C.E.P.R.E.M.A.P
JULY 1996

ON THE DIFFERENT NOTIONS OF ARBITRAGE AND EXISTENCE OF EQUILIBRIUM

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Abstract - In this paper we first give an existence of equilibrium theorem with unbounded below consumption sets, by the demand approach, assuming only that the individually rational utility set is compact. We then classify different notions of absence of arbitrage and give conditions under which they are equivalent to the existence of an equilibrium.

SUR LES DIFFERENTES NOTIONS D'ARBITRAGE ET L'EXISTENCE DE L'EQUILIBRE

Résumé - Dans ce papier, d'abord nous donnons un théorème d'existence d'un équilibre avec des ensembles de consommation non bornés inférieurement par l'approche de la demande en supposant seulement que l'ensemble des utilités individuellement rationnelles est compact. Ensuite, nous classons les différentes notions d'absence d'arbitrage et donnons des conditions qui les rendent équivalentes à l'existence d'un équilibre.

Keywords : No Arbitrage Price, Limited Arbitrage, Arbitrage Free, Viability, Market Arbitrage, Absence of Arbitrage. Individually Rational Utility Set, Quasi-Equilibrium, Equilibrium.

Mots Clés : Prix de non arbitrage, Arbitrage limité, Libre d'arbitrage, Viabilité, Arbitrage de Marché, Absence d'arbitrage, Ensemble d'utilités individuellement rationnelles, Quasi-équilibre, Equilibre.

JEL classification : D51, C62, G10.

INTRODUCTION

Since the early work of Debreu (1962), different sufficient (or necessary and sufficient) conditions have been used in the literature to prove existence of an Arrow-Debreu equilibrium when agent's consumption sets are unbounded below.

Among these conditions, one finds in particular :

i) The assumption that the individually rational attainable allocations set is compact,

ii) The assumption of existence of a "no arbitrage price" (in other words, a price for which aggregate demand exists),

iii) The assumption of absence of net trades that improve all agent's utilities.

These conditions are not always equivalent, hence one of the main purposes of this paper is to classify these assumptions and give conditions under which they are equivalent. A second purpose of the paper is to prove existence of an equilibrium under the assumption that the individually rational utility set is compact, by the demand approach. We improve existing results in the literature : our result is stronger than Nielsen's (1989) who assumes that the individually rational attainable allocations set is compact. It is also stronger than the results obtained by Brown, Werner (1995) and Dana, Le Van and Magnien (1995), who have used Negishi's approach in an infinite dimension setting but had to assume that preferred sets had non empty interior.

The paper is organised as follows :

In part one, we first set the model. We then give an example of a two-dimensional, two agents economy of which the individually rational utility set is compact while the individually rational attainable allocations is not. Lastly we prove existence of an equilibrium under the assumption that the individually rational utility set is compact.

In part two, we recall and compare various notions of absence of arbitrage for individuals and for the economy. We then compare the hypothesis of absence of arbitrage with that of compactness of the individually rational attainable utility set. Lastly we give conditions on utilities for equivalence between absence of arbitrage and existence of equilibrium.

1. EXISTENCE OF EQUILIBRIUM

1.1. The model

We consider an exchange economy $\mathcal{E}$ with m agents ($m \geq 1$). Agent i has a consumption set X_i which is a closed, convex, non-empty subset of $\mathbf{R}^\ell (\ell \geq 1)$, a utility function u_i from X_i to $\mathbf{R}$ and an initial endowment $e_i \in X_i$. We normalize the utility functions by requiring $u_i(e_i) = 0$. An *allocation* is a m-tuple $(x_1, .., x_m)$ with $x_i \in X_i$. Define $\bar{e} = \sum_i e_i$. An allocation is *attainable* if $\sum_{i=1}^{m} x_i = \bar{e}$. It is *individually rational attainable* if it is attainable and if $u_i(x_i) \geq u_i(e_i)$ for all i. We denote by A the set of all individually rational attainable allocations. The *individually rational utility set* $\mho$ is defined as follows :

$$\mho = \left\{(v_1, ..., v_m) \in \mathbf{R}^m_+ \mid \exists x \in A \text{ s.t. } 0 \leq v_i \leq u_i(x_i), \forall i\right\}.$$

A *quasi-equilibrium* is a pair $(x^*, p^*) \in A \times (\mathbf{R}^\ell \backslash \{0\})$ such that :

$$x_i \in X_i, u_i(x_i) > u_i(x_i^*) \Rightarrow p^* x_i \geq p^* x_i^* = p^* e_i.$$

(x^*, p^*) is an *equilibrium* if :

$$x_i \in X_i, u_i(x_i) > u_i(x_i^*) \Rightarrow p^* x_i > p^* x_i^* = p^* e_i.$$

An allocation $(x_1, ...x_m)$ is *Pareto-optimum* (P.O) if it belongs to A and if there exists no $(x_1', ...x_m')$ in A such that $u_i(x_i'), \geq u_i(x_i)$ for every i, with at least one strict inequality.

We list now the assumptions which might be used in the sequel of the paper.

(H1) u_i is quasi-concave, i.e. the set

$$\hat{P}_i(x_i) = \{y_i \in X_i \mid u_i(y_i) \geq u_i(x_i)\}$$

is convex for all $x_i \in X_i$;

(H2) u_i is *strictly* quasi-concave : if x_i and y_i belong to X_i and $u_i(y_i) > u_i(x_i)$ then $u_i(\lambda x_i + (1 - \lambda) y_i) > u_i(x_i)$ for all $\lambda \in [0, 1[$;

(H3) u_i is upper semi-continuous, i.e. the set $\hat{P}_i(x_i)$ is closed for all $x_i \in X_i$;

(H4) If x_i belongs to A_i (the projection of A on X_i), then the set

$$P_i(x_i) = \{y_i \in X_i \mid u_i(y_i) > u_i(x_i)\}$$

is non-empty ;

(H5) The sets $\hat{P}_i(x_i)$, for all $x_i \in X_i$, have the same asymptotic cone ;

(H6) $R_i(x_i) = \{x \in X_i \mid u_i(x) = u_i(x_i)\}$ does not contain a half-line for every $x_i \in X_i$;

(H7) Agent i has no satiation point i.e. $P_i(x_i) \neq \phi$ for all $x_i \in X_i$.

Let us recall that assumptions H2 and H3 imply H1 (see Debreu (1959)).

Let us denote by $W(x_i)$ the asymptotic cone of the set $\hat{P}_i(x_i)$ and $W_i(e_i) = W_i$ for short. Then assumption H5 means that $W_i(x_i) = W_i$ for all $x_i \in X_i$.

Notations

int(X) is the interior of X; if W is a cone, then its polar W^0 is :

$$W^0 = \{p \mid pw \leq 0, \forall w \in W\}.$$

For $x, y \in \mathbf{R}^\ell$, $]x,y[= \{\lambda x + (1-\lambda)y, \lambda \in]0,1[\}$, $[x,y] = \{\lambda x + (1-\lambda)y, \lambda \in [0,1]\}$;

If $X \subset \mathbf{R}^\ell, \bar{X}$ denotes its closure and ∂X its boundary.

1.2. An example when $\mathfrak{U}$ is compact while A is not

Now, we present an example of a finite asset market in which the set A of individually rational attainable allocations is not compact whereas the individually rational utility set $\mathfrak{U}$ is compact. We mention that in infinite dimension, there is already an example given by Cheng in L^P (Cheng, 1991). Unfortunately, one can show that in finite dimension this example fails to be true. The following seems to be the first in finite dimension.

Example 1. Consider an economy with two agents. Agent 1 has the following consumption set, $X_1 = \{(x,y) \in R^2 \mid y \geq -x/2\}$ and his utility function is defined by

(1) $u_1(x,y) = \dfrac{y+\frac{x}{2}}{x^2+\frac{x}{2}+1}$ if $(x,y) \in Z_1 = \left\{(x,y) \in X_1 \mid -\dfrac{x}{2} \leq y \leq x^2+1\right\}$,

(2) $u_1(x,y) = y - x^2$ if $(x,y) \in Z_2 = \{(x,y) \in X_1 \mid y \geq x^2+1\}$

Notice that

$$(3) \begin{cases} 0 \leq u_1 < 1 \text{ on } Z_1 \backslash Z_2 \\ u_1 = 1 \text{ on } Z_1 \cap Z_2 \\ u_1 > 1 \text{ on } Z_2 \backslash Z_1 \end{cases}$$

Clearly, u_1 is continuous on X_1.

