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ESTIMATION FROM CROSS-SECTIONS OF INTEGRATED TIME-SERIES

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ABSTRACT

ESTIMATION FROM CROSS-SECTIONS OF INTEGRATED TIME-SERIES

This paper studies under which conditions a cross-section regression yields unbiased estimates of the parameters of an individual dynamic model with fixed effects and individual-specific responses to macro shocks. We show that the OLS estimation of a system of non stationary variables on a cross-section yields estimates which converge to the true value when calendar time tends to infinity.

RESUME

Estimation en coupe de variables intégrées

Cet article étudie les conditions sous lesquelles une régression en coupe donne des estimations non biaisées des paramètres d'un modèle dynamique à effets fixes et à chocs macroéconomiques spécifiques. Nous montrons qu'une estimation en coupe par les moindres carrés linéaires d'un modèle où les variables sont non stationnaires donne des estimations non biaisées lorsque l'origine des processus tend vers moins l'infini.

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Mots Clés: contraintes de liquidité, crise de la dette, estimation.

1 Introduction

A considerable number of microeconometric studies rely on cross-section estimates. Yet, a fast growing number of other microeconometric studies use panel data and individual models involving more and more often state dependence. Were they obtained from cross-sections of such data generating processes, the former cross-section estimates would generally not be consistent. This note shows that if the underlying dynamics of the regressors are integrated, the cross-section estimates do not indeed converge to the true value when the sample size tends to infinity, but the asymptotic bias becomes eventually negligible when the origin of the individual processes is far enough remote in time. This result holds for a very general specification of the error term, allowing fixed effects, specific macro shocks and correlations between shocks. We also show that the rate of convergence depends on whether the variables are cointegrated or not in the time dimension.

These results build on other results of the time series literature, initiated by Phillips and Durlauf (1986), Park and Phillips (1988) and Park and Phillips (1989). They are more specifically related to the the literature on cointegration in panel data (see Pedroni (1996) and Pedroni (1997)).

The plan of this note is as follows. Section 2 presents the assumptions and the results. Section 3 illustrates this property by using the Summers-Heston data on annual domestic consumption and GDP for 152 countries from 1950 to 1992. By performing Monte-Carlo simulations on the calibrated model, we explore the finite sample properties of the cross section estimates. Section 4 concludes.

2 Model and Assumptions

Suppose that, at the individual level, the following linear model is true:

$$\begin{cases} y_t^h = ax_t^h + \epsilon_t^h, \\ x_t^h = bx_{t-1}^h + \eta_t^h, \end{cases} \qquad h = 1, ..., H, \quad t = 1, ..., T, \tag{1}$$

where y_t^h and x_t^h are two random variables, with $x_0^h = 0$ for all h and $|b| \leq 1$, and

$$\begin{cases} \epsilon_t^h = u^h + \rho^h u_t + u_t^h, \\ \eta_t^h = v^h + \delta^h v_t + v_t^h, \end{cases} \tag{2}$$

where u_t and v_t are two scalar macro-shocks. Notice that we allow for the presence of individual fixed effects u^h and v^h, and of individual responses to macro-shocks $\rho^h u_t$ and $\delta^h v_t$. To keep things simple, we have assumed all variables scalar. But it would be easy to generalize the following results to the case of vectors.

We make the following assumptions on error components. First, we assume that fixed effects have asymptotically finite, non zero sample variances.

Assumption 1 ***Cross-section variances.***

$$\operatorname*{plim}_{H\to\infty} \frac{1}{H}\textstyle\sum_{h=1}^{H}(u^h)^2 \in \mathbb{R}_+^*,\ \operatorname*{plim}_{H\to\infty} \frac{1}{H}\textstyle\sum_{h=1}^{H}(v^h)^2 \in \mathbb{R}_+^*,$$
$$\operatorname*{plim}_{H\to\infty} \frac{1}{H}\textstyle\sum_{h=1}^{H}(\rho^h)^2 \in \mathbb{R}_+^*,\ \operatorname*{plim}_{H\to\infty} \frac{1}{H}\textstyle\sum_{h=1}^{H}(\delta^h)^2 \in \mathbb{R}_+^*,$$

where $\mathbb{R}_+^*$ is the set of positive real numbers and where $\operatorname{plim}_{H\to\infty}$ is the limit in Probability.

