Arbitrage, Duality and Asset Equilibria

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Abstract - In finite dimension, it was proven by Werner(1987) that there exists an equilibrium under the assumption that there exists a price for which there exists " absence of free lunch " or equivalently, that the aggregate demand exists for some price. This result does not generalize to the infinite dimension. The purpose of this paper is to propose a " dual " interpretation of the notion of " absence of free lunch ". The assumption that there exists a Pareto-optimum can be viewed as the equivalent of the assumption of existence of aggregate demand while the utility weight vector associated with that Pareto-optimum being the equivalent, in the space of utility weights, of the " non-arbitrage price ". We may then define in the space of utility weights, the excess utility correspondence which has the same properties of an excess demand correspondence. As in Werner(1987), we use a generalized version of Gale-Nikaido-Debreu lemma to prove the existence of an equilibrium.

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Résumé - En dimension finie, Werner(1987) a montré qu'un équilibre existe sous l'hypothèse qu'il existe un prix pour lequel il y a "absence de free lunch" ou de façon équivalente, la demande agrégée existe. Ce résultat ne se généralise pas à la dimension infinie. L'objectif de ce papier est de proposer une interprétation "duale" de la notion d' "absence de free lunch". L'hypothèse qu'il existe un optimum de Pareto peut être vue comme celle de l'existence de la demande agrégée; les poids associés à cet optimum seront alors équivalents aux prix de "non-arbitrage". Nous définissons dans l'espace des poids associés aux optima de Pareto une correspondance d'excès d'utilité qui a les mêmes propriétés que celles de la correspondance d'excès de demande. Comme dans Werner(1987), nous utilisons une version généralisée du lemme de Gale-Nikaido-Debreu pour démontrer l'existence d'un équilibre.

Keywords : Arbitrage, Duality, Asset Equilibria, Pareto-Optimum, Utility weights, Non Arbitrage Price, Excess Utility Correspondence, Quasi-Equilibrium.

Mots Clés : Arbitrage, Dualité, Equilibres Financiers, Optimum de Pareto, Poids d'Utilité, Prix de Non Arbitrage, Correspondance d'Exces d'Utilité, Quasi-Equilibre

JEL classification : C61, C62, D51, G20

Introduction

Since the early work of Debreu (1962), equilibrium models with consumption spaces unbounded below, have been considered in different economic settings, in particular in the theory of temporary equilibrium by Green (1973) and Grandmont(1977, 1982) and in finance by Hart(1974). The problem of existence of an equilibrium (with consumption spaces unbounded below), in finite dimension, has been discussed recently by many authors (for a comparaison of hypotheses and methods, see Dana et ali(1996)) and, in infinite dimension, by Cheng (1991), Brown-Werner (1993), Chichilnisky-Heal(1993) and Dana et ali (1994).

Werner (1987) gave an existence result based on a generalized version of the Gale-Nikaido-Debreu's lemma under the hypothesis that there was at least a price, for which "there was absence of free lunch" or equivalently that aggregate demand existed for some price. As it is well-known, that condition doesn't generalize to the infinite dimension, even when consumption spaces are bounded below. Brown-Werner (1993) have later introduced "arbitrage free prices" and have studied the use of the assumption of existence of an arbitrage free price.

The purpose of this paper is to propose a "dual" approach to Werner's paper in finite and infinite dimension and a "dual" interpretation of the notion of "absence of free lunch". Indeed, we make a set of assumptions that imply there exists a Pareto-optimal allocation. This assumption can be viewed as the equivalent of the assumption of existence of aggregate demand while the utility weight vector associated with that Pareto-optimum, being the equivalent, in the space of utility weights, of the "non-arbitrage" price. We may then define in the space of utility weights, the excess utility correspondence, which has the properties of an excess demand correspondence. As in Werner (1987), we use a generalized version of Gale-Nikaido-Debreu's lemma to prove existence of an equilibrium.

