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ECONOMETRIC SPECIFICATION OF THE RISK NEUTRAL VALUATION MODEL

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Abstract

In complete markets no arbitrage opportunity implies deterministic relationships between the prices of derivative assets. These actuarial relations are incompatible with the available data and with statistical inférence. The aim of this paper is to reconcile risk neutral valuation and statistical inférence. For this purpose we justify an approach based on a stochastic risk-neutral measure.

SPECIFICATION ECONOMETRIQUE DU MODELE DE VALORISATION RISQUE NEUTRE

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Résumé

En marché complet, les conditions d'absence d'opportunité d'arbitrage impliquent des relations déterministes entre les prix des produits dérivés. Ces relations comptables se révèlent incompatibles avec les données et avec l'inférence statistique. Le but de ce papier est de réconcilier valorisations par probabilité risque neutre et études statistiques. Pour cela nous justifions l'approche consistant à introduire une probabilité risque neutre stochastique.

Keywords: Risk neutral valuation, derivative assets, asymmetric informa tion, gamma measure.

Mots clés: Valorisation risque neutre, produits dérivés, information asymétrique, mesure gamma.

JEL classification: G13, C51

I INTRODUCTION

For complete markets the assumption of no arbitrage opportunity implies the existence of a unique risk neutral measure, which may be used for pricing derivative assets [Harrison-Kreps (1979)]. As a by-product the derivative prices satisfy deterministic relationships, which are incompatible with a standard statistical analysis based on observed derivative prices. An illustration is given by the Black-Scholes model [Black-Scholes (1973)]. The price at date t of a european call with strike K and maturity date $t + H$ is a function of the current price S_t of the underlying asset and of the short term interest rate r_t :

$$P_t(K, H) = p_{BS}(S_t, r_t, K, H; \sigma) \qquad \text{(say)},$$

where σ is the volatility parameter. If at date t we observe the prices S_t, r_t and one derivative price $P_t(K_1, H)$, the implied Black-Scholes volatility, defined as the solution of :

$$P_t(K_1, H) = p_{BS}(S_t, r_t, K_1, H, \hat{\sigma}(K_1)),$$

is an estimator of σ which is infinitely accurate. If now we observe an additional derivative price $P_t(K_2, H)$, we get another infinitely accurate estimator $\hat{\sigma}(K_2)$. In practice the two estimates $\hat{\sigma}(K_1)$ and $\hat{\sigma}(K_2)$ are different and the underlying model is rejected with probability one. [1]

The empirical literature on derivative assets proposed pragmatic approaches to circumvent this basic difficulty. They are of different kinds.

i) The complete market modelling may be considered only as a benchmark used for regression models. This approach is followed when the volatility in the Black-Scholes model is estimated by ordinary least squares from derivative prices :

$$\hat{\sigma} = \arg\min_{\sigma} \sum_{j=1}^{J} [P_t(K_j, H) - p_{BS}(S_t, r_t, K_j, H; \sigma)]^2.$$

[1] The same reasoning applies in an incomplete market framework with a fixed number of factors [Hull-White (1987), Heath-Jarrow-Morton (1992)] as soon as we observe more derivatives than factors.

However the underlying model :

$$P_t(K_j, H) = p_{BS}(S_t, r_t, K_j, H; \sigma) + u_{j,t},$$

contains additional error terms $u_{j,t}$, i.e. some extra randomness which creates incompleteness and is incompatible with Black-Scholes derivations and the assumption of a unique valuation. [see Malz (1996) for an extended Black-Scholes model corrected for the smile, Bahra (1996) for a model based on a mixture of log-normal distributions].

2) Another approach consists in increasing the number of parameters in such a way that this number is always larger than the number of observed derivative prices. This literature includes all the studies on nonparametric estimation of the risk neutral density [see e.g. Hutchinson-Lo-Poggio (1994), Madan-Milne (1994), Ait-Sahalia-Lo (1995), Abken-Madan-Ramamurties (1996), Stutzer (1996)]. However, although it is possible to prove the consistency of the nonparametric estimator when the number of derivative prices increases, the distributional properties cannot be derived without introducing extra randomness, i.e. market incompleteness.

3) More flexibility may also be introduced by suppressing the complete market assumption. Then there exists an infinite number of admissible valuation measures. A part of the literature derives the deterministic bounds on derivative prices implied by the assumption of no arbitrage opportunity [see Merton (1973), Hodges (1996) who gives the constraints on the implied Black-Scholes volatilities, and Gourieroux-Scaillet- Szafarz (1997), chapter 8 for a general discussion]. Nevertheless this approach often provides large derivative price intervals not enough informative to be used in practice. This approach may be completed by introducing some prior distribution on the location of the valuation measure and deriving the posterior distribution of the admissible derivative price inside the previous interval [see Ncube (1993)].

4) Finally many econometric papers have directly specified descriptive dynamic models for the derivative prices or for these prices corrected by strike and maturity effects, i.e. for the implied Black-Scholes volatilities [see Engle-Mustafa (1992),]. The corresponding estimation methods and the predicted derivative prices are generally not compatible with the no arbitrage

implications. For instance it is known that long memory modelling for prices allows for perfect arbitrage [Rogers (1995)].

In this paper we propose to reconcile the complete market hypothesis and the statistical inference.

In section 2 we explain that the notion of market incompleteness refers to the information held and used by the market participants whereas statistical inference is concerned with the econometrician's information. We deduce that under asymmetric information the visible implication of the no arbitrage condition is the ability to obtain the derivative prices as expected discounted cash-flows with respect to a stochastic valuation measure. Therefore the extra randomness necessary to perform statistical inference is introduced via the difference between the informations.

In section 3, we study different consequences of such a modelling. We give the first and second order stochastic properties of the derivative prices. Then we predict the derivative prices and check that the predicted values satisfy the no arbitrage condition.

In section 4, we use gamma measures as a basis for specifying parametric models, and particularize this specification in section 5 to extend Black-Scholes models. It is seen that the parameters can be estimated by a generalized least squares method taking into account the correlation and heteroscasticity corresponding to the cash-flow patterns.

Finally in section 6 we consider the case of derivative assets with different horizons or with path dependent cash-flows.

II STOCHASTIC RISK NEUTRAL MEASURE

In this section and the two following ones, we consider an underlying index S_t, t varying, and derivative assets based on S. S may correspond to the price of a tradable financial asset, to a market index or to some other statistical index such as the aggregate result of the U.S. insurance companies. At date t, we only consider derivatives with a given residual maturity H, delivering an indexed cash-flow $g(S_{t+H})$ (say) at the maturity date.

2.1 Valuation in a complete framework

In a complete framework the prices at t of the derivative assets g are uniquely defined and may be represented as :

$$P_t(g) = \int_o^\infty g(s) dQ_t(s), \forall g, \tag{2.1}$$

where Q_t is the unique risk neutral measure giving at t the state prices of S_{t+H}.

2.2 Asymmetric information

The no arbitrage implication (2.1) is satisfied as soon as the market participants are sufficiently informed. Let us now assume that this condition is satisfied, but that the econometrician is less informed. Then we get :

$$P_t(g; w_t) = \int_o^\infty g(s) dQ_t(s; w_t), \forall g, \tag{2.2}$$

where w_t denotes the information available to the traders and unknown to the statistician. Then it is natural to introduce a stochastic modelling for this imperfect knowledge, i.e. to be confident in the valuation formula (2.1), but with a stochastic risk-neutral measure Q_t and stochastic derivatives prices $P_t(g)$ [see also Gourieroux-Scaillet (1997)].

This approach corresponds to the standard latent variable modelling. The latent variables are the endogenous state price Q_t and are partly observed through the derivative prices $P_t(g)$. These prices are related to the state prices by the aggregation formula (2.1), where the weights are the contingent cash-flows.