In order to check that u_1 is strictly quasi-concave, let (x_0, y_0) and $(x_1, y_1) \in X_1$ be such that $u_1(x_1, y_1) > u_1(x_0, y_0)$. We shall prove that

$$u_1(x,y) > u_1(x_0, y_0) \text{ if } (x,y) = \lambda(x_1,y_1) + (1-\lambda)(x_0,y_0), \text{ with } 0 < \lambda < 1.$$

<u>Case 1</u> : Let $(x_0, y_0) \in Z_2$. It follows from (3) that $u_1(x_1, y_1) > 1$ and from (2) that :

(4) $y_1 - x_1^2 > y_0 - x_0^2$

In this case, the claim may easily be checked.

<u>Case 2</u> : Let $(x_0, y_0) \in Z_1 \backslash Z_2$ and let $\alpha = u_1(x_0, y_0)$.

Let $C_\alpha = \{(x,y) \in Z_1 \mid u_1(x,y) > \alpha\} = \left\{(x,y) \in \mathbf{R}^2 \mid y > \alpha x^2 + \dfrac{\alpha - 1}{2}x + \alpha\right\}$.

Clearly C_α is convex and since u_1 is continuous, $(x_0, y_0) \in \partial C_\alpha$.

<u>Subcase a</u> : $(x_1, y_1) \in Z_1$. Then, from (1) and since $u_1(x_1, y_1) > \alpha$, one has $(x_1, y_1) \in C_\alpha$. It implies that

(5) $(x,y) \in C_\alpha$ (since C_α is open and convex).

If $(x,y) \in Z_1$, then (1) and (5) imply $u_1(x,y) > \alpha = u_1(x_0, y_0)$.

If $(x, y) \in Z_2$, then $u_1(x, y) \geq 1 > u_1(x_0, y_0)$ from (3).

Subcase b : $(x_1, y_1) \in Z_2 \backslash Z_1$. Since $(x_0, y_0) \in Z_1 \backslash Z_2$, there exists $(a, b) \in Z_1 \cap Z_2$ such that

(6) $](x_0, y_0); (x_1, y_1)[\cap Z_2 = [(a, b); (x_1, y_1)[$

(7) $](x_0, y_0), (a, b)[\subset C_\alpha \cap Z_1$.

Indeed, (6) follows from the convexity of Z_2 and (7) from $(x_0, y_0) \in \partial C_\alpha$ and $(a, b) \in C_\alpha$ (since $(a, b) \in Z_1 \cap Z_2$, $u_1(a, b) = \dfrac{b + \frac{a}{2}}{a^2 + \frac{a}{2} + 1} = 1 > \alpha$).

If $(x, y) \in [(a, b); (x_1, y_1)[$, then $(x, y) \in Z_2$, thus $u_1(x, y) \geq 1 > \alpha = u_1(x_0, y_0)$.

If $(x, y) \in](x_0, y_0); (a, b)[$ then, from (7), $(x, y) \in C_\alpha \cap Z_1$ hence, from (1), $u_1(x, y) > \alpha = u_1(x_0, y_0)$.

Agent 2 has the following consumption set and utility function

$X_2 = \{(x, y) \in \mathbf{R}^2 \mid (y, x) \in X_1\}$ and $u_2(x, y) = u_1(y, x)$

Consumers 1 and 2 have the same initial endowments $e_1 = e_2 = (0, 0)$.

A is not compact :

Since $u_i(e_i) = \text{Min } u_i(X_i)$ $(i = 1, 2)$, A is the set of all pairs $((x, y), (-x, -y)) \in X_1 \times X_2$. Hence

(8) $A = \left\{((x, y), (-x, -y)) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid -\frac{x}{2} \leq y \leq -2x\right\}$

Notice that $x \leq 0$ and $y \geq 0$ for all $((x, y); (-x, -y))$ in A.

$\mathcal{U}$ is bounded :

If $((x, y); (-x, -y)) \in A$, then from (8) and since $-2x \leq x^2 + 1$ for all $x \in \mathbf{R}$, one has $(x, y) \in Z_1$. Thus, from (3), $u_1(x, y) \leq 1$. Moreover

$$-\frac{(-y)}{2} \leq -x \leq -2(-y)$$

since $-\frac{x}{2} \leq y \leq -2x$. Hence $(-y,-x) \in Z_1$ and $u_2(-x,-y) = u_1(-y,-x) \leq 1$.

$\mho$ is closed:

Consider a sequence $v^n = (v_1^n, v_2^n)$ in $\mho$ which converges to $v = (v_1, v_2) \in \mathbf{R}^2$. From the definition of $\mho$, there exists a sequence $((x_1^n, y_1^n), (x_2^n, y_2^n))$ in A such that

$$0 \leq v_i^n \leq u_i(x_i^n, y_i^n) \text{ for all } n \text{ and } i = 1,2.$$

Since $\mho$ is bounded, we may assume without lost of generality that $u_i(x_i^n, y_i^n) \longrightarrow \alpha_i$, hence $0 \leq v_i \leq \alpha_i$.

Suppose first that (x_1^n) is bounded. Then (y_1^n) is also bounded since, from (8), $0 \leq y_1^n \leq -2x_1^n$. Hence, by extracting subsequences, (x_i^n, y_i^n) tends to $(x_i, y_i) \in X_i$. One has $\alpha_1 = u_1(x_1, y_1)$ and $\alpha_2 = u_2(-x_1, -y_1)$, thus $v \in \mho$.

Now suppose that (x_1^n) tends to $-\infty$ (recall that $x_1^n \leq 0$). Since $y_1^n \leq -2x_1^n \leq (x_1^n)^2 + 1$ one has $(x_1^n, y_1^n) \in Z_1$. Then since $y_1^n \leq -2x_1^n$, we deduce from (1) that $\alpha_1 = 0$. Moreover, (8) implies $-\frac{x_1^n}{2} \leq y_1^n$ or $-x_1^n \leq -2(y_1^n)$. Hence $-x_1^n \leq (-y_1^n)^2 + 1$. Then from (1), one deduces that $u_1(-y_1^n, -x_1^n)$ tends to 0 so that $\alpha_2 = 0$. Hence $v_i = u_i(e_i)(i = 1,2)$ and $v \in \mho$.

(INSERT FIGURE 1)

1.3. Existence of an equilibrium

We shall now state a theorem which is stronger than Nielsen's theorem who requires the compactness of the individually rational atttainable allocations set (see Nielsen, 1989). But, as shown in the previous section, this assumption may not be satisfied while the individually rational utility set $\mho$ is compact.

Theorem 1. *Assume H2, H3, H4 and $\mho$ compact, then there exists a quasi-equilibrium.*

The idea of the proof is as follows. We first truncate consumption sets with a ball of radius r. Using the same trick as in Florenzano-Le Van (1986), we then prove that, for r large enough, there exists a quasi-equilibrium for the truncated economy. This trick consists in adding fiat money to the model. It is used in Kajii (1996). Under the assumptions of the model, the quasi-equilibrium price of fiat money equals zero. Lastly, as $n \longrightarrow \infty$, we deduce the existence of a quasi-equilibrium $(x*, p*)$ for the original economy.

We shall use the following lemma :

Lemma 1.1- *Let $P \subseteq \mathbf{R}^\ell$ be a closed convex cone which is not a linear subspace. Let P^0, B and S denote respectively the polar of P, the unit ball and the unit sphere of $\mathbf{R}^\ell$. Let Z be an u.s.c., non-empty, compact, convex-valued correspondence from $S \cap P$ into $\mathbf{R}^\ell$ such that*

$$\forall p \in S \cap P, \exists z \in Z(p), p.z \leq 0.$$

Then there exists $\bar{p} \in S \cap P$ such that $Z(\bar{p}) \cap P^0 \neq \phi$.

