ON THE GENERIC INEFFICIENCY OF DIFFERENTIABLE MARKET GAMES *

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ABSTRACT

ON THE GENERIC INEFFICIENCY OF DIFFERENTIABLE MARKET GAMES

It is shown in this paper that differentiable market games remain generically inefficient when one makes their smooth "Strategic Outcome Function" vary. The proof is mainly based on Thom's Transversality Theorem and drops any restriction regarding the dimension of the Strategies Spaces or the rank of the Strategy-to-trade map. We complete this first result by determining the Bertrand-like non-differentiabilities inherent to most competitive market mechanisms, and in the same time we suggest a synthesis between the Cournotian and Bertrand-type approaches of Walrasian equilibrium, both developed in the recent literature.

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RESUME

SUR L'INEFFICACITE GENERIQUE DES MECANISMES DE MARCHE DIFFERENTIÂBLES.

On démontre dans ce papier que les mécanismes de marché différentiables (i.e dont la fonction de résultat est C^∞) sont génériquement inefficients au sens fort où leurs équilibres non-coopératifs sont incompatibles avec les équilibres Pareto-Optimaux de l'économie d'échange associée, et donc a fortiori avec les équilibres walrasiens de cette économie. De façon complémentaire, on met en évidence les non-différentiabilités "à la Bertrand" inhérentes à la quasi-totalité des mécanismes de marché "compétitifs" pour lesquels il existe au moins un équilibre non-coopératif simultanément walrasien. Une synthèse est alors suggérée entre les mécanismes compétitifs finis et discontinus de type Bertrand d'une part, et ceux asymptotiques et différentiables de type Cournot d'autre part, développés jusqu'à présent dans la littérature.

I. INTRODUCTION

On considering the recent literature on market games, one may notice that in almost all cases the authors endeavoured to conciliate Cournot's non-cooperative approach with Walras' theory of general equilibrium. This is particularly true of the finite competitive models of Hurwicz [1979], Schmeidler [1980] and Dubey [1982] as well as of the asymptotic studies of Postlewaite-Schmeidler [1978], Dubey-Mas-Colell-Shubik [1980] and Mas-Colell [1982]. One exception should, however, be mentioned : namely, the original attempt of Dubey [1980] who established, in the case of a finite number of agents, the generic inefficiency of Shapley-Shubik's smooth market game when consumers' preferences vary. However, this result is substantively based on an additional restrictive assumption verified by Shapley-Shubik's model : the fact that the dimension of the strategy spaces is *strictly inferior* to the number of commodities in the exchange economy considered.

This paper has two purposes :

1) The first is to show that the generic inefficiency of smooth market games in the finite case is a more general phenomenon that shown by Dubey [1980], to the extent that the only limitative assumption it requires is the differentiability of the Strategic Outcome Functions.

Our method, which is different and in a way complementary to that followed by Dubey, consists first (Sections 2 and 3) in fixing consumers' preferences so as to verify the assumptions introduced by Debreu [1972] and to make the Strategic Outcome Function describe the $C^\infty(S,\mathbb{R}^{\ell n})$ space, where S denotes the space of messages and $\mathbb{R}^{\ell n}$ that of allocations. We then get generically non Pareto-Optimal Nash equilibria.

The same result is obtained when, in a second phase (Section 4) we make the consumers' preferences vary simultaneously with the Strategic Outcome Function, which permits us this time to keep only the differentiability of utility functions and to lay aside the strict monotonicity, the strict quasi-concavity and the non-zero Gaussian curvatures for all indifference surfaces (Debreu [1972]). The method of demonstration is the same in Sections 3 and 4 : it is mainly based on a differential formulation of the Nash equilibria associated with market games, and on a Transversality Theorem established by Thom and used in its simplest form here.

2) The second purpose of the paper is in a way complementary to the first in that it is mainly concerned with determining the *Bertrand-like non-differentiabilities* inherent to most competitive market mechanisms. To do this we only have to supplement the method already used in sections 3 and 4 by introducing explicitly the *Transaction Price Functions* associated with the Strategic Outcome Functions (Section 5). We are then in a position to establish, in particular, the *discontinuity* of the mechanisms developed by Schmeidler [1980] and Dubey [1982] as a requisite of their competitivity. Furthermore, with two examples of competitive market games, one finite and discontinuous, the other asymptotical and differentiable, both dealt with by the same method as previously, we suggest a possibility of undertaking, in further research, a synthesis of all the models constructed by the authors mentioned above.

We endeavoured to use elementary mathematics as far as possible and we apologize for breaking this rule in the proof of Theorem 1, for which we briefly recall definitions that can be found in Golubitsky-Guillemin [1973] among others.

2. THE GENERAL MODEL : NOTATIONS AND DEFINITIONS

2.1. A pure exchange economy

We consider a pure exchange economy including $n \geqq 2$ consumers and $\ell \geqq 2$ commodities. Each agent $i \in I = \{1,\dots,n\}$ has a consumption set equal to $\mathbb{R}^\ell$ and a utility function $u_i : \mathbb{R}^\ell \to \mathbb{R}$ which verifies the following standard assumptions : (Debreu [1972] and Balasko [1979]) :

(A1) : u_i *is smooth and surjective*

(A2) : u_i *is differentially monotonic, i.e.* $\frac{\partial u_i}{\partial x_i^j}(x_i^j) > 0$ *for* $j = 1,\dots,\ell$.

(A3) : u_i *is strictly quasi-concave*

(A4) : *the indifference hypersurface* $u_i^{-1}(t)$ *is bounded from below for all* $t \in \mathbb{R}$

(A5) : $u_i^{-1}(t)$ *has a non-zero Gaussian curvature for all* $t \in \mathbb{R}$

If $\omega_i = (\omega_i^1,\dots,\omega_i^\ell) \in \mathbb{R}^\ell$ denotes the vector of initial endowment of agent i , the net utility function $v_i : \mathbb{R}^\ell \to \mathbb{R}$ of this agent is defined by :

$$v_i(y_i) = u_i(y_i + \omega_i) ,$$

in which $y_i \in \mathbb{R}^\ell$ is a net allocation vector.

The v_i function obviously verifies the same assumptions as u_i from which it is deduced by mere translation. In particular, for every price system $p \in \mathbb{R}^\ell_{+*}$, the maximization program :

$$\max v_i(y)$$
$$p.y \leqq 0$$

has a unique solution $g_i(p) \in \mathbb{R}^\ell$ which represents the net Walrasian demand of agent i at the price p .