III STOCHASTIC PROPERTIES OF DERIVATIVE PRICES

Let us fix the date t, and consider only the valuation problem at this date and horizon H. Then the random measure Q_t is indexed by s only. We assume the existence of the infinitesimal first and second order moments of Q_t, conditional to the exogenous information available at t, for the statistician (i.e. not including the current observed derivative prices) :

$$\begin{gathered} E_t[dQ_t(s)] = d\mu_t(s), \quad \mathrm{Cov}_t[dQ_t(s), dQ_t(s')] = \tilde{C}_t(ds, ds'), \\ V_t[dQ_t(s)] = C_t(ds). \end{gathered} \tag{3.1}$$

3.1 First and second order moments of derivative prices

By noting that the cash-flows are deterministic functions of the states and by using properties of stochastic integrals, we get :

$$\begin{aligned} E_t P_t(g) &= E_t\left[\int_o^{\infty} g(s) dQ_t(s)\right] \\ &= \int_o^{\infty} g(s) E_t dQ_t(s), \\ E_t P_t(g) &= \int_o^{\infty} g(s) d\mu_t(s), \end{aligned} \tag{3.2}$$

and :

$$\begin{aligned} \mathrm{Cov}_t\left[P_t(g), P_t(h)\right] &= \int_o^{\infty}\int_o^{\infty} g(s) h(s') \tilde{C}_t(ds, ds') \\ &+ \int_o^{\infty} g(s) h(s) C_t(ds). \end{aligned} \tag{3.3}$$

3.2 Predictions conditional to the observed derivative prices

Let us now assume that the econometrician observes at date t some derivatives prices : $P_t(g_j), j = 1, \ldots, J$. This additional information may be introduced to improve the prediction of the nonobserved option prices [2]. These predictions are :

$$\begin{aligned} & \hat{P}_t(g) \\ = \; & E_t\left[P_t(g)/P_t(g_1), \ldots, P_t(g_J)\right] \\ = \; & E_t\left[\int_o^{\infty} g(s) dQ_t(s)/P_t(g_1), \ldots, P_t(g_J))\right] \\ = \; & \int_o^{\infty} g(s) dE_t\left[Q_t(s)/P_t(g_1), \ldots, P_t(g_J)\right] \\ = \; & \int_o^{\infty} g(s) d\hat{Q}_t(s) \qquad \text{(say)}. \end{aligned} \tag{3.4}$$

This relation is of practical importance. Indeed the predicted derivative prices will also satisfy the no arbitrage condition, with the modified measure $\hat{Q}_t$. Therefore the predicted derivative prices derived from the previous modelling cannot give to the practitioner the impression that some arbitrage opportunity exists and the temptation to use strategies with leverage effects. The main practical advantage of the previous modelling is certainly

[2] Note that all the options may be traded because of the complete market assumption, and therefore the prices $P_t(g)$ are well defined.

this possibility to avoid a large financial risk taken as a consequence of a slight statistical misspecification.

Moreover the predicted state price measure $\hat{Q}_t$ gives information on the lack of information of the econometrician. The more variable is $\hat{Q}_t$ the less important is the econometrician's information. In the limit the derivative assets $g_1, \dots, g_J$ are sufficient to hedge all the randomness if and only if $\hat{Q}_t$ is deterministic.

IV THE GAMMA SPECIFICATION

It remains to propose a tractable specification for the distribution of the stochastic valuation measure. The gamma specification seems a good candidate for at least three reasons :

i) it leads to a clear factorization of the distribution into the parts corresponding to the zero-coupon price and to the risk neutral probability ;

ii) it may be located around deterministic valuation formula, such as the Black-Scholes' one ;

iii) the estimation and computation steps can be easily performed by simulation based inference methods [see Gourieroux-Monfort (1996)].

4.1 Definition

Definition 4.1 : The random measure Q_t is a gamma measure iff :
i) it has independent increments, i.e. for any set of disjoint intervals $[(a_i, a_{i+1})]$ $i = 1, \dots, I$ the variables $Q_t[(a_i, a_{i+1})]$ $i = 1, \dots, I$ are independent ;

ii) for any interval (a, b), the random variable $Q_t[(a, b)]$ follows a gamma distribution with parameters $\nu_t[(a, b)], \lambda_t$, where ν_t is a finite positive deterministic measure on $I\!R^+$ and λ_t a positive real number.

In this modelling, the lack of knowledge of the statistician is large. Because of independent increments, he/she is missing information for each state

s. This lack of knowledge is summarized by means of a scalar parameter λ_t and a functional one corresponding to the c.d.f. of ν_t.

4.2 Factorization of the distribution

The risk neutral measure Q_t may be written as :

$$Q_t(ds) = Q_t(I\!R^+)\frac{Q_t(ds)}{Q_t(I\!R^+)},$$

or :

$$Q_t(ds) = B(t, t+H)\Pi_t(ds), \tag{4.1}$$

where $B(t, t+H)$ is the zero-coupon price at date t for horizon H, and Π_t the risk neutral probability. Both are stochastic and their properties are described below [see Ferguson-Klass (1972), Ferguson (1974)].

Property 4.1 : Under the gamma specification :

i) the zero-coupon price and the risk neutral probability are independent ;

ii) $B(t, t+H)$ follows the gamma distribution with parameters $\nu_t(I\!R^+), \lambda_t$ (i.e. with mean $\nu_t(I\!R^+)/\lambda_t$ and variance $\nu_t(I\!R^+)/\lambda^2$).

iii) Π_t is a Dirichlet measure. For any partition of the positive real line $(a_i, a_{i+1}), i = 1, \dots, I,$ the random vector $\{\Pi_t[(a_i, a_{i+1})], i = 1, \dots, I\}$ follows a Dirichlet distribution with parameters $\{\nu_t[(a_i, a_{i+1})], i = 1, \dots, I\}$.

Let us recall that a Dirichlet distribution with parameters $\nu_i, i = 1, \dots, I,$ $\nu_i \geq 0$ is a distribution on the simplex $\{\pi_i, i = 1, \dots, I; \pi_i \geq 0, \sum_{i=1}^{I} \pi_i = 1\}$ with pdf :

$$f[\pi_1,\ldots,\pi_I] = \frac{\Gamma\left(\sum\limits_{i=1}^{I}\nu_i\right)}{\prod_{i=1}^{I}\Gamma(\nu_i)}\prod_{i=1}^{I}(\pi_i^{\nu_i-1}).$$

4.3 The distribution of observable prices

For statistical inference it is necessary to be more specific about the distribution of the observed derivative prices $P_t(g_j), j = 1,\ldots,J$, including possibly the zero-coupon bond ($g = 1$) and the underlying asset ($g = Id$). The moments are derived from formulas (3.2), (3.3) and the expressions of the moments of a gamma measure :

$$E_tP_t(g) = \frac{1}{\lambda_t}\int_o^\infty g(s)d\nu_t(s), \tag{4.2}$$

$$\mathrm{Cov}_t(P_t(g),P_t(h)) = \frac{1}{\lambda_t^2}\int_o^\infty g(s)h(s)d\nu_t(s). \tag{4.3}$$

The joint distribution of the derivative prices has no simple expression except through its characteristic function. We get (see appendix 1) :

$$\begin{aligned}\psi_{t,J}(v) &= \log E_t\left\{\exp\left[i\sum_{j=1}^{J}v_jP_t(g_j)\right]\right\}\\ &= \int_{I\!R^+}\log\left\{1-\frac{i}{\lambda_t}\sum_{j=1}^{J}v_jg_j(s)\right\}d\nu_t(s).\end{aligned} \tag{4.4}$$

V A Parametric specification

The gamma modelling may be particularized by selecting the average measure ν_t in a parametric family $\nu_t = \nu_t(\theta_t)$. Then the distribution of the derivative prices will only depend on a finite number of parameters, i.e. θ_t, λ_t. In particular we can select this specification in order to calibrate the expected derivative prices on some deterministic valuation formula, such as the Black-Scholes or Hull-White ones. We describe the consequence of these choices and we discuss statistical inference.

