A Model of Optimal Growth Strategy

Philippe ASKENAZY (Delta)*
Cuong LE VAN (CNRS - CEPREMAP)†

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*DELTA : UMR CNRS - EHESS - ENS.
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ABSTRACT : In this paper we present an optimal growth model for an open developing country. The latter may choose to produce consumption goods by borrowing on capital markets, or to import consumption goods by investing all its saving on capital markets. We prove that there may be two steady states. An optimal path converges either to zero or to a steady state. That depends on the levels of the initial aid and/or of the debt constraint. We prove also there exists a poverty trap if the time preference is very high.

Keywords : Optimal growth model, Euler-Lagrange equations, optimal path, value function, steady states, saddle-point, poverty trap, debt constraint.

JEL Classification numbers : C61, D92, 012,041

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Un Modèle de stratégie de croissance optimale

RESUME : Dans ce papier, nous présentons un modèle de croissance optimale pour un pays en voie de développement en économie ouverte. Ce pays peut, soit produire des biens de consommation en empruntant, soit importer ces biens en investissant son épargne sur le marché financier. Nous montrons qu'il peut y avoir deux états stationnaires. Tout sentier optimal converge vers zéro ou vers un état stationnaire, en fonction du niveau de l'aide initiale, ou/et du niveau de la contrainte de la dette. Nous montrons aussi qu'il existe une trappe de pauvreté si la préférence pour le présent est très élevée.

Mots clés : Modèle de croissance optimale, équations d'Euler-Lagrange, sentier optimal, fonction valeur, états stationnaires, point-selle, trappe de pauvreté, contrainte de la dette.

JEL numéros de classification : C61, D92, 012, 041

1 Introduction

Most of the Low Developed Countries dispose of a wealth initial stock : resources (renewable or not) in raw materials, international help, embryonic industrialization ...

According to the traditional growth theory, this countries must experience a faster growth in early stages of development, the poorer countries growing faster than richer ones.

Nevertheless, international evidence on growth rates reveals large differences in development patterns among nations (see Azariadis-Drazen (1991) or Barro-Sala-i-Martin (1995)). Some countries manage to sustain high growth rate (the Tigers for example) ; others have persistently low or negative rate of growth. Two similar countries such as Taïwan and the Philippines in the 60's exhibit very different trajectories during three decades : the Taiwan's GDP per capita is 9750 \$ in 1992 and the Philippines' GDP per capita is only 750 \$.

One explanation is that persistent differences are due to exogenous factors such as culture, social institutions, demography, market structure. Various works (see Azariadis for a very complete review) study the possibility that this differences could appear among economies with identical structures and give "historical" causes of a model of underdevelopment trap : Consumer impatience, subsistence consumption, human capital externalities, external increasing return in a decentralized economy, demographic transition with endogenous fertility... Expectations may yield multiple equilibrium growth paths (Krugman (1991)). Young (1991) (see also, Klundert-Smulders (1996)) investigates the dynamic effects of international trade with a model in which learning by doing exhibits spillovers across goods : under free trade the Low Developed Countries have rates of technical progress less than those enjoyed under autarky.

Specific economic policies allow to switch over these traps : Taïwan or Singapore imposes very high saving and investment sacrificing the consumption (Young (1995)). These policies are underoptimal for the first generation.

We explore in this paper the relaxation of a fundamental hypothesis in the traditional optimal one-sector growth model for a centralized economy : non-concavity of the production function.

The concavity is a fundamental hypothesis in the standard models ; this condition implies that the optimal intertemporal trajectory is unique and converge towards a steady state. It guarantees that the Euler equation and the transversality condition are sufficient and necessary for optimality (Carlson-Haurie-Leizarowitz (1992)).

Nevertheless, the concavity seems to be not relevant for low industrialized countries. The artisanal technology exhibits linear production capital ratio ; the transition to mass production enhances slightly the productivity of labor and capital (organizational, learning by doing, specialization effects ...). At early stages of industrialization, the returns to scale are increasing and the production function becomes concave for higher capital stocks, (see e.g. Azariadis, 1996).

Dechert-Nishimura (1983), using a discrete time model with such production function, show for a country in autarky that the optimal path converges to a steady state only if the initial capital stock is above a critical level ; otherwise it converges to zero. The time preference determines the existence and the level of critical initial capital stock.

In this paper, we extend this analysis to a country which is able to invest on international capital markets. If it borrows, then it will face a debt constraint imposed by the foreign countries. The income could be used for consumption of domestic or foreign goods, for investment in physical capital and for investment on capital markets. For example, the "oil monarchies" choose in the 80's to invest a major part of their revenues in the OECD states and import from these countries consumption goods. Our mathematical approach is different from Dechert-Nishimura (1983) by using a continuous time model.

We find as Dechert-Nishimura for the physical capital, that the wealth is necessarily monotonic and so that cycles are impossible ; nevertheless, along the optimal paths, consumption can first decrease and then increases to a steady state. We exhibit, if the time preference is higher than the real international interest rate, a poverty trap. At early stages of "development", the industrialization may be, or may be not optimal. If the debt constraint is very strong, a country must invest its saving on foreign capital markets and imports consumption goods. If the debt constraint is soft, then the country will borrow and produce consumption goods. We have a result which differs from those stated by Dechert and Nishimura. As in Dechert-Nishimura, in our model, there are two steady states. But under some assumptions they are both optimal. When the debt constraint is hard, or when the initial aid is very low, for a "poor" country, it could be optimal to jump to the low level steady state while a "rich" country converges to the high level steady state, and they have the same technology. There could be no convergence in the levels of development. If the debt constraint is now soft or if the initial aid is important, then the poor country may reach the high level steady states. Summing up, there are two ways for helping the Lower Developed Countries : offer a high level of initial aid, or release the debt constraint.

2 The Model

We consider a developing country with an initial aid D_0. At each date t, its wealth W_t is composed by capital stock k_t and saving or foreign debt S_t. S_t generates as returns, rS_t, with r real interest rate. k_t is used to produce consumption goods and capital goods for the next period. The domestic supply is equal to $f(k_t)$. If it is not sufficient to respond to the domestic demand, which is equal to consumption C_t and investment I_t, the country imports M_t. If the supply is too large in comparison with the domestic demand, then the country will export M_t.

We have the following balance equation :

$$C_t + I_t = f(k_t) + M_t$$

where

$$I_t = \dot{k}_t + \delta k_t$$

and

$$M_t = -\dot{S}_t + rS_t.$$

The consumer maximizes her intertemporal utility function :

$$\int_0^\infty u(C_t)e^{-\rho t}dt, \quad \rho > 0$$

under the constraints :

$$\dot{k}_t + C_t = f(k_t) + rS_t - \delta k_t - \dot{S}_t$$

$$S_t \geq \overline{S}(k_t) \ , \ k_t \geq 0 \ , \ \forall t \ ,$$

and

$$k_0 + S_0 = D_0, \text{ with } D_0 \text{ given.}$$

$\overline{S}(k_t)$ is the debt constraint. It depends on the capital stock k_t and, of course, is non positive.

Define $W_t = S_t + k_t$. W_t is the wealth at date t. Notice that we implicitly assume perfect substitutability between S_t and k_t. The country could "very quickly" sell capital stock k_t and imports consumption goods.

Barro, Mankiw and Sala-I-Martin (1995), Cohen and Sachs (1986) developed models in which the borrowing constraint $\overline{S}(k_t)$ could not exceed the quantity of physical capital k_t. Here, we assume $\overline{S}(k) = \overline{s} - \nu k$, with $\nu \in [0, 1[$, and $\overline{s} \leq 0$. This formulation is a combination of two interesting cases : i) the debt constraint is constant ($\nu = 0$) ; ii) it is a fraction of the capital stock k_t as in Cohen and Sachs (1986) or Barro, Mankiw and Sala-I-Martin (1995) ($\overline{s} = 0$). The feasible constraint :

$$\dot{k}_t + C_t = f(k_t) + rS_t - \delta k_t - \dot{S}_t$$

becomes :

$$\dot{W}_t + C_t = f(k_t) - (\delta + r)k_t + rW_t.$$

Define

$$D_t = \frac{W_t - \overline{s}}{1 - \nu}.$$

This constraint is transformed in :

$$\dot{D}_t + \frac{C_t}{1-\nu} = \frac{1}{1-\nu} f(k_t) - \frac{\delta + r}{1-\nu} k_t + rD_t + \frac{r\overline{s}}{1-\nu}.$$

Define

$$\widehat{C}_t = \frac{C_t}{1-\nu}$$

$$\delta' + r = \frac{\delta + r}{1-\nu}$$

$$\widehat{f}(k) = \frac{1}{1-\nu} f(k)$$

$$\widehat{u}(\widehat{C}_t) = u((1-\nu)\widehat{C}_t)$$

The problem now becomes :

$$\max_{(\widehat{C}_t, D_t k_t)} \int_0^\infty \widehat{u}(\widehat{C}_t) e^{-\rho t} dt$$

$$\text{s.t. : } \dot{D}_t + \widehat{C}_t = \widehat{f}(k_t) - (\delta' + r)k_t + rD_t + r\overline{s}/(1-\nu),$$

$$\widehat{C}_t \geq 0 \ , \ D_t \geq \frac{-\overline{s}}{1-\nu} \ , \ D_t \geq k_t \geq 0 \quad , \quad \forall t$$

D_0 is given.

In order to simplify our exposition we first assume $\overline{s} = \nu = 0$. Results obtained in this case will highlight the cases i) $\overline{s} < 0$ and $\nu = 0$; ii) $\overline{s} = 0$ and $\nu > 0$, to which is devoted paragraph 5.

