MULTIPLE STEADY STATES AND ENDOGENOUS FLUCTUATIONS WITH INCREASING RETURNS TO SCALE IN PRODUCTION[1]

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Multiple Steady States and Endogenous Fluctuations with Increasing Returns to Scale in Production

ABSTRACT. The purpose of this paper is to study the influence of the elasticity of capital-labor substitution and of the labor supply elasticity on the existence of multiple Pareto-ranked stationary equilibria, local indeterminacy, bifurcations and expectations driven fluctuations in economies with external and internal increasing returns to scale in production. This is done through general geometrical methods that do not depend upon particular specifications for preferences and technology. For the sake of concreteness, the analysis focuses on the model with heterogeneous agents and financial constraints presented in Woodford [47], and opportunely amended by Grandmont, Pintus, and de Vilder [22] to account for capital-labor substitution. Increasing returns to scale are incorporated in two different market structures:
- an economy with aggregate production externalities and perfectly competitive markets,
- an economy with increasing returns internal to the firm and imperfect competition in the product(s) market(s) only.

JEL classification numbers: C61, E32.

Keywords: capital-labor substitution, increasing returns to scale, local indeterminacy, endogenous fluctuations, local bifurcations.

Etats Stationnaires Multiples et Fluctuations Endogènes avec Rendements d'Echelle de Production Croissants

RÉSUMÉ. Le présent article étudie l'influence des élasticités de substitution capital-travail et d'offre de travail sur l'existence d'états stationnaires multiples et Pareto-ordonnés ainsi que sur l'indétermination et les bifurcations locales rendant possibles des fluctuations conduites par les anticipations, lorsque les rendements d'échelle de production sont croissants. L'analyse proposée, reposant sur une approche géométrique qui ne dépend pas de spécifications particulières des préférences et des techniques, est appliquée au modèle à agents hétérogènes de Woodford [47] dans lequel la substitution capital-travail a été introduite, suivant Grandmont, Pintus et de Vilder [22].
Les rendements d'échelle croissants sont alternativement associés à deux types de structures de marché:
- une économie parfaitement concurrentielle bénéficiant d'effets externes positifs,
- une économie de concurrence imparfaite sur le marché du bien produit par des entreprises opérant à coût marginal décroissant.

Classification du JEL: C61, E32.

Mots-clé: substitution capital-travail, rendements d'échelle croissants, indétermination locale, fluctuations endogènes, bifurcations locales.

1 Introduction

In this paper, we study the influence of the elasticity of capital-labor substitution and of the labor supply elasticity with respect to real wage on the existence of multiple Pareto-ranked stationary equilibria, local indeterminacy, bifurcations and expectations driven cycles in economies with internal and external increasing returns to scale in production. Many recent contributions which show the possibility of indeterminacy and endogenous fluctuations through the introduction of either external increasing returns (see, among the others, Benhabib and Farmer [5], Boldrin [9], Cazzavillan [12], Farmer and Guo [17]) or internal increasing returns and imperfect competition (see, among the others, Benhabib and Farmer [5], Farmer and Guo [17], Kiyotaki [31] for models with capital-labor substitution; d'Aspremont, Dos Santos Ferreira, and Gérard-Varet [2], Rivard [37] for models without productive capital) rely on particular specifications for preferences and technology. In particular, the extensive and almost exclusive use of the unit factor substitution elasticity case does not provide enough information to evaluate the sensitivity of those findings when the class of technologies is more general (see Rotemberg and Woodford [38]). This may appear a limit once it is recognized that the reported estimations of the elasticity of factor substitution are not conclusive as they actually fall within a quite large range of values.[1]

In contradistinction with the existing literature, we adopt a simple geometric method, adapted from Grandmont, Pintus, and de Vilder [22], which allows to fully characterize the behavior of the orbits around a steady state as a function of the elasticities of capital-labor substitution and of labor supply, without appealing to particular specifications for both preferences and technology. When increasing returns are small, we find that local indeterminacy, periodic or quasi-periodic endogenous fluctuations and stochastic equilibria occur when factors are highly complementary, and that local determinacy is bound to prevail when the elasticity of capital-labor substitution is slightly larger, as in models based on constant returns (see Benhabib and Laroque [7], Grandmont [19, 21], Reichlin [36], de Vilder [44, 45], Grandmont, Pintus, and de Vilder [22], Woodford [47]). More surprising, however, is the fact that when the elasticity of input substitution is increased further and made arbitrarily large, local indeterminacy and endogenous fluctuations reappear (in agreement with the unit elasticity examples reviewed above). Moreover, the higher the elasticity of factor substitution, the lower the labor supply elasticity required to get endogenous cycles.

We also provide a thorough analysis of global uniqueness versus multiplicity of stationary equilibria that does not depend on specific production and utility functions (by contrast with d'Aspremont, Dos Santos Ferreira, and Gérard-Varet [2], Benhabib and Farmer [5], Benhabib and Rustichini [8], Boldrin [9], Cazzavillan [12], Kiyotaki [31], Rivard [37], Rotemberg and Woodford [39, 40]. See Silvestre [42, 43] for an overview). It is shown that the steady state is bound to be unique when

[1]See, for instance, Hamermesh [26, Chapter 3] for a survey of direct and indirect estimates, Epstein and Denny [16] for estimates arising from a labor and capital dynamic adjustement model.

increasing returns are sufficiently high, while multiple Pareto-ranked stationary equilibria typically occur otherwise.

The main framework we investigate is the model with heterogeneous agents and financial constraints studied by Woodford [47] and, more recently, by Grandmont, Pintus, and de Vilder [22] to account for capital-labor substitution, where the length of the period can be interpreted as short. We introduce increasing returns in two different market structures:

- an economy with aggregate production externalities and perfectly competitive markets,
- an economy with increasing returns internal to the firm and imperfect competition in the product market only.[2]

The way we introduce productive externalities is close to that proposed in Baxter and King [4], Benhabib and Farmer [5], Farmer and Guo [17], and King, Plosser, and Rebelo [30]. According to this view, the private total factor productivity is positively affected by an increase in the aggregate capital and labor stocks. At least two informal arguments exist to legitimate such a formulation. The first one accounts for the contribution of aggregate capital to the externalities and is now standard: positive externalities come from learning spillovers created through the production activities (see Arrow[1]). The accumulation of productive capital increases public knowledge and hence factor productivities. The same sort of argument may be used to justify part of the positive externality coming from the aggregate labor stock through learning by doing, associated to the level of aggregate production rather than the aggregate capital stock. A second justification for labor externalities is related to the exchange process in a market economy, and in particular to the *thick market* externalities modeled in Diamond [14]. As in Murphy and alii [33] or Howitt and Mac Afee [28], we can interpret the externalities as the result of the following mechanism, taking place in the labor market.[3] The higher the level of employment, the higher the probability of a fast matching between sellers and buyers in the labor market, in which case the search costs decrease or equivalently the total factor productivity increases.

When dealing with short run fluctuations, thick market externalities may appear more relevant than those originated through learning by doing, so we should expect externalities coming from labor to be more important than capital externalities. This is precisely the kind of condition we shall use to generate indeterminacy and endogenous fluctuations for values of the elasticity of capital-labor substitution which belong to the reported range of estimated values, when labor supply is sufficiently elastic.

The rest of the paper is organized as follows. The general framework and the characterization of perfect foresight competitive equilibria with aggregate productive externalities are presented in the next

[2] In Cazzavillan and Pintus [13], the dynamics arising in the OLG economy with current consumption, endogenous labor supply and increasing returns are analyzed by applying the very same geometric approach.

[3] The argument is easily transposed to the output market and it actually constitutes the fundation of the search approach in which multiple equilibria and endogenous fluctuations have been shown to occur by those authors.

section, whereas Section 3 focuses on the steady states analysis. The local dynamics of the first model are analyzed in Section 4, while Section 5 reports the results obtained from the simulations performed on the class of the CES economies with externalities. The economy with an imperfectly competitive product market is introduced and studied in Section 6. The main conclusions are summarized in Section 7.

2 The Infinite Horizon Agents Model with Productive Externalities

In this section we present the benchmark model, following the lines set out in Woodford [47], Grandmont, Pintus, and de Vilder [22] and Pintus, Sands, and de Vilder [35]. The economy consists of two classes of agents who trade competitively, consume and have perfect foresight during their infinite lifetime.

On one side, identical agents called *workers* consume and work during each period. They supply a variable quantity of labor hours and may save a fraction of their income by holding two assets: productive capital and nominal outside money. Rental returns on capital and wage are paid by the firms at the end of the period while workers buy the unique consumption good at the outset of it. In addition, a labor market imperfection, due to incomplete or imperfect information, prevents workers from borrowing against their human capital, while productive capital is accepted as collateral to secure a loan. Therefore a financial constraint is imposed on workers. Indeed, they can't consume out of their money balances held at outset of the period as well as the returns earned on productive capital.

On the other side, identical *capitalists* run the production process, consume and save an income composed of money balances and returns on capital. Here an important assumption is made: capitalists discount future utility less than workers. In other words, the capitalists' discount factor is larger than that of workers. Therefore, at the nonautarkic steady states and nearby, capitalists end up holding the whole capital stock. The resulting nonautarkic steady states are characterized by the modified golden rule (with constant population): the stationary real rental rate on productive capital is equal to the capitalists' discount rate plus the constant depreciation rate for capital. Therefore, at these steady states and nearby, the real return on capital is positive and larger than that of money balances which is close to zero, as long as there is positive discounting and/or depreciation: capitalists choose not to hold outside money.
To summarize, the steady state (and nearby) savings structure is the following: capitalists own the whole capital stock and workers hold the entire nominal money stock, assumed to be constant over time. Finally, the financial constraint faced by workers becomes a liquidity constraint which is obvi-

ously binding at the steady states and nearby. In that framework, Woodford [47] showed that although workers have an infinite lifetime, they behave like a two period living agent: they choose optimally their labor supply for today and consequently their next period consumption demand. Equivalently, workers know the current nominal wage and the next period price for the consumption good (along an intertemporal equilibrium with perfect foresight) and choose the (unique under assumptions) optimal bundle labor today, consumption tomorrow on their offer curve. Therefore, the liquidity constraint allows one to interpret the length of the period as a quarter or a year, and eventual endogenous fluctuations occur at business cycle frequency.

The sequel of this section gives the equations which describe the economy.

2.1 The Technology with Externalities

In each period $t \in \mathbf{N}$, a perishable consumption good $y_t \geq 0$ is produced combining labor $l_t \geq 0$ and the capital stock $k_{t-1} \geq 0$ resulting from the previous period. The gross production function F exhibits constant returns to scale and allows for input substitution. We assume that each entrepreneur-capitalist benefits from positive productive externalities: while he rents l and k, the input services of labor and capital are respectively $A(K,L)l$ and $A(K,L)k$, where K and L are respectively the average capital stock and the average number of worked hours of the economy for the current period.[4] Moreover, when the average capital or labor stock goes up, the productivity of each input moves up: $A(K,L)$ is increasing in K and increasing in L, while the case where A is constant is obviously equivalent to the absence of positive externalities.

Because we assume an external effect compatible with perfect competition in all markets, each (small enough) identical entrepreneur takes K and L as given when optimizing. Of course, $K = k$ and $L = l$ in equilibrium. Therefore, the quantity of good produced is

$$y = F(A(k,l)k, A(k,l)l) = A(k,l)F(k,l) = A(k,l)lf(a). \tag{1}$$

The second and third equalities are deduced from the constant returns to scale assumption, and the latter equality defines the standard production function in intensive form defined upon the capital labor ratio $a = k/l$. On technology, we shall assume the following.

Assumption 2.1

The intensive production function $f(a)$ is continuous for $a \geq 0$, C^r for $a > 0$ and r large enough, with $f'(a) > 0$ and $f''(a) < 0$. Therefore, $\rho(a) \stackrel{\text{def}}{=} f'(a)$ is decreasing in a, while $\omega(a) \stackrel{\text{def}}{=} f(a) - af'(a)$ is increasing in a.

The total factor productivity function $A(k,l)$ is continuous on $\mathbf{R}^2_+$, C^r on $\mathbf{R}^2_{++}$ for r large enough,

[4] For notational convenience, we shall assume that the number of capitalists is equal to that of workers.

homogeneous of degree ν and can be written, therefore, as $A(k,l) = Al^{\nu}\psi(a)$, where the scaling factor $A > 0$. Moreover, $A(k,l)$ is increasing in k, i.e. the contribution of capital to the externalities is $\varepsilon_{\psi}(a) > 0$, and increasing in l, i.e. the contribution of labor to the externalities is $\nu - \varepsilon_{\psi}(a) > 0$, where $\varepsilon_{\psi}(a) \stackrel{\text{def}}{=} a\psi'(a)/\psi(a)$.

We can interpret the contribution of aggregate capital to externalities, measured by $\varepsilon_{\psi}(a)$, as coming from a learning by doing process, while the contribution of aggregate labor, i.e. $\nu - \varepsilon_{\psi}(a)$, may be resulting from learning by doing and/or thick market externalities. Resulting from these two sources of economies of scale, *social* returns to scale are equal to $1+\nu$ and are increasing, while *private* returns to scale are constant: the assumption of perfectly competitive markets can be maintained.
Firms take real rental prices of capital and labor as given and determine their input demands by equating the private marginal productivity of each input to its real price. Accordingly, the real competitive equilibrium wage is

$$\Omega = A(k,l)\omega(a) = A(k/a)^{\nu}\psi(a)\omega(a) \stackrel{\text{def}}{=} \Omega(a,k), \tag{2}$$

while the real competitive gross return on capital is

$$R = A(k,l)\rho(a) + 1 - \delta = A(k/a)^{\nu}\psi(a)\rho(a) + 1 - \delta \stackrel{\text{def}}{=} R(a,k), \tag{3}$$

where $0 \leq \delta \leq 1$ is the constant depreciation rate. Since there are increasing returns to scale, i.e. $\nu > 0$, the wage and the return on capital not only depend on the capital-labor ratio a but also on the scale of production, as expressed by the term k^{ν} in eqs. (2) and (3).

2.2 The Agents and the Intertemporal Equilibria

To complete the description of the model, we must characterize the behavior of both classes of agents.[5]
A representative worker solves the following utility optimization problem:

$$\text{maximize } \{V_2(c^w_{t+1}/B) - V_1(l_t)\} \text{ such that } p_{t+1}c^w_{t+1} = w_t l_t,\ c^w_{t+1} \geq 0,\ l_t \geq 0, \tag{4}$$

where $B > 0$ is a scaling factor, c^w_{t+1} is the next period consumption, l_t is the labor supply, $p_{t+1} > 0$ is the next period price of consumption, assumed to be perfectly foreseen, and $w_t > 0$ is the nominal wage rate, i.e. $w_t = p_t\Omega_t$. We only consider the case where leisure and consumption are gross substitutes and assume therefore the following.

Assumption 2.2

[5]See Woodford [47], Grandmont, Pintus, and de Vilder [22] or Pintus, Sands, and de Vilder [35] for more details.

The utility functions $V_1(l)$ and $V_2(c)$ are continuous for $0 \leq l \leq l^*$ and $c \geq 0$, where $l^* > 0$ is the (maybe infinite) workers' endowment of labor. They are C^r for, respectively, $0 < l < l^*$ and $c > 0$, and r large enough, with $V_1'(l) > 0$, $V_1''(l) > 0$, $\lim_{l \to l^*} V_1'(l) = +\infty$, and $V_2'(c) > 0$, $V_2''(c) < 0$. Moreover, consumption and leisure are gross substitutes, i.e. $-cV_2''(c) < V_2'(c)$.

The first order condition of the above program (4) gives the optimal labor supply $l_t > 0$ and consequently the next period consumption $c_{t+1}^w > 0$, as shown below in eq. (5):

$$v_1(l_t) = v_2(c_{t+1}^w) \text{ and } p_{t+1}c_{t+1}^w = w_t l_t, \tag{5}$$

where $v_1(l) \stackrel{\text{def}}{=} lV_1'(l)$ and $v_2(c^w) \stackrel{\text{def}}{=} cV_2'(c/B)/B$. Assumption 2.2 implies that v_1 and v_2 are increasing while v_1 is onto $\mathbf{R}_+$. Therefore, Assumption 2.2 allows one to define from eq. (5) the function $\gamma \stackrel{\text{def}}{=} v_2^{-1} \circ v_1$, whose graph is the offer curve, i.e. $c_{t+1}^w = \gamma(l_t)$. Moreover, γ is defined on a nonempty open interval, and leisure and consumption are gross substitutes, i.e. $\varepsilon_\gamma(l) > 1$. Equivalently, the labor supply is an increasing function of the real wage, whose elasticity is $\varepsilon_l(\cdot) \stackrel{\text{def}}{=} 1/(\varepsilon_\gamma(\cdot) - 1)$ at the steady state.[6]

Capitalists maximize the discounted sum of utilities derived from each period consumption. They consume $c_t^c \geq 0$ and save $k_t \geq 0$ from their income, i.e. the real gross return on capital. To fix ideas, we assume, following Woodford [47], that the capitalists' instantaneous utility function is logarithmic. Their optimal choices are

$$c_t^c = (1-\beta)R_t k_{t-1}, \; k_t = \beta R_t k_{t-1}, \tag{6}$$

where $0 < \beta < 1$ is the capitalists' discount factor and $R_t > 0$ is the real gross rate of return on capital.

As usual, equilibrium on capital and labor markets is ensured through eqs. (2) and (3). Since workers save the wage income in the form of money, the equilibrium money market condition is

$$\Omega(a_t, k_{t-1})l_t = M/p_t, \tag{7}$$

where $M > 0$ is the constant money supply and $p_t > 0$ is the current nominal price of consumption. Finally, Walras' law accounts for the equilibrium in the good market, i.e. $y_t = c_t^w + c_t^c + k_t - (1-\delta)k_{t-1}$. From the equilibrium conditions in eqs. (2), (3), (5), (6), (7), one easily deduces that the variables c_{t+1}^w, c_t^w, l_t, p_{t+1}, p_t, c_t^c and k_t are known once (a_t, k_{t-1}) are given. This implies that intertemporal equilibria are actually summarized by the dynamic behavior of the two variables a and k.

Definition 2.1

An intertemporal competitive equilibrium with perfect foresight is a sequence (k_{t-1}, a_t) of $\mathbf{R}_{++}^2$, $t = 0, 1, \ldots$, such that

$$\left\{ \begin{array}{rcl} \Omega(a_{t+1}, k_t)/a_{t+1} & = & \gamma(k_{t-1}/a_t)/k_t, \\ k_t & = & \beta R(a_t, k_{t-1})k_{t-1}. \end{array} \right. \tag{8}$$

[6] See Grandmont, Pintus, and de Vilder [22, Section 2] for more details.