Proof : See Florenzano-Le Van (1986).□

Proof of Theorem 1

Consider the truncated economy obtained by replacing agents' consumption sets $X_i, i = 1, ..., m$ by $X_i^n = X_i \cap B^n$ where B^n is the ball of radius n. Let n be large enough so that $e_i \in X_i^n$.

Let ξ_i^n and Q_i^n be the two correspondences in $S \cap (\mathbf{R}^\ell \times \mathbf{R}_+)$ defined as follows:

$$\begin{aligned} \xi_i^n(p,q) &= \{x_i \in X_i^n \mid u_i(x_i) \geq u_i(e_i) \text{ and } px_i \leq pe_i + q\}, \\ Q_i^n(p,q) &= \{x_i \in \xi_i^n(p,q) \mid y_i \in X_i^n \text{ and } u_i(y_i) > u_i(x_i) \Longrightarrow py_i \geq pe_i + q\}. \end{aligned}$$

Lemma 1.2 - *For all $i = 1, ..., m, Q_i^n$ is an u.s.c non-empty, convex compact-valued correspondence.*

Proof : see Appendix.

Let $Z^n(p,q) = \left[\sum\limits_{i=1}^{m} Q_i^n(p,q) - \bar{e}\right] \times \{-m\}, \forall (p,q) \in S \cap (\mathbf{R}^\ell \times \mathbf{R}_+)$.

It follows from lemma 1.2 that Z^n is u.s.c, non-empty, compact convex-valued.

Moreover $(p,q).x \leq 0, \forall (p,q) \in S \cap (\mathbf{R}^\ell \times \mathbf{R}_+), \forall x \in Z^n(p,q)$.

It follows from lemma 1.1 that there exists $(p^n, q^n) \in S \cap (\mathbf{R}^\ell \times \mathbf{R}_+)$ such that:

$$Z^n(p^n, q^n) \cap (\mathbf{R}^\ell \times \mathbf{R}_+)^0 \neq \phi.$$

Since $(\mathbf{R}^\ell \times \mathbf{R}_+)^0 = 0_{\mathbf{R}^\ell} \times \mathbf{R}_-$, there exists $x_i^n \in Q_i^n(p^n, q^n)$, $\forall i = 1, ..., m$, such that $\sum_{i=1}^m x_i^n = \bar{e}$. Since $x_i^n \in Q_i^n(p^n, q^n)$,we have $p^n x_i^n \leq p^n e_i + q^n$ and

(9) $u_i(x_i) > u_i(x_i^n) \Longrightarrow p^n x_i \geq p^n e_i + q^n$, $\forall i, \forall x_i \in X_i^n$.

Since $\mathcal{U}$ and S are compact, we may assume without lost of generality that

(10) $(u_i(x_i^n))_{i=1}^m \underset{n \to \infty}{\longrightarrow} v = (v_i)_{i=1}^m \in \mathcal{U}$ and $(p^n, q^n) \underset{n \to \infty}{\longrightarrow} (p^*, q^*) \in S$.

By definition of v, there exists $x^* \in A$ such that

(11) $u_i(x_i^*) \geq v_i$ for all $i = 1, ..., m$.

We will prove that (x^*, p^*) is a quasi-equilibrium of the initial economy. Indeed, let $x_i \in X_i$ be such that $u_i(x_i) > u_i(x_i^*)$. (Since $x^* \in A$, the existence of such an x_i follows from H4). Let $\lambda \in]0,1]$ and $x_i^\lambda = \lambda x_i + (1-\lambda)x_i^*$. It follows from H2 that $u_i(x_i^\lambda) > u_i(x_i^*)$. Hence, from (11) and (10), there exists n_λ , such that $x_i^\lambda \in X_i^n$ and $u_i(x_i^\lambda) > u_i(x_i^n)$ for all $n \geq n_\lambda$, hence, by (9):

$$p^n x_i^\lambda \geq p^n e_i + q^n, \forall n \geq n_\lambda.$$

When $n \to \infty$, we get :

(12) $\lambda\, p^* x_i + (1-\lambda) p^* x_i^* \geq p^* e_i + q^*, \forall \lambda \in]0,1], \forall i$

and, when $\lambda \to 0 : p^* x_i^* \geq p^* e_i + q^*, \forall i$.

Since $x^* \in A$ and $q^* \geq 0$, adding these inequalities, we get that $q^* = 0$ (so that $p^* \neq 0$) and $p^* x_i^* = p^* e_i, \forall i$.

Lastly, applying (12) with $\lambda = 1$ we obtain :

$$\forall i, \forall x_i \in X_i, u_i(x_i) > u_i(x_i^*) \Rightarrow p^* x_i \geq p^* e_i.$$

Hence, (x^*, p^*) is a quasi-equilibrium. $\square$

Remark 1.1

We will prove now that for n large enough, $q^n = 0$ and then (x^n, p^n) is a quasi-equilibrium of the truncated economy. Indeed, for n large enough, x_i^n is not a satiation point for i in X_i^n. If not there would exist a subsequence $(x_i^{n_k})_k$ with $x_i^{n_k}$ a satiation point for i in $X_i^{n_k}$.Hence, for all $x_i \in X_i$, one has $u_i(x_i^{n_k}) \geq u_i(x_i)$ for k large enough. Thus $u_i(x_i^*) \geq v_i \geq u_i(x_i)$, in contradiction with H4. We claim that $p^n x_i^n = p^n e_i + q^n$. Indeed, let $y_i \in X_i^n$ be such that $u_i(y_i) > u_i(x_i^n)$ and let $y_i^\lambda = \lambda x_i^n + (1-\lambda) y_i$ with $\lambda \in [0, 1[$. Then $u_i(y_i^\lambda) > u_i(x_i^n)$ by H2; hence $p^n y_i^\lambda \geq p^n e_i + q^n$. As $\lambda \to 1$, one has $p^n x_i^n \geq p^n e_i + q^n$, hence $p^n x_i^n = p^n e_i + q^n$. Summing over i one gets $q^n = 0$.

Remark 1.2

Observe that p^* is the limit of p^n but x^* is not necessarely the limit of $(x_i^n)_{i=1}^m$ nor is $u_i(x_i^*)$ the limit of $u_i(x_i^n)$.

Remark 1.3

In example 1, assumption H2,H3,H4 are satisfied. Hence, by Theorem 1, there exists a quasi-equilibrium.

2. RELATIONS BETWEEN DIFFERENT NOTIONS OF ARBITRAGE AND THE UTILITY SET COMPACTNESS

2.1. Useful commodity bundles

Let $\bar{u}_i = \sup_{x_i \in X_i} u_i(x_i)$.

Recall that W_i denote the asymptotic cone of $\hat{P}_i(e_i)$. $w \in W_i$ iff $e_i + tw \in X_i$ and $u_i(e_i + tw) \geq u_i(e_i), \forall t \geq 0$. Equivalently $w \in W_i$ iff $x_i + w \in X_i$ and $u_i(x_i + w) \geq u_i(e_i), \forall x_i \in \hat{P}_i(e_i)$. Equivalently $w \in W_i$ iff there exists a sequence (x_n) such that $u_i(x_n) \geq u_i(e_i)$ and a sequence $(\lambda_n) \in \mathbf{R}_+$ such that $\lambda_n x_n \to w$ and $\lambda_n \to 0$ when $n \to \infty$.

Definition 2.1 - w is W-useful if $w \in W_i$.

W_i is a closed convex cone. In order to summarize its properties, let us introduce a lemma :

Lemma 2.1. - *Let Γ be a cone in $\mathbf{R}^\ell$ and let $w \in int(\Gamma)$. Then*

$$\forall x \in \mathbf{R}^\ell, \exists t_0 \geq 0 \text{ such that } x + tw \in \Gamma, \forall t > t_0.$$

Proof : Since $w \in int(\Gamma)$, there is a ball $B(w,r)$ such that $B(w,r) \subset \Gamma$. Since Γ is a cone, $\forall t > 0, tB(w,r) = B(tw,tr) \subset \Gamma$.