Assume :

$(U1)$ u is strictly increasing ; $u(0) = 0$; $u(\infty) = \infty$.

$(U2)$ $\rho > r$.

If (C_t^*) is a solution it must be a solution to the following problem :

$$\max_{(C_t,D_t)} \int_0^\infty u(C_t)e^{-\rho t}dt$$

$$\text{s.t. :} \quad \dot{D}_t + C_t = \max\{f(k) - (\delta + r)k \mid 0 \leq k \leq D_t\} + rD_t$$

$$C_t \geq 0, \quad D_t \geq 0, \quad \forall t$$

and D_0 is given.

Define $\phi(D) = \max\{f(k) - (\delta + r)k \mid k \in [0, D]\}$.

We have therefore the following problem :

$$\max_{(C_t,D_t)} \int_0^\infty u(C_t)e^{-\rho t}dt$$

$$\text{s.t. :} \quad \dot{D}_t + C_t \leq \phi(D_t) + rD_t \ ,$$

$$C_t \geq 0, \quad D_t \geq 0, \quad \forall t$$

and D_0 is given.

Assume :

$(F1)$ f is strictly increasing, convex for $k \leq k_3$, strictly concave for $k \geq k_3$, $f(0) = 0$;

$(F2)$ There exists k_1', k_2' such that f is continuously differentiable on $[0, k_1'[$, $]k_1', k_3[$ and $]k_3, +\infty[$.

$f'(0) < r + \delta$; the left and the right derivatives at k_3 verify $f_-'(k_3) > f_+'(k_3)$.

$f'(\infty) = 0$; $f'_-(k'_1) > r + \delta$.

Under these assumptions, there exists k_1, k_2 such that $f'(k_1) = f'(k_2) = r + \delta$, $0 < k_1 < k'_1 < k_2$.

Assume furthermore :

$(F3)$ $\quad f(k_2) - (r+\delta)k_2 > 0$.

There exists, with this additional $(F3)$, $\bar{k}$, $\bar{\bar{k}}$ verifying :

$$0 < \bar{k} < k_2 < \bar{\bar{k}}$$

and

$$f(\bar{k}) = (r+\delta)\bar{k} \ , \ f(\bar{\bar{k}}) = (r+\delta)\ \bar{\bar{k}} \ .$$

Let $k(D)$ be defined as the *argmax* of ϕ, i.e. :

$$\phi(D) = f(k(D)) - (\delta + r)k(D).$$

k and ϕ are continuous by the maximum theorem. It is straightforward to check that :

i) $D = \bar{k} \Rightarrow k(D) = \{0, \bar{k}\}$ and $\phi(D) = 0$.

ii) $0 \leq D < \bar{k} \Rightarrow k(D) = 0$ and $\phi(D) = 0$.

iii) $\bar{k} < D \leq k_2 \Rightarrow k(D) = D$ and $\phi(D) = f(D) - (r+\delta)D$.

iv) $k_2 < D \Rightarrow k(D) = k_2$ and $\phi(D) = f(k_2) - (\delta + r)k_2$.

ϕ is piecewise continuously differentiable.

3 Existence of Solutions

3.1 Feasible paths

$(\widetilde{C}_t, \widetilde{D}_t)$ is said to be feasible from D_0 if $\widetilde{C}_t$ belongs to $L^1(e^{-\rho t})$, $\widetilde{D}_t$ belongs also to $L^1(e^{-\rho t})$

$$\forall t \ , \ \widetilde{C}_t + \dot{\widetilde{D}}_t \leq \phi(\widetilde{D}_t) + r\widetilde{D}_t \tag{1}$$
$$\widetilde{C}_t \geq 0 \ , \ \widetilde{D}_t \geq 0 \ ; \text{ and } \widetilde{D}_0 = D_0.$$

Proposition 1 *There exists M_1, M_2 such that :*

$$\int_0^\infty C_t e^{-\rho t} dt \leq M_1 \ , \ \int_0^\infty D_t e^{-\rho t} dt \leq M_2 \ ,$$

for every feasible (C_t, D_t) from D_0.

Proof. From (1) and the properties of ϕ, one has :

$$\dot{D}_t \leq A + rD_t \ , \text{ with } A = f(k_2) - (r+\delta)k_2.$$

Hence

$$D_t + \frac{A}{r} \leq \frac{1}{r}(A + D_0 r)e^{rt} \ ,$$

and

$$\int_0^\infty D_t e^{-\rho t} dt \leq \frac{A}{r} \int_0^\infty e^{(r-\rho)t} dt + D_0 \int_0^\infty e^{(r-\rho)t} dt = M_2.$$

From this inequality one deduces that $\lim_{t \to \infty} D_t e^{-\rho t} = 0$. Again, using (1), one gets :

$$\int_0^\infty C_t e^{-\rho t} dt + \int_0^\infty \dot{D} \ e^{-\rho t} dt \ \leq \ A \int_0^\infty e^{-\rho t} dt + r \int_0^\infty D_t e^{-\rho t} dt.$$

But

$$\int_0^\infty \dot{D}_t \ e^{-\rho t} dt = -D_0 + \rho \int_0^\infty D_t \ e^{-\rho t} dt.$$

Hence

$$\int_0^\infty C_t e^{-\rho t} dt \leq \frac{A}{\rho} + D_0 = M_1. \ \blacksquare$$

3.2 Existence of solutions in the class of bounded functions

Assume moreover :

$(U3)$ u is continuously differentiable.

Lemma 1 *Define $U(C) = \int_0^\infty u(C_t) e^{-\rho t} dt$, where $C \in L^1_+(e^{-\rho t})$. U is continuous on $L^1_+(e^{-\rho t})$ and, hence, is $\sigma(L^1, L^\infty)$ - upper semi-continuous (u.s.c.).*

Proof. We will note, to make short, L^1 (or L^1_+) instead of $L^1(e^{-\rho t})$(or $L^1_+(e^{-\rho t})$).

Let $\{C^n\}$ be a sequence in L^1_+ converging to C in L^1. We have, for any $a > 0$,

$$0 \leq \int_0^\infty u(C^n_t)e^{-\rho t}dt \leq \int_0^\infty (u(a) - au'(a))e^{-\rho t}dt + u'(a)\int_0^\infty C^n_t e^{-\rho t}dt \leq M.$$

Let α be a cluster point of $\int_0^\infty u(C^n_t)e^{-\rho t}dt$. There exists a subsequence $\{C^\nu\}$, $C^\nu \xrightarrow{L^1} C$, $C^\nu \to C$ pointwise and $C^\nu \leq g$ for some $g \in L^1$, and $\int_0^\infty u(C^\nu_t)e^{-\rho t}dt \to \alpha$.

We have always :

$$\begin{aligned} u(C^\nu_t) &\leq u(a) - u'(a)a + u'(a)C^\nu_t \\ &\leq u(a) - u'(a)a + u'(a)g. \end{aligned}$$

From Lebesgue Theorem, $\alpha = \int_0^\infty u(C_t)e^{-\rho t}dt$. Hence, $\int_0^\infty u(C^\nu_t)e^{-\rho t}dt \to \int_0^\infty u(C_t)e^{-\rho t}dt$. U is continuous in L^1_+. Since it is concave, it is weakly u.s.c. ■

Proposition 2 *Let Γ denote the set of feasible C_t from D_0. Let $g \geq 0$ be a function in L^1, and $G = \{x \mid x \leq g\}$. Then there exists a solution in $\Gamma \cap G$.*

Proof. As we have shown in the proof of Lemma 1, there exists some M such that :

$$0 \leq U(C) \leq M, \quad \forall C \in \Gamma.$$

Let $\mu = \sup\{U(C)\ ;\ C \in \Gamma \cap G\}$. There exists a sequence $\{C^n\} \subset \Gamma \cap G$ such that $U(C^n) \to \mu$. Without loss of generality, we assume that $C^n + \dot{D}^n = \phi(D^n) + rD^n$, $\forall n$. The sequence $\{C^n\}$ verifies Dunford-Pettis criterion since $0 \leq C^n \leq g$, $\forall n$, and hence $\sigma(L^1, L^\infty)$ relatively compact. We may assume $C^n \to C$, $\phi(D^n) \to \psi$, $D^n \to D$ with $\sigma(L^1, L^\infty)$. Hence $\dot{D}^n \to x$ with $\sigma(L^1, L^\infty)$.

Define $\widehat{D}(t) = \int_0^t x(u)du + D_0$. Then, we have

$$\int_0^t \dot{D}^n(u)du \to \int_0^t x(u)du, \quad \forall t,$$

i.e.

$$D^n(t) \to \widehat{D}(t), \quad \forall t.$$

From Lebesgue Theorem , $D^n \to \widehat{D}$ in L^1 and therefore, $\widehat{D} = D$. Moreover $\phi(D^n) \to \phi(D)$ in L^1 and $x = \dot{D}$.

Since, $\forall n$, $C^n + \dot{D}^n = \phi(D^n) + rD^n$, one gets $C + \dot{D} = \phi(D) + rD$, and $C \in \Gamma$.

We prove now $C \in G$. Indeed, if $C(t) > g(t)$ for some $t \in I$ with $\int_I dt > 0$, then

$$\int_I C_t e^{-\rho t} dt > \int_I g(t) e^{-\rho t} dt, \quad \text{in contradiction}$$

with

$$\int_I C^n(t) e^{-\rho t} dt \leq \int_I g(t) e^{-\rho t} dt, \quad \forall n$$

and

$$\int_I C^n(t) e^{-\rho t} dt \to \int_I C_t e^{-\rho t} dt.$$

Now, since, U is weakly continuous (Lemma 1), we have

$$\mu \leq U(C), \text{ and hence } \mu = U(C). \qquad \blacksquare$$

4 Characterization and Properties of Optimal Paths

In this paragraph we first show that in our model there would be three steady states $k_1^s < k_2^s$ and trivial one $k_3^s = 0$. We then prove that the optimal path may converge to zero (the proverty trap), or may reach k_1^s, or converge to k_2^s. The technics of the proof differs greatly from the usual ones in neoclassical models which are based on the concavity of the utility function and of the production function. Since the technology in our model is not concave, we could not apply these methods. The key tool is the monotonicity of the optimal paths, property which will be proved in this paragraph.