Second, all error components have zero cross-sectional means.

Assumption 2 ***Cross-section means.*** *For all t, for all H,*

$$\begin{array}{l}\sum_{h=1}^{H} u^h = \sum_{h=1}^{H} u_t^h = \sum_{h=1}^{H} \rho^h = 0,\\ \sum_{h=1}^{H} v^h = \sum_{h=1}^{H} v_t^h = \sum_{h=1}^{H} \delta^h = 0.\end{array}$$

This assumption may seem restrictive. Typically, one does not expect the aggregate macro-shock $\sum_{h=1}^{H} \rho^h u_t$ to vanish. Suppose, however, that $\sum_{h=1}^{H} u_t^h$ and $\sum_{h=1}^{H} \rho^h$ are not zero, then it is possible to rewrite system (1) with errors (2) satisfying assumption 2 if $y_t^h - \frac{1}{H}\sum_{l=1}^{H} y_t^l$ and $x_t^h - \frac{1}{H}\sum_{l=1}^{H} x_t^l$ are used in place of y_t^h and x_t^h. This is therefore without loss of generality that we assume all variables centered around their cross-section mean, as long as the only parameters of interest are the slope coefficients a and b.

Thirdly, idiosyncratic components u_t^h and v_t^h are mutually uncorrelated and uncorrelated with individual fixed effects.

Assumption 3 ***Cross-section covariances.*** *For all t, t',*

$$\begin{array}{llll}\frac{1}{H}\sum_{h=1}^{H} u^h u_t^h \xrightarrow{P} 0, & \frac{1}{H}\sum_{h=1}^{H} u^h v_t^h \xrightarrow{P} 0, & \frac{1}{H}\sum_{h=1}^{H} v^h u_t^h \xrightarrow{P} 0, & \frac{1}{H}\sum_{h=1}^{H} v^h v_t^h \xrightarrow{P} 0,\\ \frac{1}{H}\sum_{h=1}^{H} \delta^h u_t^h \xrightarrow{P} 0, & \frac{1}{H}\sum_{h=1}^{H} \delta^h v_t^h \xrightarrow{P} 0, & \frac{1}{H}\sum_{h=1}^{H} \rho^h v_t^h \xrightarrow{P} 0, & \frac{1}{H}\sum_{h=1}^{H} \rho^h u_t^h \xrightarrow{P} 0,\\ \frac{1}{H}\sum_{h=1}^{H} u_t^h v_{t'}^h \xrightarrow{P} 0, & & & \end{array}$$

when $H \to \infty$, where "$\xrightarrow{P}$" means "convergence in probability".

Note that macro shocks u_t and v_t can be correlated, as well as the fixed effects, u^h and v^h.

The next assumption is on the dynamics of the idiosyncratic errors u_t^h and v_t^h.

Assumption 4 *The idiosyncratic errors u_t^h and v_t^h are covariance-stationary processes with absolutely summable autocovariances and zero means.*

As far as macro shocks are concerned, we assume that they can be either I(0) or I(1). Note that, if macro shocks are covariance-stationary and $|b| = 1$, then the first equation of equation (1) defines a cointegration relationship. If v_t is I(1), then x_t^h are I(2) and whether u_t is I(0) or I(1), the first equation of equation (1) again defines a cointegration relationship. Clearly, in these cases, one already knows that system's parameters can be consistently estimated with one infinite time-series. But if $|b| = 1$, v_t is I(0) and u_t is I(1), all three variables y_t^h, x_t^h and ϵ_t^h are I(1) and y_t^h and x_t^h are not cointegrated. The cross-section dimension then becomes crucial to identify the system's parameters.

Let $a_{H,t}$ define the Ordinary Least Square estimator of a in the cross-section regression of y_t^h on x_t^h :

$$a_{H,t} = \left[\sum_{h=1}^{H} y_t^h x_t^h\right] \left[\sum_{h=1}^{H} (x_t^h)^2\right]^{-1}.$$

And let

$$\mathcal{B}_t = \operatorname*{plim}_{H\to\infty} \left[a_{H,t} - a\right] \tag{3}$$

be the asymptotic bias of cross-section-t estimate $a_{H,t}$. We now show the following proposition.