Let $x \in \mathbf{R}^\ell$, then

$$\|(x + tw) - tw\| = \|x\| < tr \text{ if } t > t_0 = \frac{\|x\|}{r};$$

hence $x + tw \in B(tw,tr) \subset \Gamma$ if $t > t_0$. □

We next summarize the properties of W_i.

Proposition 2.1 - (1) *Assume H1, H3, H7. Then $W_i \neq \{0\}$.*
(2) Assume H2 and H3. Then W_i contains a line iff $R_i(e_i)$ contains a line.

(3) Assume H2, H3, H6. Then $u_i(e_i + tw) > u_i(e_i)$, $\forall t > 0, \forall w \in W_i - \{0\}$.

(4) Assume H2, H3, H5 (resp. H6). Then $\forall w \in W_i - \{0\}, \forall x_i \in X_i$, *the map* $t \to u_i(x_i + tw)$ *is non decreasing (resp. increasing). Assume H2, H3, H5, H6, then* $W_i \neq \{0\} \Longleftrightarrow$ *H7.*

(5) Assume H1, H5, H7 and int $(W_i) \neq \phi$. *Then* $\forall w \in$ *int* (W_i), *the map* $t \to u_i(e_i + tw)$ *is non-decreasing and* $\lim_{t \to \infty} u_i(e_i + tw) = \bar{u}_i$.

Proof - To prove assertion (1), assume H1, H3, H7 and $W_i = \{0\}$. Then $\hat{P}_i(e_i)$ is compact, hence there exists s_i such that $u_i(s_i) = \sup_{x_i \in X_i} u_i(x_i)$. Hence $P_i(s_i) = \phi$ contradicting H7.

To prove (2), clearly if $R_i(e_i)$ contains a line, then W_i contains a line. Conversely if W_i contains a line, then there exists $w_i \in W_i - \{0\}$ with $-w_i \in W_i$. Therefore,

$$\forall t > 0, u_i(e_i + tw_i) \geq u_i(e_i) \text{ and } u_i(e_i - tw_i) \geq u_i(e_i).$$

If for some $t, u_i(e_i + tw_i) \neq u_i(e_i - tw_i)$, then under H2, $u_i(e_i) > \min\{u_i(e_i + tw_i), u_i(e_i - tw_i)\}$: a contradiction. Hence $u_i(e_i + tw_i) = u_i(e_i - tw_i) = u_i(e_i), \forall t > 0$ and $R_i(e_i)$ contains a line.

Proof of (3) : Assume that $u_i(e_i + tw_i) = u_i(e_i)$ for some $t > 0$ and $w_i \in W_i - \{0\}$. Let $t' > t$. If $u_i(e_i + t'w_i) > u_i(e_i + tw)$, then from H2, $u_i(e_i + tw_i) > u_i(e_i)$: a contradiction. Hence $u_i(e_i + t'w_i) = u_i(e_i), \forall t' \geq t$ contradicting H6.

Proof of (4) : Under H2, H3, H5 and H6, it follows from (3) that

$$\forall w_i \in W_i, \forall x_i \in X_i, \forall t > 0, u_i(x_i + tw_i) > u_i(x_i).$$

Let $t' > t$. Then $u_i(x_i + t'w_i) = u_i(x_i + (t' - t)w_i + tw_i) > u_i(x_i + tw_i) > u_i(x_i)$.

Hence under H2, H3, H5 and H6, if $W_i \neq \{0\}$, then $P_i(x_i) \neq \phi, \forall x_i$.

Proof of (5) : Let $\text{int}(W_i) \neq \phi$. Let $x_i \in X_i$. It follows from lemma 2.1 that there exists t_0 such that $e_i - x_i + tw \in W_i, \forall t \geq t_0$. Equivalently $e_i + tw \in x_i + W_i, \forall t \geq t_0$. Under H5, $u_i(e_i + tw_i) \geq u_i(x_i), \forall t \geq t_0$. It follows from H5, that the map $t \to u_i(e_i + tw)$ is non decreasing and that $\lim_{t \to \infty} u_i(e_i + tw) = \bar{u}_i$. □

We now generalize the notion of C-useful vectors introduced by Chichilnisky (1994).

Definition 2.2 - $w \in \mathbf{R}^\ell$ is C-useful for i if $e_i + tw \in X_i, \forall t > 0$ and $\forall x_i \in X_i, \exists t > 0$ such that $u_i(e_i + tw) \geq u_i(x_i)$.

Let C_i denote the set of C-useful vectors for i and σ_i the set of satiation consumptions ($s_i \in \sigma_i$ iff $u_i(s_i) = \bar{u}_i = \sup\limits_{x_i \in X_i} u_i(x_i)$). Clearly C_i is a cone.

In order to characterize C_i, we first prove two lemmas.

Lemma 2.2 - *Let $f : \mathbf{R}_+ \to \mathbf{R}$ be an upper semi-continuous quasi concave function. Then there exists $x_0 \in [0, \infty]$ such that f is non decreasing on $[0, x_0]$ and f is non increasing on $[x_0, +\infty[$.*

The proof of Lemma 2.2 may be found in the appendix.

Lemma 2.3 - *Assume H2 and H3. Let $w \in C_i$. Then*
i) either for some $t_0 \geq 0$, $e_i + t_0 w \in \sigma_i$.
ii) or the map $t \to u_i(e_i + tw)$ is non decreasing on $\mathbf{R}_+$ and $\lim\limits_{t\to\infty} u_i(e_i + tw) = \bar{u}_i$.

Proof - Let $w \in C_i$, then the map $t \to u_i(e_i + tw)$ is quasi-concave and upper semi-continuous on $\mathbf{R}_+$, hence by lemma 2.2 either it is non decreasing on $\mathbf{R}_+$ or there exists $t_0 \geq 0$ such that it is non decreasing on $[0, t_0]$ and non increasing on $[t_0, +\infty[$. In the last case, $u_i(e_i + t_0 w) = \bar{u}_i$. In the first case, $\lim\limits_{t\to\infty} u_i(e_i + tw)$ exists and $\lim\limits_{t\to\infty} u_i(e_i + tw) = \bar{u}_i$ since w is C-useful for i. □

We next compare W_i and C_i.

Proposition 2.2 - *Assume H1 and H3. Then*
1) If $\sigma_i \neq \phi$ then

$$C_i = \{\lambda(s_i - e_i), s_i \in \sigma_i, \lambda \in \mathbf{R}_{++}\} \text{ (hence } C_i \neq \phi).$$

2) Assume moreover H7. Then
(i) C_i is a convex cone (which may be empty),
(ii) $\bar{C}_i \subseteq W_i$,

(iii) Under H5, if int $(W_i) \neq \phi$, *then int* $(W_i) \subseteq C_i$,
(iv) $C_i \cup \{0\} = W_i$ *iff* $\lim_{t\to\infty} u_i(e_i + tw) = \bar{u}_i, \forall w \in W_i \backslash \{0\}$.

Proof - Proof of 1) : By definition of C_i, if $\sigma_i \neq \phi$, then $\forall w \in C_i$, there exists $t_0 > 0$ such that $u_i(e_i + t_0 w) = \bar{u}_i$. Hence $e_i + t_0 w \in \sigma_i$ and $C_i \subseteq \{\lambda(s_i - e_i), \lambda > 0, s_i \in \sigma_i\}$. Conversely let $s_i \in \sigma_i$. Then $u_i(e_i + s_i - e_i) = u_i(s_i) \geq u_i(x_i), \forall x_i \in X_i$, hence $s_i - e_i \in C_i$ which implies that $\{\lambda(s_i - e_i), \lambda > 0\} \subseteq C_i$, since C_i is a cone.

The proof of 2.i) is omitted. To prove (ii) let $w \in C_i$. Under H7, it follows from Lemma 2.3 that for $t \geq 0, u_i(e_i + tw) \geq u_i(e_i)$. Hence $w \in W_i$. Since W_i is closed, $\bar{C}_i \subseteq W_i$.