The application thus defined $g_i : \mathbb{R}^\ell_{+*} \to \mathbb{R}^\ell$ is a smooth diffeomorphism which verifies Walras law :

$$p.g_i(p) = 0 \text{ for every } p \in \mathbb{R}^\ell_{+*} \quad .$$

2.2. Strategic Market games

A strategic market game (mechanism) is obtained by completing the pure exchange economy $E = \{I, v = (v_i)_{i \in I}\}$ with the introduction of :

(a) a Strategy space S_i for each agent $i \in I$

In the following, we shall assume that the S_i spaces are open subsets of $\mathbb{R}^m$ ($m \in \mathbb{N}$), or more generally, submanifolds of $\mathbb{R}^m$ of the same $d \leq m$ dimension.

The Cartesian Product of all the S_i will be noted by S . Its elements $s = (s_1, \ldots, s_n)$ will be called *messages* (Hurwicz [1979]) or *selections* (Schmeidler [1980]).

(b) a Strategic Outcome Function (SOF) $z : S \to \mathbb{R}^{\ell n}$, which determines the interdependence between the agents $i \in I$ by linking every selection $s = (s_1, \ldots, s_n)$ with the final allocations $z_i(s) \in \mathbb{R}^\ell$ that result from it for every agent i ; we note : $z(s) = (z_1(s), \ldots, z_n(s)) \in \mathbb{R}^{\ell n}$. The S.O.F. z is said to be *balanced* if and only if :

$$\forall s \in S , \sum_{i \in I} z_i(s) = 0_{\mathbb{R}^\ell} .$$

We will denote B the set of all balanced S.O.F. $z : S \to \mathbb{R}^{\ell n}$. Let us consider now a mapping $\phi : S \to \mathbb{R}_+^{\ell n}$ where :

$$\forall s \in S \, , \; \phi(s) = [\phi_i^j(s)]_{i \in I, 1 \leq j \leq \ell}$$

ϕ is said to be a *Transaction Price Function* compatible with the S.O.F. z if and only if :

$$\forall i \in I \, , \forall s \in S : \sum_{1 \leq j \leq \ell} \phi_i^j(s).z_i^j(s) = 0 \, .$$

Example : In the paper by Schmeidler [1980] where for all i agents :

$$S_i = \{(p_i,q_i) \in \mathbb{R}_+^{\ell} \times \mathbb{R}^{\ell} \mid p_i q_i = 0 \text{ and } \sum_{1 \leq j \leq \ell} p_i^j = 1\} \, ,$$

the mapping $\phi(s)$ defined by :

$$\phi_i^j(s) = p_i^j \, , \; i = 1,\ldots,n \; ; \; j = 1,\ldots,\ell$$

is an obvious Transaction Price Function associated with the S.O.F. developed by this author.

Convention : Instead of adopting Schmeidler's normalization "$\sum_i \phi_i^j \equiv 1$" , we shall fix the ℓ^{th}-commodity as a numeraire in Section 5 by confining ourselves to mappings ϕ such that :

$$\phi_i^{\ell}(s) \equiv 1 \, , \qquad i = 1,\ldots,n \, .$$

(The conclusions established in Section 5 through the explicit introduction of the ϕ mappings do not depend in the least on this particular choice of normalization).

Notation : Henceforth the Strategy Spaces S_i are fixed for all agents $i \in I$. We denote $J(v,z)$ the market game defined by $I = \{1,\ldots,n\}$, $S = \underset{i \in I}{X} S_i$, the preferences $v = (v_i)_{i \in I}$ and the S.O.F. $z : S \to \mathbb{R}^{\ell n}$.

2.3. Three equilibrium concepts

Noncooperative Equilibrium : A selection $s^* = (s_1^*,\ldots,s_n^*) \in S$ is a *noncooperative equilibrium* of the mechanism $J(v,z)$ if and only if :

$$\forall\, i \in I \,,\ \forall\, s_i \in S_i \,,\ v_i \circ z_i(s^*) \geq v_i \circ z_i(s_i, s_{-i}^*) \,,$$

where : $(s_i, s_{-i}^*) = (s_1^*,\ldots,s_{i-1}^*, s_i, s_{i+1}^*,\ldots,s_n^*)$.

The set of Noncooperative Equilibria of the $J(v,z)$ *market game will be denoted* $N(v,z)$. $(N(v,z) \subset S)$.

Pareto-Optimal Equilibrium : A selection $s^* = (s_1^*,\ldots s_n^*) \in S$ is a *Pareto-Optimal equilibrium* of $J(v,z)$ if and only if the net final allocation $z(s^*) \in \mathbb{R}^{\ell n}$ is Pareto-Optimal in the ordinary microeconomic sense, i.e. if there are no $y = (y_1,\ldots,y_n) \in \mathbb{R}^{\ell n}$ allocations such that :

$$\forall\, i \in I \,,\ v_i(y_i) \geq v_i(z_i(s^*))$$

with at least one strict inequality.

The set of Pareto-Optimal Equilibria of the $J(v,z)$ *market game will be denoted* $P(v,z)$.

Walrasian or Competitive Equilibrium : A selection $s^* = (s_1^*,\ldots,s_n^*) \in S$ is a *Walrasian Equilibrium* of $J(v,z)$ if and only if the net final allocation $z(s^*) \in \mathbb{R}^{\ell n}$ is walrasian, i.e. iff there exists a price system $p^* \in \mathbb{R}^{\ell}_{++}$ such that :

(a) $\forall\ i \in I\ ,\ z_i(s^*) = g_i(p^*)$.

(b) $\sum_{i \in I} g_i(p^*) = 0_{\mathbb{R}^\ell}$.

The equilibrium condition (b) becomes evidently redundant when the Strategic Outcome Function z is *balanced* $(z \in B)$.

The set of Walrasian (Competitive) Equilibria of the $J(v,z)$ *market game will be denoted* $W(v,z)$.

We always have the inclusion : $W(v,z) \subset P(v,z) \subset S$.

3. THE GENERIC INEFFICIENCY OF NONCOOPERATIVE EQUILIBRIA IN SMOOTH MARKET GAMES : A FIRST APPROACH WHEN THE PREFERENCES ARE FIXED

In this section the preferences v_i of agents $i \in I$ are fixed once for all and they verify the assumptions *(A1)*, *(A2)*, *(A3)*, *(A4)*, *(A5)* set out in Section 2 (see Balasko [1979]).

The Strategic Outcome Function z describes the $C^\infty(S,\mathbb{R}^{\ell n})$ space which is provided with Whitney's C^∞ topology (1) . Provided with this topology, the $C^\infty(S,\mathbb{R}^{\ell n})$ space is a Baire Space (Golubitsky-Guillemin [1973]).

The following theorem is the main result of this paper :

THEOREM 1 : *(a) The set of smooth Strategic Outcome Functions* z *such that* $N(v,z) \cap P(v,z) = \phi$ *is dense in* $C^\infty(S,\mathbb{R}^{\ell n})$.