Let us define the value function :

$$V(D_0, 0) = \sup_{C_t} \int_0^{\infty} u(C_t) e^{-\rho t} dt$$

$$\begin{cases} C_t + \dot{D}_t = \phi(D_t) + rD_t, & \forall t \\ C_t \geq 0 \ , \ D_t \geq 0 \ , & \forall t \\ D_0 > 0 \text{ is given.} \end{cases}$$

We assume in this section that every optimal solution is piecewise continuous and the associated D_t is piecewise C^1.

Proposition 3 *$V(.,0)$ is non decreasing.*

Proof. See appendix. ■

Proposition 4 *Let C_t be a solution and D_t the associated path. Let t_0 be fixed. Then $V(.,0)$ is continuous at $D(t_0)$.*

Proof. See appendix. ■

Assume moreover :
$(U4)$: $u'(0) = +\infty$ (Inada condition)
We then have :

Proposition 5 *Let C_t be a solution and D_t be the associated path. Let T, T' $(T < T')$ be such that $\phi'(x)$ is continuous for every x in a neighborhood of $\{D(t)\ ;\ t \in [T, T']\}$. Then we have for $t \in [T, T']$ Euler-Lagrange equation :*

$$(E-L) \qquad -\frac{d}{dt}(u'(C_t)e^{-\rho t}) = e^{-\rho t}u'(C_t)(\phi'(D_t)+r)$$

and C,D are continuous on $[T, T']$.
Proof. See appendix. ■

Corollary 1 *Let $T, T'(T < T')$ be such that $\phi'(x)$ is continuous for $x \in \{D(t) \mid t \in [T, T']\}$, where D_t is the associated path with the optimal path C_t. Then D is monotonic on $[T, T']$.*

Proof. Assume for simplicity that $D(T_B - \varepsilon_0) = D(T_B + \varepsilon_0')$ and D is decreasing from $T_B - \varepsilon_0$ to T_B and increasing from T_B to $T_B + \varepsilon_0'$. For every $\varepsilon \in]0, \varepsilon_0]$, there exists a unique $\varepsilon' \in]0, \varepsilon_0']$ such that $D(T_B - \varepsilon) = D(T_B + \varepsilon')$.

From the maximum principle one has :

$$\begin{aligned} e^{-\rho(T_B-\varepsilon)}V(D(T_B-\varepsilon),0) &= V(D(T_B-\varepsilon), T_B-\varepsilon) \\ &= \int_{T_B-\varepsilon}^{T_B+\varepsilon'} e^{-\rho t}u(C_t)dt + e^{-\rho(T_B+\varepsilon')}V(D(T_B+\varepsilon'),0). \end{aligned}$$

Hence

$$V(D(T_B-\varepsilon),0)e^{\rho(\varepsilon'-\varepsilon)}=\int_{T_B-\varepsilon}^{T_B+\varepsilon'}e^{-\rho t}u(C_t)dt.$$

When $\varepsilon\to 0$, one has $\varepsilon'\to 0$. Since

$$V(D(T_B-\varepsilon),0)\frac{e^{\rho(\varepsilon'-\varepsilon)}}{\varepsilon'-\varepsilon}=\frac{1}{\varepsilon'-\varepsilon}\int_{T_B-\varepsilon}^{T_B+\varepsilon'}e^{-\rho t}u(C_t)dt,$$

one gets, since $V(.,0)$ is continuous on the path D_t (proposition 4), and C is continuous (proposition 5),

$$\begin{aligned}V(D_B,0)&=u(C(T_B))/\rho\\&=\int_0^{+\infty}u(C(T_B))e^{-\rho t}dt.\end{aligned}$$

The path $\widetilde{C}(t)=C(T_B)$, $\forall t$ is feasible since $\dot{D}(T_B)=0$ and $C(T_B)=\phi(D(T_B))+rD(T_B)$. It is therefore optimal for the problem with D_B as initial data. Since $\widetilde{C}(t)$ is strictly positive, it verifies Euler-Lagrange equation. But this is impossible. We obtain a contradiction. ■

Remark 1 *In our case, we obtain the monotonicity of D_t under weaker conditions than in Léonard-Van Long (1992) who assume that V is differentiable with the respect to the initial date.*

The aim of the following propositions is to show that : a) when D_0 is small then the country will never produce consumption goods, imports them in order to satisfy its consumption, and will be "under-developed" for ever (proposition 6), and b) if a country prefers strongly the present to the future then in the long term it will be "under-developed" (proposition 7).

Assume furthermore
$(U5)$ u is twice continously differentiable.

Proposition 6 *Let $\overline{k}$ be defined as in section 1, i.e.,*

$$f(\overline{k})=(r+\delta)\overline{k}.$$

There exists $\overline{D}_0<\overline{k}$ such that, if $D_0\leq\overline{D}_0$, then $C_t\to 0$ and $D_t\to 0$ when $t\to+\infty$, where C_t is the optimal path, and D_t is the associated path.

Proof. Consider Euler-Lagrange equation. Assume there exists T such that $D(T) = \overline{k}$. From corollary 1, D increases from D_0 to $\overline{k}$; $\phi(D_t) = 0$ for $t \in [0, T]$. We obtain for $t \in [0, T]$:

$$u'(C_t) = u'(C_0)e^{(\rho - r)t}. \tag{2}$$

From :

$$C_t + \dot{D}_t = rD_t \qquad \text{for} \qquad t \in [0, T],$$

we have

$$D_t = D_0 e^{rt} - e^{rt} \int_0^t C_s e^{-rs} ds$$

Since $D(T) = \overline{k}$, we have

$$\overline{k} = D_0 e^{rT} - e^{rT} \int_0^T C_s e^{-rs} ds. \tag{3}$$

Consider (3) as a relation giving C_0 as function of D_0 and T. Let us fix D_0. Differentiating (3), one gets :

$$0 = \dot{D}_T dT - e^{rT} \left(\int_0^T \frac{\partial C_s}{\partial C_0} e^{-rs} ds \right) dC_0$$

But

$$\frac{\partial C_s}{\partial C_0} = \frac{u''(C_0)}{u''(C_s)} e^{(\rho - r)s} \qquad \text{(from (2))}.$$

Hence

$$\frac{\partial C_0}{\partial T} = \frac{\dot{D}_T e^{-rT}}{\displaystyle\int_0^T \frac{u''(C_0)}{u''(C_s)} e^{(\rho - 2r)s} ds}$$

Define

$$W(T) = \int_0^T u(C_t) e^{-\rho t} dt.$$

We have

$$V(D_0, 0) = W(T) + e^{-\rho T} V(\overline{k}, 0).$$

Hence, T must maximize the function $W(t) + e^{-\rho t} V(\overline{k}, 0)$. Then

$$W'(T) = \rho e^{-\rho T} V(\overline{k}, 0). \tag{4}$$

(This is just the transversality condition of a problem with free -end-point and scrap value function ; see Léonard and Van Long, 1992).

We have :

$$
\begin{aligned}
W'(T) &= u(C_T)e^{-\rho T} + \left[\int_0^T u'(C_t)\frac{\partial C_t}{\partial C_0}e^{-\rho t}dt\right]\frac{\partial C_0}{\partial T} \\
&= u(C_T)e^{-\rho T} + \left[\int_0^T u'(C_0)\frac{u''(C_0)}{u''(C_t)}e^{(\rho-2r)t}dt\right]\frac{\dot{D}_T e^{-rT}}{\displaystyle\int_0^T \frac{u''(C_0)}{u''(C_t)}e^{(\rho-2r)t}dt} \\
&= u(C_T)e^{-\rho T} + u'(C_0)\dot{D}_T e^{-rT}.
\end{aligned}
$$

Using (4) one gets :

$$
\begin{aligned}
\rho V(\overline{k},0) &= u(C_T) + u'(C_0)\dot{D}_T e^{(\rho-r)T} \\
&\geq u'(C_0)\dot{D}_T e^{(\rho-r)T}.
\end{aligned}
$$

If $\dot{D}_t > 0$ on $[0,T]$, we have $\dot{D}_T = \overline{k} - C_T > \overline{k} - C_0 \geq \overline{k} - rD_0$, because $C_0 \leq rD_0$ since $\dot{D}(0) \geq 0$.

Hence, if D_0 is such that $\overline{k} - rD_0 > 0$, then

$$
e^{(\rho-r)T} \leq \frac{\rho V(\overline{k},0)}{(\overline{k} - rD_0)u'(rD_0)}
$$

From (U4), we obtain a contradiction if D_0 is small enough. Summing up, if D_0 is small, the path $D(t)$ could not reach $\overline{k}$. From corollary 1, $D(t)$ must be decreasing. Since $C_t + \dot{D}_t = rD_t$, we have

$$
D_t \leq \frac{C_t}{r}, \ \forall t.
$$

From (2), $u'(C_t) \to +\infty$ when $t \to +\infty$, i.e. $C_t \to 0$ (Inada condition). Hence D_t converges also to 0. ■

Proposition 7 *Assume $\rho > f'_+(k_3) - \delta$. Then for any D_0, the optimal path (C_t, D_t) converge to 0.*

Proof. Consider Euler-Lagrange equation :

$$
\begin{aligned}
-\frac{d}{dt}(u'(C_t)e^{-\rho t}) &= e^{-\rho t}u'(C_t)(\phi'(D_t) + r) \\
&< e^{-\rho t}u'(C_t)(f'_+(k_3) - \delta).
\end{aligned}
$$

Hence

$$Log\frac{u'(C_t)e^{-\rho t}}{u'(C_0)} > (\delta - f'_+(k_3))t,$$

or

$$u'(C_t) > u'(C_0)e^{(\rho+\delta-f'_+(k_3))t}.$$

Therefore, $u'(C_t) \to +\infty$ and $C_t \to 0$.