Proposition 5 *If $|b| = 1$, then, under the preceding assumptions,* $\operatorname{plim}_{t\to\infty} \mathcal{B}_t = 0$ *and the rate of convergence of $\mathcal{B}_t$ to 0 is as in the following table:*

	u_t *is I(0)*	u_t *is I(1)*
v_t *is I(0)*	$O_P(t^{-1})$	$O_P(t^{-1/2})$
v_t *is I(1)*	$O_P(t^{-3/2})$	$O_P(t^{-1})$

where $\mathcal{B}_t$ is of order $O_P(t^\alpha)$, for all real α, if for all $\varepsilon > 0$, there exists M_ε such that $\Pr\{t^{-\alpha}\mathcal{B}_t \leq M_\varepsilon\} \geq 1 - \varepsilon$ *for all t.*

If x_t^h is covariance-stationary ($|b| < 1$ and v_t is covariance stationary) then the asymptotic bias $\mathcal{B}_t$ tends in general to a non null constant when t becomes large.

We refer the reader to appendix A for a proof. Provided that x_t^h is a random walk, proposition 5 shows that a cross-section estimation yields estimates for which the asymptotic bias (when sample size H goes to infinity) tends to zero when the time origin of processes x_t^h and y_t^h is far remote. The rate of convergence of the asymptotic bias depends on the relative order of integration of ϵ_t^h and η_t^h. For example, if ε_t^h and η_t^h are stationary, the first equation of the system forms a cointegration relationship, and the rate of convergence of the asymptotic bias is of order t^{-1}. If the first equation of the system is not cointegrated but variables x_t^h and y_t^h are I(1), then the rate of convergence is only of order $t^{-1/2}$. When estimating a relationship on cross section data, we can expect the estimation to yield more precise results with older cohorts and if the estimated system is cointegrated. Moreover, proposition 5 shows that whatever the nature of the error term in (1), a cross section regression on levels always identifies the structural parameters. At the macro level, the identification relies on either a regression on levels if the system is cointegrated, or a regression on first-differences in the alternate case.

This proposition will be easily generalized to the multivariate case where y_t^h, x_t^h and the macro shocks are vectors. If b has all its eigenvalues strictly inside the unit circle, then the cross section estimates are generally biased. And when b is the identity matrix, the bias becomes negligible when t tends to infinity. The apparently intermediate case where $b - Id$, where Id is the identity matrix, is not of full rank and non null is a particular case of the case $b = Id$ after integrating a subset of the components of x_t into y_t.

Moreover, it is straightforward to show that exactly the same asymptotics apply to the case of *pooled* cross-sections, i.e. when the OLS estimator $a_{H,t}$ is the one obtained by regressing y_{t-i}^h on x_{t-i}^h in the sample $\{(x_{t-i}^h, y_{t-i}^h),\ i = 0, ..., t-1,\ h = 1, ..., H\}$. For example, $\mathcal{B}_t = O_P(t^{-1})$ when both u_t and v_t are I(0). [1]

There is a growing number of evidence that economic series are integrated even at the micro level (see Deaton and Paxson (1994) for instance). The next section assesses the empirical relevance of these results, by using real data to calibrate Monte-Carlo simulations.

3 Application and Simulations

In this section we illustrate the results established above by some Monte Carlo estimations. The motivation is to analyze the asymptotic results with real data. We use the database provided by Summers and Heston (1991) which reports annual domestic consumption and GDP for 152 countries from 1950 to 1992 to provide an empirically sensible calibration for the Monte Carlo simulations. We denote as y_t^h the logarithm of consumption, and as x_t^h the logarithm of GDP. We postulate the following statistical model for y_t^h and x_t^h:

$$\begin{cases} y_t^h = a_0 + a x_t^h + u^h + \rho^h u_t + u_t^h, \\ x_t^h = b_0 + b x_{t-1}^h + v^h + \delta^h v_t + v_t^h, \end{cases} \qquad h = 1, ..., H, \quad t = 1, ..., T, \tag{4}$$

with

$$\begin{cases} u_t = \tau_u u_{t-1} + \varepsilon_{ut}, \\ v_t = \tau_v v_{t-1} + \varepsilon_{vt}, \end{cases} \tag{5}$$

