^* = G(2+n) + N(1+n)s_1^{'*} + Ns_2^{'*}$$

$$(1+g)B^* = G(2+n) + N(1+n)s_1^{'*}(0) + Ns_2^{'*}(0) + (1+i^*-\nu)B(-1)$$

$$(2+\beta)(1+g)A^* = \left[(g-i^*)B^* - G(2+n) - [1+(1+\beta)(1+g)/(1+i^*)]Ns_2^{'*} + (1+n)Nw\right]$$

$$(2+\beta)(1+g)A^* = \left[(1+g)B^* - (1+i^*)B(-1) - G(2+n) - Ns_2^{'*}(0) - (1+\beta)(1+g)Ns_2^{'*}/(1+i^*) + (1+n)Nw\right]$$

$$s_2^{'*} + (1+i^*)A^* = s_2^{'*}(0) + (1+i^*)A(-1)$$

The solutions of this system are parameterised by the choice of the value of $s_1^{'*}$ in the way that is described in Lemma 4. Then, we add to this system, a new variable $s_1^{'*}(0)$ and a new equation $s_1^{'*}(0) = s_1^{'*}$.We easily obtain the results $B^* = B(-1)$, $A^* = A(-1)$, $s_2^{'*}(0) = s_2^{'*}$.

Then, we get equation (48). Its differentiation gives

$$N\Delta s_2^{'*} / [(1+n)\Delta n] = \left[(1+i^*)B(-1) + G + Ns_2^{'*}\right] / [1 + (1+\beta)(1+g)/(1+i^*)]$$

Assumption 3 implies that this expression is positive. We deduce equation (49) from equations (22) and (47). This equation shows that C_1^*, $C_1^*/(1+n)$ and C_2^* are increasing functions of n. Finally, we get equation (50). □