Let D_t be the associated path. We first prove there exists a sequence $\{D_{t_n}\}$ converging to 0. Assume the contrary, $D_t \geq \alpha > 0$, $\forall t$. We know that:

$$\forall \varepsilon > 0\ ,\ \exists T\ ,\ \forall t \geq T\ ,\ C_t \leq \varepsilon.$$

Define

$$\begin{array}{lll} \widetilde{C}(t) = C(t) & \text{for} & t < T \\ \widetilde{C}(t) = \varepsilon & \text{for} & t \geq T, \\ \widetilde{D}(t) = D(t) & \text{for} & t < T, \end{array}$$

and $\widetilde{D}$ verifies for $t \geq T$

$$\begin{array}{ll} \dot{\widetilde{D}}_t & = \phi(\widetilde{D}_t) + r\widetilde{D}_t - \varepsilon \\ \widetilde{D}(T) & = D(T) \geq \alpha. \end{array}$$

Choose $\varepsilon < r\alpha$.

In some interval $[T, T']$, we have $r\widetilde{D}_t - \varepsilon \geq 0$. Hence

$$\widetilde{D}(t) > \widetilde{D}(T)\ ,\ \forall t \in]T, T'].$$

Let $\overline{T}$ be the maximum T' such that $r\widetilde{D}_t - \varepsilon \geq 0\ ,\ \forall t \in]T, \overline{T}]$. Assume $\overline{T} < +\infty$. One has

$$\widetilde{D}(t) > \widetilde{D}(T)\ ,\ \forall t \in]T, \overline{T}].$$

In particular $\widetilde{D}(\overline{T}) > \widetilde{D}(T) \geqslant \alpha$.

Since $\widetilde{D}$ is continuous, there exists $\overline{T}' > \overline{T}$ such that $r\widetilde{D}_t - \varepsilon \geq 0$ for $t \in [\overline{T}, \overline{T}']$: a contradiction. Hence

$$\widetilde{D}(t) > D(T) \geq \alpha\ ,\ \forall t > T.$$

The path $\widetilde{C}_t$ is therefore feasible from D_0, and $U(\widetilde{C}) > U(C)$: a contradiction.

There exists a sequence $D_{t_1}, ..., D_{t_n}$ converging to 0. For n sufficiently large $D_{t_n} \leq \overline{D}_0$. From proposition 6, D_t must converge to 0. ■

Assume moreover :

$(U6)$: $\rho < f'_+(k_3) - \delta$

$$(U7): \quad \lim_{c \to +0} \frac{u'(C)}{u''(C)} = 0$$

$$(F4): \quad f(k_3)/k_3 > f'_+(k_3)$$

Define a *steady state* (c^s, k^s) as follows :

$$f'(k^s) = \rho + \delta$$
$$c^s = f(k^s) - \delta k^s.$$

Proposition 8 *If $f'_+(\overline{k}) > \rho + \delta$, then there exists a unique steady state (c^s_2, k^s_2) with $k_3 < k^s_2 < k_2$.*

If $f'_+(\overline{k}) \leq \rho + \delta$, then there exists two steady states (c^s_1, k^s_1), (c^s_2, k^s_2) with

$$\overline{k} \leq k^s_1 < k_3 < k^s_2 < k_2.$$

Proof. Since f is convex on $[\overline{k}, k_3]$, if $f'_+(\overline{k}) > \rho + \delta$, then $f'(x) > \rho + \delta$, $\forall x \in [\overline{k}, k_3]$. Then there exists a unique steady state.

If $f'_+(\overline{k}) \leq \rho + \delta$, since $f'_-(k_3) > \dfrac{f(k_3)}{k_3} > f'_+(k_3) > \rho + \delta$ by $(U6)$ and $(F4)$, there exists a steady state (c^s_1, k^s_1) with $\overline{k} \leq k^s_1 < k_3$. ■

(Insert figure 1)

Proposition 9 *Let $k_3 \leq D_0 < k^s_2$. Then there exists an optimal path (C_t, k_t) converging to (c^s_2, k^s_2).*

Proof. Consider the system

$$\begin{cases} \dfrac{d}{dt}(-u'(C_t)e^{-\rho t}) = u'(C_t)e^{-\rho t}(f'(D_t) - \delta) \\ \\ \dot{D}_t = f(D_t) - \delta D_t - C_t \ ; \\ \\ D_0 \text{ is given in } [k_3, k^s]. \end{cases}$$

The first equation is equivalent to :

$$\dot{C}_t = \frac{u'(C_t)}{u''(C_t)}(\delta + \rho - f'(D_t)).$$

It is straightforward to check that k_2^s is a saddle-point of this system. There exists a stable manifold (D_t, C_t). It could not intersect the axis $C = 0$ at a point $k_0 > k_3$, because, in that case, under $(U7)$ the system will have two solutions starting from $(k_0, 0)$: the stable manifold, and the half-line $(D_t, 0)$ where D_t is solution to $\dot{D}_t = f(D_t) - \delta D_t$; $D(0) = k_0$.
Hence, the stable manifold will intersect the axis $D = k_3$ at a point $C_0 \geq 0$. Then, for every $D_0 \in [k_3, k^s]$, there exists $C_0 \geq 0$ such that (D_0, C_0) belongs to the stable manifold.
We prove now the solution (D^*, C^*) of the system above with initial data (D_0, C_0) is the optimal solution.

Let (C_t) be a feasible consumption path from D_0 and

$$
\begin{aligned}
\Delta_T &= \int_0^T (u(C_t^*) - u(C_t))e^{-\rho t} dt \\
&= \int_0^T [u(\phi(D_t^*) + rD_t^* - \dot{D}_t^*) - u(\phi(D_t) + rD_t - \dot{D}_t)]e^{-\rho t} dt.
\end{aligned}
$$

We have :

$$\Delta_T \geq \int_0^T u'(C_t^*)[\phi(D_t^*) + rD_t^* - \dot{D}_t^* - \phi(D_t) - rD_t + \dot{D}_t]e^{-\rho t} dt.$$

Recall that

$$\phi(D_t^*) = f(D_t^*) - (r + \delta)D_t^* \quad \text{since} \quad D_t^* \geq \overline{k}.$$

Now,

$$f(D_t^*) - f(k_3) \geq f'(D_t^*)(D_t^* - k_3).$$

If f is convex at D_t then, by $(F4)$

$$\frac{f(k_3) - f(D_t)}{k_3 - D_t} \geq \frac{f(k_3)}{k_3} > f'_+(k_3) \geq f'(D_t^*),$$

and hence

$$f(D_t^*) - f(D_t) \geq f'(D_t^*)(D_t^* - D_t).$$

Summing up :

$$\Delta_T \geq \int_0^T u'(C_t^*)[(f'(D_t^*) - \delta)(D_t^* - D_t) - (\dot{D}_t^* - D_t)]e^{-\rho t} dt.$$

Since we have Euler-Lagrange equation :

$$\frac{d}{dt}(-u'(C_t^*)e^{-\rho t}) = u'(C_t^*)e^{-\rho t}(f'(D_t^*) - \delta)$$

we obtain

$$\Delta_T \geq -u'(C_t^*)(D_t^* - D_t)e^{-\rho t}dt \geq -u'(C_T^*)D_T^* e^{-\rho t} \to 0. \blacksquare$$

Let us consider again the dynamical system :

$$\begin{cases} \dfrac{dC}{dt} & = \dfrac{u'(C)}{u''(C)}(\rho - r - \phi'(D)) \\ \\ \dfrac{dD}{dt} & = \phi(D) + rD - C \end{cases}$$

The following phase-diagram is in figure 1.

The purpose of proposition 10 is to obtain corollary 2 stating that the low level steady state k_1^s may be optimal. This result is new : in Dechert and Nishimura (1983), the low level steady state is not optimal. We may observe also that in many papers on economic growth, their authors do not demonstrate that the steady states of their models are, or are not, optimal.

Proposition 10 *Assume* $u(C) = \dfrac{C^\sigma}{\sigma}$ *with* $\sigma \in]0,1[$, *and for* $k \in [\overline{k}, \overline{D}]$ *with* $\overline{D} < k_1^s$, *we have* $f(k) = \lambda(k - \overline{k}) + f(\overline{k})$ *where* λ *verifies* $\delta < \lambda$, $r < \lambda < \rho + \delta$.

If $\rho < 1 - \sigma + \sigma(\lambda - \delta)$, *then the optimal path* (D_t) *starting from* $\overline{D}$ *will be nondecreasing if* $\overline{D}$ *is large enough.*

Proof. See appendix. $\blacksquare$

Corollary 2 *Assume as in proposition 10. If* $\overline{D}$ *and* k_3 *are large enough, then* k_1^s *is an optimal steady state.*

Proof. First (k_1^s) verifies Euler-Lagrange equation. Consider an optimal path (D_t) starting from k_1^s. If k_3 is large enough then the optimal path could not reach k_3. In this case it will be non increasing (corollary 1). Assume it could reach $\overline{D}$. But from proposition 10 if $\overline{D}$ is large enough the optimal path starting from $\overline{D}$ must be non decreasing. We have a contradiction by corollary 1 and by the fact that $\overline{D}$ is not a steady state. Hence the optimal path starting from k_1^s will be (k_1^s). $\blacksquare$

We now summarize our results in the following theorem.