[1] This is due to the fact that the numerator of $\mathcal{B}_t$ is $O_P(t)$ and the denominator is $O_P(t^2)$. If all cross-sections from the origin to t are pooled together, then the numerator of the bias of the OLS estimator of a is still $O_P(t)$ and the denominator $O_P(t^2)$, because one can easily show that, in the present context, averaging over time produces a time process which has the same order in probability than the averaged process.

and where u^h, ρ^h, u_t^h, ε_{ut}, v^h, δ^h, v_t^h, ε_{vt} are independent normal random variables with means, respectively, 0, ρ, 0, 0, 0, δ, 0, 0, and non zero variances. Moreover, we fix $u_0 = v_0 = x_0^h = 0$.[2]

Consistent estimates of the parameters of the error-component model (4) are reported in table 1. We then use these parameters to simulate paths of log GDP and log of consumption for a large number of periods (years), for different values of b, τ_u and τ_v corresponding to the different alternative cases of proposition 5. For each set of values, we constructed 100 panel data sets with 152 "countries" and 4000 periods. Table 2 displays the different experiments.

Figure 1 displays a graph of the average absolute bias (absolute percentage deviation from the true value), as a function of the distance from the origin for the 5 cases. Coordinates are log-coordinates so that the bias paths asymptotically become straight lines. Consistent with the theory developed above, the bias is a decreasing function of time, except in the stationary case. In the stationary case, the average bias is always bigger even for small t. It appears that there exists a middle range set of values for t for which the bias is converging to zero much faster when the variables are I(2). For the "true" case, which corresponds to the estimated parameters, it takes approximately 50 years to reduce the bias by one half, and approximately 160 years to reduce it by 5. Finally, the slopes of bias paths, when t is large, are as predicted by the theory.

Table 1: Estimated Parameter Values

a	b	s.d.$(u^h)^{(1)}$	s.d.(u^t)	τ_u	s.d.(u_t^h)	s.d.(v^h)	s.d.(v^t)	τ_v	s.d.(v_t^h)
0.85	0.99	0.16	0.009	0.9	0.086	0.024	0.046	0.2	0.05

Note: "s.d." : standard deviation.

Table 2: Parameter Values Used in Experiment

Case	True convergence rate	τ_u	τ_v
$u_t \sim I(0),\ v_t \sim I(0),\ b = 0$	(0)	0	0
$u_t \sim I(0),\ v_t \sim I(0),\ b = 1$	(1/2)	1	0
$u_t \sim I(1),\ v_t \sim I(0),\ b = 1$	(1)	0.9	0.2
$u_t \sim I(0),\ v_t \sim I(1),\ b = 1$	(1)	1	1
$u_t \sim I(1),\ v_t \sim I(1),\ b = 1$	(3/2)	0	1

Note: Estimation results obtained on 200 simulated panel data sets of 152 "countries" and 800 periods. Est. refer to the average estimated convergence rate, and s.d. to its standard deviation.

[2]Note that ρ^h and δ^h do not necessarilly average to 0. Yet the OLS estimate of a implies centering of the y_t^h and x_t^h variables and thus brings this case back to the previous set-up.

4 Conclusion

In this note we have shown that the OLS estimation of a system of non stationary variables on a cross-section yields estimates which converge to the true value when calendar time tends to infinity. We emphasize the importance of the variability of idiosyncratic responses. We provide empirical evidence on the elasticity of domestic consumption to the GDP, for 152 countries, and explore the predictions of our proposition with real data. Given the variability of the different shocks in our example, the convergence of the bias to values of the order of 10% takes several decades, if not centuries, depending on the order of integration of the series. This result casts serious doubts on the conclusions that can be drawn from cross-sectional estimations when the regressors are known to be dynamic processes. To refer to only one empirical case, we can mention demand system estimation where one of the conditioning variables, total expenditure if not relative prices, is known to be highly autocorrelated, if not a random walk.