3 Steady States Analysis: Uniqueness and Multiplicity

The first step in analyzing the dynamical system (implicitly) given by eqs. (8) consists in finding its stationary points. Existence is easily established by scaling the two parameters A and B, i.e. by choosing the units of the two goods.
To study globally uniqueness and multiplicity, we shall first reduce the dimension of the problem to one. The general result states that a unique steady state is bound to prevail whenever externalities are uniformly small or uniformly large, while in the complementary case, multiple Pareto-ranked stationary equilibria may occur and may be arbitrarily close. The analysis is then applied to the standard parametric family of the CES economies. It turns out, in agreement with the general result, that the steady state is bound to be unique when externalities are sufficiently large. If externalities are very low, however, two Pareto-ranked steady states are shown to coexist generically, maybe in a very small neighborhood. In particular, the result of uniqueness obtained in the case of the Cobb-Douglas technology, for which the elasticity of capital-labor substitution equals unity, is actually a very particular configuration and indeed not a stable property: if externalities are low enough, uniqueness of the steady state does not persist after any arbitrary small perturbation, i.e. if the elasticity is as close as we want to one. On the contrary, two stationary equilibria coexist generically when the elasticity of factor substitution is different from but maybe infinitely close to one.

In view of eq. (8) and of the definition $a = k/l$, the nonautarkic steady states are the solutions $(\bar{a}, \bar{l})$ in $\mathbf{R}^2_{++}$ of $\Omega(\bar{a}, \bar{a}\bar{l}) = \gamma(\bar{l})/\bar{l}$ and $\beta R(\bar{a}, \bar{a}\bar{l}) = 1$. Equivalently, in view of eqs. (2) and (3), the steady states are given by

$$\left\{ \begin{array}{lcl} A\bar{l}^{1+\nu}\psi(\bar{a})\omega(\bar{a}) & = & \gamma(\bar{l}), \\ A\bar{l}^{\nu}\psi(\bar{a})\rho(\bar{a}) + 1 - \delta & = & 1/\beta. \end{array} \right. \tag{9}$$

We shall solve the existence issue by choosing opportunely the scaling parameters A and B, so as to ensure that one stationary solution coincides with, for instance, $(\bar{a}, \bar{l}) = (1, 1)$. This will occur if and only if $A\psi(1)\rho(1) = 1/\beta - 1 + \delta$ and $A\psi(1)\omega(1) = \gamma(1)$, or equivalently $v_2(A\psi(1)\omega(1)) = v_1(1)$. The first equality is achieved by scaling the parameter A, while the second is achieved by scaling the parameter B. Indeed, from Assumption 2.2, v_2 is decreasing in B, and the latter equation will be satisfied for some (then unique) $B > 0$ if and only if $\lim_{c\to 0} v_2(c) < v_1(1) < \lim_{c\to+\infty} v_2(c)$, or equivalently $\lim_{c\to 0} cV_2'(c) < V_1'(1) < \lim_{c\to+\infty} cV_2'(c)$.

Proposition 3.1 (Existence of the Steady State)

Under Assumptions 2.1, 2.2, and $\lim_{c\to 0} cV_2'(c) < V_1'(1) < \lim_{c\to+\infty} cV_2'(c)$, $(\bar{a}, \bar{k}) = (1, 1)$ *is a steady state of the dynamical system in eqs. (8) if and only if* $A = (1/\beta - 1 + \delta)/(\psi(1)\rho(1))$ *and* B *is the unique solution of* $A\psi(1)\omega(1)V_2'(A\psi(1)\omega(1)/B)/B = V_1'(1)$.

3.1 Uniqueness versus Multiplicity

We now study uniqueness and multiplicity of the (interior) steady states. A convenient way to deal with this issue is the following. Suppose that one can write the two-dimensional system (9) in the form $l = l_1(a)$, $l = l_2(a)$, where the two functions $l_1(a)$, $l_2(a)$ are defined and positive on the two open intervals I_1 and I_2, respectively. Finding a steady state is then equivalent to solving the much simpler one-dimensional equation $l_1(\overline{a})/l_2(\overline{a}) = 1$, where $l_1(a)/l_2(a)$ is defined on the intersection I of I_1 and I_2. At this stage, the interval I may be empty, but we know that the assumptions of Proposition 3.1 ensure that it contains $\overline{a} = 1$ (and moreover that $l_1(1) = l_2(1)$). We shall focus on two configurations, essentially because they are those which arise in the specification analyzed in Subsection 3.2, where the production and the utility functions exhibit constant elasticity.

First of all, it is clear that if $l_1(a)/l_2(a)$ is monotone (either increasing or decreasing everywhere), there exists at most one steady state (hence a unique one under the assumptions of Proposition 3.1). On the other hand, if $l_1(a)/l_2(a)$ is either single-peaked (see Fig. 1) or single-caved (see Fig. 2), there are at most two steady states. Under the assumptions of Proposition 3.1, there exist then exactely two steady states under appropriate boundary conditions (namely that $l_1(a)/l_2(a) - 1$ has the same sign when a is close to the lower and the upper bounds of the interval I), at least generically, i.e. provided that the derivative of $l_1(a)/l_2(a)$ does not vanish at $\overline{a} = 1$.

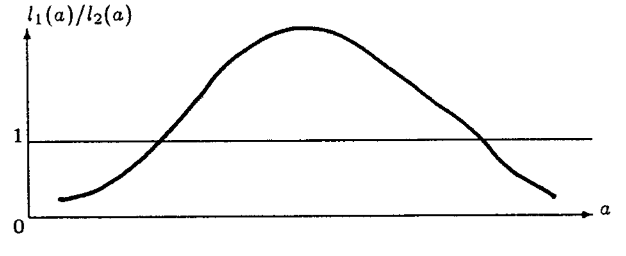


Figure 1: a single-peaked map.

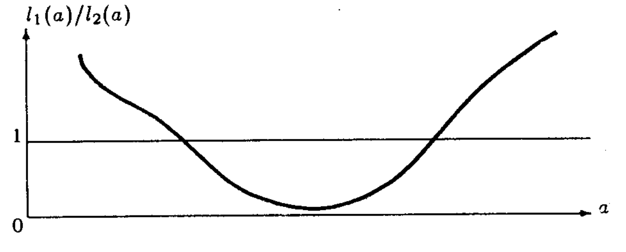


Figure 2: a single-caved map.

The use of this method requires that we put eqs. (9) in the form $l = l_1(a)$, $l = l_2(a)$. The first function $l_1(a)$ is easily obtained by solving for l the second equation of (9) and it is defined for all $a > 0$, i.e. $I_1 = (0, +\infty)$. The same procedure, however, does not work for the first equation since $\gamma(l)/l^{1+\nu}$ may not be monotone. Dividing the first equation of (9) through by the second one, yields $(1/\beta - 1 + \delta)\omega(a)/\rho(a) = \gamma(l)/l$. Since from Assumptions 2.1 and 2.2 both functions $\omega(a)/\rho(a)$ and $\gamma(l)/l$ are increasing, solving that equation for l yields the second function $l_2(a)$ we sought. It is defined, positive, differentiable and increasing, i.e. $l_2'(a) > 0$, on an open interval I_2 (which contains $\bar{a} = 1$, under the assumptions of Proposition 3.1). It follows that the multiple stationary equilibria are ordered: if $(\bar{a}_h, \bar{k}_h)$ and $(\bar{a}_l, \bar{k}_l)$ are solutions of eqs. (9), then $\bar{a}_h > \bar{a}_l$ if and only if $\bar{k}_h > \bar{k}_l$, as $k = al$.

To study the variation of $l_1(a)/l_2(a)$ on the intersection I of I_1 and I_2, it is convenient to look at its elasticity given by $\varepsilon_{l_1}(a) - \varepsilon_{l_2}(a)$, where $\varepsilon_{l_1}(a) = al_1'(a)/l_1(a)$ and $\varepsilon_{l_2}(a) = al_2'(a)/l_2(a)$ are the elasticities of $l_1(a)$ and $l_2(a)$. Indeed, the derivative of $l_1(a)/l_2(a)$ is positive if and only if $\varepsilon_{l_1}(a) - \varepsilon_{l_2}(a) > 0$ or equivalently $\varepsilon_{l_1}(a)/\varepsilon_{l_2}(a) > 1$. Straightforward computations from the two equations defining $l_1(a)$ and $l_2(a)$, i.e. $Al^\nu\psi(a)\rho(a) = 1/\beta - 1 + \delta$ and $(1/\beta - 1 + \delta)\omega(a)/\rho(a) = \gamma(l)/l$ lead to

$$\varepsilon_{l_1}(a) = (|\varepsilon_\rho(a)| - \varepsilon_\psi(a))/\nu, \; \varepsilon_{l_2}(a) = (\varepsilon_\omega(a) + |\varepsilon_\rho(a)|)/(\varepsilon_\gamma(l_s(a)) - 1), \tag{10}$$

It is perhaps a little more illuminating to restate these expressions in terms of the elasticity of factor substitution $\sigma(a)$, and the capital share in total income $0 < s(a) \stackrel{\text{def}}{=} a\rho(a)/f(a) < 1$. By definition, $1/\sigma(a)$ is the elasticity with respect to a of the ratio of the rental prices of labor and capital. In view of eqs. (2) and (3), $1/\sigma(a) = \varepsilon_\omega(a) - \varepsilon_\rho(a)$. Moreover, differentiating the Euler identity $f(a) = \omega(a) + a\rho(a)$ yields $\omega'(a) = -a\rho'(a)$. All this leads to $\varepsilon_\omega(a) = s(a)/\sigma(a)$, $|\varepsilon_\rho(a)| = (1 - s(a))/\sigma(a)$, and we finally get

$$\varepsilon_{l_1}(a) = ((1 - s(a))/\sigma(a) - \varepsilon_\psi(a))/\nu, \; \varepsilon_{l_2}(a) = 1/(\sigma(a)(\varepsilon_\gamma(l_2(a)) - 1)), \tag{11}$$

With this notation, the condition $\varepsilon_{l_1}(a) - \varepsilon_{l_2}(a) > 0$ is equivalent to

$$(1 - s(a) - \sigma(a)\varepsilon_\psi(a))(\varepsilon_\gamma(l_2(a)) - 1) > \nu. \tag{12}$$

One therefore concludes that $l_1(a)/l_2(a)$ is increasing and that there exists at most one steady state if we assume

$$(1 - s(a) - \sigma(a)\varepsilon_\psi(a))(\varepsilon_\gamma(l) - 1) > \nu \text{ for all } a, l > 0. \tag{13}$$

The above condition is satisfied (as expected in view of the results in Grandmont, Pintus, and de Vilder [22, Section 2]) in the limit case where there are no externalities, i.e. when $\nu = 0$ and $\varepsilon_\psi(a) = 0$ for all a.[7] Accordingly, it will tend to be fulfilled when the externalities are uniformly small, i.e. when ν and $\varepsilon_\psi(a)$ are small for all $a > 0$. In the constant elasticity case studied in Subsection 3.2, we shall

[7] We conclude from eqs. (11) that the function $l_1(a)$ is then infinitely elastic and, under Assumption 2.1, that the steady state is unique.

see that, apart from the Cobb-Douglas specification, eq. (13) is not met for all $a > 0$ even when $\varepsilon_\psi(a)$, which is now independant of a, is small: this is due to the fact that the capital share $s(a)$ varies then monotonically between zero and one, implying that the left-hand side of eq. (13) becomes negative from some small or large a's.

We also get that $l_1(a)/l_2(a)$ is decreasing and that there is at most one steady state in the opposite configuration

$$(1 - s(a) - \sigma(a)\varepsilon_\psi(a))(\varepsilon_\gamma(l) - 1) < \nu \text{ for all } a, l > 0. \tag{14}$$

This condition will tend to be satisfied for large externalities, i.e. when ν and $\varepsilon_\psi(a)$ large for all $a > 0$. This will be in particular the case if $\nu > \varepsilon_\gamma(l) - 1$ for all $l > 0$, or if $\sigma(a)\varepsilon_\psi(a) > 1 - s(a)$ for all $a > 0$. Finally we note that when ε_γ, ε_ψ, σ and s are constant (which will occur when both the production and the externalities functions are Cobb-Douglas while the utility functions have constant first derivatives elasticities), one is bound to be either in the case described in eq. (13) or in eq. (14), with the exception of the special case where $(1 - s - \sigma\varepsilon_\psi)(\varepsilon_\gamma - 1) = \nu$ (in the latter configuration $l_1(a)/l_2(a)$ is constant and there is, under the assumptions of Proposition 3.1, a continuum of steady states).

The above arguments can be summarized in the following proposition.

Proposition 3.2 (Uniqueness of the Steady State)

Under Assumptions 2.1 and 2.2, there is at most one steady state, i.e. at most one stationary solution $(\overline{a}, \overline{k})$ in $\mathbf{R}^2_{++}$ of the dynamical system in eqs. (8), if one of the following two conditions is satisfied:
(i) (uniformly small externalities)
$(1 - s(a) - \sigma(a)\varepsilon_\psi(a))(\varepsilon_\gamma(l) - 1) > \nu$ *for all* $a, l > 0$,
(ii) (uniformly large externalities)
$(1 - s(a) - \sigma(a)\varepsilon_\psi(a))(\varepsilon_\gamma(l) - 1) < \nu$ *for all* $a, l > 0$.
The steady state $(\overline{a}, \overline{k}) = (1, 1)$ *is then the unique interior steady state under the assumptions of Proposition 3.1.*
Condition (ii) is in particular satisfied if $\sigma(a)\varepsilon_\psi(a) > 1 - s(a)$ *for all* $a > 0$ *or if* $\nu > \varepsilon_\gamma(l) - 1$ *for all* $l > 0$. *In the special case where* ε_γ, ε_ψ, σ *and* s *are constant, either case (i) or (ii) occurs, except when* $(1 - s - \sigma\varepsilon_\psi)(\varepsilon_\gamma - 1) = \nu$. *In the latter configuration, under the assumptions of Proposition 3.1, there is a continuum of steady states.*

The above result states that one gets a unique interior steady state when externalities are uniformly small or uniformly large. The question to be addressed, now, is how many stationary equilibria one may get when these conditions are violated. We shall see that there are simple assumptions leading to the configurations described in Figs. 1 and 2, and ensuring that there are at most two steady states.

These assumptions arise naturally in the constant elasticity specifications considered in Subsection 3.2.

The picture of Fig. 2 emerges when $l_1(a)/l_2(a)$ is single-caved, i.e. when $\varepsilon_{l_1}(a) - \varepsilon_{l_2}(a)$ goes through zero at most once, and from below when it does. In view of eq. (13) and since $l_2(a)$ is increasing, we get this configuration when $1 - s(a) - \sigma(a)\varepsilon_{\psi}(a)$ is increasing in a, and $\varepsilon_{\gamma}(l)$ is nondecreasing in l. We shall see in Subsection 3.2 that this will indeed arise for small externalities when the constant elasticity of input substitution σ is less than one.

The symmetrical picture of Fig. 1, where $l_1(a)/l_2(a)$ is single-peaked, is obtained when $\varepsilon_{l_1}(a) - \varepsilon_{l_2}(a)$ goes through zero at most once, and from above when it does. This case will occur if $1-s(a)-\sigma(a)\varepsilon_{\psi}(a)$ is decreasing in a, and $\varepsilon_{\gamma}(l)$ is nonincreasing in l, and for small externalities when the constant elasticity of input substitution σ exceeds one.

Proposition 3.3 (Uniqueness versus Multiplicity of Steady States)

Let Assumptions 2.1 and 2.2 hold. There are at most two steady states, i.e. at most two stationary solutions $(\overline{a}, \overline{k})$ *in* $\mathbf{R}^2_{++}$ *of the dynamical system in eqs. (8), if one of the following conditions is satisfied:*
(i) $1 - s(a) - \sigma(a)\varepsilon_{\psi}(a)$ *is increasing in* a *and* $\varepsilon_{\gamma}(l)$ *is nondecreasing in* l.
(ii) $1 - s(a) - \sigma(a)\varepsilon_{\psi}(a)$ *is decreasing in* a *and* $\varepsilon_{\gamma}(l)$ *is nonincreasing in* l.

As mentioned at the outset of the section, one obtains then exactely two stationary states under the assumptions of Proposition 3.1, Proposition 3.3 and appropriate boundary conditions, whenever the derivative of $l_1(a)/l_2(a)$ does not vanish at $\overline{a} = 1$. In the alternative case where $\overline{a} = 1$ is a critical point of $l_1(a)/l_2(a)$, it is the unique steady state (see Fig. 3 for an illustration when $l_1(a)/l_2(a)$ is single-caved). Since the intersection between the graph of $l_1(a)/l_2(a)$ and the horizontal line at $\overline{a} = 1$

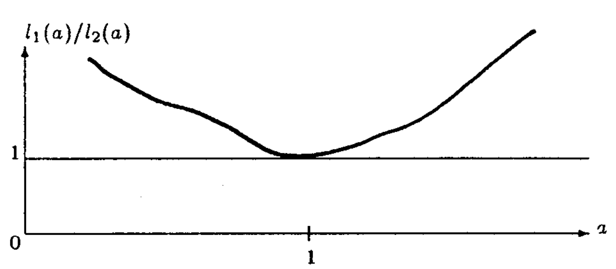


Figure 3: a non-transversal intersection leading to the unique steady state $\overline{a} = 1$.

is not transversal, in Fig. 3, uniqueness of the steady state is not a stable property, i.e. does not persist if we perturb slightly the derivative of $l_1(1)/l_2(1)$. In particular, suppose that starting from the configuration in Fig. 3 we fix the technology, i.e. $\sigma(1)$, $s(1)$ and $\varepsilon_\psi(1)$ are held fixed, while, for instance, we increase slightly $\varepsilon_\gamma(1)$ so that the derivative of $l_1(1)/l_2(1)$ is slightly positive, in view of eq. (12). This operation is licit under the assumptions of Proposition 3.1. There exists consequently a second stationary equilibria in the neighborhood and to the left of $\overline{a} = 1$ (see Fig. 4). In the

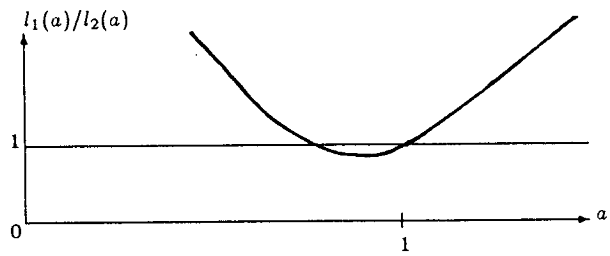


Figure 4: a transversal intersection leading to two steady states $0 < \overline{a} < 1$ and $\overline{a} = 1$.

symmetrical case where we decrease slightly $\varepsilon_\gamma(1)$, starting from the configuration in Fig. 3, another steady state appears in the right neighborhood of $\overline{a} = 1$.