Theorem 1 *Assume* $(U1), ..., (U7), (F1), ..., (F4)$. *Let* $0 \leq D_0 \leq k_2^s$.

1. *If there exists two steady state* (c_1^s, k_2^s), (c_2^s, k_2^s), *then*

   a) either the optimal path (C_t, k_t) *converges to* (c_2^s, k_2^s),

   b) or there exists T, *such that* $(C_t, k_t) = (c_1^s, k_1^s)$, $\forall t \geq T$,

   c) or the optimal path converges to 0.

2. *If there exists one steady* (c_2^s, k_2^s) *then*

   a) either the optimal converges to (c_2^s, k_2^s),

   b) or it converges to 0.

Proof.

1. Assume we have two steady states. If $D_0 \geq k_3$, then the optimal converges to (c_2^s, k_2^s) from proposition 8.

   If $D_0 = k_1^s$, we may have (c_1^s, k_1^s) as optimal path.

   If $k_1^s < D_0 < k_3$. We assume first the optimal D_t increasing. If D_t reaches k_3 at some date T, we take $C(T) = c_3$ such that (c_3, k_3) belongs to the stable manifold. (C_t, D_t) converges therefore to (c_2^s, k_2^s). Assume that D_t is increasing and could not reach k_3. Consider the dynamic equation of C_t :

   $$-\frac{d}{dt}(u'(C_t)e^{-\rho t}) = e^{-\rho t}u'(C_t)(f'(D_t) - \delta).$$

   We have

   $$u'(C_t)e^{-\rho t} = u'(C_0)\exp(-\int_0^t (f'(D_s) - \delta)ds),$$

   i.e.

   $$u'(C_t) = u'(C_0)\exp(\int_0^t (\rho + \delta - f'(D_s))ds).$$

   Hence

   $$u'(C_t) < u'(C_0)e^{(\rho+\delta-f'(D_0))t},$$

   with

   $$f'(D_0) > f'(k_1^s) = \rho + \delta,$$

   and

   $$C_t \to +\infty \quad \text{if} \quad t \to +\infty.$$

The optimal (C_t, D_t) must intersect the curve (δ) (corresponding to $\dot{D} = 0$, see fig. 1) at some date T. But when $t \geq T$, D_t will decrease and this is in contradiction with corollary 1. Then if D_t is increasing, it must reach k_3 at some date T. Assume now that D_t is decreasing. We have $D_t \leq D_0$, $\forall t$. If $D_t > k_1^s$, $\forall t$, since $\dfrac{dD}{dt} \leq 0$, we have

$$C_t \geq \phi(D_t) + rD_t > \phi(k_1^s) + rk_1^s.$$

Starting from $D_0 > k_1^s$, we have therefore :

$$\frac{dD}{dt} \leq \phi(D_0) + rD_0 - C_t \leq \phi(D_0) + rD_0 - \phi(k_1^s) - rk_1^s < 0$$

and

$$D_t - D_0 < [\phi(D_0) + rD_0 - \phi(k_1^s) - rk_1^s]t$$

which implies $D_t \rightarrow -\infty$: a contradiction. There exists T such that $D_T = k_1^s$. We may have $(C_t, D_t) = (c_1^s, k_1^s)$, $\forall t \geq T$, or D_t continues decreasing for $t \geq T$. In the latter case, D_t converges to 0. C_t converges also, in this case, to 0 (proposition 6).

2. Using the same arguments, we obtain the conclusions stated in part 2 of theorem 1. ■

Remark 2 *Consider now trajectories verifying the Euler-Lagrange and state equations. We may have many features : a trajectory may "collapse", may reach k_3 and "jump" on the stable manifold, may also, and this point is interesting, fluctuate, i.e., D_t begins to increase, then decreases, and increases again, ... (figure 2).*

(Insert figure 2)

Remark 3 *Comparison with the classical case.*

Assume that f is concave verifying $f(0) = 0$, $f'(0) > \rho + \delta$. It is easy to check that with $f'(k_2) = r + \delta$

$$\phi(D) = f(D) - (\delta + r)D \quad \text{if} \quad D \leq k_2$$

$$\phi(D) = f(k_2) - (\delta + r)k_2 \quad \text{if} \quad D \geq k_2.$$

Hence,

$$\phi(D) + rD = f(D) - \delta D \quad \textit{if} \quad D \leq k_2$$

$$\phi(D) + rD = f(k_2) - (\delta + r)k_2 + rD \quad \textit{if} \quad D \geq k_2.$$

$\phi(D) + rD$ is continuously differentiable and concave. There exists a unique steady state k^s, $f'(k^s) = \rho + \delta$. Assume $u'(0) = +\infty$, $\lim_{c \to 0} \frac{u'(c)}{u''(c)} = 0$. Then the stable manifold will start from $(0, 0)$. The phase diagram is as follows :

(Insert figure 3)

For every $0 \leq D_0 \leq k^s$, there exists an optimal path which converges to the steady state. Non optimal trajectories will "collapse". There is no fluctuations as in the case where f exhibits increasing returns for D less than some value k_3, and decreasing returns when $D \geq k_3$.

Remark 4 *Comparison with Dechert-Nishimura (1983).*

In their paper, the optimal path either converges to 0 or converges to the high level steady state ; in other words with the same technology, "poor" and "rich" countries could converge to same steady state. In our model, with the same technology, a "rich" country will converge to the high level steady state, a "poor country" could converge to the low level steady state.

5 The cases where the debt constraint is not equal to zero

In the previous parts we assumed $\overline{S} = 0$, i.e., that the country can not borrow on international market. This hypothesis is obviously unrealistic, most Lower Developed Countries (LDC) contract foreign debts to finance their investments.

In this section we provide two approaches to the debt constraint $\overline{S}(k) = \overline{s}$, $\forall k$; $\overline{S}(k) = -\nu k$ for $\nu \in]0, 1[$.

Case 1. Let $\overline{s} < 0$ be given. From section 2, we see that the optimal program is unchanged but the feasible constraint becomes :

$$C_t + \dot{D}_t = f(k_t) - (\delta + r)k_t + rD_t + r\overline{s}.$$

So, in the phase-diagram of figure 1, the curve $\frac{dD}{dt} = 0$ is vertically translated by $r\overline{s}$. We distinguish two cases :

i) if $-\overline{s} < \frac{1}{r}(f(k_1^s) - \delta k_1^s)$ then the optimal steady state is either k_1^s or k_2^s (see figures 1, 4, 5) ;

ii) if $-\overline{s} > \frac{1}{r}(f(k_1^s) - \delta k_1^s)$ then the optimal path, as in the Ramsey model, converges to the steady state k_2^s (see figure 6).

Consequently, if the foreign countries release the debt constraints, the LDC may reach the high development state.

Case 2. Assume $\overline{S}(k) = -\nu k$, with $0 < \nu < 1$. We have seen in section 2 that the program becomes :

$$\max_{(\widehat{C}_t, D_t, k_t)} \int_0^\infty \widehat{u}(\widehat{C}_t) e^{-\rho t} dt$$

$$\begin{aligned} s.t. \quad & \dot{D}_t + \widehat{C}_t = \widehat{f}(k_t) - (\delta' + r)k_t + rD_t \\ & \widehat{C}_t \geq 0 \ , \ D_t \geq k_t \geq 0 \ , \ \forall t, \\ & D_0 \text{ is given.} \end{aligned}$$

With

$$\begin{aligned} \widehat{C}_t &= \frac{C_t}{1-\nu} \\ \widehat{f}(k) &= \frac{1}{1-\nu} f(k) \\ \widehat{u}(\widehat{C}) &= u((1-\nu)\widehat{C}) \\ \delta' + r &= \frac{\delta + r}{1-\nu}. \end{aligned}$$

Mathematically, the program is quite similar to the one we studied in sections 2, 3 and 4 ; hence our previous analyses remain valid. But the steady states are now determined by the following relation :

$$\frac{1}{1-\nu} f'(k^s) = \rho + \delta' = \frac{\rho + \delta + (r - \rho)\nu}{1-\nu}$$

or :

$$f'(k^s) = \rho + \delta + (r - \rho)\nu.$$

There will be two values $k_1^s < k_2^s$ which satisfy this relation ; k_1^s is in the convex part of f, while k_2^s is in the concave part. When $\nu \to 1$, k_1^s will be smaller than $\overline{k}$ and, from our analyses in sections 2, 3 and 4, it will "disappear" as steady state of the optimal program (see proposition 8). In other words, if the foreign countries release the debt constraint (ν is close to 1) then, as in case 1, the LDC may reach the high level steady state.

APPENDIX

Proof. **of proposition 3.**

Assume $D'_0 > D_0$. Let (C_t, D_t) verify

$$C_t \geq 0, D_t \geq 0, \forall t$$

$$C_t + \dot{D}_t = \phi(D_t) + rD_t, \forall t$$

$$D(0) = D_0.$$

Let D' be a solution to

$$C_t + \dot{D}'_t = \phi(D'_t) + rD'_t$$

and

$$D'(0) = D'_0.$$

D and D' are continuous. Since $D'_0 > D_0$, $D'(t) > D(t)$ for $t \in [0, T[$. Assume $D'(T) = D(T)$. Since ϕ is increasing, we have :

$$\dot{D}'_t > \dot{D}_t , \quad \forall t \in [0, T[$$

and hence

$$D'(T) - D'_0 > D(T) - D_0,$$

which implies

$$D'(T) > D(T) + D'_0 - D_0 > D(T).$$

We have therefore

$$D'(t) > D(t) \ , \forall t.$$

(C_t, D'_t) are feasible paths with initial data D'_0. Hence $U(C) \leq V(D'_0, 0)$, for every C_t feasible from D_0, and $V(D_0, 0) \leq V(D'_0, 0)$. ■

Proof. **of proposition 4.**

$\dot{D}_t$ has constant sign in $[t_0-\varepsilon,\ t_0[$ for some $\varepsilon>0$. We have, for $\varepsilon'\in]0,\varepsilon]$,

$$V(D(t_0-\varepsilon'),0)=\int_0^{\varepsilon'}e^{-\rho t}u(C_{t+t_0-\varepsilon'})dt+e^{-\rho\varepsilon'}V(D(t_0),0)$$

by the maximum principle.