The paper is organized as follows:

In section 1, we set the model and some notations.

In section 2, we prove existence of a Pareto-optimum, we define and characterize the excess utility correspondence .

In section 3, we prove existence of a quasi-equilibrium and of an equilibrium.

In section 4, we discuss our assumptions.

In section 5, in finite dimension, we prove that the assumption of existence of a Pareto-optimum is equivalent to the assumption of existence of aggregate demand or to the assumption that there is a price, for which "there is absence of free lunch" and prove existence of a quasi-equilibrium under standard conditions.

Section 6 is devoted to the special case of differentiable utilities where the excess utility is a function.

Lastly, In section 7, we consider two examples: The C.A.P.M. and the case of L^p with Von Neumann-Morgenstern utilities.

1 The Model and Notations

We shall use the following notations. Given a subset C of $\mathbb{R}^n$, int C, ∂C, and $\bar{C}$ denote its interior, its boundary and its closure.

For a convex subset $C \subseteq \mathbb{R}^n$, $\text{int}^r C$ denotes its relative interior, when C is regarded as a subset of its affine hull.

We consider a pure exchange economy with a commodity space F assumed to be a locally convex, topological space with dual F'. There are m agents. Agent i is described by a consumption set $X_i \subseteq F$ and an initial endowment $w_i \in X_i$. The preferences of agent i are represented by a utility function $u_i : X_i \to I\!R \cup \{-\infty\}$.

Let $w = \sum_{i=1}^{m} w_i$ denote aggregate endowment. We shall make the following assumptions about agent's characteristics:

H1 X_i **is closed and convex,** $\forall i$,

H1 bis $X_i = F$, $\forall i$.

H2 $\forall i,\ u_i : X_i \to I\!R \cup \{-\infty\}$ **is strictly concave and** $u_i(w_i) = 0$,

H3 **There exists a neighborhood** $\mathcal{W} \subseteq F$ **of zero and** $w' = (w_1', ..., w_m') \in \prod_i X_i$ **with** $\sum_{i=1}^{m} w_i' = w$ **with** $u_i(w_i') > -\infty, \forall\, i$ **such that**
a) $w_1' + z \in X_1,\ u_1(w_1' + z) > -\infty,\ \forall\, z \in \mathcal{W}$,
$b)$ $u_1(w_1' + z) > u_1(w_1), \forall\, z \in \mathcal{W},\ u_i(w_i') > u_i(w_i), \forall\, i \neq 1$.

For $\varepsilon \in \mathcal{W}$, let

$$A(\varepsilon) = \left\{ (x_1, ..., x_m) \in \Pi_i X_i \mid \sum_{i=1}^{m} x_i = w + \varepsilon \right\}$$

be the set of attainable allocations when aggregate endowment is $w + \varepsilon$;

$$U(\varepsilon) = \{ v \in I\!R^m \mid v_i \leq u_i(x_i), \forall i, \text{ for some } x \in A(\varepsilon) \}$$

be the utility set,

$$V(0) = \{ z \in I\!R^m \ | u_i(w_i) \leq z_i \leq u_i(x_i), \forall i, \text{ for some } x \in A(0) \}$$

be the set of individually rational utilities.

H4 $U(0)$ **is closed,**

H4 bis $U(0)$ **is closed,** $U(0) \neq I\!R^m$.

H5 $V(0)$ **is compact.**

H6 $\forall x \in A(0),\ \exists\ (k_1, ..., k_n) \in F^m$, **such that** $u_i(x_i + k_i) > u_i(x_i),\ \forall\ i$.

H7 $\forall\ i$, **for every** $x \in A(0)$, **if** $u_i(x_i) > \inf_{x_i \in X_i} u_i(x_i)$, **then there exists** $(k_1, ..., k_{i-1}, k_{i+1}, ..., k_m)$ **such that** $u_j(k_j + x_j) > u_j(x_j), \forall\ j \neq i$ **and** $x_i - \sum_{j \neq i} k_j\ \in X_i$.