We therefore conclude that if the collection of parameters at $\overline{a} = 1$ is such that the deriative of $l_1(1)/l_2(1)$ is close to zero, or equivalently if $\varepsilon_\gamma(1)$ is close to $\varepsilon_{\gamma S}(1) \stackrel{\text{def}}{=} 1 + \nu/(1 - s(1) - \sigma(1)\varepsilon_\psi(1)) > 1$ in view of eq. (12), the two steady states which occur under the previous assumptions and additional boundary conditions are infinitely close. In Section 4 where the local dynamics around a steady state is studied, this phenemenon we just described geometrically is alternatively viewed as a *transcritical bifurcation* involving an exchange of stability between two stationary equilibria after they have merged, when $\varepsilon_\gamma(1)$ is increased and goes through $\varepsilon_{\gamma S}(1)$.

The above simple condition for arbitrarily close steady states, expressed in terms of economically relevant parameters, may be used to assess the plausibility of multiplicity in a small neighborhood. In particular, the (probably strong) restriction that the labour supply elasticity $\varepsilon_l(1) = 1/(\varepsilon_\gamma(1) - 1)$ is not larger than one at the steady state is equivalent to $\nu + \sigma(1)\varepsilon_\psi(1) \geq 1 - s(1)$.[8] Given that ν should not be too large at the aggregate, and maybe lower than the steady state labor share in income $1 - s(1)$ according to the estimates of Baxter and King [4], Caballero and Lyons [11] and Hall [25], the external effect due to the capital stock $\varepsilon_\psi(1)$ is then required to be significant.[9] For instance, if we set ν at 0.4, $s(1)$ at 0.4 and $\sigma(1)$ at 0.5, 1, or 2 (in agreement with the estimates reported in

[8] In view of the reported estimates of the labor supply elasticity, see for instance Killingsworth and Heckman [29] and Pencavel [34], the range of plausible values is quite large: it is restricted to (0,0.5) for men but is more outspread and includes one for women.

[9] Some recent empirical studies suggest, however, that external effects are almost nonexistent while returns to scale are about constant. See, for example, Basu and Fernald [3], Burnside [10].

Hamermesh [26]; see Rotemberg and Woodford [38]), the share of capital in the total externalities should be larger than or equal to 100%, 50%, or 25% respectively.

3.2 A Standard Example: the CES Economies

Naturally, one may wonder if the conditions on preferences and technology in Proposition 3.3 are not "too strong", i.e. if there exists a specification that fits into at least one of the two previous conditions. Indeed, the following mixing of some standard specifications, which will be used in the last section to perform simulations, is covered by the previous propositions. Let f be a CES production function, let A be a Cobb-Douglas externalities function homogeneous of degree ν, and let V_1 and V_2 be two utility functions with constant elasticity of marginal utility.[10] As the elasticity of capital-labor substitution and the intertemporal elasticity of consumption and leisure substitution are both constant, we shall call these economies the CES economies. In this case, $\varepsilon_{l_2}(a)$ is constant while $\varepsilon_{l_1}(a)$ is monotone.

For the standard CES production function, one has

$$f(a) = \begin{cases} (sa^{-\eta} + 1 - s)^{-1/\eta} & \text{if} \quad \eta > -1,\ \eta \neq 0, \\ a^s & \text{if} \quad \eta = 0, \end{cases} \tag{15}$$

where $0 < s < 1$, whereas the elasticity of input substitution is $\sigma = 1/(1+\eta)$. Moreover, we assume

$$A(k,l) = Ak^{\psi}l^{\nu-\psi}, \tag{16}$$

where $A > 0$, $\nu > \psi > 0$. One verifies by direct inspection that the capital share in total income is given here by

$$s(a) = s/(s + (1-s)a^{\eta}). \tag{17}$$

The real wage $\Omega(a,k)$ and the real gross return $R(a,k)$ which appear in eqs. (8) are, in view of eqs. (2) and (3), accordingly

$$\Omega(a,k) = A(k,k/a)(1-s(a))f(a),\ R(a,k) = A(k,k/a)s(a)f(a)/a + 1 - \delta. \tag{18}$$

Moreover, in the case at hand, one obtains $|\varepsilon_\rho(a)| = (1-s(a))/\sigma$ and $\varepsilon_\omega(a) = s(a)/\sigma$.
As for the workers, we shall assume

$$V_1(l) = \frac{l^{1+\alpha_1}}{1+\alpha_1},\ V_2(c/B) = \frac{(c/B)^{1-\alpha_2}}{1-\alpha_2}, \tag{19}$$

where $B > 0$, $\alpha_1 > 0$, $0 < \alpha_2 < 1$. As easily shown, one has $\gamma(l) = v_2^{-1} \circ v_1(l) = Bl^{\gamma}$, with $\gamma = (1+\alpha_1)/(1-\alpha_2) > 1$. The functions V_1 and V_2 satisfy the assumptions of Proposition 3.1, and one can verify that the following choices of the scaling parameters

$$A = (1/\beta - 1 + \delta)/s,\ B = (1/\beta - 1 + \delta)(1-s)/s, \tag{20}$$

[10] i.e. two so-called Constant Relative Risk Aversion utility functions, in the context of uncertainty.

guarantee that $(\overline{a}, \overline{k}) = (1, 1)$ is a steady state, whatever the values of γ and η.

The share of capital in total income $s(a) \stackrel{\text{def}}{=} a\rho(a)/f(a)$ coincides with s if and only if $\overline{a} = 1$, i.e. at the persistent steady state, or $\eta = 0$ and $a > 0$, i.e. at the Cobb-Douglas case $\sigma = 1$. From eqs. (11), we derive

$$\varepsilon_{l_1}(a) = ((1 - s(a))/\sigma - \psi)/\nu, \; \varepsilon_{l_2}(a) = 1/((\gamma - 1)\sigma), \tag{21}$$

Therefore, three cases have to be covered, according to $s(a)$ decreasing, constant or increasing.

$\eta > 0$ ($\sigma < 1$): in that case, $s(a)$ decreases from one to zero (see eq. (17)). If $(1-\sigma\psi)(\gamma-1) < \nu$, the case (ii) of uniformly large externalities in Proposition 3.2 applies, and the steady state $(\overline{a}, \overline{k}) = (1, 1)$ is unique. For smaller externalities, i.e. when $(1 - \sigma\psi)(\gamma - 1) > \nu$, case (i) of Proposition 3.3 is obtained and there are at most two steady states. One can verify by direct inspection that the single-caved function $l_1(a)/l_2(a)$ goes then to $+\infty$ when a tends to zero or to $+\infty$. So, there must be exactely two steady states, provided that the derivative of that function does not vanish at $\overline{a} = 1$, i.e. provided that $(1 - s - \sigma\psi)(\gamma - 1) \neq \nu$.

$\eta = 0$ ($\sigma = 1$): in that case, $s(a)$ is constant and equal to s, and therefore we are either in case (i) or (ii) of Proposition 3.2, and the steady state $(\overline{a}, \overline{k}) = (1, 1)$ is unique. In the case where $(1 - s - \psi)(\gamma - 1) = \nu$, there is a continuum of steady states.

$\eta < 0$ ($\sigma > 1$): in that case, $s(a)$ increases from zero to one (see eq. (17)). If externalities are large, i.e. $(1 - \sigma\psi)(\gamma - 1) < \nu$, the steady state $(\overline{a}, \overline{k}) = (1, 1)$ is unique. For smaller externalities, i.e. $(1 - \sigma\psi)(\gamma - 1) > \nu$, the function $l_1(a)/l_2(a)$ is single-peaked and goes then to zero as a tends to zero or to $+\infty$. One has therefore exactely two steady states, if $(1 - s - \sigma\psi)(\gamma - 1) \neq \nu$.

The next proposition summarizes the previous discussion about the number of steady states in the CES economies.

Proposition 3.4 (Uniqueness and Multiplicity in the CES Economies)

Assume (15), (16), (19) and (20). Then the following cases hold.

1. *$\eta = 0$, i.e. $\sigma = 1$:*
   If $\nu \neq (\gamma - 1)(1 - s - \psi)$, then $(\overline{a}, \overline{k}) = (1, 1)$ is the unique steady state in $\mathbf{R}^2_{++}$ of the dynamical system in eqs. (8).

2. *$\eta \neq 0$, i.e. $\sigma \neq 1$:*
   If externalities are large, i.e. $\nu > (\gamma - 1)(1 - \psi\sigma)$, then $(\overline{a}, \overline{k}) = (1, 1)$ is the unique stationary solution $(\overline{a}, \overline{k})$ in $\mathbf{R}^2_{++}$ of the dynamical system in eqs. (8).

If externalities are small, i.e. $\nu < (\gamma - 1)(1 - \psi\sigma)$*, then, generically, i.e. when* $\nu \neq (\gamma - 1)(1 - s - \psi\sigma)$*, there are two stationary solutions* $(\overline{a}, \overline{k})$ *in* $\mathbf{R}^2_{++}$ *of the dynamical system in eqs. (8).*

If we consider the results stated in Proposition 3.4, it appears that the Cobb-Douglas case $\sigma = 1$ is very peculiar and indeed structurally unstable, not only in this model but, it should be expected, also in most dynamical models with increasing returns: we can find parameters constellations for which uniqueness of the steady state does not persist after any arbitrary small C^∞ perturbation. In particular, in view of the inequality $\nu < (\gamma - 1)(1 - \psi\sigma)$ in Proposition 3.4, i.e. *when there are small externalities*, multiplicity is bound to occur if σ is (arbitrarily) close to (but different from) one, whenever $\sigma < 1/\psi$, and if the labor supply elasticity $1/(\gamma - 1)$ is arbitrarily low. More precisely, for very low values of the externality parameters, the two conditions for multiplicity in Proposition 3.4, part 2, are met for an elasticity of labour supply $1/(\gamma - 1)$ below one (γ larger than 2) and $0 < \sigma < 1.5$ ($\sigma \neq 1$). As shown at the end of Subsection 3.1, however, the two steady states are arbitrarily close if and only if externalities are substantially higher, when a labor supply elasticity lower than one is required. The same conclusion holds here for the CES economies as a particular class: the two stationary equilibria are arbitrarily close if the derivative of $l_1(1)/l_2(1)$ is slightly different from zero, i.e. if γ is sufficiently close to $\gamma_S = 1 + \nu/(1 - s - \sigma\psi) > 1$ in view of eqs. (12) and (21), and the restriction that the labor supply elasticity is not larger than one at $\overline{a} = 1$ is equivalent to $\nu + \sigma\psi \geq 1 - s$. It means that a significant proportion of the total externalities is required to be originated from the capital stock when the steady states are arbitrarily close.

3.3 Welfare Analysis of the Multiple Steady States

Due to the presence of productive externalities, we expect the First Welfare Theorem to fail and some intertemporal equilibria to Pareto-dominate others. Indeed, this is proved for the multiple steady states exhibited in the previous subsections.

Proposition 3.5 (Pareto-Ranked Steady States)

Under the assumptions of Proposition 3.3, let $(\overline{a}_h, \overline{k}_h)$ *and* $(\overline{a}_l, \overline{k}_l)$ *in* $\mathbf{R}^2_{++}$ *be two stationary solutions of eqs. (8). It follows that* $\overline{a}_h > \overline{a}_l$ *if and only if* $\overline{k}_h > \overline{k}_l$*, and therefore that* $(\overline{a}_h, \overline{k}_h)$ *strongly Pareto-dominates* $(\overline{a}_l, \overline{k}_l)$*.*

Proof: the ordering of the two vectors $(\overline{a}_h, \overline{k}_h)$, $(\overline{a}_l, \overline{k}_l)$ follows from the fact that $l_2(a)$ is increasing, in view of Assumptions 2.1 and 2.2. From eqs. (6) and (9), the capitalists' steady state consumption is $c^c = (1 - \beta)k/\beta$. Therefore, capitalists strictly prefer $(\overline{a}_h, \overline{k}_h)$ to $(\overline{a}_l, \overline{k}_l)$. The fact that workers strictly prefer the highest steady state follows from the observation that the contemporaneous workers' utility function is strictly quasi-concave and from the fact that the slope of the offer curve is positive. As a result, workers strictly prefer $(\overline{a}_h, \overline{k}_h)$ to $(\overline{a}_l, \overline{k}_l)$, since $\overline{l}_h = \overline{k}_h/\overline{a}_h > \overline{l}_l = \overline{k}_l/\overline{a}_l$. It follows that both capitalists and workers strictly prefer the highest steady state, i.e. that $(\overline{a}_h, \overline{k}_h)$ strongly Pareto-dominates $(\overline{a}_l, \overline{k}_l)$. □

Even without any information about the stability of each steady state, Proposition 3.5 shows that although agents have perfect foresight, they face a serious coordination problem, at least if cooperative behavior is ruled out. *Even though the steady states are strongly Pareto-ranked*, we cannot predict, at this stage, which stationary equilibrium the economy will end up with. However, the next section will bring out the information about the (local) stability property at each of the fixed points of the map defined by eqs. (8).

4 Local Dynamics and Bifurcation Analysis

We wish to study the dynamics of eq. (8) around one of its interior stationary points. These equations define locally a dynamical system of the form $(a_{t+1}, k_t) = G(a_t, k_{t-1})$ if the derivative of $\Omega(a,k)/a$ with respect to a does not vanish at the steady state, or equivalently if $\varepsilon_\psi(\overline{a}) + \varepsilon_\omega(\overline{a}) - 1 - \nu \neq 0$. Then, the usual procedure to study the local stability of the steady states is to use the linear map associated to the Jacobian matrix of G, evaluated at the fixed point under study. In view of the Hartman-Grobman Theorem[11], this procedure is indeed valid if the Jacobian at the steady state is invertible and has no eigenvalue on the unit circle. From now on, the following notation is used: for example, $\varepsilon_{R,a} \stackrel{\text{def}}{=} (\overline{a}\,\partial R(\overline{a},\overline{k})/\partial a)/R(\overline{a},\overline{k})$ denotes the elasticity of $R(a,k)$ with respect to a evaluated at the steady state $(\overline{a},\overline{k})$ under study. Similarly, $\varepsilon_{R,k}$ is the elasticity of $R(a,k)$ with respect to k at the steady state. Analogous interpretations hold for $\varepsilon_{\Omega,a}$ and $\varepsilon_{\Omega,k}$.

Proposition 4.1 (Linearized Dynamics around a Steady State)

Let $\varepsilon_{R,k}$, $\varepsilon_{R,a}$, $\varepsilon_{\Omega,k}$, $\varepsilon_{\Omega,a}$, ε_γ and ε_ψ be the elasticities of the functions $R(a,k)$, $\Omega(a,k)$, $\gamma(l)$ and $\psi(a)$ evaluated at a steady state $(\overline{a},\overline{k})$ of the dynamical system in eqs. (8). Assume $\varepsilon_{\Omega,a} \neq 1$, i.e. $\varepsilon_\omega + \varepsilon_\psi \neq 1 + \nu$.
The linearized dynamics for the deviations $da = a - \overline{a}$, $dk = k - \overline{k}$ is determined by the linear map

$$\left\{\begin{array}{rcl} da_{t+1} & = & -\frac{\varepsilon_\gamma + \varepsilon_{R,a}(1+\varepsilon_{\Omega,k})}{\varepsilon_{\Omega,a}-1} da_t + \frac{\overline{a}}{\overline{k}} \frac{\varepsilon_\gamma - (1+\varepsilon_{\Omega,k})(1+\varepsilon_{R,k})}{\varepsilon_{\Omega,a}-1} dk_{t-1}, \\ dk_t & = & \frac{\overline{k}}{\overline{a}} \varepsilon_{R,a} da_t + (1+\varepsilon_{R,k}) dk_{t-1}. \end{array}\right. \tag{22}$$

[11] exposed, e.g. , in Grandmont [20, Theorem B.4.1], Guckenheimer and Holmes [23, Theorem 1.4.1].

The associated Jacobian matrix evaluated at the steady state under study has trace T and determinant D, where

$$T = T_1 - \frac{\varepsilon_\gamma - 1}{\varepsilon_{\Omega,a} - 1}, \quad with \quad T_1 = 1 + \frac{|\varepsilon_{R,a}| - 1 - \varepsilon_{R,k} + \varepsilon_{R,k}\varepsilon_{\Omega,a} + \varepsilon_{\Omega,k}|\varepsilon_{R,a}|}{\varepsilon_{\Omega,a} - 1},$$

$$D = \varepsilon_\gamma D_1, \quad with \quad D_1 = \frac{|\varepsilon_{R,a}| - 1 - \varepsilon_{R,k}}{\varepsilon_{\Omega,a} - 1}.$$

Moreover, $T_1 = 1 + D_1 + \Lambda$, *with* $\Lambda = \frac{\varepsilon_{R,k}\varepsilon_{\Omega,a} + \varepsilon_{\Omega,k}|\varepsilon_{R,a}|}{\varepsilon_{\Omega,a}-1}$.

4.1 A Geometrical Method of Local Stability and Bifurcation Analysis

Confronted with the complicated expression of the Jacobian in Proposition 4.1, one may think that tedious computations are necessary to study the local dynamics. Indeed, we shall show that this way of analysis is not unavoidable. As in Grandmont, Pintus, and de Vilder [22], we shall use a geometrical method to get a very simple picture of how local (in)determinacy, local bifurcations and endogenous fluctuations emerge as a function of the fundamental parameters of the system. This procedure aims at studying how the trace and the determinant of the Jacobian, i.e. respectively the sum and the product of the roots of the associated characteristic polynomial $Q(\lambda) \overset{\text{def}}{=} \lambda^2 - \lambda T + D$, vary in the (T, D) plane when one changes continuously the values of some parameters of the model. In particular, if the trace and the determinant are such that the point (T, D) lies in the triangle ABC represented in Fig. 5, we conclude that the eigenvalues have modulus less than one and therefore that the steady state is asymptotically stable, i.e. locally indeterminate in the present context where the capital stock is the single predetermined variable. On the contrary, in the complementary region of the triangle ABC in the (T, D) plane, the stationary state is locally determinate, i.e. either a saddle when $|T| > |1 + D|$, or a source (see Fig. 5).
To simplify matters, we shall assume that a steady state exists and has been set at $(\overline{a}, \overline{k}) = (1, 1)$ through the scaling procedure stated in Proposition 3.1.