Proof of 2.iii) : It follows from proposition 2.1, (5) that $\lim_{t\to\infty} u_i(e_i + tw) = \bar{u}_i, \forall w \in$ int (W_i). Hence int $(W_i) \subseteq C_i$.

2.iv) follows from Lemma 2.3. □

Remark 2.1 - 1) If $\sigma_i \neq \phi$, then the inclusion $C_i \subseteq W_i$ may not hold as shown by the following example :

Let $X_i = \mathbf{R}$,$u_i : \mathbf{R} \to \mathbf{R}$ be defined by $u_i(x) = -(x-1)^2 + 1$ and $e_i = 0$. Then $\sigma_i = \{1\}$, $C_i = \mathbf{R}_{++}$ but $W_i = \{0\}$.

2) C_i may be empty as shown in the following example :
$X_i = \{(x,y) \mid x \geq 0, 0 \leq y \leq \sqrt{x}\}, u_i(x,y) = y$ and $e_i = (0,0)$. Then $W_i = \{\lambda(1,0), \lambda \geq 0\}$ and $C_i = \phi$.

2.2. Different notions of arbitrage

Let us now introduce difference notions of arbitrage.

Definitions 2.3 - A price vector $p \in \mathbf{R}^\ell$ is a "no arbitrage price" for i if $p.w_i > 0, \forall w_i \in W_i - \{0\}$.

A price vector $p \in \mathbf{R}^\ell$ is "limited arbitrage" for i, if $pw > 0, \forall w \in \bar{C}_i \cap \sum$ where $\sum$ denotes the unit sphere of $\mathbf{R}^\ell$.

A price vector $p \in \mathbf{R}^\ell$ is "arbitrage free" for i if for all sequence (x_n) in $\mathbf{R}^\ell$ such that $e_i + x_n \in X_i, \forall n, \lim_{n\to\infty} u_i(e_i + x_n) = \bar{u}_i$ and $\lim_{n\to\infty} px_n$ exists, then $\lim_{n\to\infty} px_n > 0$.

A price vector $p \in \mathbf{R}^\ell$ is "viable" if the problem $\begin{cases} \max\ u_i(x) \\ px \leq pe_i \\ x \in X_i \end{cases}$ has a solution.

Let $S_i = \left\{p \in \mathbf{R}^\ell \mid pw_i > 0, \forall w_i \in W_i - \{0\}\right\}$ denote the set of no-arbitrage prices.

Let W_i^0 denote the polar of $W_i : W_i^0 = \left\{p \in \mathbf{R}^\ell \mid pw \leq 0, \forall w \in W_i\right\}$.

Proposition 2.3 - *Assume H1 and H3. Then* $S_i = int\ (-W_i^0)$ *and* $S_i \neq \phi \Leftrightarrow W_i$ *contains no line,* $\Leftrightarrow R_i(e_i)$ *contains no line.*

Proof - Let $\Phi_i(p) = \min_{w \in W_i \cap \sum} p.w$ where $\sum$ is the unit-sphere. Then Φ_i is continuous and by definition of the polar set, $p \in -W_i^0$ iff $\Phi_i(p) \geq 0$. Hence $\Phi_i(p) > 0$ implies $p \in$ int $(-W_i^0)$.

Conversely if $p \in$ int $(-W_i^0), \Phi(p+z) \geq 0, \forall z \in B(0,\varepsilon)$ with $\varepsilon > 0$. Hence $pw \geq zw$, $\forall w \in W_i, \forall z \in B(0,\varepsilon)$ and hence $pw > 0$ $\forall w \in W_i$ and $p \in S_i$.

Since W_i is closed, $(W_i^0)^0 = W_i$. Hence, by Rockafellar's corollary 14.6.1, if W_i contains no line then int $(W_i^0) \neq \phi$.

Conversely if W_i contains a line, $S_i = \phi$ (if not $pw_i > 0, -pw_i > 0$ for some p and $w_i \neq 0$, a contradiction). □

In the following proposition, we study the relation between the various notions of arbitrage that have been introduced.

Proposition 2.4 - *1) Assume H1 and H3. If* $p \in S_i$, *then* p *is viable.*
2) Assume H2, H3, H5 and H6. Then $p \in S_i$ *iff* p *is viable.*

Proof - (1) is obvious. To prove 2) let p be viable. Let $x_i(p)$ be an optimal solution to the demand problem. By proposition 2.1, $u_i(x_i(p)+w_i) > u_i(x_i(p)), \forall w_i \in W_i - \{0\}$; hence $pw_i > 0$ which implies that p is no arbitrage. □

Proposition 2.5 - *1) Assume H1, H3 and H7. If* $p \in S_i$, *then* p *is arbitrage free and limited arbitrage.*

2) Assume H1, H3, H7 and $\lim_{n\to\infty} u_i(e_i + tw) = \bar{u}_i, \forall w \in W_i - \{0\}$. *Then:* $p \in S_i \Longleftrightarrow$ *p is limited arbitrage* $\Leftrightarrow$ *p is arbitrage free* $\Leftrightarrow$ *p is viable.*

3) Assume H1, H3, H5 and int $(W_i) \neq \phi$, *then :*

$$\textit{limited arbitrage} \Leftrightarrow \textit{no arbitrage}$$

Proof -To prove 1), let $p \in S_i$. If p is not arbitrage free, there exists a sequence (x_n) such that

$$\lim_{n\to\infty} u_i(e_i + x_n) = \bar{u}_i \text{ and } \lim_{n\to\infty} px_n \leq 0.$$

Under H7, $u_i(e_i + x_n) \geq u_i(e_i)$ for n large enough and (x_n) is unbounded.

Indeed, assume the contrary: (x_n) is bounded. One may assume that the sequence $(u_i(e_i + x_n))$ is increasing and $x_n \to \bar{x}$. Let $I_n = \{x \mid u_i(e_i + x) \geq u_i(e_i + x_n)\}$. By H3, I_n is closed. Since $x_j \in I_n$ if $j \geq n$, we have $\bar{x} \in I_n, \forall n$. In other words, $u_i(e_i + \bar{x}) \geq u_i(e_i + x_n)$, $\forall n$: a contradiction with H7.

We may assume without lost of generality that $\dfrac{e_i + x_n}{\|x_n\|} \to w \in W_i - \{0\}$.

But then $pw = \lim_{n\to\infty} p\dfrac{x_n}{\|x_n\|} \leq 0$ contradicting the assumption that $p \in S_i$.

Under H1, H3 and H7, by proposition 2.2, $\bar{C}_i \subseteq W_i$. Hence if $p \in S_i$, then $pw > 0, \forall w \in \bar{C}_i \cap \sum$, implying that p is limited arbitrage.

To prove 2) : By proposition 2.2, we have $W_i = C_i \cup \{0\}$, hence $p \in S_i$ iff p is limited arbitrage.

If $\lim_{t\to\infty} u_i(e_i + tw) = \bar{u}_i$ for every $w \in W_i - \{0\}$, and if p is arbitrage free, we have $\lim_{t\to\infty} t\ p\ w > 0$; hence $pw > 0$.Thus if p arbitrage free then $p \in S_i$. The converse is true by 1).

Let us prove that if $\lim_{t\to\infty} u_i(e_i + tw) = \bar{u}_i, \forall w \in W_i - \{0\}$, then p viable implies $p \in S_i$. Since $u_i(e_i + tw) \to \bar{u}_i$, one has $u_i(e_i + tw) > u_i(x_i(p))$, for t large enough.

Hence $p(e_i + tw) > pe_i$. Under H7, $px_i(p) = p.e_i$, thus $pw > 0$. By proposition 2.4.1) the converse is true.

To prove 3) : by proposition 2.2, under H5 and if int $(W_i) \neq \phi$, we have $W_i = \bar{C}_i$. Thus $p \in S_i$ iff p is limited arbitrage. □

We next give an example of a price that is viable, arbitrage free, limited arbitrage but which is *not* no arbitrage, because H6 is not satisfied.