In section four, we shall comment in detail our assumptions. Let us already remark that **H1 -H2** are standard. As far as **H3** is concerned, it is usual to assume that $(w_1, ..., w_m)$ is not weakly Pareto-optimal, in other words that there exists $w' \in A(0)$ such that $u_i(w'_i)\ > u_i(w_i), \forall\ i$. If not, it is well known that there exists a price p such that $((w_1, ..., w_m), p)$ is a quasi-equilibrium. Assumption **H3** is stronger: it implies that $(w_1, ..., w_m)$ is not weakly Pareto-optimal and therefore that int $V(0) \neq \emptyset$. Moreover **H3** implies that X_1 has non empty interior and that u_1 is continuous on int X_1. **H6** means that if $x = (x_1, x_2, ...) \in A(0)$, then agent i is not satiated at x_i.

Let V be a closed convex set of $I\!R^m$, then $V_\infty = \left\{x \in I\!R^m \mid x = \lim_n \lambda_n x_n, \lambda_n \geq 0,\ \lambda_n \to 0, x_n \in V\right\}$ is its asymptotic cone and $V^\circ = \{y \in I\!R^m \mid v \cdot y \leq 0\ ,\ \ \forall v \in V\}$ its polar cone.

Since we do not assume $U(\varepsilon)$ closed, we shall need to consider $\bar{U}_\infty(\varepsilon)$ the asymptotic cone of $\bar{U}(\varepsilon)$ and $\bar{U}^\circ_\infty(\varepsilon)$ the polar of the asymptotic cone $\bar{U}_\infty(\varepsilon)$.

For further use, let us remark the following: For $a \in I\!R^m$, let

$$V(a, 0) = \{v \in I\!R^m \mid a_i \leq v_i \leq u_i(x_i), \forall i, x \in A(0)\}.$$

Proposition 1.1.: *Assume* **H1-H2-H4-H5** , *then* $V(a, 0)$ *is compact,* $\forall\ a$.

Proof: Since $U(0)$ is closed, $V(a, 0)$ is closed, $\forall\ a$. Hence it is bounded iff its asymptotic cone equals $\{0\}$. $V(a, 0)_\infty = U(0)_\infty \bigcap I\!R^m_+ = V(0)_\infty = \{0\}$ by **H5**.■

It then follows from Proposition 1.1. that any subset of the utility set which is bounded below is bounded above.

For sake of completeness, we recall two definitions:

A pair $(\bar{x}, \bar{\psi}) \in \prod_i X_i \times F' - \{0\}$ is an quasi-equilibrium if
i) $\forall\, i, u_i(x_i) > u_i(\bar{x}_i)$, implies $\bar{\psi} \cdot x_i \geq \bar{\psi} \cdot w_i$,
ii) $\sum\limits_i \bar{x}_i = w$.

A pair $(\bar{x}, \bar{\psi}) \in \prod_i X_i \times F' - \{0\}$ is an equilibrium if
i) $\forall\, i, u_i(\bar{x}_i) \geq u_i(x_i)$, for every x_i such that $\bar{\psi} \cdot x_i \leq \bar{\psi} \cdot w_i$,
ii) $\sum\limits_i \bar{x}_i = w$.

2 The Excess Utility Correspondence

In this section, we define the Excess Utility Correspondence which, as we shall show, has all the properties of an Excess Demand Correspondence in finite dimension, except that it is defined on an open subset of the unit simplex. In order to find its domain, we first show that the support function of the utility set $U(\varepsilon)$ is subdifferentiable and that it has a domain D independent of (ε) and with non empty interior. We then show that int$D$ is the set of strictly positive weights that support Pareto-optima. We may then define on int$D$, the Excess Utility Correspondence as the set of transfers per unit of weight at Pareto-optima.