(b) The set of smooth and balanced S.O.F. z *such that* $N(v,z) \cap P(v,z) = \phi$ *is dense in* $B \cap C^\infty(S,\mathbb{R}^{\ell n})$.

Corollary 1 : The set of *smooth* S.O.F. z (respectively *smooth and balanced*) such that $N(v,z) \cap W(v,z) = \phi$ is *dense* in $C^\infty(S,\mathbb{R}^{\ell n})$ (respectively *dense* in $B \cap C^\infty(S,\mathbb{R}^{\ell n})$) .

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(1) It is recalled (Golubitsky-Guillemin [1973]) that a sequence of smooth mappings $f_m \in C^\infty(S,\mathbb{R}^{\ell n})$ converges towards $f \in C^\infty(S,\mathbb{R}^{\ell n})$ in the sense of Whitney's C^∞ topology if and only if there exists a compact $K \subset S$ such that :

(a) f_m and their differentials of any order converge uniformly on K respectively towards f and the differentials $D_k f$.

(b) $f_m \equiv f$ on $S \setminus K$ except perhaps for a finite number of m indices.

Remark 1 : The almost certain incompatibility established in this corollary between the noncooperative equilibria of smooth mechanisms $J(v,z)$ and the Walrasian equilibria of the associated exchange economy does not follow from any ad hoc assumption of price rigidity and, more generally, from any limitation of the strategy spaces S_i or of the consumption sets $X_i \subset \mathbb{R}^{\ell}$ of the agents $i \in I$ (1) : it translates more simply the *frictional character* inherent to most *smooth* market games, in particular when their smoothness results from a *differenciation of the products* as in Hotelling [1929] or from an *imperfect information* of the economic agents (2).

Remark 2 : In the light of the foregoing, Hurwicz's mechanism in [1979] seems to us *exceptional* as it is both *competitive* $(N(v,z) = W(v,z) \neq \phi)$ and *smooth*. Its exceptional character could not proceed from a genericity result concerning only the preferences v , all the more since the dimension of Hurwicz's "Price-Quantity" Strategy Spaces is necessarily greater than the number of commodities ℓ , contrary to the Shapley-Shubik's model considered in Dubey [1980]. We shall now explain why, in the genericity concerning the S.O.F. z , it is unnecessary to make any restrictive assumption regarding the dimension of the Strategy Spaces S_i .

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(1) See in this connection Y.Younès [1982]. The case where the X_i are strictly included in $\mathbb{R}^{\ell}$ is treated in our concluding remark.

(2) J. Bertrand [1883] explains very clearly why in the absence of frictions in the market, especially when the product is homogeneous ("sources of identical quality") the duopoly mechanism becomes discontinuous and therefore competitive (i.e. $N(v,z) \subset W(v,z)$). This last point will be dealt with in our Section 5.

Proof of Theorem 1 :

(a) Consider any smooth mechanism $J(v,z)$ where $z \in C^{\infty}(S,\mathbb{R}^{\ell n})$, and let be $s^{*} \in N(v,z) \cap P(v,z)$:

1. The Pareto-Optimality of $s^{*} \in P(v,z)$ may be written :

$$(\#)^{P} : \frac{v'_{ij}}{v'_{i\ell}}(z_i(s^{*})) = \frac{v'_{1j}}{v'_{1\ell}}(z_1(s^{*})) \ , \ 1 \leq j \leq \ell-1 \ , \ 2 \leq i \leq n \ ,$$

where v'_{ij} denotes the partial derivative $\frac{\partial v_i}{\partial z^j}$ and $z(s^{*}) = (z_1(s^{*}),\ldots,z_n(s^{*})) \in \mathbb{R}^{\ell n}$.

2. On the other hand the fact that $s^{*} \in N(v,z)$ implies the following first order necessary conditions :

$$\forall \ i \in I \ , \ \frac{\partial}{\partial s_i}(v_i.z_i)(s^{*}) = 0_{\mathbb{R}^d} \ , \text{ where } d = \dim S_i \ .$$

In other words :

$$(\#)^{N} : \sum_{1\leq j\leq \ell} v'_{ij}(z_i(s^{*})).\ \frac{\partial z_i^j}{\partial s_i}(s^{*}) = 0_{\mathbb{R}^d} \ , \ 1 \leq i \leq n \ ,$$

where $\frac{\partial z_i^j}{\partial s_i}(s^{*}) \in \mathbb{R}^d$ denotes the gradient vector of the partial mapping : $s_i \in S_i \rightarrow z_i^j(s_i,s^{*}_{-i}) \in \mathbb{R}$, for all i and for all j .

The proof is then based on the following intuition : it is known (Balasko [1979]) that whenever preferences v_i verify the assumptions *(A1)* to *(A5)* set forth above, the set Q of Pareto-Optimal allocations is a strict submanifold of $\mathbb{R}^{\ell n}$ diffeomorphic to $\mathbb{R}^{\ell} \times \mathbb{R}^{n-1}$. It may therefore be easily admitted that, generically on $z \in C^{\infty}(S,\mathbb{R}^{\ell n})$, the $(\#)^{P}$ system has a rank a least equal to one in $s \in S$. Consider now Nash system $(\#)^{N}$: it includes $n.d = \dim S$ equations in $s \in S$, and may become of $n.d$ maximal rank through an infinitesimal perturbation of $z \in C^{\infty}(S,\mathbb{R}^{\ell n})$; moreover, the presence of the terms of first order $\frac{\partial z_i^j}{\partial s_i}$ provides the $(\#)^N$ system with an additional degree of liberty toward the $(\#)^P$ system which includes only terms of order zero in $z_i(s)$ (1). It is therefore clear that generically on $z \in C^{\infty}(S,\mathbb{R}^{\ell n})$, the global system $(\#) = (\#)^P$ and $(\#)^N$ will include at least (nd+1) independent equations in the S space of dimension nd and then will have no solution $s^* \in S$, which is the desired result (1) .

This intuitive reasoning involves simultaneously the strategies $s \in S$, the allocations $x = z(s) \in Q \subset \mathbb{R}^{\ell n}$, and the partial derivatives $\frac{\partial z_i^j}{\partial s_i}(s) \in \mathbb{R}^d$. More strictly, the proof of Theorem 1 requires the use of a transversality result stronger than the one usually used, precisely Thom's Transversality Theorem on jets of order one (1-jets) whose definition we shall now recall :

<u>Definition</u> : *Let* X *and* Y *be smooth manifolds.* ($X = S$, $Y = \mathbb{R}^{\ell n}$) . *If* $x_0 \in X$ *and* $f \in C^1(X,Y)$, *the 1-jet of* f *in* x_0 *designates the triplet* $j^{-1}f(x_0) = (x_0, f(x_0), df(x_0))$, *where* $df(x_0)$ *denotes the differential of* f *in* x_0 . *(See Milnor [1965], § 1).*

The set of 1-jets $\sigma = j^1 f(x)$,where f describes $C^1(X,Y)$ and x describes X,is denoted $J^1(X,Y)$. According to Golubitsky-Guillemin [1973], $J^1(X,Y)$ is a smooth manifold such that for every $f \in C^1(X,Y)$, the mapping $j^1 f : x \to j^1 f(x)$ is differentiable from X to $J^1(X,Y)$.