Example 2.1 - Let $X_i = \{(x,y) \in \mathbf{R}^2 \mid x \geq 0 \text{ and } 0 \leq y \leq \sqrt{x}\}, u_i(x,y) = y$ and $e_i = (0,0)$. Then $W_i = \{\lambda(1,0), \lambda \in \mathbf{R}_+\}$ and $S_i = \{p = (p_1,p_2) \in \mathbf{R}^2 \mid p_1 > 0\}$. Let $p = (0,1)$. $p \notin S_i$, but p is viable since the problem

$$\begin{cases} \max\ y \\ y \leq 0 \\ (x,y) \in X_i \end{cases}$$

has a solution. p is arbitrage free. Indeed if $u_i(x_n,y_n) = y_n \rightarrow \infty$ then $p(x_n,y_n) = y_n \rightarrow \infty$. p is limited arbitrage since $C_i = \phi$ (remark 2.1.2).

In this example, H1, H3, H5, H7 are fulfilled, but H6 is violated. One can also remark that $W_i \neq \bar{C}_i$.

2.3. No arbitrage, market-arbitrage and compactness of the utility set

A price vector $p \in \mathbf{R}^\ell$ is a "no arbitrage" price for the economy $\mathcal{E}$ if $pw_i > 0$, $\forall w_i \in W_i - \{0\}, \forall i = 1,...,m$. Equivalently $p \in \mathbf{R}^\ell$ is a "no arbitrage" price for $\mathcal{E}$ iff $p \in \bigcap_{i=1}^{m} S_i$.

A vector $(w_1,...,w_m) \in \prod_{i=1}^{m} W_i$ with $w_i \neq 0$ for some i is a "market arbitrage" if $\sum_{i=1}^{m} w_i = 0$.

There is "no market arbitrage" if $\sum_{i=1}^{m} w_i = 0$ with $w_i \in W_i, \forall i$, implies $w_i = 0, \forall i$.

Proposition 2.6 - *Assume H1, H3 and that $R_i(e_i)$ contains no line. Then the following are equivalent:*

a) There exists no market arbitrage,

b) $\sum_{i=1}^{m} W_i$ *contains no line,*

c) $\bigcap_{i=1}^{m} S_i \neq \phi$.

Proof : $\underline{a \Rightarrow b}$ - Assume that $\sum_{i=1}^{m} W_i$ contains a line. Then there exists $y \neq 0, (w_1, ..., w_m)$ and $(w'_1, ..., w'_m)$ such that

$$y = (w_1 + ... + w_m) \text{ and } -y = (w'_1 + ... + w'_m)$$

Hence $\sum_{i=1}^{m}(w_i + w'_i) = 0$. Since there is no market arbitrage, $w_i + w'_i = 0, \forall i$. From proposition 2.3, W_i contains no line, hence $w_i = 0, \forall i$, contradicting $y \neq 0$.

$\underline{b \Rightarrow c}$ - Let us first remark that if $\sum_{i=1}^{m} W_i$ contains no line, then $\sum_{i=1}^{m} W_i$ is closed (Debreu, 1959). Hence, from Rockafellar (1970), corollary 16.4.2, one has $\sum_{i=1}^{m} W_i = (\bigcap_{i=1}^{m} W_i^0)^0$ and from Rockafellar (1970), Corollary 16.4.1, $\bigcap_{i=1}^{m} W_i^0$ has non empty interior. Since int $(\bigcap_i W_i^0) = \bigcap_i$ int $(W_i^0) = -\bigcap_{i=1}^{m} S_i$ (proposition 2.3), we have $\bigcap_{i=1}^{m} S_i \neq \phi$.

$\underline{c \Rightarrow a}$ - The proof which is obvious is omitted. □

Proposition 2.7 - *Assume H1 and H3. Then there is no market arbitrage iff A compact.*

If furthermore, $R_i(e_i)$ contains no line, $\forall i$, then

$\bigcap_{i=1}^{m} S_i \neq \phi \Leftrightarrow$ *no market arbitrage* $\Leftrightarrow$ *A compact* $\Rightarrow \mathcal{U}$ *compact.*

Proof - Let A_∞ denote the asymptotic cone of A :

$$A_\infty = \left\{ (w_1, ..., w_m) \in \prod_{i=1}^{m} W_i \mid \sum_{i=1}^{m} w_i = 0 \right\}$$

Hence $A_\infty = \{0\}$ iff there is no market arbitrage. Since A is closed and convex, $A_\infty = \{0\}$ iff A compact. The implication: A compact $\Rightarrow \mho$ compact follows from H3. The rest of the assertion follows from the proposition 2.6. □

Observe that in example 1, $\mho$ is compact while A is not because $R_i(e_i)$ contains a line.

2.4. Arbitrage and equilibrium

Proposition 2.8 - *(1) Assume H1, H3, H5 and H6. Then* $\bigcap_{i=1}^{m} S_i \neq \phi \Leftrightarrow A$ *compact* $\Leftrightarrow \mho$ *compact* $\Leftrightarrow$ *Existence of a P.O.* $\Leftrightarrow$ *No market arbitrage.*

(2) Assume furthermore H2 and H4. Then any previous assertion implies existence of a quasi-equilibrium.

(3) Assume furthermore

(*) $\forall i$, *inf* $pX_i < pe_i, \forall p \in \bigcap_i \tilde{S}_i, p \neq 0.$

Then any previous assertion is equivalent to existence of an equilibrium.

Assume furthermore : $\forall i$, *int* $(W_i) \neq \phi$. *Then any previous assertion is equivalent to limited arbitrage or to existence of an equilibrium*

Proof - (1) From proposition 2.7, it suffices to show that under H5, H6, $\mho$ compact implies existence of a P.O. which in turn implies no market arbitrage. Assume $\mho$ compact. Let $g : \mho \to \mathbf{R}$ be defined by $g(v) = \sum_{i=1}^{m} v_i$. Since g is continuous, g has a maximum $v^* = (v_1^*, ..., v_m^*) \in \mho$. Hence there exists $x^* \in A$ such that $v_i^* = u_i(x_i^*)$. Clearly x^* is a P.O.

Let $x^* = (x_1^*, ..., x_m^*)$ be a P.O and assume that there exists $w^* = (w_1^*, ..., w_m^*)$ a market arbitrage. Then $w_i^* \neq 0$ for some i. Under H5, H6 $u_i(x_i^* + w_i^*) \geq u_i(x_i^*), \forall i$, with a strict inequality for some i, contradicting the assumption that x^* is a P.O.

(2) follows from theorem 1.

(3) If (x^*, p^*) is a quasi-equilibrium, then $p^* \in \cap \bar{S}_i$ since $\bar{S}_i = -W_i^0$ by proposition 2.3. Hence $\inf p^* X_i < p^* e_i$. Then (x^*, p^*) is an equilibrium.

Conversely if (x^*, p^*) is an equilibrium, under H2, H3, H4, H5 and H6

$$\forall i,\ u_i(x_i^* + w_i) > u_i(x_i^*), \forall w_i \in W_i - \{0\}\,.$$

Hence $p^* w_i > 0$, $\forall w_i \in W_i - \{0\}$, $\forall i$, equivalently $p^* \in \bigcap_{i=1}^{m} S_i$ which therefore is non empty.

The last claim follows from 3) in proposition 2.5. □

We exhibit examples which show that if, in proposition 2.8, either assumption H6 or condition (*) is dropped, then we may have "pathological" situations.

Example 2.2 - A two agents economy which has an equilibrium but no arbitrage-price. H2, H3, H4, H5 are fulfilled but not H6.

Let $X_1 = \{(x, y) \in \mathbf{R}^2 \mid y \geq 0\}$ and let $u_1(x, y) = y$ and $e_1 = (0, 0)$.

Let $X_2 = \{(x, y) \in \mathbf{R}^2 \mid x \in [0, 1]\,, y \geq 0\}$ and let $u_2(x, y) = y$ and $e_2 = (0, 0)$.

$W_1 = \{(x, y) \in \mathbf{R}^2 \mid y \geq 0\}$. Obviously $S_1 = \phi$.

$W_2 = \{\lambda(0, 1), \lambda \geq 0\}$, hence $S_2 = \{(x, y) \mid y > 0\}$. $S_1 \cap S_2 = \phi$: there is no arbitrage-price.