5.1 An extended Black-Scholes formula

In the standard Black-Scholes model there is one factor equal to the price of the underlying asset. The dynamics of this factor F_t is a geometric brownian motion, in particular :

$$\log F_t \sim N[(\mu - \frac{\sigma^2}{2})t + \log F_0, \sigma^2 t]. \tag{5.1}$$

Under the risk neutral probability this factor is also a geometric brownian motion :

$$dF_t = rF_t dt + \sigma F_t dW_t, \tag{5.2}$$

where (W_t) is a standard brownian motion. Then the associated risk neutral probability for horizon H, π_t^{BS} (say), is the log-normal distribution with parameters $\left(r - \frac{\sigma^2}{2}\right) H + \log F_t, \sigma^2 H$. We can now introduce this deterministic risk neutral measure inside the gamma modelling, after replacing r by r_t.

Definition 5.1 : The Black-Scholes gamma model corresponds to the choice : $\nu_t(ds) = \lambda_t \exp(-r_t H)\pi_t^{BS}(ds), \lambda_t = \lambda,$ and $\log F_t \sim N[(\mu - \frac{\sigma^2}{2})t + \log F_0, \sigma^2 t]$.

Since the mean of this gamma measure is the Black-Scholes measure $\exp(-r_t H)\pi_t^{BS}$ this specific choice locates the expected derivative prices on the standard Black-Scholes formula, while allowing for some variability around this formula. The fit is summarized by the scalar parameter λ. When λ tends to infinity the gamma model becomes a deterministic one, $F_t = S_t$ (since, if $\lambda = +\infty, S_t = \int s dQ_t(s) = \exp(-r_t H) \int s d\pi_t^{BS}(s) = F_t$) and the Black-Scholes specification is satisfied, both for the historical dynamics of S_t and for the deterministic relationship of the derivative prices.

For a general λ it is useful to distinguish the price S_t and the factor F_t. Since the econometrician does not possess all the information, there is no reason why he/she will know the factor. Secondly if $F_t = S_t$, the no arbitrage condition will imply :

$$S_t = P_t(Id) = \int_o^\infty s dQ_t(s),$$

where Q_t is function of S_t. Therefore the price of the underlying asset would be defined as the solution of a complicated implicit equation.

Note however that the factor may always be considered as the expected price, since :

$$\begin{aligned} ES_t &= E\int_o^\infty s dQ_t(s) \\ &= \exp(-r_t H)\int_o^\infty s d\pi_t^{BS}(s) \\ &= F_t. \end{aligned}$$

The moments of the price of european calls are given in the property below.

Property 5.2 : Let us denote $P_t(H,K)$ the price at t of the european call with strike K, maturing at $t+H$. Under the Black-Scholes gamma model, we get conditionally to the factor F_t :

i) $E_t P_t(H,K) = \exp(-r_t H) E_{\pi_t^{BS}}[(S_{t+H}-K)^+/F_t]$

$= F_t\Phi[D_t(K)+\sigma\sqrt{H}] - K\exp(-r_t H)\Phi[D_t(K)]$

ii) $\mathrm{Cov}_t[P_t(H,K), P_t(H,L)]$

$= \dfrac{1}{\lambda}\exp(-r_t H) E_{\pi_t^{BS}}[(S_{t+H}-K)^+(S_{t+H}-L)^+/F_t]$

$= \dfrac{1}{\lambda}\{KL\exp(-r_t H)\Phi[D_t(L)] - (K+L)F_t\Phi[D_t(L)+\sigma\sqrt{H}]$

$+F_t^2\exp(r_t H+\sigma^2 H)\Phi[D_t(L)+2\sigma\sqrt{H}],\}$ for $L \geq K$.

where Φ is the cdf of the standard normal distribution and :

$$D_t(K) = \frac{1}{\sigma\sqrt{H}}\log\frac{F_t}{K\exp(-r_t H)} - \frac{1}{2}\sigma\sqrt{H}.$$

Proof : see appendix 2.

By contruction the expected derivative prices coincide with the Black-Scholes valuations. More interesting are the expressions of the second order moments. Indeed we observe both heteroscedasticity with respect to the strike and correlation. This is easily understood.

Let us consider two european calls with strikes $L \geq K$. The corresponding cash flows are such that : $(S_{t+H} - K)^+ \geq (S_{t+H} - L)^+$, and by arbitrage arguments the two random prices $P_t(H, K)$ and $P_t(H, L)$ will satisfy the inequality $P_t(H, K) \geq P_t(H, L)$. The gamma model takes into account such a constraint. Then this constraint will imply variances depending on the strike K. Moreover when K tends to L, we expect $P_t(H, K)$ to tends to $P_t(H, L)$, and the correlation between the two prices to tend to one. This is only possible if the covariance depends on the two strikes K and L.

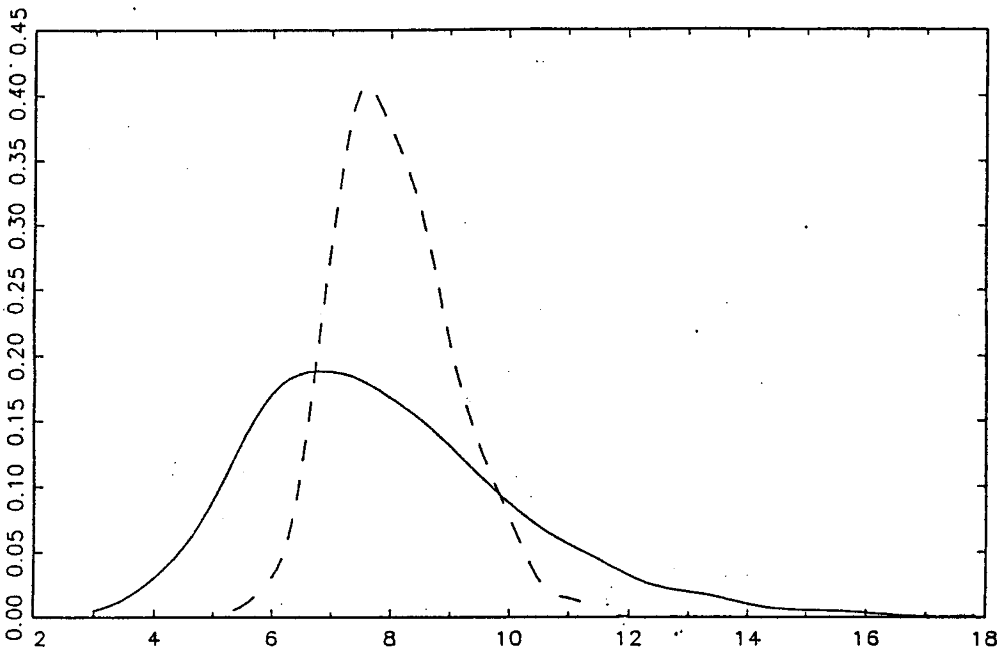
5.2 Monte-Carlo studies

Some further properties on the distribution of derivative prices may be illustrated by simulations.

The distribution of a derivative price conditional to the factor F_t is a complicated mixture of gamma distributions. It is generally asymmetric, mainly because of the asymmetric weights $(s - K)^+$ which are introduced. In figure 5.1 we give the conditional distribution of a european call with strike $K = 5$, residual maturity $H = 10$. The volatility of the underlying price is $\sigma = 0.3$, and the conditioning values are $F_t = 10, r_t = 0.1$. We have fixed two variability levels $\lambda = 100$ and $\lambda = 50$ respectively. These distributions are derived from 1 000 simulations of a gamma measure [see Cheng-Feast (1979), (1980) and appendix 3].