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(1) The absence of terms of an order different from 1 in v_i in both the $(\#)^P$ and $(\#)^N$ systems justifies on the contrary Dubey's additional assumption : $\dim S_i = d \leq \ell - 1$ to ensure the generic inefficiency of Nash Equilibria of the J(v,z) mechanism when the S.O.F. z is fixed and only the preferences v vary.

Now Thom's Transversality Theorem adapted to the particular case of 1-jets reads as follows : (Golubitsky-Guillemin, op. cit., p. 54) :

<u>THOM'S TRANSVERSALITY THEOREM</u> : *Let* X *and* Y *be two smooth manifolds, and* W *a submanifold of* $J^1(X,Y)$. *The set*

$$T_W = \{f \in C^\infty(X,Y) \mid j^1 f \text{ is transversal to } W\}$$

is dense (1) *in* $C^\infty(X,Y)$ *for the* C^∞*-Whitney Topology.*

This result applies to our proof in the following way : Consider the submanifold W of $J^1(S,\mathbb{R}^{\ell n})$ defined by :

$$W = \{j^1 f(s) \in J^1(S,\mathbb{R}^{\ell n}) \text{ such that :}$$

$$(\#)^P : \left| \begin{array}{l} \frac{v'_{ij}}{v'_{i\ell}}(f_i(s)) = \frac{v'_{1j}}{v'_{1\ell}}(f_1(s)) \ , \ 1 \leq j \leq \ell - 1 \ , \ 2 \leq i \leq n \ , \\ \text{where } f(s) = (f_1(s),\ldots,f_n(s)) \in \mathbb{R}^{\ell n} \end{array} \right.$$

$$\text{and } (\#)^N : \left| \begin{array}{l} \sum_{1 \leq j \leq \ell} v'_{ij}(f_i(s)).\frac{\partial f_i^j}{\partial s_i}(s) = 0_{\mathbb{R}^d} \ , \ 1 \leq i \leq n \ , \\ \text{where } \frac{\partial f_i^j}{\partial s_i}(s) \text{ is the gradient of the partial mapping} \\ s_i \to f_i^j(s_i,s_{-i})\} \ . \end{array} \right.$$

According to Balasko [1979], the $(\#)^P$ relations are equivalent to : "$f(s) \in Q$" , where Q is a submanifold of $\mathbb{R}^{\ell n}$ diffeomorphic to $\mathbb{R}^\ell \times \mathbb{R}^{n-1}$, i.e. of codimension $\ell n - (\ell+n-1)$ *greater or equal to 1* when $n \geq 2$ and $\ell \geq 2$ which was assumed at the beginning of Section 2.

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(1) More precisely T_W is a countable intersection of dense open subsets of $C^\infty(X,Y)$, therefore a G_δ-dense subset of $C^\infty(X,Y)$ which, provided with the Whitney Topology, is a Baire Space.

On the whole, the codimension of W in $J^1(S,\mathbb{R}^{\ell n})$ is greater than (nd+1) , where nd is the rank of the system $(\#)^N$ which is *independent* from $(\#)^P$ in $J^1(S,\mathbb{R}^{\ell n})$ because $(\#)^P$ does not include terms of order 1 in f.

Let us now consider a Strategic Outcome Function $z \in C^\infty(S,\mathbb{R}^{\ell n})$ such that $N(v,z) \cap P(v,z) \neq \phi$, and let s^* belong to this intersection ; in the light of the above, the necessary conditions $(\#)^N$ and $(\#)^P$ will be verified by the 1-jet $j^1z(s^*) \in J^1(S,\mathbb{R}^{\ell n})$; in other words we have $j^1z(s^*) \in W$.

Therefore : $N(v,z) \cap P(v,z) \neq \phi \Rightarrow \text{Im } j^1z \cap W \neq \phi$, which is equivalent to the implication :

$$\text{Im } j^1z \cap W = \phi \quad \Rightarrow \quad N(v,z) \cap P(v,z) = \phi \quad ,$$

where Im j^1z denotes the image $j^1z(S)$ of the set S of messages by the differentiable mapping $j^1z : S \to J^1(S,\mathbb{R}^{\ell n})$.

Let us now apply Thom's Transversality Theorem :
For every z belonging to a (G_δ)-dense subset of $C^\infty(S,\mathbb{R}^{\ell n})$, the mapping j^1z is *transversal* to the submanifold W . (we note : $j^1z \pitchfork W$) . This means :

- either

$$(1)\ \forall s \in S\ ,\ T_{j^1z(s)}(J^1(S,\mathbb{R}^{\ell n})) = T_{j^1z(s)} W \oplus (dj^1z)_s\ (T_sS)\ .$$

(T_xX denotes the Tangent Space to the manifold X at the point $x \in X$; and $(dj^1z)_s$ denotes the differential of j^1z at the point $s \in S$) ;

- or

(2) $\text{Im } j^1z \cap W = \phi$.

In order to show that (1) is impossible here, we will reason on dimensions ; in fact if we note :

$$d_1(s) = \dim T_{j^1z(s)}(J^1(S,\mathbb{R}^{\ell n}))$$

$$d_2(s) = \dim T_{j^1z(s)} W$$

$$d_3(s) = \dim (dj^1z)_s (T_sS) ,$$

we obtain :

$$d_1(s) = \dim J^1(S,\mathbb{R}^{\ell n})$$

$$d_2(s) = \dim W$$

$$d_3(s) \leqq \dim T_sS = \dim S = nd .$$

with : $\operatorname{codim}_{J^1(S,\mathbb{R}^{\ell n})} W = d_1(s) - d_2(s) \geqq nd + 1 > nd$.

Therefore the following strict inequality :

$$d_1(s) - d_2(s) > d_3(s) ,$$

which eliminates possibility (1) , is always verified here and leads finally to the conclusion that the set $T_W = \{z \in C^\infty(S,\mathbb{R}^{\ell n}) \mid \operatorname{Im} j^1z \cap W = \phi\}$ is *dense* in $C^\infty(S,\mathbb{R}^{\ell n})$.