Moreover, $A = \{((x_1, y_1), (x_2, y_2)) \in X_1 \times X_2 \mid x_1 + x_2 = 0, y_1 + y_2 = 0\}$,

$$= \{((x, 0), (-x, 0)) \in \mathbf{R}^4 \mid x \in [0, 1]\};$$

hence A is compact.

Let $p^* = (0, 1)$. We claim that (e_1, e_2, p^*) is an equilibrium. Indeed :

$$\begin{aligned} u_1(x, y) &> u_1(0, 0) \Leftrightarrow y > 0 \Leftrightarrow p^*.(x, y) > 0 \\ u_2(x, y) &> u_2(0, 0) \Leftrightarrow y > 0 \Leftrightarrow p^*.(x, y) > 0. \end{aligned}$$

Example 2.3 A two agents economy which satisfies H2, H3, H4, H5, H6 and not (*). It has a quasi-equilibrium which is not P.O and no equilibrium.

Let

$$X_1 = \left\{(x,y) \in \mathbf{R}^2_+ \mid x \in [0,1], y \in [0,2]\right\}, u_1(x,y) = x,\ e_1 = (0,1),$$

and $X_2 = \left\{(x,y) \in \mathbf{R}^2_+ \mid x \leq 1\right\}, u_2(x,y) = y, e_2 = (0,0).$

Obviously H2, H3, H5, H6 are satisfied. One can easily check that $S_1 = \mathbf{R}^2; S_2 = \{(p_1,p_2) \mid p_2 > 0\}$ and hence $S_1 \cap S_2 = S_2$. The individually rational attainable allocations set is :

$$A = \left\{((0,y_1),(0,y_2)) \in \mathbf{R}^4_+ \mid y_1 + y_2 = 1\right\}.$$

Thus H4 is fulfilled. Let $\bar{p} = (1,0)$. Then $(e_1, e_2, \bar{p})$ is a quasi-equilibrium but it is not P.O since, with $a_1 = a_2 = (0, \frac{1}{2})$, one has $(a_1, a_2) \in A, u_1(a_1) = u_1(e_1)$ and $u_2(a_2) = \frac{1}{2} > u_2(e_2)$.

We prove now that in this economy there is no equilibrium. Assume (a_1^*, a_2^*, p^*) is an equilibrium. We have $a_1^* = (0, y_1^*)$ and $a_2^* = (0, 1 - y_1^*)$. Let $p^* = (p_1^*, p_2^*)$. We have :

$$p^* a_1^* = p^* e_1 = p_2^* \text{ and } p^* a_2^* = p^* e_2 = 0 = p_2^*(1 - y_1^*).$$

Let $y > 1 - y_1^*$. Then $u_2(0,y) = y > u_2(a_2^*) = 1 - y_1^*$ hence $p_2^* y > p^* e_2 = 0$. Therefore $p_2^* > 0$. Let $x > 0$; then $u_1(x,0) = x > u_1(a_1^*) = 0$, hence $p_1^* x > p^* e_1 = p_2^* > 0$. This is a contradiction since x can be chosen arbitrarily close to 0.

$(*)$ is not satisfied since $0 = \bar{p}.e_1 = \inf \bar{p}X_1$ and $0 = \bar{p}.e_2 = \inf \bar{p}X_2$.□

2.5. The concept of arbitrage in the literature

a) Individual absence of arbitrage

The cone W_i has been used in Nielsen (1989) where elements of W_i are called "directions of improvement". It has been introduced by Werner (1987) who assumes H5 and calls elements of W_i "useful vectors".

The cone C_i has been introduced by Chichilnisky (1993, 1994, 1995) when $X_i = \mathbf{R}^\ell_+$ or $\mathbf{R}^\ell$. We extend her concept to any closed convex consumption set.

The cone S_i of no-arbitrage prices for i has been introduced by Werner (1987) under H5, while the cone of limited arbitrage prices for i is called the "market cone" by Chichilnisky (1993,1994,1995). Viable prices have been introduced by Werner (1987). Lastly "arbitrage free prices" have been defined by Brown-Werner (1995).

b) absence of arbitrage for the economy

Prices that we call "no arbitrage" prices for the economy have been used in many papers, in particular in Page (1984), Werner (1987) and Page and Wooders (1993).

Limited arbitrage is defined in Chichilnisky (1993, 1994, 1995) and Chichilnisky and Heal (1993).

The concept of "no market arbitrage" goes back to Debreu (1962). It was, at least, used later by Hart (1974) in an asset equilibrium model, then by Milne (1979), Hammond (1983), Page (1987), Nielsen (1989) ("positive semi-independence of directions of improvement") and Page and Wooders (1995) who use the terminology "no unbounded arbitrage".

c) Absence of arbitrage, aggregate demand and existence of equilibrium

All papers mentionned before, prove that the absence of some kind of arbitrage implies existence of an equilibrium and sometimes give necessary and sufficient conditions for existence of an equilibrium.

The proof of existence of an equilibrium by the demand approach when consumption sets are not bounded below goes back to Green (1973) and Grandmont (1978, 1982) who prove existence of a temporary equilibrium. Grandmont has kindly provided to us the following lemma which is a corollary of his "Market equilibrium lemma" (see "Grandmont" (1982)).

Disaggregated market equilibrium lemma

Let (D_i) be a collection of open convex cones of $\mathbf{R}^\ell$ for $i = 1, ..., m$. For each i, let Z_i be an upper semi-continuous, non empty, convex, compact valued *correspondence from D_i into $\mathbf{R}^\ell$. Assume that every Z_i is homogenous of degree zero and satisfies the following boundary condition:*

if $\{p^k\} \in D_i$ and $\frac{p^k}{\|p^k\|} \to \bar{p} \in \partial D_i$ then $pz^k \to +\infty$ for every $p \in D_i$ and for every z^k in $Z_i(p^k)$.

Assume finally Walras' Law :

$$\forall p \in \bigcap_i D_i,\ p. \sum_i Z_i(p) = 0.$$

Then there exists a market equilibrium, i.e. p^* such that $0 \in \sum_i Z_i(p^*)$, iff $\bigcap_i D_i \neq \phi$. □

Werner (1987) has proven another corollary of Grandmont's market equilibrium lemma. He deduces from it a sufficient condition for existence of an equilibrium.

We will show that statement 3) of our proposition 2.8 ($\bigcap_i S_i \neq \phi \Leftrightarrow$ existence of equilibrium) may be deduced from Disaggregated Market Equilibrium Lemma. But observe that the sufficient conditions for the existence of an equilibrium in our Theorem 1 are weaker than the ones of this statement.

Proposition 2.9 - *Assume every u_i concave, H2, H3, H4, H6 and (*). For $p \in \mathbf{R}^\ell$, define $Z_i(p) = argmax\ \{u_i(x) \mid x \in X_i,\ px \leq pe_i\}$. Then :*
(i) $Z_i(p) \neq \phi \Leftrightarrow p \in S_i$
(ii) Z_i is convex, compact valued and u.s.c.
(iii) If $p^k \to \bar{p} \in \partial S_i \backslash \{0\}$, then $\forall z^k \in Z_i(p^k)$, the sequence (z^k) is unbounded or, equivalently, $p.z^k \to +\infty, \forall p \in S_i$.
(iv) $\sum_i Z_i$ verifies Walras' law.

Proof :
(i) That is one of the statements in Proposition 2.4.
(ii) Its proof is easy.
(iii) If (z^k) is bounded then (z^k) could be assumed to converge to $z \in Z_i(\bar{p})$. From (i), $\bar{p} \in S_i$: a contradiction since S_i is open.

For every $p \in S_i$ we have $p.z^k \leq \| p \| \| z^k \|$. Hence, $p.z^k \to +\infty \Rightarrow \| z^k \| \to +\infty$. Conversely, assume $\| z^k \| \to +\infty$ and $(p.z^k)$ bounded above. Then $z^k / \| z^k \| \to z \in W_i \setminus \{0\}$ and we have $p.z \leq 0$: a contradiction since $p \in S_i$.