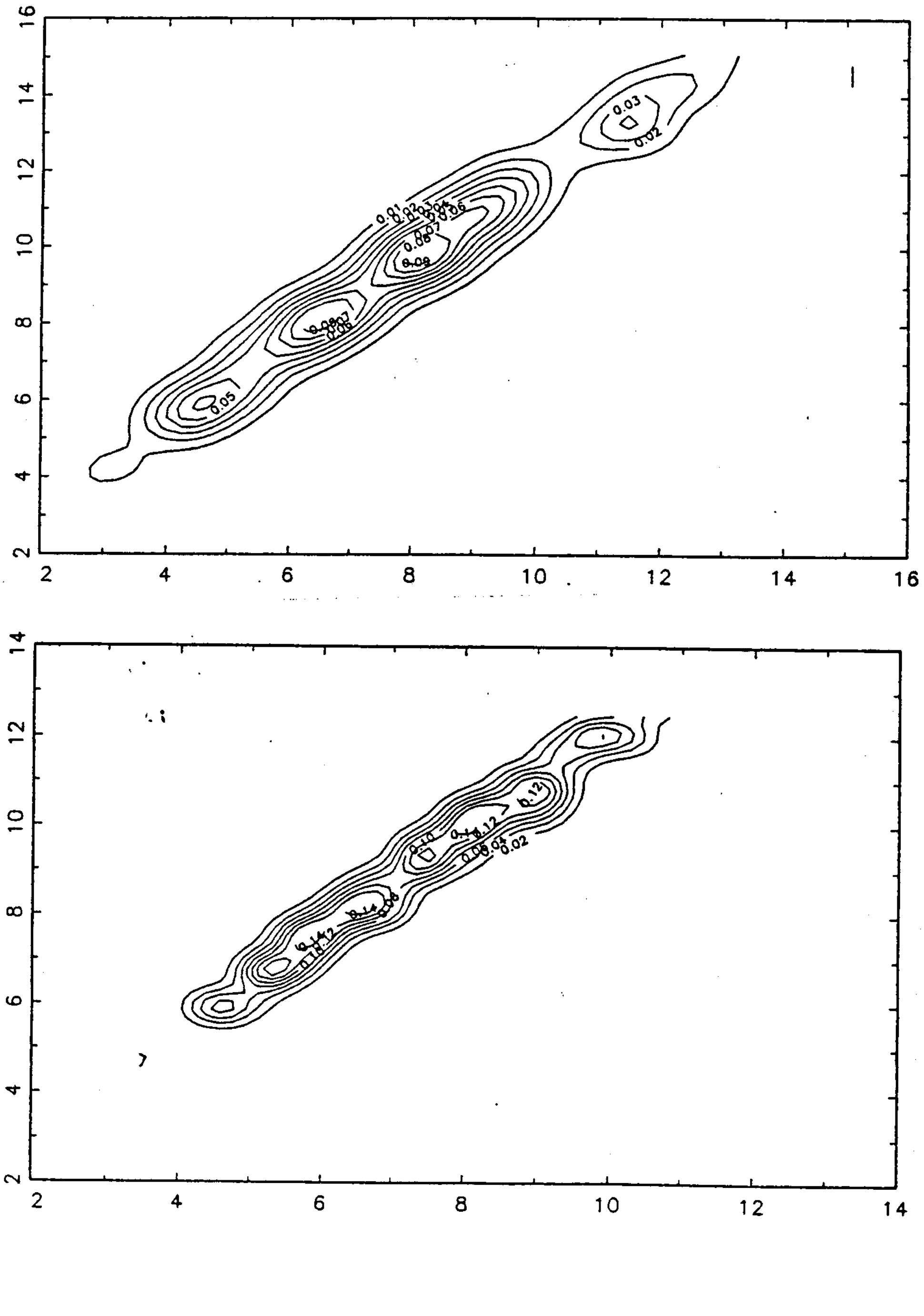
The two distributions are centered at the standard Black-Scholes price equal to 8.22.

Figure 5.1 : Distribution of the price of the european call conditional to F_t



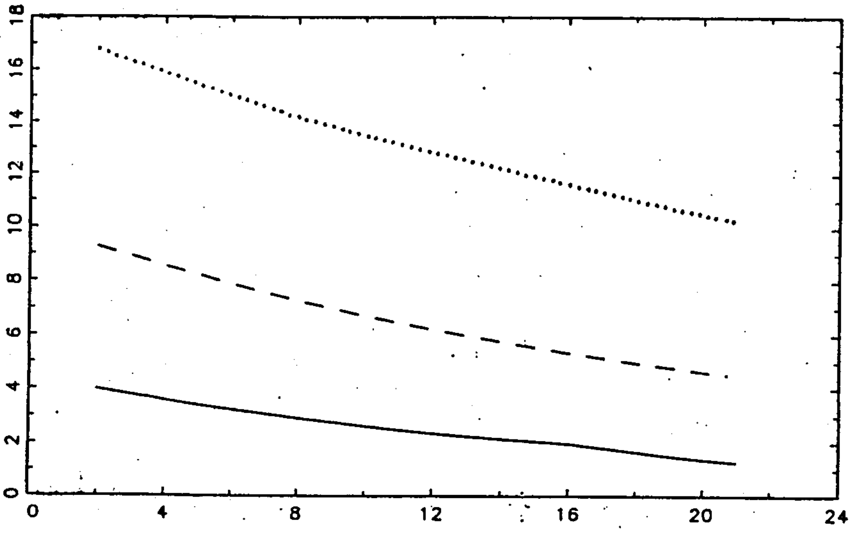
From a practical point of view, it is useful to also simulate joint distributions of derivative prices. They will reveal high asymmetries, but also nonlinearities and bounded support because of the inequality constraints coming from the no arbitrage conditions. In figure 5.2 we give typical iso-density curves for the joint distribution of $(P_t(H,K), S_t)$, conditional to $F_t = 10$. The support is limited by the curves $p_1(s) = s$ and $p_2(s) = (s - KB(t,t+H))^+$ and restricted to positive values (since $S_{t+H} \geq (S_{t+H} - K)^+ \geq S_{t+H} - K$ implies $S_t \geq P_t(H,K) \geq S_t - KB(t,t+H)$, and since $P_t(H,K) \geq 0$). The highest level are around the Black-Scholes valuation formula. The complicated patterns of the iso-density curves explain the difficulty to propose values at risk (VaR) for a portfolio containing both the underlying and the derivative assets.

Figure 5.2 : Joint distribution of the prices $S_t, P_t(H;K)$, conditional to $F_t(\lambda = 50, 100)$



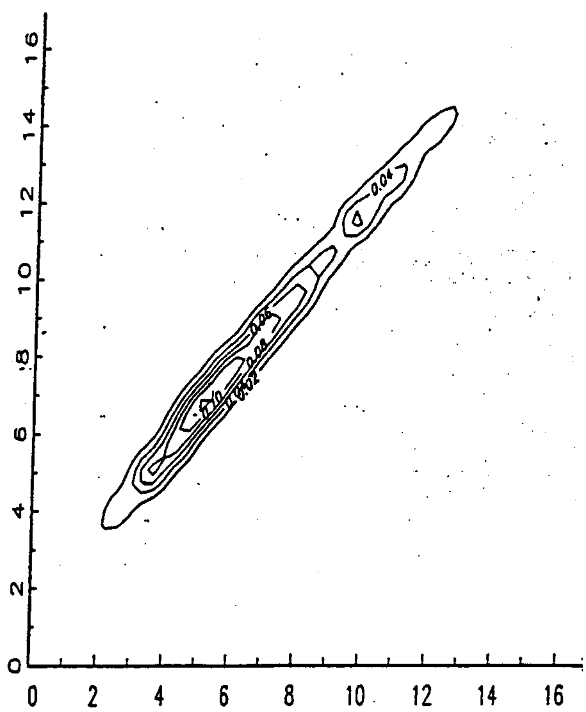
Finally in figure 5.3, we give confidence bounds on the derivative price as a function of the strike, with a variability parameter $\lambda = 50$. The two bounds are computed to leave a probability of 2.5% below and above respectively. As expected the confidence interval is asymmetric with respect to the Black-Scholes price. The introduction of only one variability parameter allows for a regular heteroscedasticity.