The general part (a) of Theorem 1 is thus proved. To prove the second part (b) on *balanced* outcome functions, it suffices merely :

1) To replace in the foregoing the $\mathbb{R}^{\ell n}$ set of all the allocations by the subspace $\mathcal{B}_0 \subset \mathbb{R}^{\ell n}$ of *balanced allocations* $(y_1,\ldots,y_n) \in \mathbb{R}^{\ell n}$ such that $\sum_{i\in I} y_i = 0_{\mathbb{R}^\ell}$.

2) To use the diffeomorphism (also proved in Balasko [1979]) between the set $Q_0 = \mathcal{B}_0 \cap Q$ of *balanced* Pareto-Optimal allocations and the space $\mathbb{R}^{n-1}$. As the codimension of Q_0 in $\mathcal{B}_0$ is always greater than 1 , the proof of part (b) is the same as that of part (a) above.

This completes the proof of Theorem 1. $\square$

4. THE GENERIC INEFFICIENCY OF SMOOTH MARKET GAMES J(v,z) WHEN THE S.O.F.z AND THE PREFERENCES v VARY SIMULTANEOUSLY

In this section, we assume that the v_i utility functions describe the $C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})$ space which is also provided with Whitney's C^{∞}-Topology.

THEOREM 2 : *The set of smooth couples* (v,z) *such that* $N(v,z) \cap P(v,z) = \phi$ *is dense in the Product-Space* $[C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})]^n \times C^{\infty}(S,\mathbb{R}^{\ell n})$.

The proof of this theorem is similar to that of Theorem 1 : Let us consider any couple (v_0,z_0) where $v_0 = (v_{0i})_{i \in I} \in [C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})]^n$ and $z_0 \in C^{\infty}(S,\mathbb{R}^{\ell n})$, and let $V_0 \times Z_0$ be an open neighbourhood of (v_0,z_0) in the Product-Space $[C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})]^n \times C^{\infty}(S,\mathbb{R}^{\ell n})$. It is clear that, by perturbing v_0 very slightly, we may find $v \in V_0$ such that the $(\#)^P$ system :

$$(\#)^P : \frac{v'_{ij}}{v'_{i\ell}}(x_i) = \frac{v'_{1j}}{v'_{1\ell}}(x_1) \ , \ 1 \leq j \leq \ell - 1 \ , \ 2 \leq i \leq n \ ,$$

defines, in $x = (x_1,\ldots,x_n)$, a submanifold Q of $\mathbb{R}^{\ell n}$ of *codimension greater or equal to 1*. From here on the proof follows closely that of Theorem 1 and ensures the existence of a Strategic Outcome Function $z \in Z_0$ such that $N(v,z) \cap P(v,z) = \phi$. Q.E.D. .□.

Remark : We also obtain, as in Theorem 1 (b), that the *smooth and balanced* market games J(v,z) such that $N(v,z) \cap P(v,z) = \phi$ are *dense* in the set of all smooth and balanced mechanisms. This very strong result about generic inefficiency of smooth market games will now be completed by a determination of non-differentiabilities and especially discontinuities inherent to most mechanisms J(v,z) such that $N(v,z) \cap W(v,z) = \phi$, in particular when these mechanisms are competitive.

5. NON-DIFFERENTIABILITIES IN MARKET GAMES $J(v,z)$ SUCH THAT $N(v,z) \cap W(v,z) \neq \phi$. (1)

Preferences v_i of agents $i \in I$ are fixed again so as to verify assumptions *(A1)* to *(A5)*.

In the following we shall confine ourselves to Strategic Outcome Functions $z : S \to \mathbb{R}^{\ell n}$ which are quasi-differentiable in the sense that, for all $i \in I$, the partial mappings $s_i \to z_i(s_i,s_{-i})$ are differentiable except on a *finite* or *countable* subset of S_i.$[z(s) = (z_1(s),\ldots,z_n(s))]$.

We shall then select Transaction-Price Functions $\phi : S \to \mathbb{R}_+^{\ell n}$ that are compatible with the S.O.F. z and allow at most only the z non-differentiabilities. From an economic point of view, it seems natural to submit the ϕ transaction-prices to the following minimum condition :

(C) *If* $s^* \in W(v,z)$, *then for all* $i \in I$:

$z_i(s^*) \neq 0_{\mathbb{R}^\ell} \Rightarrow \phi_i(s^*) = \hat{p}$ *where* $\hat{p}$ *is the Walrasian price defined by* :
$z_i(s^*) = g_i(\hat{p})$, $i = 1,\ldots,n$.

----

(1) This section extends a result of Benassy. See Benassy [1984], Section 4.

This condition, verified in particular in Schmeidler [1980] and Dubey [1982] , is mathematically compatible with Walras'law : $p.g_i(p) \equiv 0$, $i = 1,\dots,n$. (1)

<u>Lemma</u> : *Let* $z : S \to \mathbb{R}^{\ell n}$ *be a quasi-differentiable S.O.F. and* ϕ *a transaction-Price Function compatible with* z *and verifying condition* (C).

If z *is differentiable at point* $s^* \in N(v,z) \cap W(v,z)$, *then the following conditions* $(\#)^W$ *are necessarily verified at that point* :

$$(\#)^W : \sum_{1\leq j\leq \ell-1} z_i^j(s^*).\frac{\partial \phi_i^j}{\partial s_i}(s^*) = 0_{\mathbb{R}^d} , \quad 1 \leq i \leq n ,$$

where $\frac{\partial \phi_i^j}{\partial s_i}$ *denotes the gradient of the partial mapping* $s_i \to \phi_i^j(s_i,s_{-i})$.

Proof : Let $s^* \in N(v,z) \cap W(v,z)$ be a differentiability point of the S.O.F. z ; $s^* \in N(v,z)$ verifies the first order necessary conditions :

$$(\#)^N : \sum_{1\leq j\leq \ell} v'_{ij}(z_i(s^*)).\frac{\partial z_i^j}{\partial s_i}(s^*) = 0_{\mathbb{R}^d} , \quad 1 \leq i \leq n .$$

On the other hand we have :

$$z_i^\ell(s) = - \sum_{1\leq j\leq \ell-1} \phi_i^j(s).z_i^j(s)$$

----

(1) The existence of a *Transaction-Price Function* $\phi = (\phi_1,\dots,\phi_n)$ verifying condition (C) can be proved very simply in the following way :

1. On a neighbourhood of each $s^* \in W(v,z)$ such that $z_i(s^*) \neq 0_{\mathbb{R}^\ell}$ by applying the *Implicit Functions Theorem* to the mapping ψ_i defined by :

   $$\psi_i(\phi_i^j,s) = \phi_i^j.z_i^j(s) + \sum_{\substack{1\leq k\leq \ell-1 \\ k\neq j}} \hat{p}^k.z_i^k(s) + z_i^\ell(s) , \text{ where } z_i^j(s^*) \neq 0 .$$

   We then get a unique ϕ_i mapping defined on a neighbourhood of s^* and such that $\phi_i^k(s) = \hat{p}^k$ if $k \neq j$.