(iv) We prove that $p.Z_i(p) = p.e_i$ for every $p \in S_i$. Assume the contrary: $p.Z_i(p) < p.e_i$. Let $w_i \in W_i \setminus \{0\}$ and $t > 0$. Then $u_i(Z_i(p) + tw_i) > u_i(Z_i(p))$ (proposition 2.1 (4)). For t sufficiently small one has $p.(Z_i(p) + tw_i) < p.e_i$, a contradiction. □

APPENDIX

1. Proof of lemma 1.2

For n large enough, $e_i \in \xi_i^n(p,q)$, which is therefore non empty. It is convex since u_i verifies H2, H3 (and hence, H1; Debreu, 1959) and compact, since X_i^n is compact and u_i verifies H3. Since u_i verifies H3, the correspondence ξ_i^n has closed graph. Clearly

$$(*)\ Q_i^n(p,q) \subseteq \xi_i^n(p,q) \subseteq X_i^n \qquad \forall n \geq 1, \forall (p,q) \in S \cap (\mathbf{R}^\ell \times \mathbf{R}_+).$$

$Q_i^n(p,q) \neq \phi$ for n large enough since it contains the set of maximizers of u_i on $\xi_i^n(p,q)$. It is convex valued since u_i verifies H2 and H3. Let us prove that the correspondence Q_i^n has also closed graph. Let $(p^k,q^k) \to (p,q)$ and $x_i^k \in Q_i^n(p^k,q^k), x_i^k \to x_i$. Then $x_i \in \xi_i^n(p,q)$ since ξ_i^n has closed graph. Let $y_i \in X_i^n$ be such that $u_i(y_i) > u_i(x_i)$. Since u_i verifies H3, $u_i(y_i) > u_i(x_i^k)$ for k large enough, hence $p^k y_i \geq p^k e_i + q^k$ which implies in the limit that $py_i \geq pe_i + q$. Hence $x_i \in Q_i^n(p,q)$ and Q_i^n has closed graph. It follows from (*) that it is u.s.c.

2. Proof of lemma 2.2

Let $\alpha = \sup \{f(t) \mid t \geq 0\}$. There are two cases :

(i) $f(t_0) = \alpha$ for some $t_0 \geq 0$;
(ii) $f(t) < \alpha, \forall t \geq 0$.

Case (i) Let $0 \leq t \leq t' \leq t_0$. By quasi-concavity, $f(t') \geq f(t)$ i.e. f is non decreasing on $[0,t_0]$.

Let $t_0 \leq t \leq t'$. Again, by quasi-concavity, $f(t) \geq f(t')$ i.e. f is non increasing for $t \geq t_0$.

(ii) We will prove that, in this case, f is non decreasing. Let (t'_n) be a sequence such that $\lim f(t'_n) = \alpha$. Since f is u.s.c. and the supremum is not attained, we may assume $\{t'_n\}$ increasing and $t'_n \to +\infty$. Define a sequence (t_n) as follows:

$$f(t_0) = \max \{f(t) \mid t \in [0,t'_0]\},$$

and , for all $n \geq 1$

$$f(t_n) = \max\{f(t) \mid t \in [t_{n-1}, t'_n]\}.$$

Such a sequence exists since f is u.s.c. We observe that :

a) the sequence $f(t_n)$ is non decreasing,
b) $f(t_n) \geq f(t'_n), \forall n$, and hence $f(t_n) \to \alpha$,
c) (t_n) is non decreasing and $t_n \to +\infty$,
d) $f(t_{n-1}) \leq f(t) \leq f(t_n), \forall t \in [t_{n-1}, t_n]$ by a), the quasi-concavity of f and the definition of t_n.

Now, let $0 \leq t \leq t'$. If $t_n \leq t \leq t' \leq t_{n+1}$,from d) one gets $f(t') \geq f(t)$, by quasi-concavity. If $t_{n-1} \leq t \leq t_n$ and $t_{n'-1} \leq t' \leq t_{n'}$ with $n' > n$, then $f(t') \geq f(t)$ from d) and a). The proof is now complete. □

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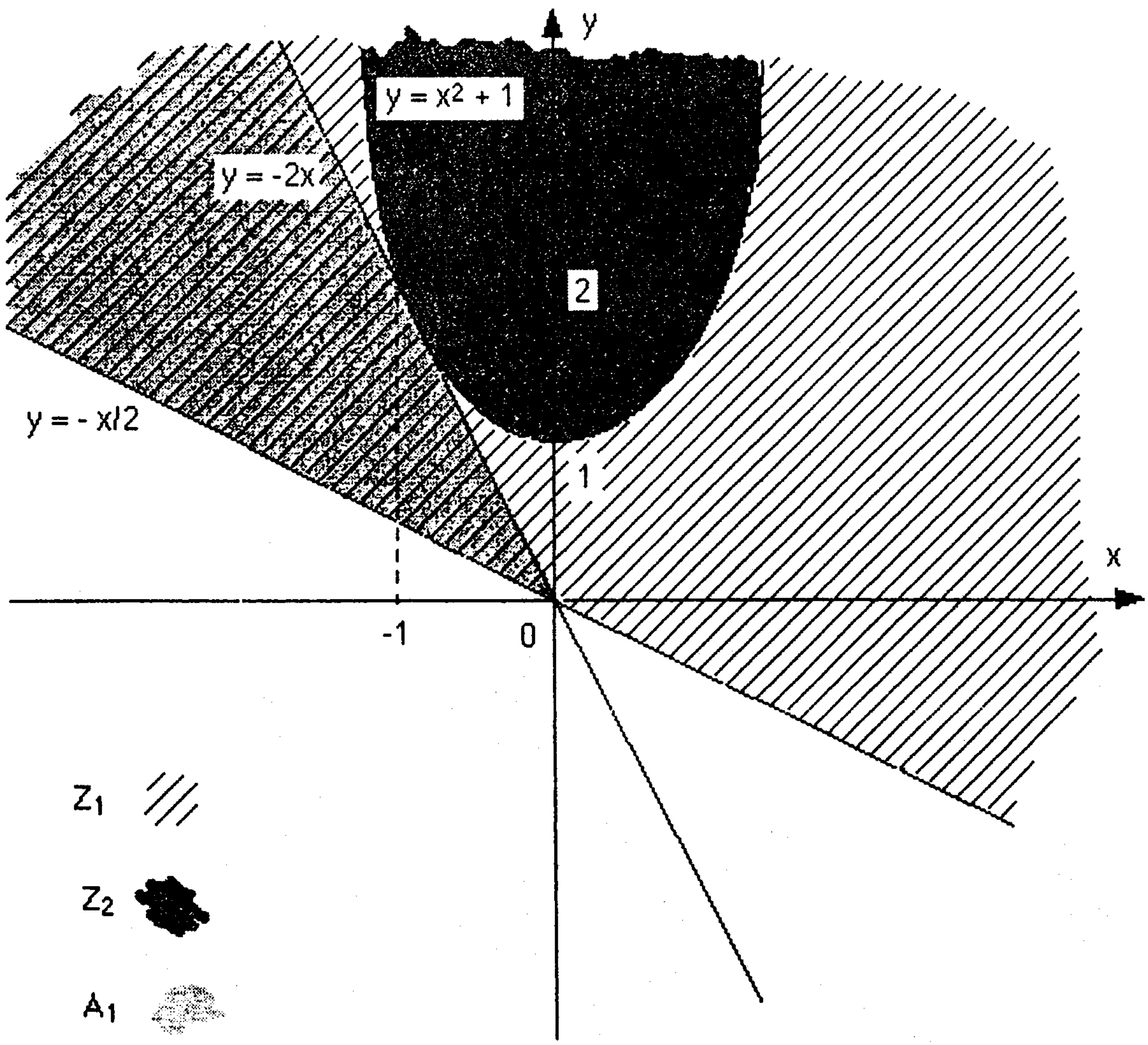

Consumption set and individually rational attainable allocations set of agent 1
(characteristics of agent 2 are symmetric with respect to the first diagonal of the ones of agent 1)

- figure 1 -