A first valuable feature of this method appears if, at the steady state, we fix the technology (i.e. $\varepsilon_{R,k}$, $\varepsilon_{R,a}$, $\varepsilon_{\Omega,k}$, $\varepsilon_{\Omega,a}$) and vary the parameter representing workers' preferences $\varepsilon_\gamma > 1$. In other words, consider the parametrized curve $(T(\varepsilon_\gamma), D(\varepsilon_\gamma))$ when ε_γ describes $(1, +\infty)$. The direct inspection of the expressions of T and D in Proposition 4.1 shows that this locus is a half-line Δ that starts from (T_1, D_1) when ε_γ is close to 1, and whose slope is $1 + \varepsilon_{R,k} - |\varepsilon_{R,a}|$, as shown in Fig. 5. The value of $\Lambda = T_1 - 1 - D_1$, on the other hand, represents the deviation of the generic point (T_1, D_1) from the line (AC) of equation $D = T - 1$, in the (T, D) plane. The examples presented in Fig. 5 illustrate the types of local bifurcations one should (generically) expect when ε_γ varies, the technology being fixed, so that $(T(\varepsilon_\gamma), D(\varepsilon_\gamma))$ describes Δ. Suppose that for some value of ε_γ larger than 1, Δ intersects the line (AC). At this point of intersection, one root of the characteristic polynomial $Q(\lambda) = \lambda^2 - \lambda T + D$

is equal to 1, i.e. $Q(1) = 1 - T + D = 0$ (see Fig. 5). If the analysis is applied to the steady state $(\overline{a}, \overline{k}) = (1,1)$, which is known to persist under the assumptions of Proposition 3.1, one generally expects an exchange of stability between that steady state and another one through a transcritical bifurcation. Moreover, if the half-line Δ intersects the interior of the segment $[BC]$ in Fig. 5, the two eigenvalues of the Jacobian in Proposition 4.1 are complex conjugate and have modulus 1 at the intersection point. In this case, therefore, the family of maps G_{ε_γ}, parametrized by ε_γ, generically undergoes a Hopf bifurcation. When Δ crosses the line (AB), one of the eigenvalues is equal to -1, i.e. $Q(-1) = 1 + T + D = 0$, at the crossing point and one generically expects a flip bifurcation when ε_γ varies (see Fig. 5). For a thorough treatment of local bifurcations, see e.g. Grandmont [20, Section C], Guckenheimer and Holmes [23, Section 3.5], Wiggins [46, Section 3.2].[12]

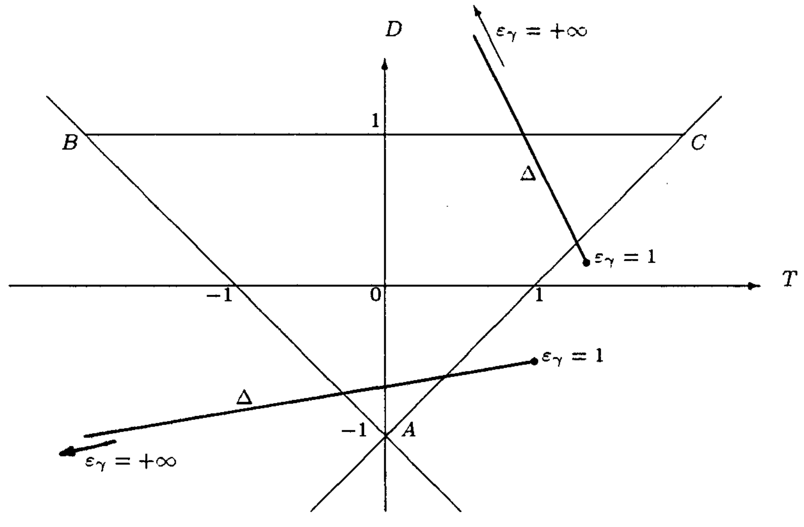


Figure 5: the half-line Δ in the (T, D) plane.

The core of the method we shall use consists of locating the half-line Δ in the plane (T, D), i.e. its origin (T_1, D_1) and its slope $1 + \varepsilon_{R,k} - |\varepsilon_{R,a}|$, as a function of the parameters of the system. The parameters we shall focus on are the depreciation rate for the capital stock $0 \leq \delta \leq 1$, the capitalist's discount factor $0 < \beta < 1$, the share of capital in total income $0 < s = \overline{a}\rho(\overline{a})/f(\overline{a}) < 1$, the externality parameters $\nu > 0$ and $\varepsilon_\psi = \varepsilon_\psi(\overline{a}) > 0$, and the elasticity of input substitution $\sigma = \sigma(\overline{a}) > 0$, all evaluated at the steady state $(\overline{a}, \overline{k})$ under study. We have therefore to relate the elasticities of the functions Ω and R, with respect to a and k, to these parameters. By definition, $1/\sigma(a)$ is the elasticity with respect to a of the ratio of the rental prices of labor and capital. In view of eqs. (2) and (3),

[12]The present approach enables one to verify directly whether the modulus of the bifurcating eigenvalue, i.e. the real eigenvalue in absolute value or the square root of the complex eigenvalues product, $\sqrt{D(\varepsilon_\gamma)}$ here, goes through one at a non-zero speed when the bifurcation parameter is increased. It is clearly the case for the model we are focusing on when ε_γ is taken as the bifurcation parameter.

$1/\sigma(a) = \varepsilon_\omega(a) - \varepsilon_\rho(a)$. Moreover, the derivative of the Euler identity, i.e. the zero profit condition $f(a) = \omega(a) + a\rho(a)$ with respect to a yields $\omega'(a) = -a\rho'(a)$. All this leads to

$$\begin{aligned} \varepsilon_{\Omega,a} &= s/\sigma - \nu + \varepsilon_\psi, & |\varepsilon_{R,a}| &= \theta((1-s)/\sigma + \nu - \varepsilon_\psi), \\ \varepsilon_{\Omega,k} &= \nu, & \varepsilon_{R,k} &= \theta\nu, \end{aligned} \tag{23}$$

where $\theta \stackrel{\text{def}}{=} 1 - \beta(1-\delta) > 0$ and all these expressions are evaluated at the steady state. Therefore, from eq. (23) and Proposition 4.1, we derive

$$\begin{aligned} D_1 &= (\theta(1-s) - \sigma(1+\theta\varepsilon_\psi))/(s - \sigma(1+\nu-\varepsilon_\psi)), & \Lambda &= \theta\nu/(s-\sigma(1+\nu-\varepsilon_\psi)), \\ T_1 &= 1 + D_1 + \Lambda, & slope_\Delta &= 1 + \theta(\varepsilon_\psi - (1-s)/\sigma). \end{aligned} \tag{24}$$

Our aim now is to locate the origin (T_1, D_1) and the slope of the half-line Δ when the capitalists' discount rate β, as well as the technological parameters δ, s, ν and ε_ψ at the steady state are fixed, whereas the elasticity of factor substitution σ is made to vary. In the benchmark case with *no externalities* $\nu = \varepsilon_\psi = 0$, studied by Grandmont, Pintus, and de Vilder [22], the origin (T_1, D_1) of Δ is located on the line (AC), i.e. $\Lambda \equiv 0$ (see Fig. 6). In particular, under appropriate assumptions on the parameters, D_1 is a decreasing function of σ, from some value between zero and one to $-\infty$ as σ is increased from 0 to s, and from $+\infty$ to 1 as σ goes from s to $+\infty$. Moreover, the slope of Δ increases from $-\infty$ to 1 as σ moves from 0 to $+\infty$ (see Fig. 6). The immediate implication of this geometrical representation is that indeterminacy and endogenous fluctuations can emerge only for low values of σ, i.e. for $\sigma < \sigma_I$, and indeed for σ less than 0.2 when the other parameters are calibrated, while, on the contrary, local determinacy is bound to prevail and no fluctuations occur for larger values of σ (see Fig. 6).

We are now going to show that in the case with *positive externalities*, the origin (T_1, D_1) of Δ still describes part of a line Δ_1 which is steeper than (AC) and intersects (AC) at a point I whose ordinate is between zero and one, as shown in Fig. 7. In particular, D_1 decreases from a positive value less than one while the slope of Δ increases from $-\infty$ when σ goes up. Indeterminacy, therefore, is still expected for σ close to zero and indeed disappears even more quickly in comparison with the no externality case. In addition, Δ crosses (AC) when σ is sufficiently low, leading generally to an exchange of stability between two steady states when ε_γ varies. More importantly, however, is the fact that for large σ's, the origin of Δ is such that D_1 is between zero and one (see Fig. 7). Therefore, Δ intersects the triangle ABC where the steady state is asymptotically stable: stochastic equilibria can be constructed, for $\sigma > \sigma_{H2}$ and ε_γ close to one, in any small neighborhood of the locally indeterminate stationary state. Moreover, Δ also crosses the interior of $[BC]$ when σ is large enough and, accordingly, a Hopf bifurcation is generally expected, generating periodic and quasiperiodic dynamics around the steady state, when ε_γ is increased. Therefore, *both indeterminacy and endogenous fluctuations are possible when the elasticity of factor substitution is sufficiently high.*

To prove formally these claims, our first task is to show that the point (T_1, D_1) as a function of σ indeed describes part of a line Δ_1. From the fact that $T_1(\sigma) = 1 + D_1(\sigma) + \Lambda(\sigma)$ and $D_1(\sigma)$ are fractions

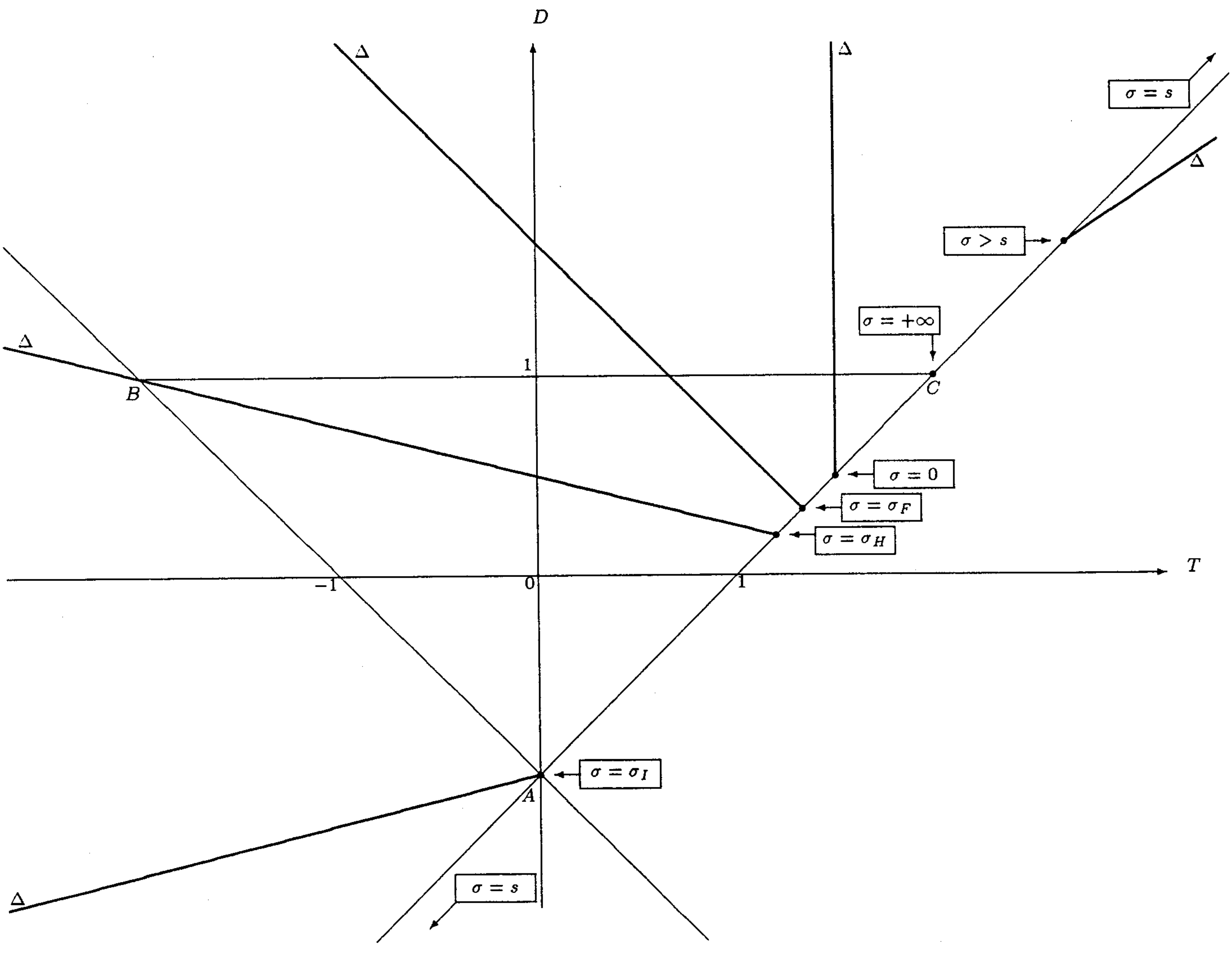


Figure 6: the local dynamics regimes in the (T, D) plane when there are *no externalities*.

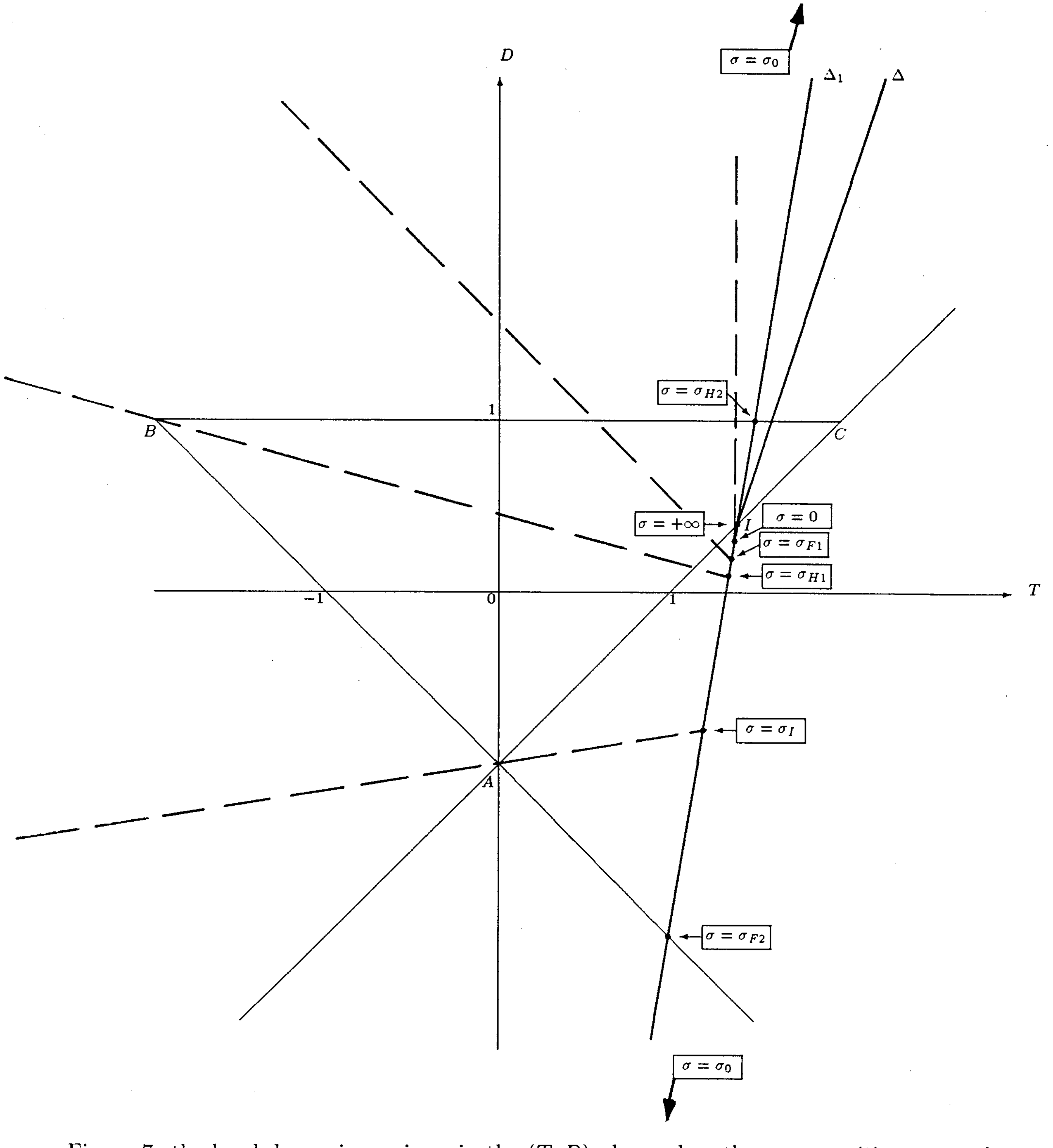


Figure 7: the local dynamics regimes in the (T, D) plane when there are *positive externalities.*

of first degree polynomials in σ with the same denominator (see eq. (24)), we conclude that the ratio of their derivatives $D_1'(\sigma)/T_1'(\sigma)$ or $D_1'(\sigma)/(D_1'(\sigma)+\Lambda'(\sigma))$ is independent of σ.[13] Straightforward computations show that the slope of Δ_1 is

$$slope_{\Delta_1} = \frac{D_1'(\sigma)}{T_1'(\sigma)} = \frac{\theta(1-s)(1+\nu-\varepsilon_\psi)-s(1+\theta\varepsilon_\psi)}{\theta(1-s+\nu)(1+\nu-\varepsilon_\psi)-s(1+\theta\varepsilon_\psi)}. \tag{25}$$

From eq. (24), we conclude that $\Lambda(\sigma)$ vanishes when σ goes to infinity. It follows that Δ_1 intersects the line (AC) at a point I of coordinates $(T_1(+\infty), D_1(+\infty))$, where $D_1(+\infty) = (1+\theta\varepsilon_\psi)/(1+\nu-\varepsilon_\psi) > 0$ (see Fig. 7). We shall focus throughout on the configuration presented in Fig. 7 where $D_1(+\infty) < 1$ and the slope of Δ_1 is greater than 1. We shall ensure the latter condition by imposing, as in the no externality case, that $T_1(\sigma)$ is a decreasing function, i.e. $T_1'(\sigma) = D_1'(\sigma)+\Lambda'(\sigma) < 0$. Indeed, since $\Lambda(\sigma)$ is increasing (see eq. (24)), it follows that $D_1(\sigma)$ is then decreasing and, accordingly, that the slope of Δ_1 is greater than one. From the denominator in eq. (25), $T_1'(\sigma) < 0$ is equivalent to $\theta(1-s+\nu)/s < (1+\theta\varepsilon_\psi)/(1+\nu-\varepsilon_\psi)$. The conditions that $D_1(+\infty) < 1$ and that the slope of Δ_1 is greater than one are therefore equivalent to

$$\frac{\theta(1-s+\nu)}{s} < D_1(+\infty) = \frac{1+\theta\varepsilon_\psi}{1+\nu-\varepsilon_\psi} < 1. \tag{26}$$

The right inequality is equivalent to $\varepsilon_\psi(1+\theta) < \nu$. On the other hand, the expression $D_1(+\infty)$ is an increasing function of ε_ψ. Therefore, to ensure that $D_1(+\infty)$ satisfies the left inequality in eq. (26) for all $0 < \varepsilon_\psi < \nu/(1+\theta)$, it is necessary and sufficient that it satisfies it with a weak inequality sign for $\varepsilon_\psi = 0$, or equivalently that $P(\nu) \stackrel{\text{def}}{=} \theta\nu^2 + \theta(2-s)\nu + \theta(1-s) - s \leq 0$. Under the assumption $\theta(1-s) < s$, this puts an upper bound on the externality parameter ν: it cannot exceed the unique positive zero ν^* of the polynomial $P(\nu)$. From eq. (26), $\theta(1-s+\nu^*) < s$, or $\nu^* < (s-\theta(1-s))/\theta$, by construction. All this indeed proves the following lemma.