2. The mappings ϕ_i thus constructed are then extended to the whole manifold S by applying a *Partition of Unit* theorem (Golubitsky-Guillemin, op. cit.).

for every $s \in S$ according to the definition of ϕ (Section 2) .

By deriving we obtain :

$$\frac{\partial z_i^{\ell}}{\partial s_i}(s) = - \sum_{1\leq j\leq \ell-1} \phi_i^j(s) . \frac{\partial z_i^j}{\partial s_i}(s) - \sum_{1\leq j\leq \ell-1} z_i^j(s) . \frac{\partial \phi_i^j}{\partial s_i}(s) .$$

By replacing $\frac{\partial z_i^{\ell}}{\partial s_i}(s)$ by this equivalent expression in $(\#)^N$, we obtain :

$$(\#)^N \quad : \quad \sum_{1\leq j\leq \ell-1} \left(\frac{v'_{ij}}{v'_{i\ell}}(z_i(s^*)) - \phi_i^j(s^*)\right) . \frac{\partial z_i^j}{\partial s_i}(s^*) = \sum_{1\leq j\leq \ell-1} \frac{\partial \phi_i^j}{\partial s_i}(s^*) . z_i^j(s^*) , \ 1\leq i\leq$$

The lemma results then immediately from condition (C) on ϕ and from the implication :

$$s^* \in W(v,z) \Rightarrow \frac{v'_{ij}}{v'_{i\ell}}(z_i(s^*)) = \hat{p}^j \quad \text{where } z_i(s^*) = g_i(\hat{p}) \ , \ i=1,\ldots,n;j=1,\ldots\ell-1. .\square .$$

This lemma has three important consequences :

1. *At first, it determines the non-differentiabilities at the points* $s^* \in N(v,z) \cap W(v,z)$ *which do not verify the conditions* $(\#)^W$.
   Thus, for example, without going into the details of the competitive market games of Schmeidler [1980] and Dubey [1982], the simple fact that the *Transaction-Prices* $\phi_i^j(p,q)$ associated with these mechanisms *are equal to the quoted prices* p_i^j (such that $\frac{\partial \phi_i^j}{\partial p_i^j} \equiv 1$ and $\frac{\partial \phi_i^j}{\partial p_i^k} \equiv 0$ for $k \neq j$) makes necessary the non-differentiability of their Strategic Outcome Functions z at the points $s^* \in N(v,z) = N(v,z) \cap W(v,z)$ such that $z(s^*) \neq 0_{\mathbb{R}^{\ell n}}$. This *non-differentiability* is however equivalent, in each of these two mechanisms, to a *discontinuity* (See Schmeidler and Dubey, op. cit.).

2. *Moreover, the preceding lemma enables us to establish directly the generic non-competitivity of smooth market games* already obtained as a corollary of Theorem 1 : for if we consider any equilibrium $s^* \in N(v,z) \cap W(v,z)$, it necessarily verifies the system :

$$(\mathcal{D}) \quad \begin{cases} \sum_{1 \leq j \leq \ell-1} \left(\frac{v'_{ij}}{v'_{i\ell}}(z_i(s^*)) - \hat{p}^j\right) . \frac{\partial z_i^j}{\partial s_i}(s^*) = 0_{\mathbb{R}^d} \ , \ i = 1,\ldots,n \ . \\ \frac{\partial z_i^\ell}{\partial s_i}(s^*) = - \sum \hat{p}^j . \frac{\partial z_i^j}{\partial s_i}(s^*) \quad \text{for all agents } i \text{ such that } z_i(s^*) \neq 0_{\mathbb{R}^\ell} . \end{cases}$$

where $\hat{p}$ is uniquely determined by $z_i(s^*) = g_i(\hat{p})$, $i = 1,\ldots,n$.

In other words the 1-jet $j^1 z(s^*)$ belongs to the submanifold Z of $J^1(S,\mathbb{R}^{\ell n})$ defined by $(\mathcal{D})$ and whose codimension in $J^1(S,\mathbb{R}^{\ell n})$ is obviously superior or equal to $(nd+1)$. The genericity of $N(v,z) \cap W(v,z) = \phi$ when z varies obtains then by applying Thom's Transversality Theorem to the submanifold Z . ▫

3. *At last, the lemma and its proof suggest a simple and general method for the study of competitive mechanisms.*

Consider in particular the two following examples : the first one, with a *finite* number of agents, relies on Bertrand-type *discontinuities* (see [1883]), whereas the second one — *differentiable* — corresponds to the limit situation of an *infinite* number of "small" agents $i \in I$ whose individual influence on their own Transaction-Price $\phi_i(s)$ is *negligeable.*

Example 1 : (Dubey [1982]) .

We consider a partial market ($j = 1$, $\ell = 2$) . For each agent $i = 1,\dots,n$ ($n \geq 4$) , a strategy s_i consists of a *Price-Quantity* couple $s_i = (p_i,q_i) \in \mathbb{R}_+^* \times \mathbb{R} = S_i$, where q_i designates a net quantity of commodity 1 *offered* ($q_i < 0$) or *purchased* ($q_i > 0$) by agent i , and p_i the *minimal offer price* or the *maximal demand price* quoted by this agent for this quantity.

Given a selection $(p,q) = [(p_1,q_1),\dots,(p_n,q_n)] \in S = \underset{i \in I}{X} S_i$, the *Aggregate Offer Curve* $O(p,q)$ (respectively the *Aggregate Demand Curve* $D(p,q)$) is constructed by classifying individual strategies (p_i,q_i) in order of increasing offer prices (respectively in order of decreasing demand prices) so as to give the priority to the sellers who quote the lowest prices and to the purchasers who quote the highest prices (Figure 1).