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Appendix A. Proof of Theorem 5

Replacing y_t^h by $ax_t^h + \epsilon_t^h$ in the formula for $a_{H,t}$ yields:

$$a_{H,t} = a + \left[\sum_{h=1}^{H} \epsilon_t^h x_t^h\right]\left[\sum_{h=1}^{H}(x_t^h)^2\right]^{-1}.$$

Moreover, let

$$\theta_t(L) = 1 + bL + ... + b^{t-1}L^{t-1},$$

where L is the lag-operator. Then:

$$\begin{aligned} x_t^h &= \theta_t(L)\eta_t^h \\ &= \theta_t(1)v^h + \delta^h\theta_t(L)v_t + \theta_t(L)v_t^h. \end{aligned}$$

Step 1 : computation of bias $\mathcal{B}_t = \text{plim}_{H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H}\epsilon_t^h x_t^h\right]\left[\frac{1}{H}\sum_{h=1}^{H}(x_t^h)^2\right]^{-1}$.

First, we have

$$\begin{aligned} \frac{1}{H}\sum_{h=1}^{H}\epsilon_t^h x_t^h = {} & \theta_t(1)\left[\frac{1}{H}\sum_{h=1}^{H}u^h v^h\right] + \left[\frac{1}{H}\sum_{h=1}^{H}u^h\delta^h\right][\theta_t(L)v_t] + \frac{1}{H}\sum_{h=1}^{H}u^h\left[\theta_t(L)v_t^h\right] \\ & +\theta_t(1)\left[\frac{1}{H}\sum_{h=1}^{H}\rho^h v^h\right]u_t + \left[\frac{1}{H}\sum_{h=1}^{H}\rho^h\delta^h\right]u_t[\theta_t(L)v_t] + \frac{1}{H}\sum_{h=1}^{H}\rho^h\left[\theta_t(L)v_t^h\right]u_t \\ & +\theta_t(1)\left[\frac{1}{H}\sum_{h=1}^{H}u_t^h v^h\right] + \left[\frac{1}{H}\sum_{h=1}^{H}u_t^h\delta^h\right][\theta_t(L)v_t] + \frac{1}{H}\sum_{h=1}^{H}u_t^h\left[\theta_t(L)v_t^h\right]. \end{aligned}$$

Hence, for all t, assumption 3 implies that:

$$\begin{aligned} \frac{1}{H}\sum_{h=1}^{H}\epsilon_t^h x_t^h = {} & \theta_t(1)\left[\frac{1}{H}\sum_{h=1}^{H}u^h v^h\right] + \left[\frac{1}{H}\sum_{h=1}^{H}u^h\delta^h\right][\theta_t(L)v_t] \\ & +\theta_t(1)\left[\frac{1}{H}\sum_{h=1}^{H}\rho^h v^h\right]u_t + \left[\frac{1}{H}\sum_{h=1}^{H}\rho^h\delta^h\right]u_t[\theta_t(L)v_t] + o_P^H(1) \end{aligned}$$

where $o_P^H(1)$ is a random variable which converges to 0 in probability when H tends to infinity.

Similarly,

$$\frac{1}{H}\sum_{h=1}^{H}(x_t^h)^2 = \theta_t(1)^2\left[\frac{1}{H}\sum_{h=1}^{H}(v^h)^2\right] + \left[\frac{1}{H}\sum_{h=1}^{H}(\delta^h)^2\right][\theta_t(L)v_t]^2 + \frac{1}{H}\sum_{h=1}^{H}\left[\theta_t(L)v_t^h\right]^2$$
$$+ 2\theta_t(1)\left[\frac{1}{H}\sum_{h=1}^{H}v^h\delta^h\right][\theta_t(L)v_t] + 2\theta_t(1)\frac{1}{H}\sum_{h=1}^{H}v^h\left[\theta_t(L)v_t^h\right]$$
$$+ 2\frac{1}{H}\sum_{h=1}^{H}\delta^h\left[\theta_t(L)v_t^h\right][\theta_t(L)v_t].$$