Let us first introduce the support function of $U(\varepsilon)$: $\forall \lambda \in I\!R^m, \forall \varepsilon \in \mathcal{W}$, let

$$h(\lambda, \varepsilon) = \sup_{z \in U(\varepsilon)} \sum_i \lambda_i z_i.$$

Define $D(\varepsilon) = \operatorname{dom} h(\lambda, \varepsilon) = \{\lambda \in I\!R^m \mid h(\lambda, \varepsilon) < \infty\}$.

Since $\bar{U}_\infty(\varepsilon) \supset I\!R^m_-$, it has non empty interior. It follows from Aubin p.34, proposition 9 that

$$D(\varepsilon) = \bar{U}^\circ_\infty(\varepsilon) \subset (I\!R^m_-)^\circ = I\!R^m_+.$$

Proposition 2.1.
a) *Assume* **H1-H2**. *Then* $\forall \varepsilon \in \mathcal{W}$, $h(\cdot, \varepsilon)$ *is convex and lower semi-continuous and* $\forall \lambda \in D(\varepsilon)$, $h(\lambda, \cdot)$ *is concave,*
b) *Assume* **H1-H2-H3a**. *Then* $\forall \varepsilon \in \mathcal{W}$, $\varepsilon' \in \mathcal{W}$, $D(\varepsilon) = D(\varepsilon')$.
c) *Assume* **H1-H2-H3**. *Then* $\forall \lambda \in D(\varepsilon)$, $h(\lambda, \cdot)$ *is continuous.*

Proof: $h(\cdot, \varepsilon)$ is convex, and lower semi-continuous as a support function (see Rockafellar, 1970, theorem 13.2.). $h(\lambda, \cdot)$ is trivially concave, moreover $\forall \lambda \geq 0, \forall \varepsilon \in \mathcal{W}$,

$$h(\lambda, \varepsilon) \geq \sum_{i \neq 1} \lambda_i u_i(w'_i) + \lambda_1 u_1(w'_1 + \varepsilon) > -\infty.$$

Assume that $h(\lambda, \varepsilon) < \infty$ and there exists $\varepsilon' \in \mathcal{W}$ such that $h(\lambda, \varepsilon') = \infty$. Let $\varepsilon'' \in \mathcal{W}$ be such that $\varepsilon = t\varepsilon' + (1-t)\varepsilon''$ with $t \in]0, 1[$. We then have:

$$+\infty > h(\lambda, \varepsilon) \geq t\, h(\lambda, \varepsilon') + (1-t) h(\lambda, \varepsilon'') = +\infty,$$

since $h(\lambda, \varepsilon'') > -\infty$, a contradiction. Hence $h(\lambda, \varepsilon'') < +\infty$ and $D(\varepsilon') = D(\varepsilon)$.

Let us lastly prove that $h(\lambda, \cdot)$ is continuous in $\mathcal{W}$.

$$h(\lambda, \varepsilon) \geq \sum_{i \neq 1} \lambda_i u_i(w_i') + \lambda_1 u_1(w_1' + \varepsilon) \geq \sum_i^m \lambda_i u_i(w_i) = 0,$$

by assumptions **H2, H3**. Since the function $h(\lambda, \cdot)$ is concave, finite valued and bounded below, it is continuous in $\mathcal{W}$. ■

In view of proposition 2.1.c, we shall simply denote by $D = \text{dom}\, h(\cdot, \varepsilon)$ and by U_∞ the asymptotic cone of $\bar{U}(\varepsilon)$.