Since C is bounded in $[t_0-\varepsilon',\varepsilon]$, we have $V(D(t_0-\varepsilon'),0)\to V(D(t_0),0)$ when $\varepsilon'\to 0$. By the same way, $V(D(t_0+\varepsilon),0)\to V(D(t_0),0)$ when $\varepsilon\to 0$. ■

Proof. of proposition 5.

Let us recall that $C_t=\phi(D_t)+rD_t-\dot{D}_t$. Since D is continuous and $\dot{D}$ is piecewise continuous on $[T,T']$ and $u'(0)=+\infty$ we have $C_t\geq\alpha>0$, $\forall t\in[T,T']$, for some α. Let h be piecewise C^1 on $[T,T']$ with $h(T)=h(T')=0$. Let T_i be the intervals in $[T,T']$ on which $\dot{D}$ is continuous. There exists $\varepsilon>0$ such that

$$\max_i\sup_{t\in T_i}\{|h(t)|+\left|\dot{h}(t)\right|\}\leq\varepsilon\Rightarrow\phi(D_t+h_t)+r(D_t+h_t)-\dot{D}_t-\dot{h}_t>0\ \forall t\in[T,T'].$$

Let $\lambda\in\mathbb{R}$. Define

$$\theta_h(\lambda)=\int_T^{T'}u(\phi(D+\lambda h)-r(D+\lambda h)-(\dot{D}+\lambda\dot{h}))e^{-\rho t}dt.$$

θ is differentiable at $\lambda=0$. We have :

$$\int_0^\infty u(\phi(D+\lambda h)-r(D+\lambda h)-(\dot{D}+\lambda\dot{h}))e^{-\rho t}dt-\int_0^\infty u(\phi(D)+rD-\dot{D})e^{-\rho t}dt$$
$$=\theta_h(\lambda)-\theta_h(0).$$

If λ is sufficiently small, $\theta_h(\lambda)\leq\theta_h(0)$. Hence $\theta_h'(0)=0$; this implies :

$$\int_T^{T'}[u'(C_t)(\phi'(D)+r)h-u'(C_t)\dot{h}]e^{-\rho t}dt=0. \tag{5}$$

Let

$$\psi(t)=\int_T^t u'(C_s)(\phi'(D_s)+r)e^{-\rho s}ds,t\in[T,T']$$

and

$$a(t)=\psi(t)+K$$

where K is defined by

$$\int_T^{T'}[u'(C_s)e^{-\rho s}+\psi(s)]ds+K(T'-T)=0.$$

Now,

$$\int_T^{T'} [u'(C_t)(\phi'(D_t) + r)]he^{-\rho t}dt = \int_T^{T'} a'(t)h(t)dt$$

$$= -\int_T^{T'} a(t)h'(t)dt,$$

since $h(T) = h(T') = 0$.

(5) becomes :

$$-\int_T^{T'} (a(t) + u'(C_t)e^{-\rho t})\dot{h}(t)dt = 0, \tag{6}$$

for every h, piecewise C^1 verifying $h(T) = h(T') = 0$.

Now, take

$$h = -\int_T^t (u'(C_s)e^{-\rho s} + \psi(s))ds - K(t - T),\ t \leq T'.$$

One gets, from (6) and the very definitions of a and K :

$$\int_T^{T'} (a(t) + u'(C_t)e^{-\rho t})^2 dt = 0.$$

Hence

$$a(t) = -u'(C_t)e^{-\rho t}\ , \text{ for } t \in [T, T'].$$

Since a is continuous on $[T, T']$, C is also continuous on this interval. $\dot{a}$ is also continuous, and

$$\dot{a}(t) = u'(C_t)(\phi'(D_t) + r)e^{-\rho t} = -\frac{d}{dt}(u'(C_t)e^{-\rho t})$$

which is Euler-Lagrange equation.

Since $C_t = \phi(D_t) + rD_t - \dot{D}_t$, $\dot{D}$ is also continuous on $[T, T']$. ∎

Proof. of proposition 10.

Let $D_0 = \overline{D}$. If the optimal path is decreasing between 0 and T where T is defined by $D_T = \overline{k}$, then we have the following equations

$$\dot{D}_t + C_t = \lambda(D_t - \overline{k}) + f(\overline{k}) - \delta D_t \tag{7}$$

$$-\frac{d}{dt}(u'(C_t)e^{-\rho t}) = e^{-\rho t}u'(C_t)(\lambda - \delta). \tag{8}$$

Integrating (8) one gets :

$$C_t = C_0 e^{\frac{\rho-\lambda+\delta}{\sigma-1}} t. \tag{9}$$

Integrating (7) we have, with $f(\overline{k}) = (r+\delta)\overline{k}$:

$$\begin{aligned} D_t = D_0 e^{(\lambda-\delta)t} \quad &+ \frac{r+\delta-\lambda}{\delta-\lambda}\overline{k}(1-e^{(\lambda-\delta)t}) \\ &-e^{(\lambda-\delta)t}C_0\left(\frac{\sigma-1}{\rho-\sigma(\lambda-\delta)}\right)\left(e^{\frac{\rho-\sigma(\lambda-\delta)}{\sigma-1}t}-1\right). \end{aligned} \tag{10}$$

Let $V(D,0)$ denote the value function. We have as in proposition 4 :

$$V(D_0,0) = \int_0^T u(C_t)e^{-\rho t}dt + e^{-\rho T}V(\overline{k},0).$$

T must verify :

$$\frac{d}{dt}\left(\int_0^T u(C_t)e^{-\rho t}dt\right) - \rho e^{-\rho T}V(\overline{k},0) = 0. \tag{11}$$

Tedious computations yield (as in the proof of proposition 6) :

$$e^{\rho T}\frac{d}{dt}\left(\int_0^T u(C_t)e^{-\rho t}dt\right) = u(C_T) + u'(C_0)\dot{D}_T e^{(\rho-(\lambda-\delta))T}$$

with

$$\dot{D}_T = r\overline{k} - C_0 e^{\frac{\rho-\lambda+\delta}{\sigma-1}T}. \tag{12}$$

Using (9) and substituting in (11) C_t by its expression, we obtain :

$$\rho V(\overline{k},0) = C_0^\sigma\left(\frac{1}{\sigma}-1\right)e^{(\frac{\rho-\lambda+\delta}{\sigma-1})\sigma T} + C_0^{\sigma-1}r\overline{k}\, e^{(\rho-\lambda+\delta)T}. \tag{13}$$

Since $\dot{D}(0) < 0$, we have :

$$(\lambda-\delta)D_0 + (r+\delta-\lambda)\overline{k} - (\lambda-\delta)C_0 < 0. \tag{14}$$

Hence when $D_0 \to +\infty$, then $C_0 \to \infty$.

From (13), we have also $T \to +\infty$, and $C_0 e^{(\frac{\rho-\lambda+\delta}{\sigma-1})T}$ is bounded. Let us go back to (10) and take $t = T$. Then :

$$\overline{k} = D_0 e^{(\lambda-\delta)T} + \frac{r+\delta-\lambda}{\delta-\lambda}\overline{k}(1-e^{(\lambda-\delta)T})$$

$$-e^{(\lambda-\delta)T} C_0 \left(\frac{\sigma-1}{\rho-\sigma(\lambda-\delta)}\right)\left(e^{\frac{\rho-\sigma(\lambda-\delta)}{\sigma-1}T}-1\right),$$

or

$$\overline{k}\ e^{(\delta-\lambda)T} = D_0 + \frac{r+\delta-\lambda}{\lambda-\delta}\overline{k} + \frac{r+\delta-\lambda}{\delta-\lambda}\overline{k}\ e^{(\delta-\lambda)T}$$

$$-C_0 \left(\frac{\sigma-1}{\rho-\sigma(\lambda-\delta)}\right)\left(e^{\frac{\rho-\sigma(\lambda-\delta)}{\sigma-1}T}-1\right).$$

Observe that $\rho - \sigma(\lambda-\delta) > \rho - \lambda + \delta > 0$ and therefore $C_0 e^{\frac{\rho-\sigma(\lambda-\delta)}{\sigma-1}T}$ is bounded since $C_0 e^{\frac{\rho-\lambda+\delta}{\sigma-1}T}$ is also bounded.