Figure 5.3 : The pricing interval as a function of the strike



Simulated derivative prices may also be considered unconditionally, i.e after integrating out the factor. We give in figure 5.4 the joint distribution of $[S_t, P_t(H,K)]$ for $\lambda = 100$ which is around the standard deterministic Black-Scholes relation.

Figure 5.4 : Joint distribution of the prices $S_t, P_t(H, K)$

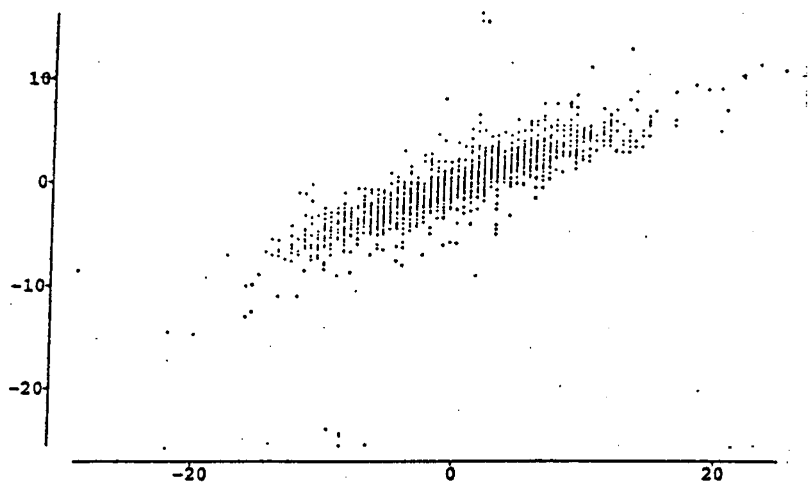


It is seen on the graphs that the distributions may present several modes, whose number is generally larger on the conditional joint distributions than on the unconditional one and on the marginal ones. [In our experiments both marginal distributions of the spot and option prices have only one mode]. It results mainly from the nonlinear cash-flows of a european call. Indeed the distribution of the call price at maturity date, when $\lambda \to +\infty$, corresponds to the distribution of $(S_{t+H} - K)^+$, i.e. to a mixture of a point mass at zero and of a log-normal distribution. Therefore it may admit one or two modes, and this property is also satisfied when the prediction is performed earlier. Moreover it is interesting to note that the variability of the option and asset prices, together with waves appearing in the iso-density curves of the distribution function seem to exist in practice. We give in figure 5.5 such an empirical joint distribution [3] for future on DAX and option on DAX computed on a subperiod of two months in 1992. However some other

[3] We thanks P. Bossaerts for providing this observed distribution.

given to explain such a shape, such as the discreteness of the prices, the jumps between bid and ask prices...

Figure 5.5 : Empirical joint distribution



5.3 Least squares approach

The parameters of a Black-Scholes gamma model can be consistently estimated in various ways, date by date when J tends to infinity or over all the period when the total number of observed derivative prices with the same maturity H tend to infinity.

For instance we may use a simulated nonlinear least squares approach [see Gourieroux-Monfort (1996)].

The moments have to be marginalized with respect to the unobservable factor F_t, practically by simulation. With obvious notations we get :

$$\begin{cases} E_t P_t(H, K_j) = g_{BS}(r_t, K_j, H, \sigma; \mu), \\ \mathrm{Cov}_t\{P_t(H, K_j), P_t(H, K_l)\} = \sigma_{BS}(r_t, K_j, K_l, H, \sigma; \mu). \end{cases}$$

In a first step the volatility and trend parameters may be estimated by :

$$(\hat{\mu}_{t,J}, \hat{\sigma}_{t,J}) = \arg\min_{\mu,\sigma} \sum_{j=1}^{J} [P_t(H, K_j) - g_{BS}(r_t, K_j, H, \sigma; \mu)]^2, \tag{5.3}$$

However these estimators may not be very accurate because of the strike effects which imply heteroscedasticity and correlation. They can be improved by simulated quasi-generalized least squares. The second step estimator is defined as :

$$(\hat{\hat{\mu}}_{t,J}, \hat{\hat{\sigma}}_{t,J}) = \arg\min_{\mu\sigma} \quad [P_t(H, K) - g_{BS}(r_t, K, H, \sigma; \mu)]' \textstyle\sum_{BS}^{-1}(r_t, H, \hat{\sigma}_{t,J}; \hat{\mu}_{t,J})$$

$$[P_t(H, K) - g_{BS}(r_t, K, H, \sigma, \mu)],$$

where $P_t(H, K) = (P_t(H, K_1), \ldots, P_t(H, K_J))'$, and the other notations are defined accordingly. The previous estimation method is different from the usual calibration method based on the Black-Scholes model. First the underlying price is introduced as the price of a particular derivative asset with $K_j = 0$. Second the Black-Scholes formula is marginalized with respect to the factor. This explains why we have a joint estimation of μ and σ as soon as an asymmetric information exists. Finally the correction by the weights $\sum_{BS}^{-1}$ is crucial in practice, since the estimations are usually performed date by date, and the structure of the observed derivatives by strike is time varying. This problem is similar to the determination of the term structure of interest rates from data on bonds, i.e. portfolios of the non observable zero-coupon bonds.

Then the λ parameter is estimated by :

$$\hat{\lambda}_{t,J} = \frac{1}{J}\left[\hat{v}_t' \Sigma_{BS}^{-1}(r_t, H, \hat{\sigma}_{t,J}, \hat{\mu}_{t,J})\hat{v}_t\right]^{-1}, \tag{5.4}$$

$$\text{where :} \quad \hat{v}_t = P_t(H, K) - g_{BS}(r_t, K, H, \hat{\hat{\sigma}}_{t,J}, \hat{\hat{\mu}}_{t,J}). \tag{5.5}$$

5.4 Extended Hull-White model

The approach followed for extending the Black-Scholes model may be considered as a general modelling principle to derive dynamics compatible with the no arbitrage condition. While it corresponds to an incomplete market framework, we may study the case of the Hull-White modelling in order to capture the smile effect. This model is a two factors model, usually the price of the underlying asset and the volatility. Their historical dynamics is given by the differential system :

$$\begin{cases} dF_{1t} = \tilde{\mu} F_{1t} dt + F_{2t} F_{1t} dW_t^1, \\ \\ dF_{2t} = \alpha F_{2t} + \beta F_{2t} dW_t^2, \end{cases} \tag{5.6}$$

where (W^1, W^2) are two independent brownian motions. Then we can center the gamma measure on a particular risk neutral probability associated with the Hull-White model assuming a null risk premium on the volatility. For instance we can follow the same approach as in section 5.1 after replacement of π_t^{BS} by the distribution π_t^{HW} corresponding to the log-normal distribution with parameters :

$$r_t H - \frac{1}{2} \int_t^{t+H} F_{2u} du + \log F_{1t}, \int_t^{t+H} F_{2u} du,$$

marginalised with respect to the conditional distribution of $F_{2u}, u \geq t$ given F_{2t}. We get a distribution of π_t^{HW} which depends on F_{1t}, F_{2t}. As for the extended Black-Scholes modelling the prices of the derivatives depend on two kinds of randomness corresponding to the gamma measure and to the factors respectively.

VI Extension to the multihorizon case

The previous approach can be extended to the case where we study derivative assets with different maturities or with path dependent cash-flows. We first recall the constraints implied by the no arbitrage opportunity condition ; then we discuss different specifications for the stochastic risk neutral probability and the stochastic term structure.

6.1. Security prices

Let us consider a security g giving cash-flows at dates $t+1, t+2, \ldots, t+H$, and such that the cash-flow of date $t+h$ depends on the values $s_{t+1}, \ldots, s_{t+h}$ taken by an index s. We use the following notations :

$$s_{\underline{t+h}} = (s_{t+1}, \ldots, s_{t+h}),$$

and $g_h(s_{\underline{t+h}})$ is the cash-flow at date $t+h$.