And by assumption 3

$$\frac{1}{H}\sum_{h=1}^{H}v^h\left[\theta_t(L)v_t^h\right] \xrightarrow{P} 0 \text{ and } \frac{1}{H}\sum_{h=1}^{H}\delta^h\left[\theta_t(L)v_t^h\right][\theta_t(L)v_t] \xrightarrow{P} 0$$

when H tends to infinity. Moreover, by assumption 3:

$$\frac{1}{H}\sum_{h=1}^{H}\left[\theta_t(L)v_t^h\right]^2 \xrightarrow{P} \sum_{j=0}^{t}\phi_j\gamma_j$$

where

$$\begin{aligned}\gamma_j &= E\left[v_t^h v_{t-j}^h\right]\\ &= \operatorname*{plim}_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}v_t^h v_{t-j}^h,\end{aligned}$$

and where

$$\phi_j = 2b^j\left[1+b^2+...+b^{2(t-j)}\right],\ j=1,...,t.$$

It follows that

$$\operatorname*{plim}_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}(x_t^h)^2 = \theta_t(1)^2\operatorname*{plim}_{H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H}(v^h)^2\right] + \operatorname*{plim}_{H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H}(\delta^h)^2\right][\theta_t(L)v_t]^2$$
$$+ 2\theta_t(1)\operatorname*{plim}_{H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H}v^h\delta^h\right][\theta_t(L)v_t] + \sum_{j=0}^{t}\phi_j\gamma_j + o_P^H(1).$$

Step 2 : computation of limit bias $\operatorname{plim}_{t\to\infty}\mathcal{B}_t$ **when** $b=1$.

We now show that $\mathcal{B}_t \xrightarrow{P} 0$ if $b=1$. When $b=1$, $\theta_t(1)=t$ and

$$\theta_t(L)v_t = \sum_{j=0}^{t}v_{t-j},\quad \theta_t(L)v_t^h = \sum_{j=0}^{t}v_{t-j}^h.$$

Thus

$$\frac{1}{H}\sum_{h=1}^{H}\epsilon_t^h x_t^h = t\left[\frac{1}{H}\sum_{h=1}^{H}u^h v^h\right] + \left[\frac{1}{H}\sum_{h=1}^{H}u^h\delta^h\right]\left[\sum_{j=0}^{t}v_{t-j}\right] \tag{A.1}$$
$$+ t\left[\frac{1}{H}\sum_{h=1}^{H}\rho^h v^h\right]u_t + \left[\frac{1}{H}\sum_{h=1}^{H}\rho^h\delta^h\right]u_t\left[\sum_{j=0}^{t}v_{t-j}\right] + o_P^H(1)$$

and

$$\frac{1}{H}\sum_{h=1}^{H}(x_t^h)^2 = t^2\left[\frac{1}{H}\sum_{h=1}^{H}(v^h)^2\right] + \left[\frac{1}{H}\sum_{h=1}^{H}(\delta^h)^2\right]\left[\sum_{j=0}^{t}v_{t-j}\right]^2 \tag{A.2}$$
$$+ 2t\left[\frac{1}{H}\sum_{h=1}^{H}v^h\delta^h\right]\left[\sum_{j=0}^{t}v_{t-j}\right] + \sum_{j=0}^{t}\phi_j\gamma_j + o_P^H(1)$$

We first prove three lemmas which will be helpful in deriving the final results.

Lemma A1 *If v_t is a covariance stationary variable, then $t^{-1/2}\sum_{j=0}^{t}v_{t-j} = O_P(1)$. If v_t is integrated of order one, then $t^{-3/2}\sum_{j=0}^{t}v_{t-j} = O_P(1)$.*

Proof: See Park and Phillips (1988). ■

Lemma A2 *With the notation defined above,*

$$\frac{1}{t}\sum_{j=0}^{t}\phi_j\gamma_j = O(1).$$

Proof: The lemma is a simple consequence of

$$\frac{1}{t+1}\sum_{j=0}^{t}\phi_j\left|\gamma_j\right| \leq 2\sum_{j=0}^{t}\left|\gamma_j\right| \leq 2\sum_{j=0}^{\infty}\left|\gamma_j\right| < \infty.$$

since $(t+1)^{-1}\phi_j \leq 2$ and where the last inequality follows from the fact that v_t^h is covariance stationary. ■

Lemma A3

$$\frac{1}{t^{\omega}}u_t\sum_{j=1}^{t}v_{t-j} \xrightarrow{P} 0$$

in one of the three cases defined below,

1. *u_t and v_t are I(0), with $\omega = 1$.*

2. *u_t is I(1) and v_t is I(0), with $\omega = 3/2$.*

3. u_t is I(0) and v_t is I(1), with $\omega = 2$.