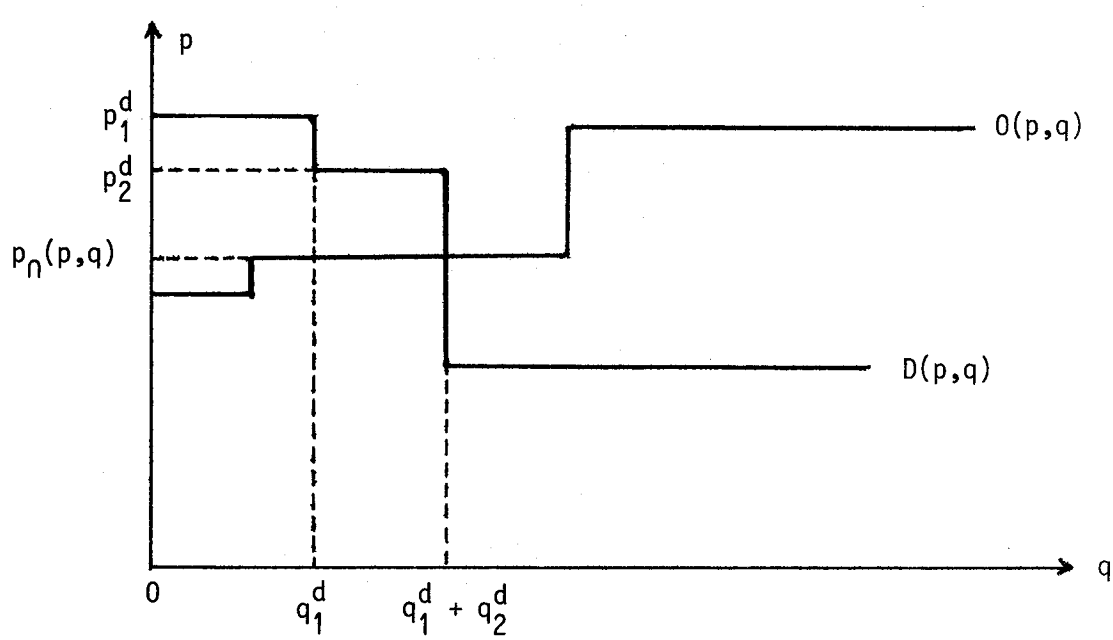

Figure 1

The Strategic Outcome Function $z(p,q) = [z_1(p,q),\dots,z_n(p,q)]$ is then defined as follows :

- If $D(p,q) \cap O(p,q) = \phi$, there is no trade : $z(p,q) = 0_{\mathbb{R}^{2n}}$.
- If $D(p,q) \cap O(p,q) \neq \phi$, we note $p_\cap(p,q)$ the *intersection price* of these two aggregated curves and we state, for all agents $i \in I$, that $z_i(p,q) = [z_i^1(p,q) , z_i^2(p,q)]$

  where :

  . $q_i > 0$ and $p_i > p_\cap(p,q) \Rightarrow z_i^1(p,q) = q_i$

  . $q_i > 0$ and $p_i < p_\cap(p,q) \Rightarrow z_i^1(p,q) = 0$

  . $q_i < 0$ and $p_i < p_\cap(p,q) \Rightarrow z_i^1(p,q) = q_i$

  . $q_i < 0$ and $p_i > p_\cap(p,q) \Rightarrow z_i^1(p,q) = 0$

  . $p_i = p_\cap(p,q) \Rightarrow$ agent i is rationed in proportion of his demand if $q_i > 0$ or of his offer if $q_i < 0$.

(On Figure 1, *the offers* at price $p_\cap(p,q)$ are being rationed) . Having thus determined the non-numeraire components $z_i^1(p,q)$ we fix the *numeraire* components $z_i^2(p,q)$ by Transaction Prices :

. $\phi_i^1(p,q) = p_i$ for $i = 1,\dots,n$ (1) .

Consider now an *active* noncooperative equilibrium $(p^*,q^*) \in S$, for which there exist at least two active buyers and two active sellers.

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(1) In other words we suppose that *each agent transacts at the price he quoted* and that the differences between demand and offer prices are retained by a broker (see Dubey [1982], concluding Remark 3).

(a) *All the prices* p_i^* *are equal to* $p_\cap(p^*,q^*)$.

Proof : Suppose that some agent i quoted a price $p_i^* \neq p_\cap(p^*,q^*)$ and obtained $z_i(p^*,q^*) \neq 0_{\mathbb{R}^2}$. According to the lemma's proof above, we necessarily have : (1)

$$(1) \quad \left(\frac{v'_{i1}}{v'_{i2}}(z_i(p^*,q^*)) - p_i^*\right). \frac{\partial z_i^1}{\partial q_i}(p^*,q^*) = 0$$

$$(2) \quad \left(\frac{v'_{i1}}{v'_{i2}}(z_i(p^*,q^*)) - p_i^*\right). \frac{\partial z_i^1}{\partial p_i}(p^*,q^*) = 1 \times z_i^1(p^*,q^*) \ .$$

Now, from the above conventions about the S.O.F. z :

$$p_i^* \neq p_\cap(p^*,q^*) \quad \text{and} \quad z_i(p^*,q^*) \neq 0_{\mathbb{R}^2} \Rightarrow \frac{\partial z_i^1}{\partial q_i}(p^*,q^*) = 1 \neq 0$$

Consequently, by (1) :

$\frac{v'_{i1}}{v'_{i2}}(z_i(p^*,q^*)) = p_i^*$, which, by (2), leads to the absurdity :

$$0 \times \frac{\partial z_i^1}{\partial p_i}(p^*,q^*) = z_i^1(p^*,q^*) \neq 0 \ .$$

This establishes (a).

(b) (p^*,q^*) *is a Walrasian equilibrium.*

Proof : Suppose for a moment that $(p^*,q^*) \notin W(v,z)$. Proposition (a) together with the *activity* assumption on (p^*,q^*) and the above convention about shortage sharing at price $p_\cap(p,q)$ make then *necessary* for the partial derivatives $\frac{\partial z_i^1}{\partial q_i}(p^*,q^*)$ *to exist and to be non-zero* for all $i \in I$.

----

(1) The partial mappings $q_i \mapsto z_i(p_i^*,q_i,(p^*,q^*)_{-i})$ and $p_i \mapsto z_i(p_i,q_i^*,(p^*,q^*)_{-i})$ are clearly differentiable at (p^*,q^*) if $p_i^* \neq p_\cap(p^*,q^*)$.

We deduce by the still valid relation (1) :

$$\frac{v'_{i1}}{v'_{i2}}(z_i(p^*,q^*)) = p_i^* = p_n(p^*,q^*) \text{ , i.e. } z_i(p^*,q^*) = g_i(p_n(p^*,q^*))$$

for all $i \in I$. We thus obtain : $(p^*,q^*) \in W(v,z)$, as the S.O.F. z is *balanced.*

To avoid this contradiction, Proposition (b) must then necessarily be verified. ▫

Example 2 : (Shubik [1973], Shapley-Shubik [1977], Dubey-Mas-Colell-Shubik [1980]) .

We consider the limit case where the set of agents I is a continuum. (For example $I = [0,1]$).

There are two commodities (ℓ=2) . For each agent $i \in I$, a strategy $(s_1(t),s_2(t)) \in S_t = \mathbb{R}_+^2$ consists of an offer of commodity 1 ($s_1(t) \geq 0$) and an offer of numeraire 2 ($s_2(t) \geq 0$) .