When there is only one horizon H the price $P_t(g)$ of derivative g at date t is given by formula (2.2).

$$\begin{aligned} P_t &= \int_o^\infty g(s) dQ_t(s) \\ &= Q_t(I\!R^+) \int_o^\infty g(s) \frac{dQ_t(s)}{Q_t(I\!R^+)} \\ &= B(t, t+H) E_{\Pi_t}(g) \end{aligned}$$

where $B(t, t+H) = Q_t(I\!R^+)$ is the zero coupon price and $\Pi_t = \dfrac{Q_t}{Q_t(I\!R^+)}$ is the risk neutral probability.

The absence of arbitrage opportunity assumption leads to the following generalization :

$$P_t(g) = \sum_{h=1}^{H} B(t, t+h) E_{\Pi_t}(g_h) \tag{6.1}$$

where $B(t, t+h), h = 1, \ldots, H$ are the zero coupon prices at various horizons and Π_t is the risk neutral probability on $I\!R^{+H}$.

By arbitrage arguments we have :

$$B(t,t+h) = \Pi_{\tau=1}^{h} B(t,t+\tau-1,t+\tau),$$

where $B(t,t+\tau-1,t+\tau)$ is the forward price, fixed at date t, which will be paid at date $t+\tau-1$ in order to obtain one dollar at date $t+\tau$.

6.2 Stochastic specifications

Now under asymmetric information, i.e. a complete market for some participants and some lack of knowledge of the econometrician, we may introduce stochastic specifications for the $B(t,t+\tau-1,t+\tau), \tau = 1,\dots,H$ and Π_t.

We assume that the forward prices, denoted by $B_{t,\tau}, \tau = 1,\dots,H$, are independent variables and we introduce the notations :

$$\begin{aligned} m_{t,\tau} &= EB_{t,\tau}, \\ M_{t,\tau} &= EB_{t,\tau}^2, \qquad \tau = 1,\dots H. \end{aligned}$$

More precise specifications will be proposed below.

As far as the random probability Π_t is concerned, we adopt a specification which is a natural extension of the one used above, namely a multivariate Dirichlet random probability. Such a random probability is defined by a finite measure ν_t on $I\!R^{+H}$ (in our case) also denoted by $\alpha\pi_t$, where π_t is a deterministic probability distribution and α a positive real number. The mean of the random measure Π_t is π_t and the random measure Π_t converges to π_t when $\alpha \to \infty$. Moreover Π_t is assumed to be independent of the $B_{t,\tau}, \tau = 1,\dots,H$.

For a Dirichlet measure, denoted by $\Pi_t \sim \mathcal{D}_i(\alpha\pi_t)$, it can be shown [see Ferguson (1973) or Rolin (1992)] that for any functions f, f' belonging to $L_2(\pi_t)$:

$$E\left[E_{\Pi_t}(f)\right] = E_{\pi_t}(f), \tag{6.2}$$

$$\operatorname{cov}\left[E_{\Pi_t}(f), E_{\Pi_t}(f')\right] = \frac{1}{\alpha+1}\operatorname{cov}_{\pi_t}(f,f'), \tag{6.3}$$

and, therefore :

$$E\left[E_{\Pi_t}(f).E_{\Pi_t}(f')\right] = \frac{1}{\alpha+1}E_{\pi_t}(ff') + \frac{\alpha}{\alpha+1}E_{\pi_t}(f)E_{\pi_t}(f'). \qquad (6.4)$$

6.3 Moments of the prices

From the previous results we directly derive the first and second order moments of the derivatives prices. We get :

$$E\left[P_t(g)\right] = \sum_{h=1}^{H}\left(\Pi_{\tau=1}^{h} m_{t,\tau}\right) E_{\pi_t}(g_h). \qquad (6.5)$$

If we consider two securities g and g', we have :

$$\mathrm{Cov}\left[P_t(g), P_t(g')\right] = \sum_{h=1}^{H}\sum_{k=1}^{H} \mathrm{Cov}\left[B(t,t+k)E_{\Pi_t}(g_h), B(t,t+k)E_{\Pi_t}(g'_k)\right] (6.6)$$

where (assuming $h \leq k$) :

$$\begin{aligned}
&\mathrm{Cov}\left[B(t,t+h)E_{\Pi_t}(g_h), B(t,t+k)E_{\Pi_t}(g'_k)\right] \\
&\quad = \left(\Pi_{\tau=1}^{k} M_{t,\tau}\right)\left(\Pi_{\tau=h+1}^{k} m_{t,\tau}\right)\left[\frac{1}{\alpha+1}E_{\pi_t}(g_h g'_k) + \frac{\alpha}{\alpha+1}E_{\pi_t}(g'_k)\right] \\
&- \left(\Pi_{\tau=1}^{h} m_{t,\tau}^2\right)\left(\Pi_{\tau=h+1}^{k} m_{t,\tau}\right) E_{\pi_t}(g_h)E_{\pi_t}(g'_k) \qquad (6.7)
\end{aligned}$$

(where, in the case $k = h$, we use the convention $\prod_{\tau=h+1}^{k} m_{t,\tau} = 1$), or equivalently,

$$\begin{aligned}
&\left(\Pi_{\tau=h+1}^{k} m_{t,\tau}\right)\left\{\frac{1}{\alpha+1}\mathrm{Cov}_{\pi_t}(g_h, g'_k)\Pi_{t=1}^{h} M_{t,\tau} + E_{\pi_t}(g_h)E_{\pi_t}(g'_k)\right. \\
&\qquad\qquad \left.\left[\Pi_{\tau=1}^{h} M_{t,\tau} - \Pi_{\tau=1}^{h} m_{t,\tau}^2\right]\right\}. \qquad (6.8)
\end{aligned}$$

6.4. Specification of the zero-coupon prices

From (6.7) [or (6.8)] the first two moments of any set of securities are specified as soon as the moments $m_{t,\tau}$ and $M_{t,\tau}$ are.

But the risk analysis requires a knowledge of the whole distribution of derivative prices and therefore of the forward zero-coupon prices $B_{t,\tau}$.

Three specifications are considered below. They lead to the same structure of the first and second order moments. Their differences concern the constraints implied on the forward rates, their coherency with respect to maturity marginalisation and their practical implementation.

i) **The gamma specification**

This specification is the direct extension of the model of section 4.

We assume that : $B_{t,\tau} \sim \gamma(\alpha_\tau, \alpha_\tau / m_{t,\tau}), \tau = 1, \dots, H$, with $m_{t,\tau} \in \mathbb{R}^+, \alpha_\tau \in \mathbb{R}^+$. The mean of $B_{t,\tau}$ is $m_{t\tau}$; its variance, $m^2_{t,\tau}/\alpha_\tau$, converges to zero when $\alpha_\tau \to \infty (m_{t,\tau}$ fixed). In this specification we have :
$M_{t,\tau} = m^2_{t,\tau} \dfrac{\alpha_\tau + 1}{\alpha_\tau}$.

If we consider the case of one horizon (by convention $H = 1$) and if we take $\alpha_1 = \alpha$ the product $B_{t,1}\Pi_t$ follows the gamma measure $\gamma\left[\alpha\pi_t(.), \alpha/m_{t,\tau}\right]$, and we are back in the specification considered above in the one horizon case.

A Gamma-Dirichlet model is thus characterized by the selected horizon H and the set of parameters :

$$[H, (m_{t,\tau}, \alpha_\tau)_{\tau=1,\dots,H}, \alpha, \pi_t]$$

If the horizon is reduced, the marginalized model corresponding to the new horizon $H_o < H$ has the same form and is characterized by :

$$[H_o, (m_{t\tau}, \alpha_\tau)_{\tau=1,\dots,H_o}, \alpha, \pi_{t,H_o}],$$

where π_{t,H_o} is the marginal distribution of π_t corresponding to the first H_o coordinates.

ii) **The beta specification**

In this specification the forward coupon prices $B_{t,\tau}$, and therefore the zero coupon prices $B(t, t+\tau), \tau = 1, \dots, H$, automatically belong to $]0, 1[$.