Proof: Lets define the variables $\tilde{u}_t$ and $\tilde{v}_t$ such as

1. if u_t and v_t are I(0), $\tilde{u}_t = u_t$ and $\tilde{v}_t = \frac{1}{t}\sum_{j=0}^{t} v_{t-j}$.
2. if u_t is I(1) and v_t is I(0), $\tilde{u}_t = u_t/t^{1/2}$ and $\tilde{v}_t = \frac{1}{t}\sum_{j=0}^{t} v_{t-j}$.
3. if u_t is I(0) and v_t is I(1), $\tilde{u}_t = u_t$ and $\tilde{v}_t = \frac{1}{t^2}\sum_{j=0}^{t} v_{t-j}$.

Then from lemma A1 we have $\tilde{v}_t \xrightarrow{P} 0$ when t tends to infinity.

As $\tilde{u}_t$ is covariance stationary, $E\tilde{u}_t{}^2$ is equal to some finite constant σ^2 independent of t. Chebyshev's inequality then applies to show that:

$$\Pr\{|\tilde{u}_t| > \alpha\} \leq \frac{\sigma^2}{\alpha^2}$$

for all positive α. Moreover, it is true that

$$|\tilde{u}_t| \leq \alpha \text{ and } |\tilde{u}_t|\,|\tilde{v}_t| \geq \alpha\beta \Longrightarrow |\tilde{v}_t| \geq \beta.$$

Hence

$$\Pr\{|\tilde{v}_t| \geq \beta\} \geq \Pr\{|\tilde{u}_t| \leq \alpha \text{ and } |\tilde{u}_t|\,|\tilde{v}_t| \geq \alpha\beta\}$$

for all positive numbers α and β. Then remark also that

$$\Pr\{|\tilde{u}_t| > \alpha\} \geq \Pr\{|\tilde{u}_t| > \alpha \text{ and } |\tilde{u}_t|\,|\tilde{v}_t| \geq \alpha\beta\}.$$

Consequently

$$\Pr\{|\tilde{u}_t|\,|\tilde{v}_t| \geq \alpha\beta\} \leq \Pr\{|\tilde{v}_t| \geq \beta\} + \Pr\{|\tilde{u}_t| > \alpha\}.$$

For all positive number ϵ, the convergence of $|\tilde{v}_t|$ to 0 in probability makes possible to find β such as

$$\Pr\{|\tilde{v}_t| \geq \beta\} \leq \epsilon/2.$$

And, choosing $\alpha = \sqrt{\frac{2\sigma^2}{\epsilon}}$ yields that

$$\Pr\{|\tilde{u}_t|\,|\tilde{v}_t| \geq \alpha\beta\} \leq \epsilon.$$

This achieves to show that

$$\tilde{u}_t\tilde{v}_t \xrightarrow{P} 0$$

when t tends to infinity. ■

Using the lemmas A1 through A3, we are now able to calculate the limit bias $p\lim_{t\to\infty} \mathcal{B}_t$.

If both u_t and v_t are I(0), dividing equation (A.1) by t and applying lemmas A1 through A3, gives

$$\operatorname*{plim}_{H\to\infty} \frac{1}{t}\frac{1}{H}\sum_{h=1}^{H} \epsilon_t^h x_t^h = \operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} u^h v^h + \left[\operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \rho^h v^h\right] u_t + o_P^t(1)$$

where $o_P^t(1)$ is a random variables which converges to 0 in probability when t tends to infinity. Dividing equation (A.2) by t^2 gives,

$$\operatorname*{plim}_{H\to\infty} \frac{1}{t^2}\frac{1}{H}\sum_{h=1}^{H}(x_t^h)^2 = \operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H}(v^h)^2 + o_P^t(1). \tag{A.3}$$

Hence,

$$\mathcal{B}_t = \frac{1}{t}\frac{\text{plim}_{H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H} u^h v^h\right] + \text{plim}_{H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H} \rho^h v^h\right] u_t}{\text{plim}_{H\to\infty\ H\to\infty}\left[\frac{1}{H}\sum_{h=1}^{H}(v^h)^2\right]} + o_P^t(t^{-1}) \tag{A.4}$$