Let

$$\partial_\varepsilon h(\lambda, \varepsilon) = \{z \in F' \mid h(\lambda, \varepsilon) - h(\lambda, \varepsilon') \geq z(\varepsilon - \varepsilon'),\ \forall\, \varepsilon' \in \mathcal{W} + \varepsilon\}.$$

Proposition 2.2. *Assume* **H1-H2-H3**.

a) $\forall \lambda \in D$, $\forall \varepsilon \in \mathcal{W}$, $\partial_\varepsilon h(\lambda, \varepsilon)$ *is non empty, convex and weakly compact,*

b) $\forall \lambda \in$ *int* D, *there exists a weakly compact subset* K *of* F' *such that* $\partial_\varepsilon h(\lambda, \varepsilon) \subseteq K$, $\forall \lambda' \in V(\lambda)$ *a neighborhood of* λ,

c) *The correspondence* $\lambda \in$ *int* $D \to \partial_\varepsilon h(\lambda, 0)$ *is upper hemi- continuous.*

Proof: Since by proposition 2.1., $h(\lambda, \cdot)$ is continuous in $\mathcal{W}$, $\partial_\varepsilon h(\lambda, \varepsilon)$ is non empty, convex and weakly compact. To prove b), we have

$$h(\lambda', 0) - h(\lambda', \varepsilon) \geq \partial_\varepsilon h(\lambda', 0)(-\varepsilon),$$

$$h(\lambda', 0) - h(\lambda', -\varepsilon) \geq \partial_\varepsilon h(\lambda', 0)(\varepsilon).$$

Let $V(\lambda)$ be a compact neighborhood of λ such that $V(\lambda) \subseteq \text{int}\, D$. Since $h(\cdot, 0)$ is continuous in $V(\lambda)$, $|h(\lambda', 0)| \leq M$. We also have:

$$h(\lambda', \varepsilon) \geq 0,\ \forall \lambda' \in V(\lambda), \quad \forall\, \varepsilon \in \mathcal{W}.$$

Hence

$$|\partial_\varepsilon h(\lambda', 0)\varepsilon| \leq h(\lambda', 0) \leq M,\ \forall \lambda' \in V(\lambda),\ \forall\, \varepsilon \in \mathcal{W}.$$

Lastly to prove c), since $h(\cdot, 0)$ and $h(\cdot, \varepsilon)$ are continuous on int D, the correspondence $\lambda \in \text{int}\, D \to \partial_\varepsilon h(\lambda, 0)$ has a closed graph and from b) is upper hemi- continuous for the weak topology of F. ■

Let us next characterize D. Let

$$\Phi(\lambda) = \{\sup \lambda \cdot p \mid p \in U_\infty, ||p|| = 1\}.$$

Proposition 2.3. *Assume* **H1-H2-H3a-H4-H5**

a) $D = \{\lambda \in I\!R^m_+ \mid \Phi(\lambda) \leq 0\}$, *b) int $D \neq \phi$, and int $D = \{\lambda \in I\!R^m_+ \mid \Phi(\lambda) < 0\}$.*

Proof: Since $D = U^\circ_\infty \subseteq I\!R^m_+$, the proof of a) is obvious. To prove b), let us assume that int $D = \phi$. Then U_∞ contains a line $\{tv,\ t \in I\!R\}$. Let $I_1 = \{i \mid v_i > 0\}$, $I_2 = \{i \mid v_i < 0\}$ and $I_3 = \{i \mid v_i = 0\}$. Without lost of generality, we may assume that $I_1 \neq \phi$. Let $i_0 \in I_1$. Let $\bar{z} \in V(0)$ be such that $\bar{z}_{i_0} = \max\{z_{i_0}, z \in V(0)\}$. There exists $x \in A(0)$, $x' \in A(0)$ such that

$$\forall\, i, \quad \bar{z}_i + v_i \leq u_i(x_i)$$
$$\bar{z}_i - v_i \leq u_i(x'_i)$$