From (14) we have

$$k\ e^{(\delta-\lambda)T} < C_0 + C_0\frac{\sigma-1}{\rho-\sigma(\lambda-\delta)} + C_0\frac{1-\sigma}{\rho-\sigma(\lambda-\delta)}e^{\frac{\rho-\sigma(\lambda-\delta)}{\sigma-1}T}. \qquad (15)$$

We have

$$1 + \frac{\sigma-1}{\rho-\sigma(\lambda-\delta)} = \frac{\rho-\sigma(\lambda-\delta)+\sigma-1}{\rho-\sigma(\lambda-\delta)}.$$

Hence, if $\rho < 1-\sigma+\sigma(\lambda-\sigma)$, the second member of (15) goes to $-\infty$ when $C_0 \to +\infty$: a contradiction. ■

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FIGURE 1

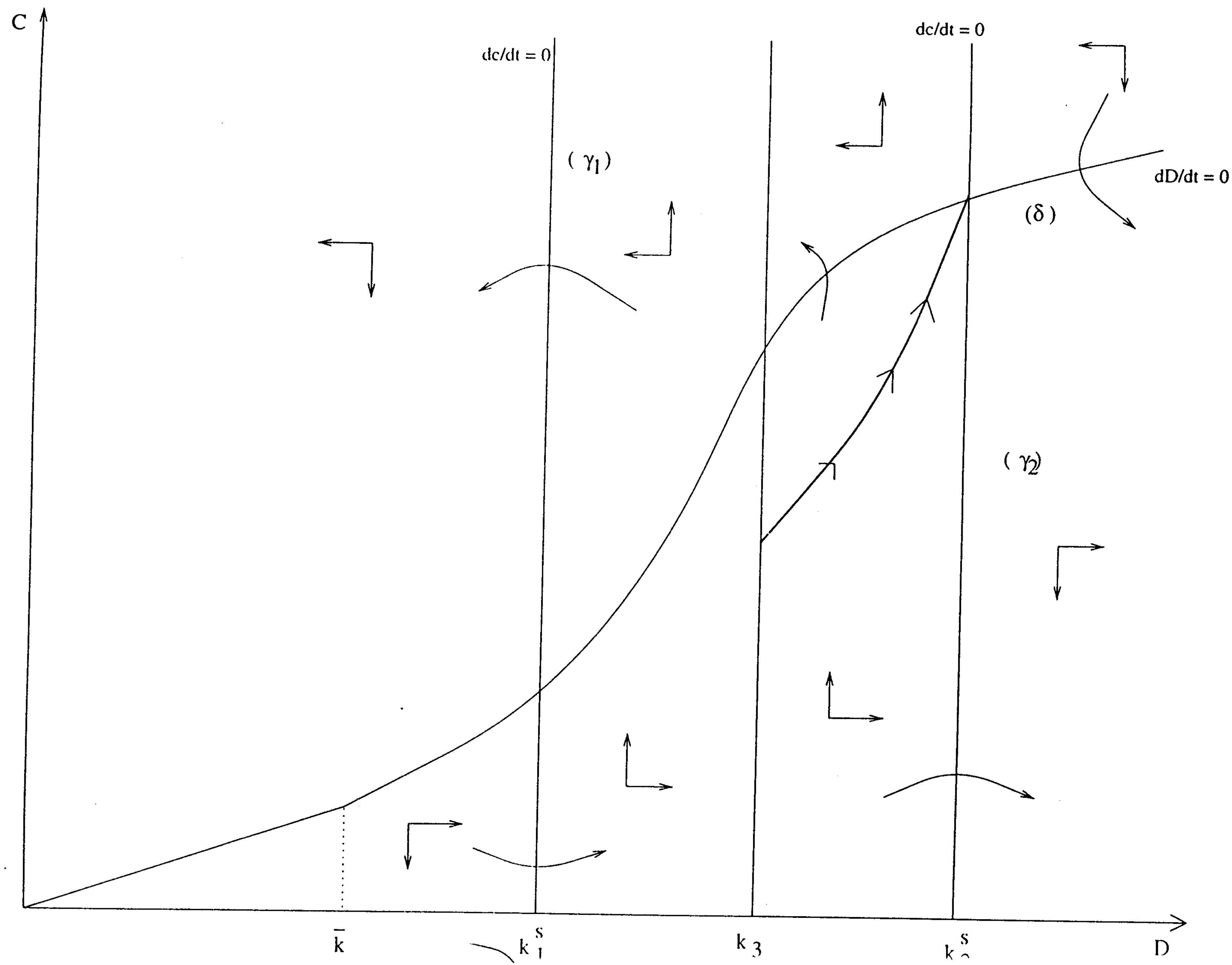

FIGURE 2

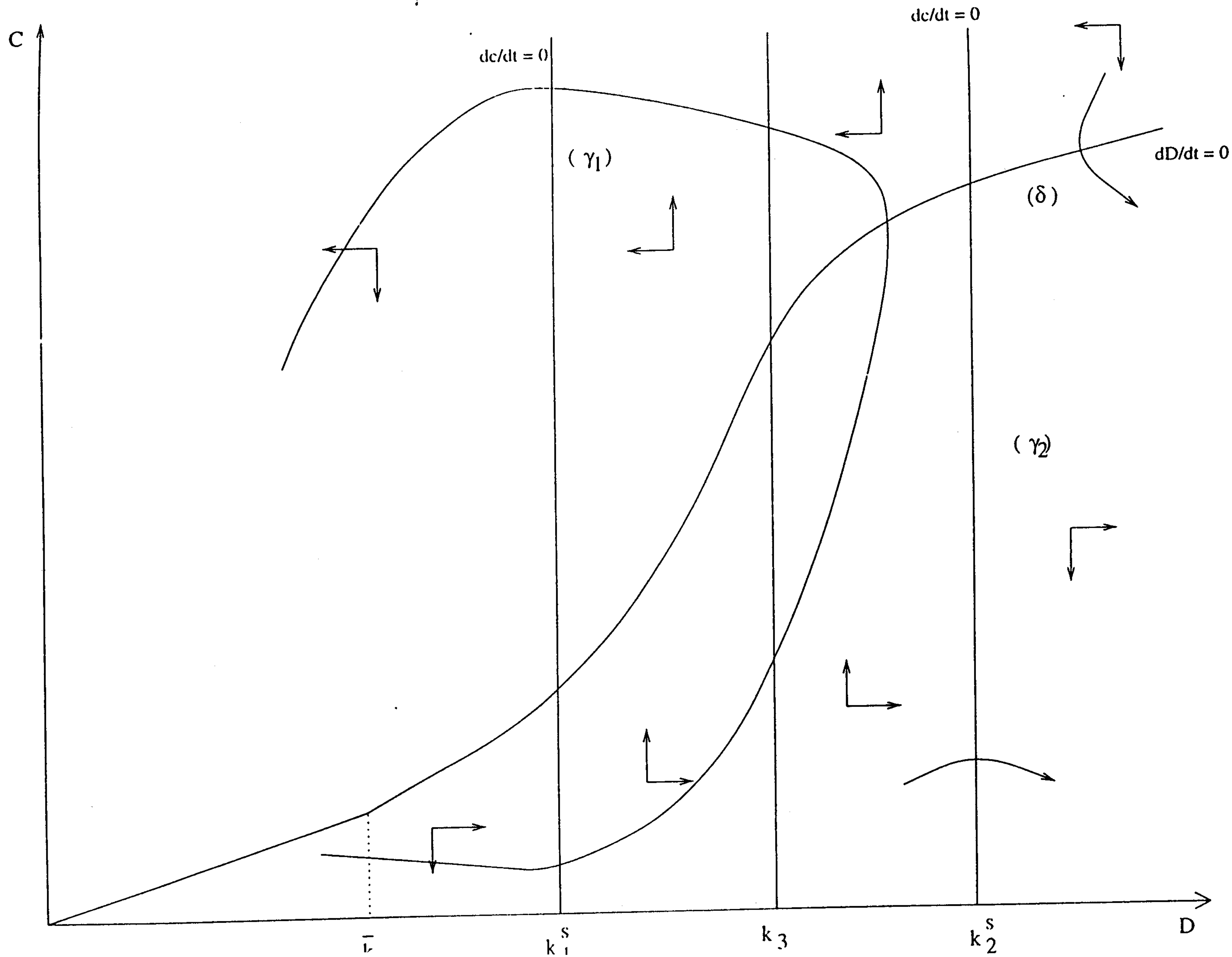

FIGURE 3

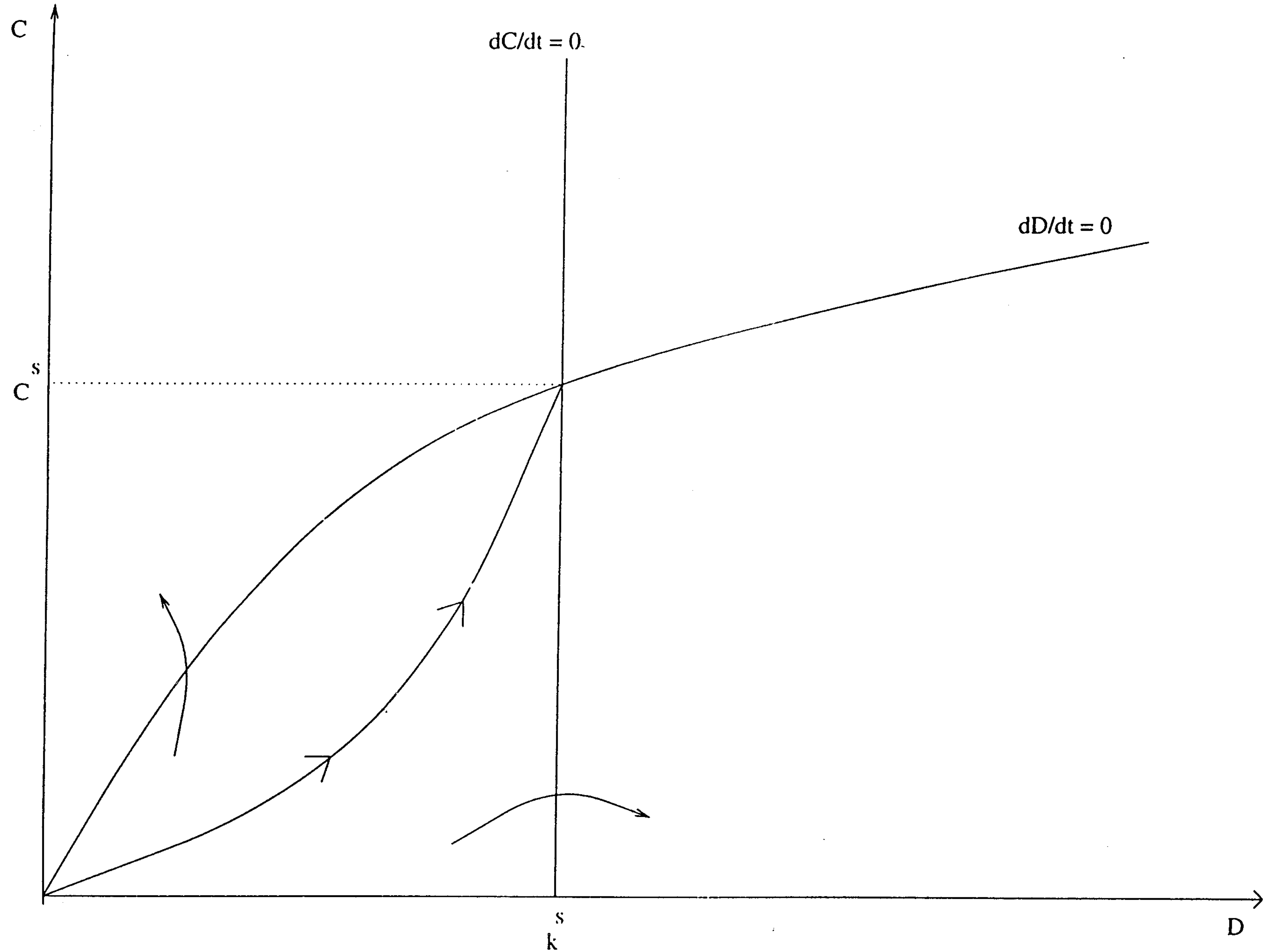

FIGURE 4

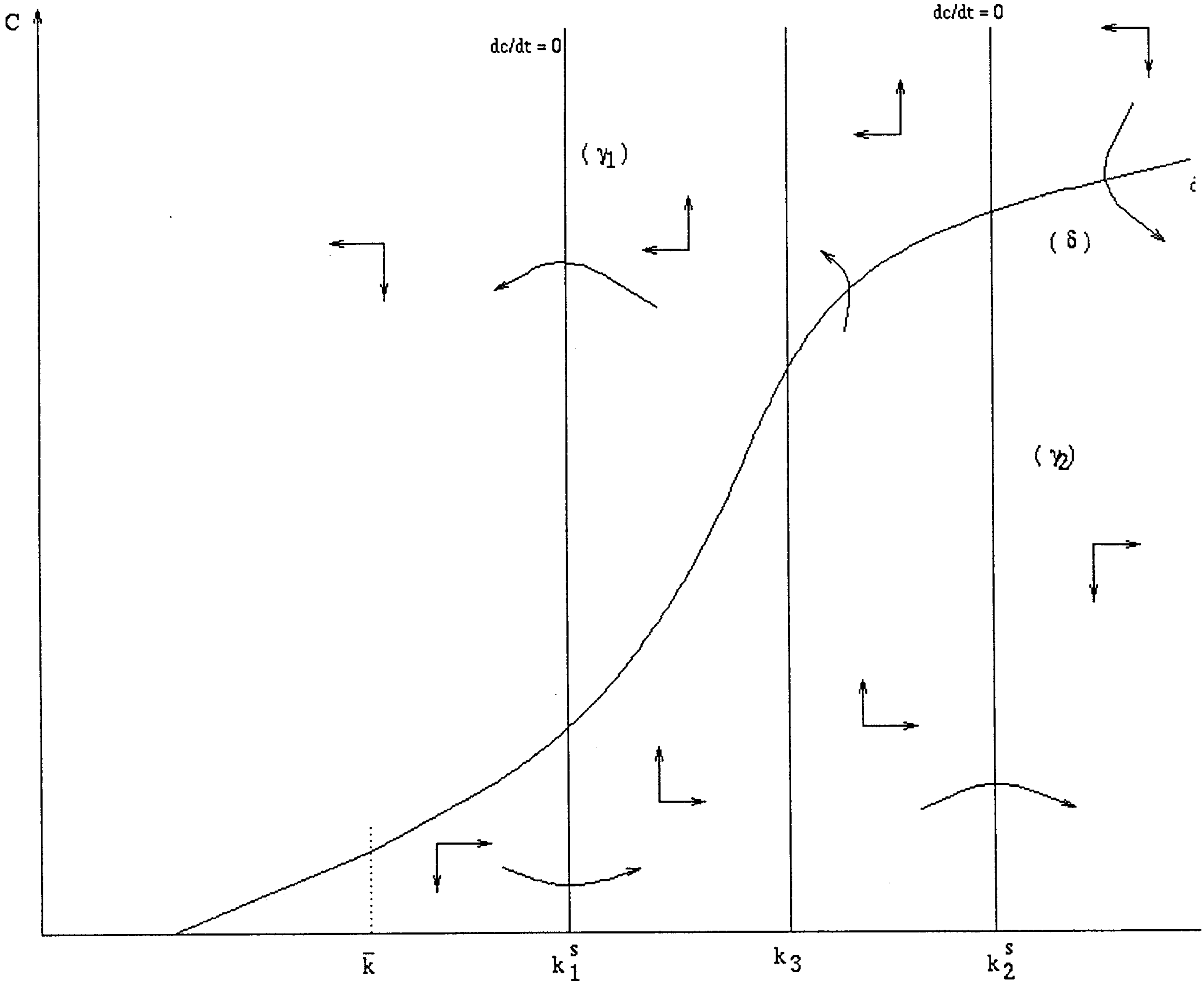

FIGURE 5

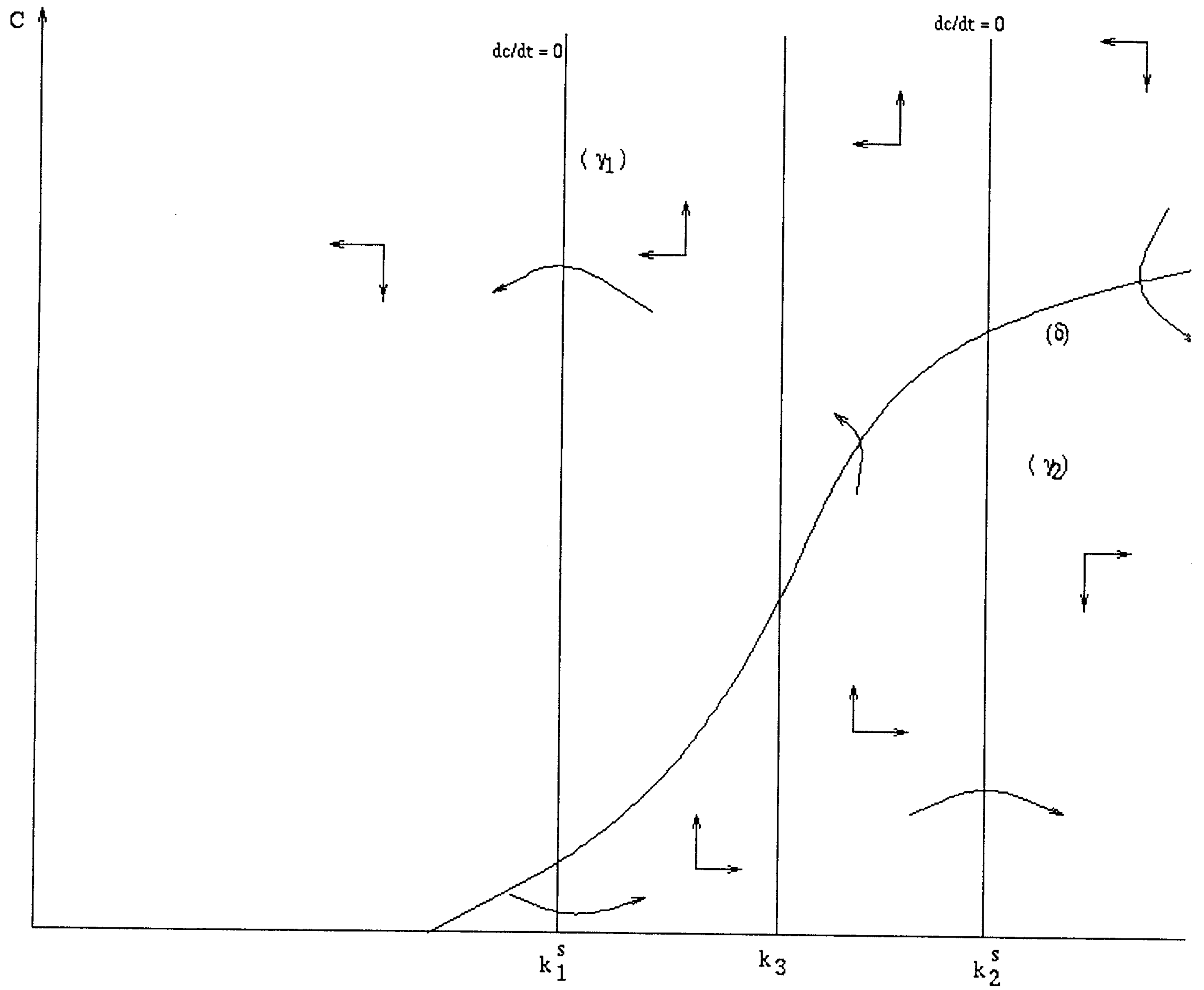

FIGURE 6