Lemma 4.1

Assume $\theta(1-s) < s$ and $\varepsilon_\psi(1+\theta) < \nu$, i.e. $D_1(+\infty) < 1$. Moreover, let $\nu < \nu^* < (s-\theta(1-s))/\theta$, where ν^* is the unique positive root of $\theta\nu^2+\theta(2-s)\nu+\theta(1-s)-s = 0$. Then the slope of the line Δ_1 is greater than one and the functions $T_1(\sigma)$, $D_1(\sigma)$ are decreasing for every $0 < \varepsilon_\psi < \nu/(1+\theta)$.

To obtain the configuration of Fig. 7, it is therefore sufficient to impose the restrictions on the parameters in Lemma 4.1.[14] None of these conditions is very restrictive when $\theta = 1-\beta(1-\delta)$ is small, which is bound to be the case when the period is short since β is then close to one and δ is

[13] The numerator of the derivative of a homographic function $f(x) = (a+bx)/(c+dx)$ is in fact independent of x, since $f'(x) = (bc-ad)/(c+dx)^2$.

[14] The geometrical method can be applied as well to every case out of Lemma 4.1.

close to zero. In that case, $\varepsilon_\psi(1+\theta) < \nu$ will be satisfied if there is an even moderately positive contribution of labor to the externalities, as measured by $\nu - \varepsilon_\psi$, while, on the other hand, the upper bound ν^* is large when θ is small.

From the assumptions of Lemma 4.1, one also gets all the necessary information to appraise the variations of $(T_1(\sigma), D_1(\sigma))$ as well as of the slope of Δ when σ moves from 0 to $+\infty$. In particular, $T_1(0)$ and $D_1(0) = \theta(1-s)/s$ are positive and the corresponding point is below I on the line Δ_1. As σ increases from 0, $T_1(\sigma)$ and $D_1(\sigma)$ are decreasing and tend to $-\infty$ when σ tends to $\sigma_0 = s/(1+\nu-\varepsilon_\psi)$ from below.[15] When σ increases from $\sigma = \sigma_0$ to $+\infty$, $T_1(\sigma)$ and $D_1(\sigma)$ are still both decreasing, from $+\infty$ to $(T_1(+\infty), D_1(+\infty))$, which is represented by the point I in Fig. 7. On the other hand, the intersection of Δ_1 with $[BC]$ is characterized by $D_1(\sigma) = 1$ which leads, in view of eq. (24), to $\sigma_{H2} \stackrel{\text{def}}{=} (s - \theta(1-s))/(\nu - (1+\theta)\varepsilon_\psi)$. In addition, the slope of Δ as a function of σ increases monotonically from $-\infty$ to $1 + \theta\varepsilon_\psi > 1$ as σ moves from 0 to $+\infty$, and vanishes when $D_1(\sigma) = 0$. Moreover, the half-line Δ is above Δ_1 when $\sigma < \sigma_0$, and below it when $\sigma > \sigma_0$.

The only remaining feature to be determined in order to carry out our geometrical analysis is when the slope of Δ is equal to one. From eq. (24), the value of σ for which $slope_\Delta = 1$ is $\sigma_S = (1-s)/\varepsilon_\psi$. That feature is important because for $\sigma < \sigma_S$, the half-line Δ crosses the line (AC) in the (T, D) plane and therefore a transcritical bifurcation generally occurs when ε_γ is made to vary, whereas Δ does not cross (AC) and no transcritical occurs when $\sigma > \sigma_S$. In order to fix ideas, we shall focus on the case where $\sigma_S > \sigma_{H2}$, which can be restated as

Assumption 4.1

The slope of the half-line Δ for $\sigma = \sigma_{H2} \stackrel{\text{def}}{=} (s - \theta(1-s))/(\nu - (1+\theta)\varepsilon_\psi)$ is less than one, i.e. $\varepsilon_\psi < \nu(1-s)$.

Given that the value of s is about 0.4 in industrialized economies, this requires that the relative contribution of capital in the overall externalities, i.e. ε_ψ/ν does not exceed 0.6, or equivalently that the contribution of labor $(\nu - \varepsilon_\psi)/\nu$ is larger than 0.4. Accordingly, this assumption does not appear implausible and, at any rate, its influence on the geometric analysis is only peripheral. We give a few hints in Remark 4.1 on how to modify the analysis when Assumption 4.1 is violated.

All these geometrical features are summarized in Table 1.

The direct inspection of table 1 and of Fig. 7 shows that indeterminacy occurs first for low values of σ: as σ increases from zero, $D_1(\sigma)$ moves down from $0 < D_1(0) < 1$ while the slope of Δ goes up

[15] When $\sigma = \sigma_0$, the function $\Omega(a,k)/a$ of a has a critical point, i.e. its derivative with respect to a vanishes, and the dynamical system derived from eqs. (8) is not defined.

σ	0		$\frac{\theta(1-s)}{1+\theta\varepsilon_\psi}$		$\sigma_0 = \frac{s}{1+\nu-\varepsilon_\psi}$		σ_{H2}	$\sigma_S = \frac{1-s}{\varepsilon_\psi}$	$+\infty$
Slope of Δ	$-\infty$	$\nearrow$	0		$\nearrow$		$\nearrow$	1	$1+\theta\varepsilon_\psi$
Position of Δ			above Δ_1		$\|\|$			below Δ_1	
D_1	$\frac{\theta(1-s)}{s}$	$\searrow$	0	$\searrow$	$-\infty\|\|+\infty$	$\searrow$	1	$\searrow$	$\frac{1+\theta\varepsilon_\psi}{1+\nu-\varepsilon_\psi}$
Λ	$\frac{\theta\nu}{s}$		$\nearrow$		$+\infty\|\|-\infty$			$\nearrow$	0

Table 1: variations and limits of the critical expressions involved in the geometrical analysis, when σ varies.

from $-\infty$. Therefore, the half-line Δ crosses the triangle ABC whenever $\sigma < \sigma_I$, where σ_I is such that Δ goes through the point A (see Fig. 7). In particular, different local bifurcations are expected to occur: a Hopf bifurcation for $\sigma < \sigma_{F1}$, since the half-line Δ crosses in that case the interior of $[BC]$; a Hopf followed by a flip bifurcation when $\sigma_{F1} < \sigma < \sigma_{H1}$, since Δ intersects then both the segment $[BC]$ and the line (AB); finally a flip bifurcation, when $\sigma_{H1} < \sigma < \sigma_I$, since Δ crosses then (AB). The critical values of σ defining these regimes are respectively defined as the value of σ such that the slope of Δ is equal to -1 (for σ_{F1}), and the value for which Δ goes through the point B (for σ_{H1}), in Fig. 7. These configurations arising for low values of σ are described in cases 1, 2 and 3 of Proposition 4.2 below and are similar to those analyzed in the no externality case. The expressions of the critical values σ_{F1}, σ_{H1} and σ_I are given in the proof of that proposition, in Appendix.
In the case of positive externalities, a new regime occurs when $\sigma_I < \sigma < \sigma_{F2}$: the steady is generically either a saddle or a source and undergoes a (reverse) flip bifurcation since Δ intersects (AB) (see Fig. 7 and case 4 of Proposition 4.2). The critical value σ_{F2} is such that the origin of Δ, i.e. $(T_1(\sigma), D_1(\sigma))$ belongs to the line (AB) (see the Appendix).
For higher values of σ, in fact for $\sigma_{F2} < \sigma < \sigma_0$ and for $\sigma_0 < \sigma < \sigma_{H2}$, the steady state is generically either a saddle or a source and there are no endogenous fluctuations in its neighborhood (see Fig. 7). This configuration is reviewed in case 5 of Proposition 4.2.
The only qualitative difference occuring when positive externalities are considered is that the half-line Δ crosses the line (AC) for some value $\varepsilon_\gamma = \varepsilon_{\gamma S}$. This implies that there generically exist two steady states when ε_γ is close enough to $\varepsilon_{\gamma S}$, and that a transcritical bifurcation involving an exchange of stability between the two steady states is generally expected to occur.

Interesting regimes, however, arise when σ is increased further, i.e. when $\sigma > \sigma_{H2}$, since $D_1(\sigma)$ is then positive but less than one: for ε_γ close enough to one, the point (T, D) lies in the triangle ABC (see Figs. 7, 8 and 9). Accordingly, the steady state is locally indeterminate and there exist stochastic equilibria in any of its small neighborhoods. As far as the Hopf bifurcation is concerned, since the slope of Δ is positive when σ is slightly larger than σ_{H2}, Δ is bound to cross the interior of $[BC]$ (see Fig. 8). From Proposition 4.1, this occurs when $D = \varepsilon_\gamma D_1 = 1$, i.e. for $\varepsilon_{\gamma H} \overset{\text{def}}{=} 1/D_1$. Therefore, a Hopf bifurcation is generically expected for a fixed σ slightly larger than σ_{H2}, when ε_γ goes through $\varepsilon_{\gamma H}$. Since the slope of Δ tends to $1 + \theta\varepsilon_\psi > 1$ as σ goes to infinity, the same phenomenon should be observed for large σ's (see Fig. 9). One can show analytically (see the proof of Proposition 4.2

in Appendix) that in fact, the half-line Δ crosses the interior of $[BC]$, and therefore that a Hopf bifurcation should emerge, for every fixed $\sigma > \sigma_{H2}$: deterministic as well as stochastic expectations driven fluctuations should generically occur when the elasticity of factor substitution is large. Two subcases arise depending upon whether the slope of Δ is less than one ($\sigma < \sigma_S$), in which case Δ crosses (AC) and a transcritical bifurcation occurs when ε_γ goes through $\varepsilon_{\gamma S}$ (see Fig. 8), or whether the slope of Δ exceeds one ($\sigma > \sigma_S$), in which case there is no transcritical bifurcation (see Fig. 9). To these regimes correspond cases 6 and 7 of Proposition 4.2.

In view of the preceding discussion, one can therefore order seven possible different dynamic regimes as σ is made to vary from 0 to $+\infty$. Among those, the last two ones are most interesting since they show the compatibility of indeterminacy and endogenous fluctuations with a high enough elasticity of factor substitution. We summarize, in Proposition 4.2 below, the characteristics of each regime.

Indeterminacy and expectations driven fluctuations for large σ's, i.e. for all $\sigma > \sigma_{H2}$, are plausible phenomena only if the lower bound σ_{H2} is not too high and if $\varepsilon_{\gamma H}$ is not too low, i.e. if labor supply is not too elastic. From the expressions of $\sigma_{H2} = (s - \theta(1-s))/(\nu - (1+\theta)\varepsilon_\psi)$ and $\varepsilon_{\gamma H} = 1/D_1 = (s - \sigma(1+\nu-\varepsilon_\psi))/(\theta(1-s) - \sigma(1+\theta\varepsilon_\psi))$, one sees that imposing arbitrary upper bounds σ^* and ε_l^* on σ_{H2} and $\varepsilon_{lH} = 1/(\varepsilon_{\gamma H} - 1)$ is equivalent to set $\varepsilon_\psi(1+\theta) < \nu - (s-\theta(1-s))/\sigma^*$ and $\varepsilon_\psi(1+\theta) < \nu - ((s\varepsilon_l^* - \theta(1-s)(\varepsilon_l^*+1))/\sigma + 1 + \theta\varepsilon_\psi)/\varepsilon_l^*$, for $\sigma_{H2} < \sigma < \sigma^*$. The latter restriction implies the former, and both are much stronger than the condition $\varepsilon_\psi(1+\theta) < \nu$ needed in Lemma 4.1 to establish the configuration of Fig. 7. Being the time period short, i.e. $\theta = 1 - \beta(1-\delta) \approx 0$, the latter inequality approximately reduces to $\nu - \varepsilon_\psi > s/\sigma + 1/\varepsilon_l^*$, for $\sigma_{H2} < \sigma < \sigma^*$.
In particular, if one wishes to get local indeterminacy and the occurence of a Hopf bifurcation for σ which belongs to the estimated range, which includes of course the Cobb-Douglas production function (see Hamermesh [26], Rotemberg and Woodford [38]), it is necessary, in view of the above inequality, to stick on the assumption of a highly elastic labor supply when the externalities are not irrealisticly large. Since the social increasing returns to scale $1+\nu$ should be less than 1.6, according to the estimates of Baxter and King [4], Caballero and Lyons [11], Hall [25], and the capital share $s \approx 0.4$, endogenous fluctuations are then compatible with a value of σ not too far from one when the externality from labor is substantially higher than the externality from capital (see Benhabib and Farmer [5], Farmer and Guo [17]).[16]

Proposition 4.2 (Local Stability and Bifurcations of the Steady States)

[16]In fact, to assign a positive value for ε_ψ is not uncontroversial since a number of studies deny the evidence of an even moderate *short term* external effect due to the aggregate physical capital stock (learning by doing), while they leave open the possibility of increasing returns "due to something else", see Benhabib and Jovanovic [6]. On the other hand, Basu and Fernald [3], and Burnside [10], among the others, find negligible external effects and almost constant returns to scale in U.S. manufacturing.

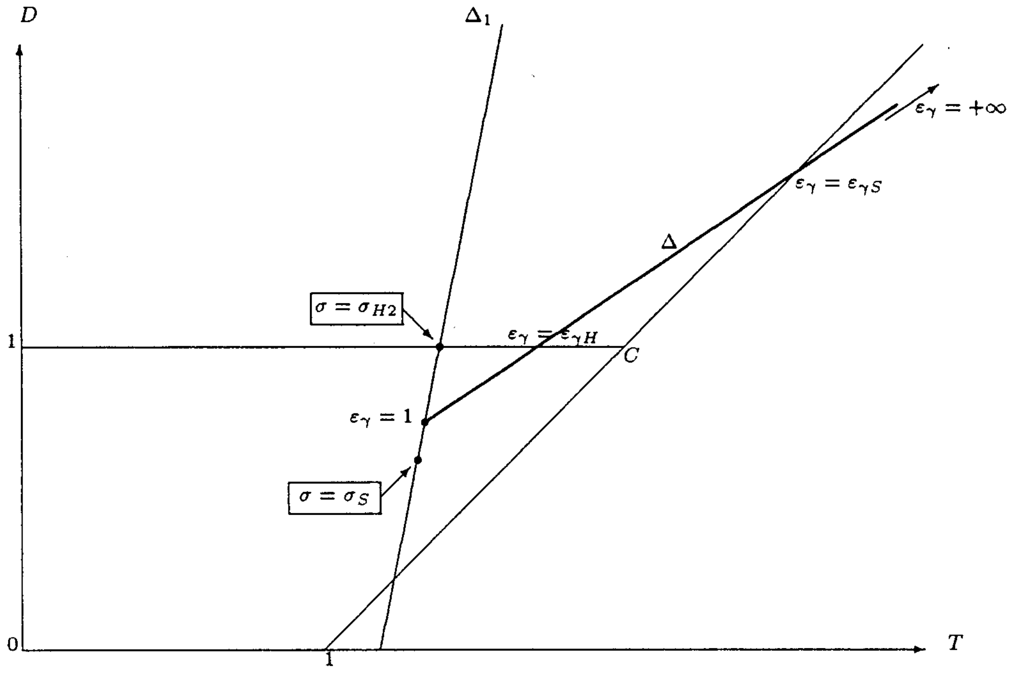


Figure 8: case 6 ($\sigma_{H2} < \sigma < \sigma_S$) of the analysis.

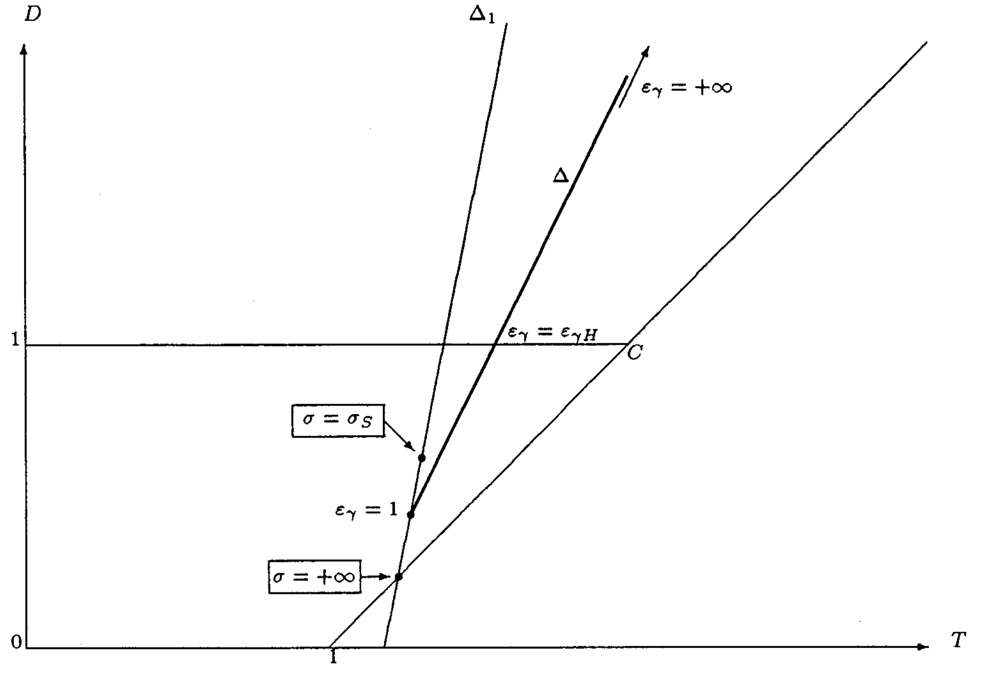


Figure 9: case 7 ($\sigma_S < \sigma$) of the analysis.

Consider a steady state that is assumed to be set at $(\bar{a}, \bar{k}) = (1,1)$ through the procedure in Proposition 3.1. Under the assumptions of Lemma 4.1, and under Assumptions 2.1, 2.2 and 4.1, the following generically holds.[17]

1. $0 < \sigma < \sigma_{F1}$: *the steady state is a saddle when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma S}$, *where* $\varepsilon_{\gamma S}$ *is the value of* ε_γ *for which the half-line* Δ *crosses the line* (AC). *At* $\varepsilon_\gamma = \varepsilon_{\gamma S}$, *one characteristic root goes through* 1 *and the steady state is a sink for* $\varepsilon_{\gamma S} < \varepsilon_\gamma < \varepsilon_{\gamma H}$, *where* $\varepsilon_{\gamma H}$ *is the value of* ε_γ *for which* Δ *crosses* $[BC]$. *The steady state undergoes a Hopf bifurcation at* $\varepsilon_\gamma = \varepsilon_{\gamma H}$, *and is a source when* $\varepsilon_\gamma > \varepsilon_{\gamma H}$.