Let $i \in I_1$. If $x_i \neq x'_i$,

$$\bar{z}_i < u_i(\frac{x_i + x'_i}{2}).$$

If $x_i = x'_i$, then

$$\bar{z}_i < \bar{z}_i + v_i \leq u_i(x_i) = u_i(\frac{x_i + x'_i}{2}).$$

Similarly, for $i \in I_2$, $\bar{z}_i < u_i(\frac{x_i+x'_i}{2})$ and for $i \in I_3$, $\bar{z}_i \leq u_i(\frac{x_i+x'_i}{2})$.
Let $z' \in I\!R^m$ be defined by $z'_i = u_i(\frac{x_i+x'_i}{2}), \forall\, i$. Then $z' \in V(0)$ and $z'_{i_0} > \bar{z}_{i_0}$ a contradiction. Hence U_∞ contains no line and int $D \neq \phi$. Lastly since Φ is continuous,

$$\text{int } D = \{\lambda \in I\!R^m_+ \mid \Phi(\lambda) < 0\}. \blacksquare$$

Lemma 2.4.: *Let $C(\lambda) = \{u \in U(0) \mid \lambda \cdot u \geq 0\}$. Assume* **H1-H2-H3a-H4**. *Then the correspondence C : int $D \rightarrow I\!R^m_+$ is convex, compact valued and u.h.c. Assume furthermore* **H3b**, *then it is continuous.*

The proof may be found in the appendix.■

Proposition 2.5.
a) Assume **H1-H2-H3a**. *Let $\lambda \in D$. If $h(\lambda, 0) = \lambda \cdot v$, then there exists $x \in A(0)$ such that $h(\lambda, 0) = \sum_i \lambda_i u_i(x_i)$. If $\lambda_i > 0$, v_i is unique and there exists a unique x_i such that $v_i = u_i(x_i)$.*
b) Assume **H1-H2-H3a-H4**. *Then int $D = \{\lambda >> 0 \mid \exists v \in U(0),\ h(\lambda, 0) = \lambda \cdot v\} = \{\lambda >> 0 \mid \exists x \in A(0),\ h(\lambda, 0) = \sum_i \lambda_i u_i(x_i)\}$.*
c) Assume **H1-H2-H3-H4**. *Then the map int $D \rightarrow$ Argmax$\{\lambda \cdot u \mid u \in U(0)\}$ is continuous.*

Proof: If $h(\lambda, 0) = \lambda \cdot v$, then there exists $x \in A(0)$ such that $v_i \leq u_i(x_i)$. Hence $h(\lambda, 0) = \sum_i \lambda_i u_i(x_i)$.
If $\lambda_i > 0$, $v_i = u_i(x_i)$ and x_i is unique since the $u'_i s$ are strictly concave.
To prove b) let $\lambda \in$ int D. Then by lemma 2.4., $C(\lambda)$ is compact and $h(\lambda, 0) = \sup\{\lambda \cdot u \mid u \in C(\lambda)\} = \lambda \cdot v$ for some $v \in U(0)$. Since $\lambda \in$ int D, $\lambda >> 0$.
Conversely let us assume that $h(\lambda, 0) = \lambda \cdot v$ for some $\lambda >> 0$ and $v \in U(0)$. If $\lambda \notin$ int D, then there exists $p \in U_\infty - \{0\}$ such that $\lambda \cdot p = 0$. But then $h(\lambda, 0) = \lambda \cdot v = \lambda \cdot (v + p)$ which contradicts a). Hence $\lambda \in$ int D.

Lastly to prove c), we use the fact that $h(\lambda, 0) = \sup\{\lambda \cdot u \mid u \in C(\lambda)\} = \lambda \cdot v$, the lemma and the maximum theorem. ■

The next two propositions will used in sections three and six of the paper and may be skipped at first reading.

Proposition 2.6. *Assume* **H1bis-H2-H3a-H4-H6** *or* **H1-H2-H3a-H4-H7**. *Then:*
a) $\partial U(0) = \{v \mid \exists\, \lambda \in \text{int } D \text{ such that } h(\lambda, 0) = \lambda \cdot v\}$,
b) $\tilde{x} \in A(0)$ *is Pareto-optimal iff there exists* $\lambda \in \text{int } D$ *such that* $h(\lambda, 0) = \sum_i \lambda_i u_i(\tilde{x}_i)$.