If u_t is I(1) and v_t is I(0), dividing equation (A.1) by $t^{3/2}$ and applying lemmas A1 through A3, gives

$$\operatorname*{plim}_{H\to\infty} \frac{1}{t^{3/2}}\frac{1}{H}\sum_{h=1}^{H} \epsilon_t^h x_t^h = \frac{u_t}{t^{1/2}}\left[\operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \rho^h v^h\right] + o_P^t(1)$$

Dividing equation (A.2) by t^2 and using equation (A.3) gives

$$\mathcal{B}_t = \frac{1}{t^{1/2}}\frac{\frac{u_t}{t^{1/2}}\,\text{plim}_{H\to\infty\ H\to\infty}\frac{1}{H}\sum_{h=1^H} \rho^h v^h}{\text{plim}_{H\to\infty\ H\to\infty}\frac{1}{H}\sum_{h=1}^{H}(v^h)^2} + o_P^t(t^{-1/2}) \tag{A.5}$$

If u_t is I(0) and v_t is I(1), dividing equation (A.1) by $t^{3/2}$ and applying lemmas A1 through A3, gives

$$\operatorname*{plim}_{H\to\infty} \frac{1}{t^{3/2}}\frac{1}{H}\sum_{h=1}^{H} \epsilon_t^h x_t^h = \left[\operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} u^h\delta^h\right]\frac{1}{t^{3/2}}\left[\sum_{j=0}^{t} v_{t-j}\right] + \left[\operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \rho^h v^h\right]\frac{u_t}{t^{1/2}} + o_P^t(1)$$

and dividing equation (A.2) by t^3 gives,

$$\operatorname*{plim}_{H\to\infty} \frac{1}{t^3}\frac{1}{H}\sum_{h=1}^{H}(x_t^h)^2 = \left[\operatorname*{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H}(\delta^h)^2\right]\frac{1}{t^3}\left[\sum_{j=0}^{t} v_{t-j}\right]^2 + o_P^t(1).$$

Hence,

$$\mathcal{B}_t = \frac{1}{t^{3/2}} \frac{\frac{1}{t^{3/2}} \left(\sum_{j=0}^{t} v_{t-j}\right) \text{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} u^h\delta^h + \left[\text{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \rho^h v^h\right] \frac{u_t}{t^{1/2}}}{\frac{1}{t^3}\left(\sum_{j=0}^{t} v_{t-j}\right)^2 \text{plim}_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \left(\delta^h\right)^2} + o_P^t(t^{-3/2})$$

where $o_P^t(t^{-3/2})$ is a random variables such that $t^{3/2} o_P^t(t^{-3/2})$ converges to 0 in probability when t tends to infinity.

If u_t and v_t are I(1), dividing equation (A.1) by t^2 and applying lemmas A1 through A3, gives

$$\underset{H\to\infty}{\text{plim}} \frac{1}{t^2}\frac{1}{H}\sum_{h=1}^{H} \epsilon_t^h x_t^h = \frac{u_t}{t^2}\left[\sum_{j=0}^{t} v_{t-j}\right]\left[\underset{H\to\infty}{\text{plim}} \frac{1}{H}\sum_{h=1}^{H} \rho^h\delta^h\right] + o_P^t(1)$$

and dividing equation (A.2) by t^3 gives,

$$\underset{H\to\infty}{\text{plim}} \frac{1}{t^3}\frac{1}{H}\sum_{h=1}^{H} (x_t^h)^2 = \frac{1}{t^3}\left[\underset{H\to\infty}{\text{plim}} \frac{1}{H}\sum_{h=1}^{H} (\delta^h)^2\right]\left[\sum_{j=0}^{t} v_{t-j}\right]^2 + o_P^t(1).$$

Hence,

$$\mathcal{B}_t = \frac{1}{t} \frac{\frac{u_t}{t^2}\left(\sum_{j=0}^{t} v_{t-j}\right) \text{plim}_{H\to\infty} \left[\frac{1}{H}\sum_{h=1}^{H} \rho^h\delta^h\right]}{\frac{1}{t^3}\left(\sum_{j=0}^{t} v_{t-j}\right)^2 \text{plim}_{H\to\infty\ H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \left(\delta^h\right)^2} + o_P^t(t^{-1}).$$

This finishes to show proposition 5. ■

Figure 1 : Log of Absolute Bias