2. $\sigma_{F1} < \sigma < \sigma_{H1}$: *the steady state is a saddle when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma S}$, *becomes a sink when* $\varepsilon_{\gamma S} < \varepsilon_\gamma < \varepsilon_{\gamma H}$; *one eigenvalue goes through* 1 *at* $\varepsilon_\gamma = \varepsilon_{\gamma S}$. *Then the steady state undergoes a Hopf bifurcation at* $\varepsilon_\gamma = \varepsilon_{\gamma H}$ *and is a source when* $\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma F}$, *where* $\varepsilon_{\gamma F}$ *is the value of* ε_γ *for which* Δ *crosses the line* (AB). *A flip bifurcation occurs at* $\varepsilon_\gamma = \varepsilon_{\gamma F}$ *and the steady state is a saddle when* $\varepsilon_\gamma > \varepsilon_{\gamma F}$.

3. $\sigma_{H1} < \sigma < \sigma_I$: *the steady state is a saddle when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma S}$, *a sink when* $\varepsilon_{\gamma S} < \varepsilon_\gamma < \varepsilon_{\gamma F}$ *and one eigenvalue crosses* 1 *at* $\varepsilon_\gamma = \varepsilon_{\gamma S}$. *A flip bifurcation occurs at* $\varepsilon_\gamma = \varepsilon_{\gamma F}$ *and the steady state is a saddle if* $\varepsilon_\gamma > \varepsilon_{\gamma F}$.

4. $\sigma_I < \sigma < \sigma_{F2}$: *the steady state is a saddle when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma F}$, *undergoes a flip bifurcation at* $\varepsilon_\gamma = \varepsilon_{\gamma F}$ *and is a source when* $\varepsilon_{\gamma F} < \varepsilon_\gamma < \varepsilon_{\gamma S}$. *At* $\varepsilon_\gamma = \varepsilon_{\gamma S}$, *one characteristic root goes through* 1, *and the steady state is a saddle when* $\varepsilon_\gamma > \varepsilon_{\gamma S}$.

5. $\sigma_{F2} < \sigma < \sigma_{H2}$ *and* $\sigma \neq \sigma_0$: *the steady state is a source when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma S}$. *One eigenvalue crosses* 1 *at* $\varepsilon_\gamma = \varepsilon_{\gamma S}$ *and the steady state is a saddle when* $\varepsilon_\gamma > \varepsilon_{\gamma S}$.

6. $\sigma_{H2} < \sigma < \sigma_S$: *the steady state is a sink when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma H}$, *undergoes a Hopf bifurcation at* $\varepsilon_\gamma = \varepsilon_{\gamma H}$ *and is a source when* $\varepsilon_{\gamma H} < \varepsilon_\gamma < \varepsilon_{\gamma S}$. *At* $\varepsilon_\gamma = \varepsilon_{\gamma S}$, *one eigenvalue goes through* 1 *and the steady state is a saddle when* $\varepsilon_\gamma > \varepsilon_{\gamma S}$.

7. $\sigma_S < \sigma$: *the steady state is a sink when* $1 < \varepsilon_\gamma < \varepsilon_{\gamma H}$, *undergoes a Hopf bifurcation at* $\varepsilon_\gamma = \varepsilon_{\gamma H}$ *and is a source when* $\varepsilon_\gamma > \varepsilon_{\gamma H}$.

Remark 4.1 In Assumption 4.1, we imposed that the slope of Δ is less than one for $\sigma = \sigma_{H2}$, i.e. that $\sigma_{H2} < \sigma_S$, in order to simplify the exposition by focusing on a single configuration. One easily adapts cases 5, 6 and 7 of Proposition 4.2 when alternative assumptions are made as, for instance, $\sigma_0 < \sigma_S < \sigma_{H2}$. Using the expressions of these critical values, one gets that the latter condition is

[17] The expressions σ_{F1}, σ_{H1}, σ_I, σ_{F2}, $\varepsilon_{\gamma S}$, and $\varepsilon_{\gamma F}$ (or how to compute them) are given in the proof of the proposition in Appendix. The expressions $\varepsilon_{\gamma H} = (s - \sigma(1 + \nu - \varepsilon_\psi))/(\theta(1-s) - \sigma(1 + \theta\varepsilon_\psi))$, $\sigma_0 = s/(1 + \nu - \varepsilon_\psi)$, $\sigma_{H2} = (s - \theta(1-s))/(\nu - (1+\theta)\varepsilon_\psi)$ and $\sigma_S = (1-s)/\varepsilon_\psi$ were given in the text.

equivalent to the less restrictive inequality $(\varepsilon_\psi - 1 + s)/(1-s) < \nu < \varepsilon_\psi/(1-s)$. Since, as in the analysis of the text, Δ cannot cross (AC) and a transcritical bifurcation cannot occur when $\sigma > \sigma_S$, the steady state being generically either a sink or a source, cases 1, 2, 3 and 4 of Proposition 4.2 remain then unchanged, while the configuration of case 5 occurs for $\sigma_{F2} < \sigma < \sigma_S$. When $\sigma_S < \sigma < \sigma_{H2}$, the steady state is a source for all $\varepsilon_\gamma > 1$, and finally, case 7 occurs for $\sigma_{H2} < \sigma$. The investigation of the different regimes when $\sigma_S < \sigma_0$, which can be performed by adopting the very same tools, is left to the reader.

4.2 Sensitivity Analysis of Coordination Failures and Endogenous Fluctuations

The purpose of this section is to present a short numerical sensitivity analysis to ease the comparison with the results obtained in other dynamic models where self-fulfilling business cycles are shown to exist (see Schmitt-Grohe [41]).

We shall begin by considering the standard parameter values based on yearly data: we fix the depreciation rate $\delta = 0.1$, the share of capital in total income $s = 1/3$, and the capitalists' discount factor $\beta = 0.99$. The social returns to scale are set at 1.5 (i.e. $\nu = 0.5$), a value consistent with the range considered in several works (see Baxter and King [4], Farmer and Guo [17], Hornstein [27] or Rotemberg and Woodford [40]). Finally, we set $\varepsilon_\psi = 0.1$, i.e. a strong enough externality scale effect which is of course compatible with Assumption 2.1 and the assumptions of Lemma 4.1. Given that configuration, we conclude, in view of Proposition 4.2, that:
- there exist multiple arbitrarily close steady states if $\sigma < \sigma_S \approx 6.7$, and if ε_γ is close enough to $\varepsilon_{\gamma S} \approx 1.88$ when σ is close enough to 1, i.e. if the elasticity of labor supply $\varepsilon_l = 1/(\varepsilon_\gamma - 1)$ is around 1.13,
- a Hopf bifurcation, and therefore deterministic and stochastic expectations driven fluctuations, is expected if $\sigma > \sigma_{H2} \approx 0.67$, and if ε_γ is close enough to $\varepsilon_{\gamma H} \approx 1.14$ when σ is close enough to 1, i.e. if the labor supply elasticity ε_l is close to 7.

This exercise suggests that although fluctuations occur around the Cobb-Douglas case ($\sigma = 1$) if the labor supply elasticity with respect to the real wage is sufficiently high, multiplicity arises even if that elasticity is close to unity.[18]

As recently reviewed in Schmitt-Grohe [41], models of aggregate fluctuations due to self-fulfilling expectations require a rather high elasticity of labor supply (above 4) for empirically plausible degree of social returns to scale (below 2), in the Cobb-Douglas case. To follow her comparison, we now

[18] In view of Subsection 4.1, the two steady states are indeed arbitrarily close if ε_γ is arbitrarily close to $\varepsilon_{\gamma S}$, since a transcritical bifurcation occurs for this value. See Section 5 for an application to the CES economies.

take the benchmark parameter values based on quarterly data, that is $\delta = 0.025$, $\beta = 0.988$, $s = 0.42$ and $\sigma = 1$. We then set ν and ε_ψ such that Assumption 2.1 and the assumptions of Lemma 4.1 are satisfied. In order to compare with the sensitivity analysis provided in Schmitt-Grohe [41] for four different models, we compute ν_{min} defined as the minimal value of increasing returns necessary to obtain endogenous fluctuations, i.e. the Hopf bifurcation point $\varepsilon_\gamma = \varepsilon_{\gamma H}$, when the only source of external effects at the steady state is labor ($\varepsilon_\psi = 0$). The results are reported in table 2. The first

	$s = 0.42$ $\varepsilon_\gamma = 1.25$	$s = 0.42$ $\varepsilon_\gamma = 1$	$s = 0.3$ $\varepsilon_\gamma = 1.25$	$s = 0.3$ $\varepsilon_\gamma = 1$
$\sigma = 1$	0.64	0.40	0.52	0.27

Table 2: the necessary degree of increasing returns for endogenous fluctuations ν_{min}, when $\sigma = 1$ and $\varepsilon_\psi = 0$.

column shows that the model requires a degree of social increasing returns which is 30% less than that in the two models with decreasing marginal costs and constant markup reviewed in Schmitt-Grohe [41]. Moreover, the third column of table 2 shows that even if one accepts the calibration adopted by Benhabib and Farmer [5], or Farmer and Guo [17] (which implies an infinitely elastic labor supply), the degree of increasing returns necessary to obtain endogenous fluctuations is only 1.28, i.e. a 15% less than what needed in those contributions.[19]

Since our local dynamics analysis does not depend on the Cobb-Douglas technology but relies instead on a more general technology, it allows us to go one step further and to assess whether endogenous fluctuations are compatible with other equally plausible values of the elasticity of factor substitution. Table 3 is the counterpart of table 2 when the elasticity of capital-labor substitution is set at 0.5, 1.5 and 2, instead of one, these values being consistent with the reported range of estimates (see Hamermesh [26]).

This exercise shows that even if the elasticity of factor substitution is increased, endogenous fluctuations require a high elasticity of labor supply with respect to the real wage, at the steady state. When σ is higher, however, but still belongs to the estimated range, the level of increasing returns to scale due to labor needed to obtain endogenous fluctuations is significantly lower, especially when the capital share is close to its minimal admissible value.

[19]If we compare table 2 in Schmitt-Grohe [41] and table 2, first column supra, we see that ν_{min} is slightly smaller than the minimal value required to get indeterminacy of the steady state in the two models with constant marginal costs and variable markup reported in the latter reference: a modified version of Gali [18] and Rotemberg and Woodford [40]. Although we do not provide a detailed sensitivity analysis, it can be easily checked that the bifurcation values for σ and ε_γ are not very sensitive to variations in δ or β since they are multiplicative constants in the definition of steady states (see eqs. (9)). The bifurcation values are rather more sensitive to variations in ν, ε_ψ and s because they "act like powers" in eqs. (9). This fact is already observable in table 2.

	$s = 0.42$ $\varepsilon_\gamma = 1.25$	$s = 0.42$ $\varepsilon_\gamma = 1$	$s = 0.3$ $\varepsilon_\gamma = 1.25$	$s = 0.3$ $\varepsilon_\gamma = 1$
$\sigma = 0.5$	1.04	0.8	0.79	0.55
$\sigma = 1.5$	0.51	0.27	0.43	0.18
$\sigma = 2$	0.45	0.2	0.38	0.14

Table 3: the necessary degree of increasing returns for endogenous fluctuations ν_{min}, when $\sigma = 0.5, 1.5, 2$ and $\varepsilon_\psi = 0$.

The first generation of calibrated models of expectations driven fluctuations (see Benhabib and Farmer [5], Farmer and Guo [17], Gali [18], Rotemberg and Woodford [40]) has exploited local indeterminacy of the steady state to generate stochastic fluctuations. However, it is a very well known result that local indeterminacy is only a sufficient condition for the existence of stochastic equilibria. Indeed, *nonlinearity* matters a lot in assessing the possibility of stochastic equilibria driven by shocks to expectations. As shown in Grandmont, Pintus, and de Vilder [22], it is possible constructing stochastic equilibria around a stationary point, not only arbitrarily near a locally indeterminate steady state, but also *along local bifurcations*. These techniques can be equally applied to our model, up to the straightforward introduction of endogenous uncertainty, to show that stochastic equilibria exist along the Hopf bifurcation. Of course, the support of those equilibria depends critically on the direction and stability of that local bifurcation, and these features require information about higher order derivatives of the map G. In the next section we shall characterize the Hopf bifurcation through the simulations based on the constant elasticity case.

5 Simulations of the CES Economies with Externalities

We now present the simulations based on the class of the CES economies defined by eqs. (15), (16), (19) and (20), in subsection 3.2, building on the parameter values used in subsection 4.2. In the general setting of the previous section, we cannot easily infer the nature of the local bifurcations encountered. On the contrary, the map G associated with eqs. (8) in the CES case (see subsection 3.2) is easily derived and will next be studied. One could apply to these nonlinear maps the methods for simplifying the study of a dynamical system: the center manifold theory to reduce opportunely the dimension of the system so as to determine more easily the stability of a periodic orbit when some eigenvalues have modulus one, and the method of normal forms to derive the nature of the local bifurcations encountered. The references given above, page 20, provide an exposition of these powerful tools. Because of our concern for briefness and simplicity, we used instead the program DUNRO [15], by D.J. Sands and R.G. de Vilder, to check the direction and stability of each bifurcation of the fixed point $(\overline{a}, \overline{k}) = (1, 1)$ appearing in cases 6 and 7 of Proposition 4.2.

5.1 Multiple Steady States and the Transcritical Bifurcation

In Proposition 4.2, cases 1 to 6, where $\sigma < \sigma_S$, show that one eigenvalue of the Jacobian matrix evaluated at $(\overline{a}, \overline{k}) = (1,1)$ crosses 1, when ε_γ goes through $\varepsilon_{\gamma S}$. Using DUNRO [15], we found that a *transcritical* bifurcation of the fixed point $(\overline{a}, \overline{k}) = (1,1)$ actually occurs.[20] To illustrate this bifurcation[21], we now fix $\nu = 0.6$, $\psi = 0.1$ and consider two examples.

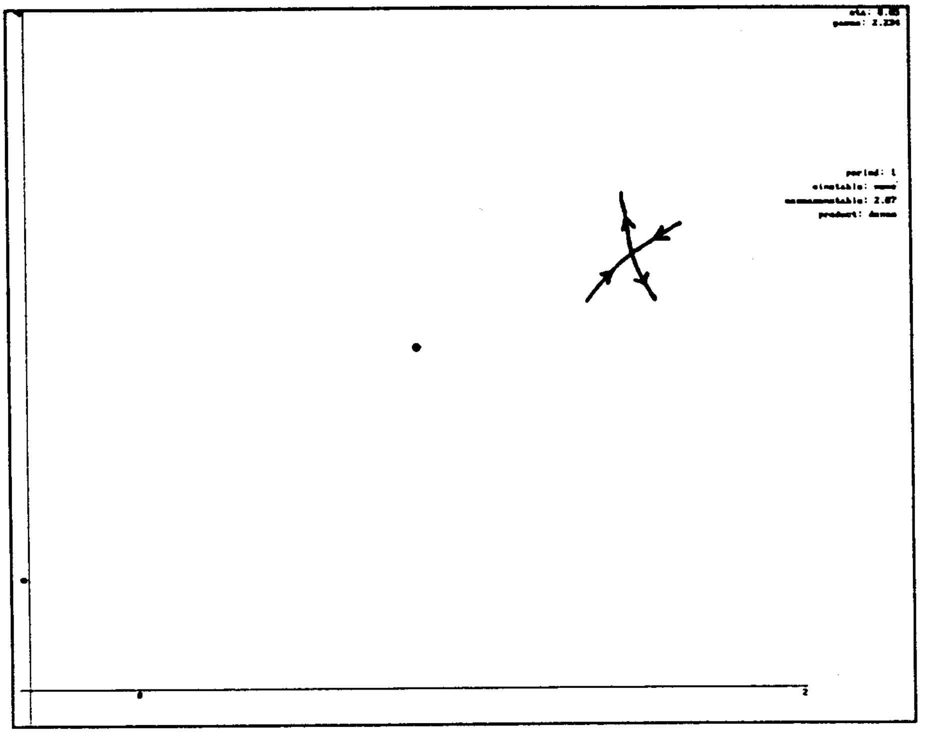
Example 1: $\eta = 0.05$ ($\sigma \approx 0.95$).
In that configuration, we observe that for ε_γ slightly smaller than $\varepsilon_{\gamma S}$, $(1,1)$ is a source (see Fig. 8, case 6 of Proposition 4.2). In the (k, a) plane, a second positive fixed point is located north-east of $(1,1)$ and is a saddle (see Fig. 10 (a)). As the bifurcation parameter ε_γ is increased towards $\varepsilon_{\gamma S}$, the saddle approches the source $(1,1)$ from north-east. The two steady states actually merge at the bifurcation point $\varepsilon_\gamma = \varepsilon_{\gamma S}$, where one eigenvalue is equal to 1. Finally, for ε_γ larger than $\varepsilon_{\gamma S}$, *the two positive fixed points have exchanged stability*: $(1,1)$ is henceforth a saddle while the second steady state located south-west of $(1,1)$ is a source (see Fig. 10 (b)).

Example 2: $\eta = -0.05$ ($\sigma \approx 1.05$).
In view of Fig. 8, and case 6 of Proposition 4.2, $(1,1)$ is a source when ε_γ is slightly smaller than $\varepsilon_{\gamma S}$ and becomes a saddle after the transcritical bifurcation. The second positive fixed point is located south-west of $(1,1)$ and is a saddle before the transcritical bifurcation, becomes a source through the bifurcation and moves towards north-east of $(1,1)$ when ε_γ increases from $\varepsilon_{\gamma S}$, (see Fig. 11 (a) and Fig. 11 (b)).
These two examples illustrate how special is the case $\sigma = 1$ (the Cobb-Douglas production function), as far as the number of stationary equilibria is concerned: a slight departure from this specification originates multiple steady states. From a more technical point of view, these two examples also allow the comparison between two possible ways of proving multiplicity. Proposition 3.4 explores a global approach, while Proposition 4.2 uses the local dynamics information to provide simple sufficient conditions for arbitrarily close steady states (generically speaking).