We assume that : $B_{t,\tau} \sim Be[\alpha_\tau m_{t,\tau}, \alpha_\tau(1-m_{t,\tau})]$, with $m_{t,\tau} \in [0,1], \alpha_\tau \in I\!R^+$.

The mean of $B_{t,\tau}$ is $m_{t,\tau}$, its variance , $m_{t,\tau}(1-m_{t,\tau})/(\alpha_\tau+1)$, converges to zero when $\alpha_\tau \to \infty(m_{t,\tau}$, fixed), and we have :

$$M_{t,\tau} = m_{t,\tau}^2 + \frac{m_{t,\tau}(1-m_{t,\tau})}{\alpha_\tau+1} = m_{t,\tau}\frac{\alpha_\tau m_{t,\tau}+1}{\alpha_\tau+1}.$$

In this Beta-Dirichlet specification the marginalization due to a decrease of the horizon H works in the same way as in the Gamma-Dirichlet specification. If we denote by α_τ^γ and α_τ^β, the α_τ parameters appearing respectively in the Gamma-Dirichlet and the Beta-Dirichlet specification, these specifications are second-order observationnally equivalent if :

$$\frac{m_{t,\tau}(1-m_{t,\tau})}{\alpha_\tau^\beta+1} = \frac{m_{t,\tau}^2}{\alpha_\tau^\gamma}$$

$$\text{i.e.} \quad \alpha_\tau^\beta = \frac{\alpha_\tau^\gamma(1-m_{t,\tau})}{m_{t,\tau}} - 1,$$

which is positive as soon as α_τ^γ is sufficiently large.

iii) The Log-Normal specification

We assume that :
$B_{t,\tau} \sim LN(\log m_{t,\tau} - \frac{\sigma_\tau^2}{2}, \sigma_\tau^2), m_{t,\tau} \in I\!R^+, \sigma_\tau^2 \in I\!R^+$.

The mean of $B_{t,\tau}$ is $m_{t,\tau}, M_{t,\tau} = EB_{t,\tau}^2 = m_{t,\tau}^2 \exp(\sigma_t^2)$, and the variance of $B_{t,\tau}$, i.e. $m_{t,\tau}^2[\exp(\sigma_t^2)-1]$, converges to zero when $\sigma_\tau^2 \to 0$ (or $\alpha_\tau = \frac{1}{\sigma_\tau^2} \to +\infty$, if we want a parameterization similar to the previous one).

This Log-Normal Dirichlet is characterized by :

$$[H, (m_{t\tau}, \sigma_\tau^2)_{\tau=1,\dots,H}, \alpha, \pi_t]$$

As before, the marginalization due to a decrease of the horizon ($H_o < H$) leads to a specification of the same class defined by :

$$[H_o, (m_{t\tau}, \sigma_\tau^2)_{\tau=1,\dots,H_o}, \alpha, \pi_{t,H_o}]$$

Another advantage of this specification, which is not shared by the previous ones, is that it is stable with respect to a marginalization due to a time aggregation. For instance if $H = p\tilde{H}(p \in \mathbb{N})$ and if the relevant dates (for the cash flows and the influential index values) are $t + p\tau, \tau = 1, \dots, \tilde{H}$, the marginalized model remains Log-Normal Dirichlet and is characterized by :

$$[\tilde{H}, (\tilde{m}_{t,\tau}, \tilde{\sigma}_{\tau}^2)_{\tau=1,\dots,\tilde{H}}, \alpha, \tilde{\pi}_t],$$

where

$$\tilde{m}_{t,\tau} = \Pi_{h=t+p(\tau-1)+1}^{t+p\tau} m_{t,h}, \tilde{\sigma}_{\tau}^2 = \sum_{h=p(\tau-1)+1}^{p\tau} \sigma_h^2,$$

and $\tilde{\pi}_t$ is the marginal distribution of π_t corresponding to the coordinates indexed by $p\tau, \tau = 1, \dots, \tilde{H}$.

The Log-Normal Dirichlet specification is second order observationnally equivalent to the Gamma-Dirichlet specification if :

$$\begin{aligned} m_{t,\tau}^2[\exp(\sigma_\tau^2) - 1] &= \frac{m_{t,\tau}^2}{\alpha_\tau},\ \tau = 1, \dots, H, \\ \text{i.e. if}\, \sigma_\tau^2 &= \log(1 + \frac{1}{\alpha_\tau}), \tau = 1, \dots, H. \end{aligned} \tag{6.9}$$

6.5 A further extension of the Black-Scholes formula

As in section 5.1 we assume that there is an underlying factor F_t. Its dynamics under the standard Black-Scholes risk neutral probability corresponds to a geometric brownian motion :

$$dF_t = rF_t dt + \sigma F_t dW_t,$$

whereas it corresponds to :

$$dF_t = \mu F_t dt + \sigma F_t d\tilde{W}_t,$$

under the historical probability.

The associated risk neutral probability for discrete horizons $h = 1, \dots, H$ is the conditional probability distribution of $(F_{t+1}, \dots, F_{t+H})$ given F_t ; this distribution is given by :

$$\log F_{t+h} - \log F_{t+h-1} \sim N[r - \frac{\sigma^2}{2}, \sigma^2], h = 1, \dots, H,$$

independently, and is denoted by $\pi^{BS}(F_t, r, \sigma^2, H)$ or π_t^{BS}.

A Log-Normal Dirichlet model, generalizing the standard multihorizon Black-Scholes model is characterized by :

$$\begin{aligned} m_{t,\tau} &= \exp(-r_t), \\ \sigma_\tau^2 &= \sigma_0^2, \\ \pi_t &= \pi^{BS}(F_t, r_t, \sigma^2, H), \\ \mu &\in I\!R^+, \end{aligned}$$

$$\log F_t - \log F_{t-1} \sim N[\mu - \frac{\sigma^2}{2}, \sigma^2].$$

We deduce that : $M_{t,\tau} = \exp(-2r_t + \sigma_0^2)$.

So, we can compute the covariance of any pair of security prices. For instance let us consider the prices of two european call options with maturities ℓ and $k \geq \ell$ and strikes L and K, respectively.

The cash flow functions are, respectively :

$$\begin{cases} g_h = 0 \quad \forall h \neq l \\ g_l = (s_l - L)^+, \end{cases}$$

$$\begin{cases} g'_h = 0 \quad \forall h \neq k \\ g'_k = (s_k - K)^+. \end{cases}$$

Formula (6.5) gives :

$$E[P_t(g)] = \exp(-r_t l) E_{\pi_t}(S_\ell - L)^+ = p_{BS}(F_t, r_t, l, L, \sigma^2)$$

where p_{BS} denotes the Black-Scholes price, and similarly for g'.

Formula (6.7) give :

$$\begin{aligned}
&\operatorname{Cov}[P_t(g), P_t(g')] = \\
&\exp[-(l+k)r_t + l\sigma_o^2][\frac{1}{\mu+1} E_{\Pi_t}(S_\ell - L)^+(S_k - K)^+ + \frac{\mu}{\mu+1} E_{\Pi_t}(S_\ell - L)^+ E_{\Pi_t}(S_k - K)^+] \\
&-\exp[-(l+k)r_t] E_{\Pi_t}(S_\ell - L)^+ E_{\Pi_t}(S_k - K)^+ \\
&= \frac{1}{\mu+1} \exp[-(l+k)r_t + l\sigma_o^2] E_{\Pi_t}(S_\ell - L)^+(S_k - K)^+ \\
&\qquad + p_{BS}(F_t, r_t, l, L, \sigma^2) p_{BS}(F_t, r_t, k, K, \sigma^2)[\frac{\mu}{\mu+1} \exp(l\sigma_o^2) - 1]. \qquad (6.10)
\end{aligned}$$

If we look at the special where $l = k = H$, and where the Log-Normal-Dirichlet is calibrated in order to get the one horizon gamma model of section 5, i.e. if we choose $\sigma_o^2 H = \log(1+\frac{1}{\mu})$ (see 6.9), $\mu = \lambda \exp(-r_t H)$, the previous covariance becomes :

$$\begin{aligned}
& \frac{1}{\mu+1} \exp(-2Hr_t) \frac{\mu+1}{\mu} E_{\Pi_t}(S_H - L)^+(S_H - K)^+ \\
+ \;& p_{BS}(F_t, r_t, H, L, \sigma^2) p_{BS}(F_t, r_t, H, K, \sigma^2)[\frac{\mu}{\mu+1} \frac{\mu+1}{\mu} - 1] \\
= \;& \frac{\exp(-r_t H)}{\lambda} E_{\Pi_t}(S_H - L)^+(S_H - K)^+,
\end{aligned}$$

which is the formula of proposition 5.2.