At every point $s = (s_1,s_2) = [(s_1(t),s_2(t))_{t\in I}] \in (\mathbb{R}_+^2)^{[0,1]} = S$ such that $\int_I s_1(t)\,dt$ and $\int_I s_2(t)\,dt > 0$, we define the S.O.F. $z : S \to (\mathbb{R}^2)^{[0,1]}$

by :

$$z_s^1(t) = \frac{\int_I s_1}{\int_I s_2} \cdot s_2(t) - s_1(t)$$

$$z_s^2(t) = \frac{\int_I s_2}{\int_I s_2} \cdot s_1(t) - s_2(t)$$

for all $t \in I$, and we extend to the case $\int_I s_1(t)\,dt = 0$ or $\int_I s_2(t)\,dt = 0$ by writing : $z_s^1(t) = - s_1(t)$ and $z_s^2(t) = - s_2(t)$ in that case.

The mapping $\phi : S \to [\mathbb{R}_+^2]^{[0,1]}$ defined, for all $t \in I$, by :

$$\left| \begin{array}{l} \phi_s^1(t) = \dfrac{\int_I s_2}{\int_I s_1} = \phi_s^1 \text{ (with } \phi_s^1 = 0 \text{ when } \int_I s_1 = 0 \text{ or } \int_I s_2 = 0) \\ \phi_s^2(t) \equiv 1 \end{array} \right.$$

is an evident Transaction Price Function compatible with z , and verifies :

$$\frac{\partial \phi_s^1}{\partial s_1(t)} = \frac{\partial \phi_s^2}{\partial s_2(t)} \equiv 0 \quad \text{for all } t \in I .$$

(The necessary condition $(\#)^W$ of the lemma is in particular verified here).

Consider now an *active* noncooperative equilibrium $s^* = (s_1^*, s_2^*) \in N(v,z)$ such that $\int_I s_1^*$ and $\int_I s_2^*$ are *strictly positive*.

According to lemma's proof, we have for all $t \in I$:

$$\left(\frac{v'_{t1}}{v'_{t2}}(z_{s^*}(t)) - \phi_{s^*}^1\right) . \underbrace{\frac{\partial z_s^1(t)}{\partial s_1(t)}(s^*)}_{= -1} = \underbrace{\frac{\partial \phi_s^1(t)}{\partial s_1(t)}}_{= 0} \times z_{s^*}^1(t) = 0$$

Therefore : $\dfrac{v'_{t1}}{v'_{t2}}(z_{s^*}(t)) = \phi_{s^*}^1$, i.e. $z_{s^*}(t) = g_t(\phi_{s^*})$ for all $t \in I$.

As the mechanism z is *balanced*, we automatically have :

$$\sum_{t \in I} g_t(\phi_{s^*}) = 0_{\mathbb{R}^2} \text{ , i.e. } s^* \in N(v,z) \cap W(v,z) \neq \phi \ .$$

Remark : These two examples extend easily to any number of commodities ℓ . It suffices to separate the global market $(1 \leq j \leq \ell)$ into $(\ell-1)$ partial markets $[j,\ell]_{1\leq j\leq \ell-1}$ by supposing as in Dubey [1982] or in Dubey-Mas Colell-Shubik [1980] that the final allocation $z_i^j(s)$ and the Transaction-Price $\phi_i^j(s)$ do not depend on strategies s_i^k quoted by agent i on the other markets $k \neq j$.

$$\text{In other words : } \frac{\partial z_i^j}{\partial s_i^k} = \frac{\partial \phi_i^j}{\partial s_i^k} \equiv 0 \quad \text{for } 1 \leq k \neq j \leq \ell-1$$

and for all i , where $s_i = (s_i^1,\ldots,s_i^{\ell-1})$ and $s = (s_i)_{i\in I}$.

6. A CONCLUDING REMARK

In the foregoing we supposed that the consumption sets X_i of all the agents $i \in I$ were equal to the whole space $\mathbb{R}^{\ell}$, thus doing away with the individual non-feasibility problems that generally relate to the study of market games (Schmeidler [1980]). Let us now assume that the X_i are *strictly included* in $\mathbb{R}^{\ell}$: will the preceding genericity results be still relevant in that case ? For answering this question it will be convenient to distinguish between the following two situations :

(a) *All the* X_i *are open in* $\mathbb{R}^{\ell}$.

In this case, the necessary first order conditions for a selection s^* to belong to the intersection $N(v,z) \cap P(v,z)$ remain the same as in the proofs of Theorems 1 and 2 above. These results are therefore still valid.

(b) *There is at least one agent* $i \in I$ *whose consumption set* X_i *has a non-empty boundary* ∂X_i

In this case the Pareto-Optimality of an allocation $z(s^*) \in \partial X_i$ may no longer be expressed through $(\#)^P$ equations (see Sections 3 and 4 above), but rather through *inequations* of the Smale [1974] type.

The proof of Theorem 1 may well then not be valid, with however the following exception : the important case where ∂X_i is included in a *finite or countable* union of submanifolds V_i^k of dimension $(\ell-1)$. For example, if $X_i = \mathbb{R}_+^{\ell}$, the boundary ∂X_i is included in the finite union of hyperplanes $\{x^j = 0\}$, $j = 1,\ldots,\ell$.

Consider now the systems :

$$(S_i^k) \quad \left| \begin{array}{l} (\#)^N \ : \ \sum_{1\leq j\leq \ell} v'_{ij}(z_i(s^*)) . \dfrac{\partial z_i^j}{\partial s_i}(s^*) = 0_{\mathbb{R}^d} \ , \ i = 1,\ldots,n \ . \\ \\ \text{and} \quad z_i(s^*) \in V_i^k \ . \end{array} \right.$$

Each of these systems (S_i^k) defines an submanifold X_i^k of $J^1(S,\mathbb{R}^{\ell n})$, of codimension superior or equal to : nd + codim $_{\mathbb{R}^\ell} V_i^k$ = nd+1 . We know then by Thom's Transversality Theorem, that for z belonging to a dense open subset 0_i^k of $C^\infty(S,\mathbb{R}^{\ell n})$, the system S_i^k has no solution s^* . As the *countable intersection* of dense and open subsets 0_i^k is itself *dense* in the *Baire Space* $C^\infty(S,\mathbb{R}^{\ell n})$, we are sure that generically on $z \in C^\infty(S,\mathbb{R}^{\ell n})$ the Nash allocations $z(s^*)$ where s^* verifies $(\#)^N$ will belong to $\mathring{X}_i$ (the interior of X_i) .

We are thus brought back to the previous case (a) where Theorems 1 and 2 apply.

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