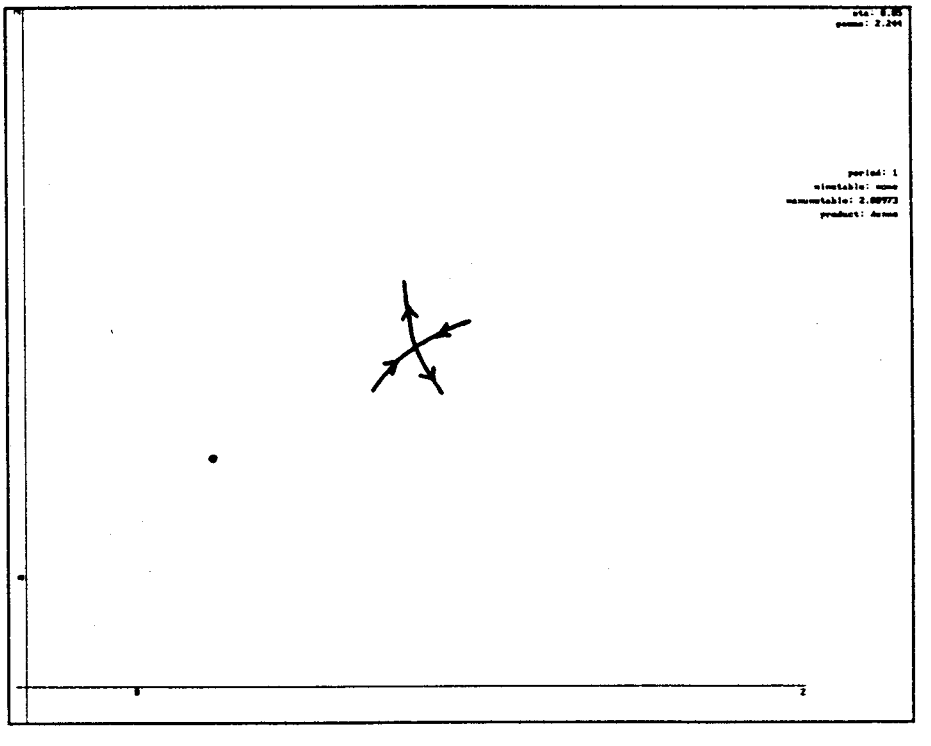
In Subsections 3.2 and 4.2 we argued that the possibility of multiple steady states cannot be rejected through the empirical exercise of calibration. Therefore, multiplicity of steady states, a feature both theoretically possible and empirically plausible, is responsible for coordination failures. For instance, two *locally determinate* positive steady states may coexist in a small neighborhood, as the two previous examples illustrated. Moreover, the two stationary points are strongly Pareto-ranked: the highest is strictly preferred by both classes of agents, as shown above in Proposition 3.5. Even under perfect

[20]Because we assume that eq. (20) holds, the steady state $(\overline{a}, \overline{k}) = (1,1)$ persists whatever the values of γ and σ are. Moreover, Proposition 3.4 showed that there exist at most two steady states. Therefore, we should expect that neither the generic saddle-node bifurcation nor the pitchfork bifurcation can occur.

[21]See e.g. Grandmont [20, Subsection C.2], Wiggins [46, Subsection 3.2A] for a theoretical treatment and bifurcation diagrams.

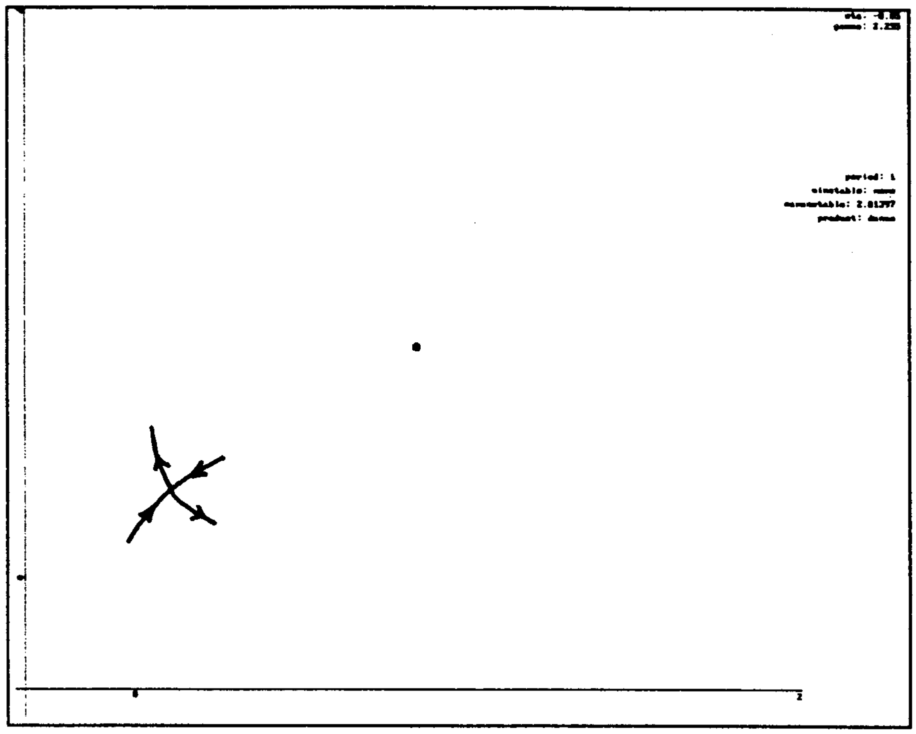


(a)

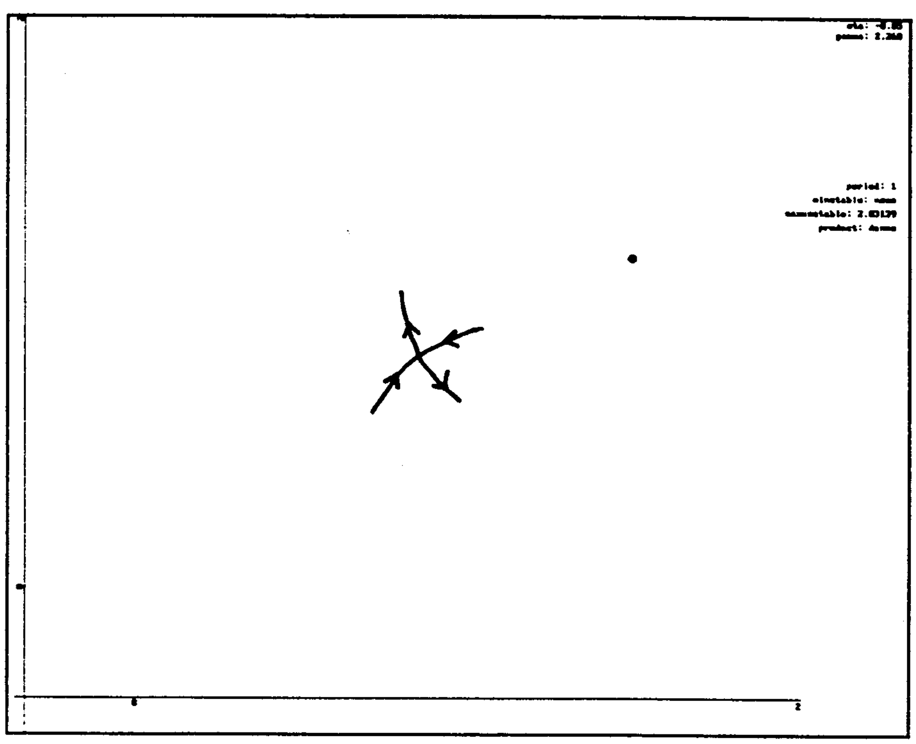


(b)

Figure 10: the example 1 ($\eta = 0.05$) in the (k, a) plane. The multiple steady states, (a) before the transcritical bifurcation, i.e. for ε_γ close enough to but smaller than $\varepsilon_{\gamma S}$, (b) after the transcritical bifurcation, i.e. for ε_γ close enough to but larger than $\varepsilon_{\gamma S}$.



(a)



(b)

Figure 11: the example 2 ($\eta = -0.05$) in the (k, a) plane. The multiple steady states, (a) before the transcritical bifurcation, i.e. for ε_γ close enough to but smaller than $\varepsilon_{\gamma S}$, (b) after the transcritical bifurcation, i.e. for ε_γ close enough to but larger than $\varepsilon_{\gamma S}$.

foresight, agents' actions could lead to a Pareto-dominated stationary equilibrium. With such strong implications for welfare, the study of the appropriate policy required to restore the Pareto-efficiency of the stationary equilibrium is an open and interesting issue.[22]

5.2 The Hopf Bifurcation and Expectations Driven Fluctuations

We used the program DUNRO [15] to determine the direction and stability of the Hopf bifurcation[23], at $\varepsilon_\gamma = \varepsilon_{\gamma H}$ (see cases 6 and 7 in Figs. 8 and 9 and in Proposition 4.2). We keep the values $\nu = 0.6$, $\psi = 0.1$ and give two examples.

Example 3: $\eta = 0.05$ ($\sigma \approx 0.95$).
Proposition 4.2 shows that slightly before the Hopf bifurcation, $(1,1)$ is a sink, i.e. it is (locally) asymptotically stable, while slightly after the bifurcation, $(1,1)$ is a source (see Fig. 8). Fig. 12 reveals that for ε_γ larger than and close to $\varepsilon_{\gamma H}$, $(1,1)$ is surrounded by an attracting and invariant closed curve, homeomorphic to a circle, created through the Hopf bifurcation. In that configuration, the Hopf bifurcation is *supercritical.*

Example 4: $\eta = 0$ ($\sigma = 1$).
In that case, using DUNRO [15], we found that the Hopf bifurcation is now *subcritical*: a repelling and invariant closed curve surrounds the asymptotically stable fixed point $(1,1)$ before the Hopf bifurcation, i.e. for $\varepsilon_\gamma < \varepsilon_{\gamma H}$ and $|\varepsilon_\gamma - \varepsilon_{\gamma H}|$ small. Slightly after the bifurcation, $(1,1)$ is a source.

As already mentioned above, the existence of deterministic and stochastic expectations driven fluctuations depends critically on the direction and stability of the Hopf bifurcation. We refer the reader to Grandmont, Pintus, and de Vilder [22, Section 3], for a geometrical method of proving the existence of stochastic equilibria around a locally indeterminate steady state as well as along local bifurcations. Appealing to the results of these latter authors, we next characterize those regions in the (k,a) plane where endogenous fluctuations occur.
In the example 3, the steady state $(\overline{a}, \overline{k}) = (1,1)$ is locally indeterminate before the supercritical Hopf bifurcation: infinitely many nondegenerate stochastic equilibria with a compact support can be constructed in any arbitrarily small neighborhood of the sink. After the supercritical Hopf bifurcation, deterministic endogenous fluctuations occur in any arbitrarily small neighborhood of the steady state, containing or not the invariant and attracting closed curve. From Grandmont, Pintus, and de Vilder [22], we know that, similarly, there exist infinitely many nondegenerate stochastic equilibria

[22] though not trivial because the capital stock is a predetermined variable and therefore makes uneasy an eventual once and for all adjustement.

[23] See e.g. Grandmont [20, Subsection C.4], Wiggins [46, Subsection 3.2C] for a theoretical treatment and bifurcation diagrams.

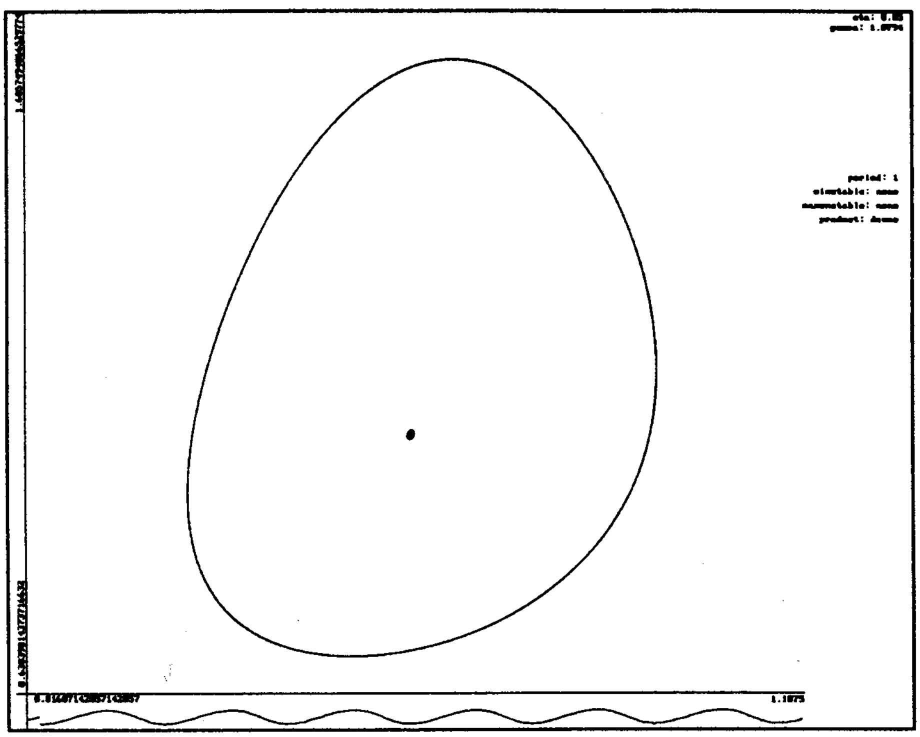


Figure 12: the example 3 ($\eta = 0.05$) in the (k, a) plane. Quasiperiodic dynamics on the attracting invariant closed curve created through a supercritical Hopf bifurcation, i.e. for $\varepsilon_\gamma > \varepsilon_{\gamma H}$ and $|\varepsilon_\gamma - \varepsilon_{\gamma H}|$ small.

with a compact support *containing in its interior the attracting closed curve*, in Fig. 12. Intuitively, one ought to expect that (infinitely many) stochastic equilibria should be constructed in any set M such that $G(M) \subset \text{Int}\, M$: the restriction of the map G to that set is *contracting.*[24] In particular, such sets exist and are easily located when the fixed point is asymptotically stable and when the invariant closed curve is attracting. respectively before and (slightly) after the supercritical Hopf bifurcation, as the characterization of the example 3 has just showed.

In the example 4, the Hopf bifurcation is subcritical: there exist infinitely many nondegenerate stochastic equilibria with a compact support *contained in the interior of the repelling closed curve*, when $\varepsilon_\gamma < \varepsilon_{\gamma H}$ and $|\varepsilon_\gamma - \varepsilon_{\gamma H}|$ is small. Bounded deterministic endogenous fluctuations only occur *on* this latter repelling set. therefore solely before the subcritical Hopf bifurcation. After this bifurcation. the steady state is locally determinate: no endogenous fluctuations occur in any of its small neighborhoods.[25]

6 Internal Increasing Returns to Scale and Imperfect Competition

In this section, we modify the model introduced in Section 2 in only one of its dimensions: the unique good is now produced under *internal* increasing returns to scale. In order for the market structure to be compatible with internal increasing returns, we introduce imperfect competition in the product market, while money and production factors are still traded in a perfectly competitive way.

The critical assumption we adopt is that costless entry and exit eliminate the possibility of "profits" once factors are remunerated, so that the dynamical system in eqs. (8) still summarizes the

[24]If such a set M is closed and connected, it is a *trapping region.* In practice, this notion is used to locate the associated *attracting set.* Indeed, finding a trapping region is equivalent to finding a *Liapunov function.* On the asymptotic behavior of dynamical systems, see, for instance, Guckenheimer and Holmes [23, Section 1.6], Wiggins [46, Subsection 1.1H]. However, it is shown in Grandmont, Pintus, and de Vilder [22] that one doesn't necessarily need to find a trapping region to characterize stationary stochastic equilibria.

[25]With the local dynamics analysis presented in Section 4 as a starting point, one may wish to get information about the *global* dynamics, i.e. about the orbits away from the steady state. In particular, the techniques presented in de Vilder [44, 45], Pintus, Sands, and de Vilder [35] should be useful to study the global bifurcations arising on and the break up of the invariant closed curve, as well as the possible homoclinic bifurcations. However, one should consider with care global results for they occur in regions that may be too far from a steady state to ensure that the dynamical system actually results from the model presented in section 2. Indeed, we shall recall that the capitalists' and workers' behaviors are corner solutions. Nevertheless, as in Pintus, Sands, and de Vilder [35], one could control the inequalities that must be checked along any orbit quiet far from the steady state with the remaining degrees of freedom, as e.g. the intertemporal leisure substitution elasticity or the workers' discount factor. However, again for briefness concern, we do not explore such interesting lines. We propose to the reader the two previous references for an applied study of complicated global fluctuations expected to occur in two-dimensional dynamical systems with, at least, two parameters.

intertemporal equilibria. Ignoring "Ford effects", it follows that the markup is constant and that the elasticities of factor rental prices depend, as above, only on technology, although increasing returns are now represented by a single parameter. In particular, it turns out that a slight change in the parameters representing increasing returns leads one back to the previous elasticities of factor prices: *it is now as if each factor originated an external effect proportional to his share in total income*, given the constant level of social, or equivalently internal, increasing returns. Consequently, as far as the steady states and local dynamics analysis are concerned, the above methods apply equally well to the model with an imperfectly competitive product market as a particular case of the model with productive externalities, and yield qualitatively similar results, i.e. multiple Pareto-ranked steady states and endogenous fluctuations for large elasticities of factor substitution.

Especially, we consider Cournot oligopolistic competition between the entrepreneurs-capitalists: each producer knows the aggregate inverse demand and chooses the optimal amount to produce, taking as given the quantities chosen by all the other competitors. We introduce imperfect competition through the Cournotian approach only for simplicity of the exposition, being aware that one may criticize the absence of an apparent price-setting behavior. Indeed, our presentation shouldn't suffer from that since we could alternatively equip the model with differentiated intermediate goods, each produced by a single entrepreneur who sets its price, and used as imperfect substitutes to produce or to consume the unique final good. One recognizes here the Chamberlinian monopolistic competition extensively used in the recent literature on the macroeconomics of imperfect competition, together with the convenient CES aggregator for quantities and its dual prices aggregator. With respect to the steady states and local dynamics analysis, this wouldn't change the properties of the model that we are going to derive.[26] We could alternatively adapt the formulation by d'Aspremont, Dos Santos Ferreira, and Gérard-Varet [2], i.e. the Cournotian monopolistic competition in the product market. This modelling involves differentiated goods, each produced by several firms, each taking as given the quantities of the competitors in its sector (Cournot oligopolistic competition within a sector) as well as the prices of the other goods (monopolistic competition between sectors) and knowing the inverse demand. Of course, we would impose, as for instance in Gali [18], that every worker purchases all goods and consume a composite commodity, and that every capitalist within a sector buys all goods in order to increase his (composite) capital stock necessary, together with labor, to produce the differentiated product of his sector. This framework distinguishes between the respective price elasticities of the aggregate and sectoral demands and makes possible a suitable relatively large number of firms within each sector even if increasing returns are relatively high. By adopting it, we would nevertheless certainly make the notation heavier without adding substance to the results concerning the steady states and the dynamics of the present model that are going to appear.

[26]To be more precise, we should assume that profits are zero, at least in the long-run, in consequence of costless entry and exit.