In the general formula (6.10) we have to compute the term $E_{\Pi_t}(S_\ell - L)^+(S_k - K)^+$. If $l = k$, this quantity has a closed form given in (5.2). Otherwise if, for instance $l < k$ we have :

$$\begin{aligned}
& E_{\Pi_t}\left\{(S_\ell - L)^+ E_{\Pi_t}\left[(S_k - K)^+ / S_\ell\right]\right\} \\
= \;& \exp\left[(k-l)r_t\right] E_{\Pi_t}\left[(S_\ell - L)^+ p_{BS}(S_\ell, r_t, k-l, K, \sigma^2)\right].
\end{aligned}$$

Finally :

$$
\begin{aligned}
&\operatorname{Cov}[P_t(g), P_t(g')] = \omega_{BS}(F_t, r_t, l, L, k, K, \sigma^2, \sigma_o^2, \mu) \quad \text{(say)} \\
&= \frac{1}{\mu+1} \exp(-2lr_t + l\sigma_o^2) E_{\Pi_t}[(S_\ell - L)^+ p_{BS}(S_\ell, r_t, k-l, K, \sigma^2)] \\
&\quad + p_{BS}(F_t, r_t, l, L, \sigma^2) p_{BS}(F_t, r_t, k, K, \sigma^2)[\frac{\mu}{\mu+1} \exp(l\sigma_o^2) - 1] \qquad (6.11)
\end{aligned}
$$

In the expectation the relevant probability is :

$$
LN\left[\left(1 - \frac{\sigma^2}{2}\right) l + \log F_t, \sigma^2 l\right].
$$

Therefore the covariances can be easily evaluated by simulations.

7. Concluding remark

In this paper we have tried to reconcile the standard pricing theories based on the complete market assumption and the econometric implementation, which requires enough randomness to derive non degenerate distributions for the estimators. The proposed solution is to suppose asymmetric information for the market participants and the econometrician. It leads to a pricing formula with a stochastic pricing measure. We have also explained how to locate the stochastic risk neutral probability measures around some standard ones, such as the Black-Scholes probability measure.

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Appendix 1
Characteristic function of the derivative prices

We have :

$$\begin{aligned}
\psi_{t,J}(v) &= \log E_t\{\exp i \sum_{j=1}^{J} v_j P_t(g_j)\} \\
&= \log E_t\{\exp i \sum_{j=1}^{J} v_j \int_o^\infty g_j(s) dQ_t(s)\} \\
&= \log E_t\{\exp i \textstyle\int_o^\infty (\sum_{j=1}^{J} v_j g_j(s)) dQ_t(s)\} \\
&= \textstyle\int_o^\infty d\varphi_t(s, i \sum_{j=1}^{J} v_j g_j(s)),
\end{aligned}$$

where $d\varphi_t(s,v)$ is the second characteristic function of $dQ_t(s)$, and since the gamma measure has independent increments.

The result follows :

$$\psi_{t,J}(v) = \int_o^\infty \log \left[1 - \frac{i}{\lambda_t} \sum_{j=1}^{J} v_j g_j(s) \right] d\nu_t(s).$$

Appendix 2
Second order moments of derivative prices

We get for : $L \geq K$:

$$
\begin{aligned}
&\mathrm{Cov}_t[P_t(H,K), P_t(H,L)] \\
&= \frac{\exp(-r_t H)}{\lambda} \underset{\pi_t^{BS}}{E} \left[(S_{t+H}-K)^+ (S_{t+H}-L)^+ / F_t\right] \\
&= \frac{\exp(-r_t H)}{\lambda} \underset{\pi_t^{BS}}{E} \left[(S_{t+H}-K)(S_{t+H}-L) \mathbb{1}_{S_{t+H}>L} / F_t\right] \\
&= \frac{\exp(-r_t H)}{\lambda} \underset{\pi_t^{BS}}{E} \left[(S_{t+H}-K)(S_{t+H}-L) \mathbb{1}_{\varepsilon > -D_t(L)} / F_t\right] \\
&= \frac{\exp(-r_t H)}{\lambda} \Big\{ KL\, P[\varepsilon > -D_t(L)] \\
&\qquad (K+L) F_t E\left[\exp\left\{(r_t - \frac{\sigma^2}{2})H + \sigma\sqrt{H}\varepsilon\right\} \mathbb{1}_{\varepsilon > -D_t(L)}\right] \\
&\qquad + F_t^2 E\left[\exp\left\{(2r_t - \sigma^2)H + 2\sigma\sqrt{H}\varepsilon\right\} \mathbb{1}_{\varepsilon > -D_t(L)}\right] \Big\}. \\
&= \frac{\exp(-r_t H)}{\lambda} \Big\{ KL\, \Phi[D_t(L)] - (K+L) F_t \exp(r_t H) \Phi[D_t(L) + \sigma\sqrt{H}] \\
&\qquad + F_t^2 \exp(2 r_t H + \sigma^2 H) \Phi[D_t(L) + 2\sigma\sqrt{H}] \Big\}.
\end{aligned}
$$

Appendix 3
Simulation of a gamma process

When the Q measure is a gamma measure $\gamma(\nu(.), \lambda)$, we know that $\Pi(.) = \dfrac{Q(.)}{Q(I\!R)}$ is a Dirichlet process and that Π and $Q(I\!R)$ are independent. We can then simulate a gamma measure by simulating a Dirichlet process, and by multiplying it by an independent drawing in the gamma distribution with parameters $(\nu(I\!R), \lambda)$.

The following result gives a property of Dirichlet processes which can be useful for simulations (see Florens and Rolin (1994)).

Proposition 1 : Let $\Pi(.)$ the Dirichlet process associated with the gamma measure $\gamma(\nu(.), \lambda)$, where ν is a continuous finite measure, then :

$$\Pi(.) = \sum_{i \geq 1} x_i \delta_{y_i}(.) \ a.e.,$$

where :

. δ_{y_i} is the Dirac measure on y_i,
. the sequence (x_i) and (y_i) are independent,
. (y_i) is an iid infinite sample of $\dfrac{\nu(.)}{\nu(I\!R)}$,
. $x_i = v_i \prod_{j=1}^{i-1}(1 - v_j)$, where (v_j) is an infinite iid sample of the Beta distribution with parameters 1 and $\nu(I\!R)$.

This representation of a Dirichlet process permits to simulate it easily by considering the approximation :

$$\Pi(.)^I = \sum_{i=1}^{I} x_i \delta_{y_i}(.),$$

or

$$\Pi(.)^I = \frac{1}{\sum_{i=1}^{I} x_i} \sum_{i=1}^{I} x_i \delta_{y_i}(.).$$

In both cases, we have :

$$\lim_{I \to +\infty} \mathbb{E} \sup_{B \in \mathcal{B}(\mathbb{R})} |\Pi(B) - \Pi^I(B)| = 0,$$

$$\lim_{I \to +\infty} \mathrm{Var} \ \sup_{B \in \mathcal{B}(\mathbb{R})} |\Pi(B) - \Pi^I(B)| = 0.$$