Proof: To prove a) clearly $\{v \mid \exists\, \lambda >> 0, \text{ such that } h(\lambda, 0) = \lambda \cdot v\} \subseteq \partial U(0)$. Conversely, if $v \in \partial U(0)$, then $U(0)$ being closed, there exists $\lambda \in I\!R^m_+$ such that $h(\lambda, 0) = \lambda \cdot v$. Since $v \in U(0)$, there exists $x \in F^m$, such that $v_i \leq u_i(x_i)$, $\forall\, i$. Hence $h(\lambda, 0) = \sum_i \lambda_i u_i(x_i)$. Assume that $\lambda_k = 0$ for some k and $\lambda_j > 0$. Assume **H1bis,H6**. Let k_j be given by **H6** and $x' \in A(0)$ be defined by $x'_i = x_i$ if $i \neq \{k, j\}$, $x'_k = x_k - k_j$ and $x'_j = x_j + k_j$. Then $h(\lambda, 0) < \sum_i \lambda_i u_i(x'_i)$, a contradiction. By proposition 2.5. b, $\lambda \in$ int D . The proof is similar if one assumes **H1,H7**.
To prove b) if $h(\lambda, 0) = \sum_i \lambda_i u_i(\tilde{x}_i)$, for some $\tilde{x} \in F^m$ and $\lambda \in$ int D. Then as $\lambda >> 0$, $\tilde{x}$ is Pareto-optimal. Conversely if $\tilde{x}$ is Pareto-optimal, then there exists $\lambda \in I\!R^m_+$ such that $h(\lambda, 0) = \sum_i \lambda_i u_i(\tilde{x}_i)$. If $\lambda_k = 0$ for some k, by the same proof as a), we get a contradiction.■

Proposition 2.7.
a) Assume **H1-H2-H3a-H4-H5-H6-H7**. *Let* $\lambda_n \in \text{int } D \to \lambda \in \partial D$, *then for some* i, $\lambda_i = 0$. *Moreover if*
$(u_i(x_i(\lambda_n)))_{i=1}^m \to (v_i)_{i=1}^m$, *then* $v_i = \inf_{X_i} u_i(x_i)$, $\forall\, i$ *such that* $\lambda_i = 0$.
b) Assume **H1bis-H2-H3a-H4-H5-H6**. *If* $\lambda_n \in \text{int } D \to \lambda \in \partial D$, *then* $u_i(x_i(\lambda_n)) \to -\infty$ *for some* i.

Proof of a): Since $v_i = \lim_n u_i(x_i(\lambda_n))$, $\quad v_i \geq \inf_{X_i} u_i(x_i)$. As,

$$\sum_i \lambda_i^n u_i(x_i(\lambda_n)) \geq \sum_i \lambda_i^n z_i, \quad \forall n, \forall z \in U(0),$$

$\lambda \cdot v \geq \lambda \cdot z$, $\quad \forall z \in U(0)$ and $\lambda \cdot v = h(\lambda, 0)$. If $\lambda >> 0$, then by proposition 2.5.b, $\lambda \in$ int D, a contradiction. Hence for some i, $\lambda_i = 0$. There exists $(x'_i)_{i=1}^m$ such that $v_i \leq u_i(x'_i)$, $\forall\, i$. Let i be such that $\lambda_i = 0$. Then if $u_i(x'_i) > \inf_{X_i} u_i(x_i)$, by assumption **H7**, there exists $(x''_i)_{i=1}^m$ such that $\sum_i \lambda_i u_i(x''_i) > h(\lambda, 0)$ a contradiction. Hence $u_i(x'_i) = \inf_{X_i} u_i(x_i)$ which implies that $v_i = \inf_{X_i} u_i(x_i)$.
Proof of b): Let us first remark that under **H1 bis**, $\inf_{X_i} u_i(x_i) = -\infty$. Assume on the contrary that $\forall\, i$, $\exists\, m_i$ such that $u_i(x_i(\lambda_n)) \geq m_i$. Then by proposition 1.1., $\forall\, i$, $\exists\, M_i$ such that $u_i(x_i(\lambda_n)) \leq M_i$. Hence w.l.o.g., we may assume that there exists $v \in U(0)$ such that $u_i(x_i(\lambda_n)) \to v_i$, $\forall\, i$. Since $(u_i(x_i(\lambda_n)))_{i=1}^m \in \partial U(0)$, $\forall\, n$ and $\partial U(0)$ is closed, $v \in \partial U(0)$. By proposition 2.6.a, $h(\lambda, 0) = \lambda \cdot v$ with $\lambda >> 0$ which by proposition 2.5., contradicts the fact that $\lambda \in \partial D$.■