6.1 The Technology with Increasing Returns and the Symmetric Cournot Equilibria

Each identical entrepreneur-capitalist is endowed with the same technology producing $y_t \geq 0$ from a capital stock $k_{t-1} \geq 0$ and a labor stock $l_t \geq 0$, possibly in variable proportion. The main difference from the technology in Subsection 2.1 is that the gross production production F now exhibits internal increasing returns to scale and, moreover, is homogeneous of degree $e > 1$. Therefore, the quantity of good produced is

$$y = AF(k,l) = Al^e f(a). \tag{27}$$

The second equality defines the production in intensive form defined upon the capital labor ratio $a = k/l$. On technology, we shall assume the following.[27]

Assumption 6.1

f is a continuous function of $a \geq 0$, C^r for $a > 0$ and r large enough, with $f'(a) > 0$.

The optimal amount of each factor is determined by a Cournot competitor in the following manner. Each firm knows the inverse demand function $p(y_i + y_{-i})$, where $y_i \geq 0$ and $y_{-i} \geq 0$ are respectively the residual demand adressed to the firm i and the aggregate production resulting from the decisions of all the other competitors, whereas the map p gives the associated price of the good. Taking y_{-i} as given, the firm i determines its factor demands $l_i \geq 0$ and $k_i \geq 0$ by maximizing the profit $p(AF(k_i, l_i) + y_{-i})AF(k_i, l_i) - wl_i - rk_i$, where w and r are respectively the nominal rental prices of labor and capital, taken as given by firm i under the assumption of perfectly competitive factor markets. As usually, the critical information needed for a Cournot competitor is the elasticity of the inverse demand function: this elasticity determines the level of the price over marginal cost markup. Indeed, we shall consider *symmetric* Cournot-Nash equilibria[28] and ignore the so-called Ford effects reflecting the dependence of the demand to the income paid by the firms: the price-elasticity of the aggregate demand for the good equals -1 at the steady state and nearby, expressing the fact that the total nominal revenue, i.e. the stock of money and the returns on capital, is taken as given by the firms. Accordingly, at a symmetric equilibrium involving n firms, the familiar first order conditions of that problem are therefore

$$P = (1 - 1/n)AF_k(\cdot),\ \Omega = (1 - 1/n)AF_l(\cdot), \tag{28}$$

[27]The properties of the function f differ from those presented in Assumption 2.1. In particular, the second derivative f'' is not necessarily negative, and is indeed positive for a high enough increasing returns level e.

[28]Using the same kind of arguments as those in d'Aspremont, Dos Santos Ferreira, and Gérard-Varet [2, Lemma 3], one proves that any intertemporal equilibria is symmetric in quantities relatively to the active firms.

where P and Ω denote respectively the real competitive rental prices of capital and labor.[29] In presence of market power in the product market[30], leading to a price higher than the perfectly competitive one, it should come as no surprise that the real factor prices are lower than the corresponding marginal productivities.

Together with the necessary condition (28), the nonnegative profit condition for each active firm is required for a positive production.[31] Indeed, we shall assume that costless entry and exit push the level of profit towards zero[32]. i.e. $AF(k,l) = Pk + \Omega l$. Under this assumption, one easily deduces from the homogeneity of the production function, hence the Euler relation $eF(\cdot) = kF_k(\cdot) + lF_l(\cdot)$, and the conditions (28) that the level of increasing returns determines then the number of active firms to be $\epsilon/(e-1)$. Without loss of generality and only for convenience, we shall suppose that the number of entrepreneurs-capitalists actually equals $e/(e-1)$, and equals also the number of workers.[33] We derive without any difficulty the respective real rental prices of capital and labor arising from the symmetric Cournot equilibrium on the product market and the perfectly competitive equilibria on the factor markets.

$$\begin{aligned} P(a,k) &\stackrel{\text{def}}{=} A(k/a)^{e-1}\rho(a)/e, \quad \text{where} \quad \rho(a) = f'(a), \\ \Omega(a,k) &\stackrel{\text{def}}{=} A(k/a)^{e-1}\omega(a)/e, \quad \text{where} \quad \omega(a) = ef(a) - a\rho(a). \end{aligned} \tag{29}$$

As in the case of the previous technology with externalities, a remarkable consequence of increasing returns is that both marginal productivities depend on the scale of production, i.e. on k in eqs. (29). We shall show that this is of importance for the number of steady states and the dynamics of the model.

Apart from the technology and the product market structure, the remaining part of the model is identical. In particular, the behaviors of workers and capitalists exposed in subsection 2.2 are still valid. More importantly, Definition 2.1 of the intertemporal equilibria in general terms still holds in the economy with an imperfectly competitive product market. The results about the number of steady states and the local dynamics presented in Sections 3 and 4 for the competitive economy with externalities relied heavily on the elasticities of the functions P and Ω with respect to a and k. Although these functions now depend on a new parameter e and on the differently defined functions ρ and ω, in eqs. (29), one expects that the methods in Sections 3 and 4 apply equally well here,

[29] Of course, one can alternatively consider the dual minimizing cost problem, since the homogeneity of the production function allows one to derive easily the total cost function $C(y,w,r) = y^{1/e}c(w,r)$, for given w and r. The well-known outcome of this procedure is the equality between marginal revenue and marginal cost.

[30] Market power is measured, for example, by the Lerner index, i.e. the ratio of the price minus the marginal cost over the price $1/n$. or alternatively by the markup $n/(n-1)$.

[31] It is easily shown that when the first order and the nonnegative profit conditions are satisfied, then the Hessian matrix is negative definite or, alternatively, the second derivative of the profit function with respect to y is negative: the solution is globally unique.

[32] By now, there seems to be no empirical evidence of pure profits for the european and american economies.

[33] The limit case of constant returns to scale $e = 1$ implies an infinite number of firms and therefore the absence of market power in the product market: perfect competition would reign in all markets, hence the results of Grandmont, Pintus, and de Vilder [22] would apply to that case.

with the only algebraic difference that those important elasticities will now depend differently on our prime parameters δ, β, s, σ, and depend also on e. Indeed, we shall show that this is the case and that the results concerning the number of stationary equilibria and the local dynamics of the model are qualitatively the same. Moreover, we shall show that a slight change of parameters makes the preceding results for the perfectly competitive economy with externalities directly transposable to the present economy with an imperfectly competitive product market: the deep reason for the new phenomena exhibited in this paper is the presence of increasing returns and not the product market structure.[34]

6.2 Steady States, Local Dynamics and Bifurcation Analysis

Preliminary computations are required to get the equivalence announced just above. The objective is again to express the elasticities of the functions P and Ω, with respect to a and k, in terms of our parameters. From the definition $1/\sigma(a) = \varepsilon_\omega(a) - \varepsilon_\rho(a)$ (see eqs. (29)) and the derivative of the Euler identity with respect to a yielding $\omega'(a) + a\rho'(a) = (e-1)\rho(a)$, we derive $\varepsilon_\omega(a) = (e-1)s(a) + s(a)/\sigma(a)$ and $\varepsilon_\rho(a) = (e-1)s(a) - (1-s(a))/\sigma(a)$. Together with eqs. (29) and the definition $R(\cdot) = P(\cdot) + 1 - \delta$, this leads to the following expressions evaluated at a steady state.

$$\begin{aligned} \varepsilon_{\Omega,a} &= s/\sigma - (1-s)(e-1), & |\varepsilon_{R,a}| &= \theta(1-s)(1/\sigma + e - 1), \\ \varepsilon_{\Omega,k} &= e-1, & \varepsilon_{R,k} &= \theta(e-1), \end{aligned} \tag{30}$$

where $\theta \overset{\text{def}}{=} 1 - \beta(1-\delta)$.

From the direct constrast of eqs. (23) with eqs. (30), one easily deduces that the expressions are actually identical if the change of parameters

$$e - 1 = \nu, \; (1-s)(e-1) = \nu - \varepsilon_\psi, \text{ hence } s(e-1) = \varepsilon_\psi, \tag{31}$$

is made in eqs. (30). This change of parameters is quite natural since it amounts to identify the level of *internal* increasing returns in the present economy to the level of *social* increasing returns in the economy with externalities, i.e. e to $1+\nu$. Moreover, the external effect due, respectively, to the aggregate capital and labor stocks is identified here to the contribution of each factor to the internal increasing returns proportional to its share in total production, i.e. to $s(e-1)$ and $(1-s)(e-1)$. But we have seen in Sections 3 and 4 that each contribution separately matters, at the *aggregate* level, to account for the existence of multiple steady states and expectations driven aggregate fluctuations. Therefore, using an homogeneous technology with increasing returns gives one degree of freedom less than introducing increasing returns through aggregate productive externalities: in the latter case, each factor contributes to the social increasing returns independently of the other. In conclusion,

[34] This type of equivalence has already been established by Benhabib and Farmer [5] for a Cobb-Douglas technology in the traditional infinitely-lived agents model.

one expects that the change of parameters in eqs. (31) makes the analysis of the preceding sections directly transposable as a special case to the present economy with an imperfectly competitive product market.

Concerning the steady states analysis of the present model, one easily obtains similar results as those in Section 3 when the change of parameters in eqs. (31) is applied. In particular, Assumptions 2.2 and 6.1 ensure the existence of the steady state $(\overline{a}, \overline{k}) = (1, 1)$ if and only if $A = (1/\beta + \delta - 1)/\rho(1)$, while $B > 0$ is the solution of $v_2(A) = v_1(1)$, uniquely determined if $\lim_{x \to 0} xV_2'(x) < V_1'(1) < \lim_{x \to +\infty} xV_2'(x)$. Similarly, Propositions 3.2, 3.3 still apply here after using eqs. (31) and redefining $\varepsilon_{l_1}(a) = -\varepsilon_\rho(a)/(e-1)$ and $\varepsilon_{l_2}(a) = \varepsilon_\omega(a)/(\varepsilon_\gamma(\cdot) - 1)$. The CES example in the present economy is easily adapted from Subsection 3.2 and yields a counterpart of Proposition 3.4: there are exactely two interior steady states if $e - 1 < (\gamma - 1)(1 - s\sigma(e-1))$ and if $e - 1 \neq (\gamma - 1)(1 - s - s\sigma(e-1))$. Finally, Proposition 3.5 still holds and is proved along the same arguments since the function $\omega(a)/\rho(a)$ is still increasing in a: the multiple steady states of the imperfectly competitive economy are still ordered and Pareto-ranked. Summarizing, the main result that there may exist, under plausible parameter values, multiple Parero-ranked steady states in the economy with externalities remains true in the economy with internal increasing returns and imperfect competition.

The second theme of the present paper is dynamics. Since the dynamical system in eqs. (8) represents also the intertemporal equilibria of the economy with an imperfectly competitive product market, the associated linearized map is still given by Proposition 4.1. Therefore, one deduces again the existence of the half-line Δ described in the (T, D) plane when ε_γ increases, and of the line Δ_1 generated when σ increases: the two key ingredients of the geometrical method of analyzing the dynamics around a steady state are equally useful to study locally the orbits of this model. Therefore, one adapts Lemma 4.1, table 1 and the Appendix by using eqs. (31) and gets the counterpart of Proposition 4.2: through the Hopf bifurcation, deterministic and stochastic expectations driven fluctuations are shown to exist (generically) for an arbitrary high elasticity of capital-labor substitution, in the imperfectly competitive economy.

7 Conclusion

This paper shows that the introduction of increasing returns to scale in the model originates multiple steady states, and expectations driven fluctuations at business cycle frequency for an arbitrarily high elasticity of capital-labor substitution. This result is obtained through a geometric approach of the local dynamics analysis that does not depend on the model, and allows one to appraise the sensitivity of self-fulfilling cycles to some important parameters whose estimates are not conclusive.

Although our results are primarily obtained in a relatively simple model because perfect competition is maintained. it is shown that the same tools are useful to study the model with internal increasing returns and imperfect competition in the product market, and that they yield qualitatively similar results. Therefore, this paper suggests that the contribution of increasing returns to scale in proving the emergence of deterministic and stochastic endogenous fluctuations at business cycle frequency can be made clear in the Woodford's model with capital-labor substitution, with a theoretical and methodological care as well as a pertinence concern.

Appendix: Proof of Proposition 4.2

The Hopf Bifurcation in Cases 6 and 7: $\sigma > \sigma_{H2}$

In these cases, the appraisal of the values of σ for which a Hopf bifurcation occurs at $\varepsilon_\gamma = \varepsilon_{\gamma H}$ is not immediate. Table 1, Figs. 8 and 9 show that, by continuity, Δ intersects $[BC]$ in its interior for σ slightly larger than σ_{H2}, for σ close enough to σ_S (where the slope of Δ is close to 1), and for arbitrarily large values of σ. However, in the between these small neighborhoods, geometrical arguments cannot account for the occurence of the Hopf bifurcation whatever $\sigma > \sigma_{H2}$. To see why, fix σ in $(\sigma_{H2}, +\infty)$ and draw a line Σ joining the associated point (T_1, D_1) on Δ_1 and the point C, i.e. $(2, 1)$. In this case, the slope of Σ is $(1 - D_1)/(2 - T_1)$. The critical inequality to check, which is in fact a necessary condition for a Hopf bifurcation to occur at $\varepsilon_\gamma = \varepsilon_{\gamma H}$, is that the slope of Δ is bigger than that of Σ, i.e. that $T(\varepsilon_{\gamma H}) < 2$. An ambiguity remains to be solved, as shown in table 1: when σ increases from σ_{H2}, D_1 decreases and the slope of Δ increases, but maybe with not enough speed to ensure that Δ always crosses $[BC]$ in its interior, at $\varepsilon_\gamma = \varepsilon_{\gamma H}$. We next solve the equality between the slope of Δ and that of Σ in σ, whose solutions are the roots of a second degree polynomial $Q(\sigma)$, and show that the slope of Δ is greater than that of Σ whenever $\sigma > \sigma_{H2}$.
If we equate the slope of Δ and that of Σ, with $\theta \overset{\text{def}}{=} 1 - \beta(1-\delta)$, we get the equality $1 + \theta(\varepsilon_\psi - (1 - s)/\sigma) = (1 - D_1)/(2 - T_1)$. If we replace by their expressions D_1 and T_1, and if we manipulate the resulting equality, we end up with $Q(\sigma) = a\sigma^2 + b\sigma + c = 0$. The coefficients of $Q(\sigma)$ are

$$\begin{aligned} a &= \varepsilon_\psi(\nu - (1+\theta)\varepsilon_\psi) > 0, \\ b &= s(\nu - \varepsilon_\psi) + \varepsilon_\psi(1 - s + \theta(2(1-s) + \nu)) > 0, \\ c &= (1-s)(s - \theta(1 - s + \nu)) > 0. \end{aligned}$$

The signs are derived from the assumptions of Lemma 4.1 and allow one to conclude that the roots of $Q(\sigma)$ are either real negative or complex. Therefore, $Q(\sigma)$ doesn't change sign when $\sigma > \sigma_{H2}$: the slope of Δ is always larger than that of Σ for every $\sigma > \sigma_{H2}$, i.e. a Hopf bifurcation is expected to occur for every $\sigma > \sigma_{H2}$.

The Local Bifurcation Values

We define $\theta \overset{\text{def}}{=} 1 - \beta(1-\delta)$, $\alpha \overset{\text{def}}{=} \theta(1-s)/s$, and $s_\Delta \overset{\text{def}}{=} 1 + \theta(\varepsilon_\psi - (1-s)/\sigma)$ the slope of the half-line Δ.

An eigenvalue of 1*:*
The condition $s_\Delta = 1$ gives the bifurcation value $\sigma_S = (1-s)/\varepsilon_\psi$.

Secondly, the equation of the line (AC) in Fig. 7, i.e. $1 - T + D = 0$, yields the second bifurcation value $\varepsilon_{\gamma S} = 1 + \nu/(1 - s - \varepsilon_\psi \sigma)$.

An eigenvalue of −1: the flip bifurcation.
The equality $s_\Delta = -1$ allows one to derive $\sigma_{F1} = \theta(1-s)/(2+\theta\varepsilon_\psi)$.
The line (AB) in Fig. 7, of equation $1 + T + D = 0$, yields $\varepsilon_{\gamma F} = (s(2+\alpha) + \theta\nu - \sigma(2 + 2\nu + (\theta - 2)\varepsilon_\psi))/(\sigma(2+\theta\varepsilon_\psi) - \theta(1-s))$.
The condition $\varepsilon_{\gamma F} = 1$ gives the last flip bifurcation value $\sigma_{F2} = (s(2+\alpha-\theta) + \theta(1+\nu))/(4 + 2\nu + (2\theta - 2)\varepsilon_\psi)$.

A pair of eigenvalues of modulus 1: the Hopf bifurcation.
The equality of the slope Δ and that of the line joining (T_1, D_1) and the point B in Fig. 7, i.e. $(-2, 1)$, is $s_\Delta = (1 - D_1)/(-2 - T_1)$. It yields a polynomial $Q_H(\sigma) \stackrel{\text{def}}{=} a\sigma^2 + b\sigma + c$, whose roots contain the bifurcation value σ_{H1}. In the configuration of Lemma 4.1, it is easily shown in Fig. 7 that there must exist two distinct real roots, and that σ_{H1} is the lowest. The coefficients of $Q_H(\sigma)$ are

$$\begin{aligned} a &= 4 + 4(\nu - \varepsilon_\psi) + \theta\varepsilon_\psi(4 + 3\nu + (\theta - 3)\varepsilon_\psi), \\ b &= -s(4 + \theta\varepsilon_\psi(3+\alpha) + \alpha(4 + 3\nu + (\theta-3)\varepsilon_\psi)) - \theta\nu(1 + \theta\varepsilon_\psi), \\ c &= \alpha s(s(3+\alpha) + \theta\nu). \end{aligned}$$

The condition $D_1 = 1$ yields $\sigma_{H2} = (s - \theta(1-s))/(\nu - (1+\theta)\varepsilon_\psi)$.
Finally, the bifurcation value $\varepsilon_{\gamma H} = (s - \sigma(1 + \nu - \varepsilon_\psi))/(\theta(1-s) - \sigma(1+\theta\varepsilon_\psi))$ follows from $D = 1$.

Indeterminacy of the steady states:
If we equate the slope of Δ and that of the line joining (T_1, D_1) and the point A, in Fig. 7, we derive $s_\Delta = (1 + D_1)/T_1$. This leads to a polynomial $Q_I(\sigma) \stackrel{\text{def}}{=} a\sigma^2 + b\sigma + c$,

$$\begin{aligned} a &= \theta\varepsilon_\psi(2 + \nu + (\theta - 1)\varepsilon_\psi), \\ b &= -\alpha s(2 + \nu + (\theta-1)\varepsilon_\psi) - \theta\nu - \theta\varepsilon_\psi(s(1+\alpha) + \theta\nu), \\ c &= \alpha s(s(1+\alpha) + \theta\nu). \end{aligned}$$

Again, a geometrical argument shows that the roots of $Q_I(\sigma)$ must be real and distinct, and that the lowest is σ_I.

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