For $\lambda \in$ int D, let

$$v(\lambda) = \text{Argmax}\{\lambda \cdot u \mid u \in U(0)\}$$

and let $x_i(\lambda)$ be such that $v_i(\lambda) = u_i(x_i(\lambda))$.

We may now define the **excess utility** correspondence. For $\lambda \in \text{int } D$, $\forall i = 1, ..., m$, let

$$E_i(\lambda) = \{\frac{z \cdot (x_i(\lambda)) - w_i)}{\lambda_i}, \ z \in \partial_\varepsilon h(\lambda, 0)\}.$$

Proposition 2.8. *Assume* **H1-H5**
For $\lambda \in$ int D, E is a convex, compact, non empty valued, upper hemi-continuous correspondence, which satisfies $\lambda \cdot E(\lambda) = 0$ (Walras-law).

Proof: Clearly E is convex and non empty valued. Let $O(\lambda)$ be a compact neighborhood of $\lambda \in$ int D. Then $\forall\, i, \forall\, z \in \partial_\varepsilon h(\lambda', 0), \lambda' \in O(\lambda)$,

$$v_i(\lambda') = u_i(x_i(\lambda')) - u_i(w_i) \geq \frac{z \cdot (x_i(\lambda') - w_i)}{\lambda_i}.$$

Hence

$$t_i \leq \max_{O(\lambda)} |v_i(\lambda')| \leq A_i, \quad \forall t_i \in \ E_i(\lambda'), \ \forall \lambda' \in O(\lambda)$$

for some A_i, by proposition 2.5.c. Since $\sum_i \lambda'_i E_i(\lambda') = 0$, $\exists B_i \in I\!R$ such that

$$t_i \geq B_i, \forall\, t_i \in E_i(\lambda'), \ \lambda' \in O(\lambda).$$

Hence $E(\lambda')$ has values in a fixed compact set of $I\!R^m$ for $\lambda' \in O(\lambda)$.
Let us finally prove that E has closed graph. Let $\lambda_n \to \lambda \in$ int D and $\frac{z_n \cdot (x_i(\lambda_n) - w_i)}{\lambda_i^n} \to \xi_i$, $\forall i, z_n \in \partial_\varepsilon h(\lambda_n, 0)$.

$$\forall i, \quad v_i(\lambda_n) - v_i(\lambda) \geq \frac{z_n \cdot (x_i(\lambda_n) - x_i(\lambda))}{\lambda_i^n}.$$

Since by proposition 2.2, $\partial_\varepsilon h(\lambda, 0)$ is u.h.c., some subsequence of z_n converges weakly to $\bar{z} \in \partial_\varepsilon h(\lambda, 0)$. Hence since $v_i(\lambda_n) \to v_i(\lambda)$, by proposition 2.5.c., we get

$$\bar{z} \cdot x_i(\lambda) \geq \sum_i \text{limsup } z_n \cdot x_i(\lambda_n) \geq \text{limsup} \sum_i z_n \cdot x_i(\lambda_n) = \bar{z} \cdot w,$$

hence $\bar{z} \cdot x_i(\lambda) = \text{limsup } z_n \cdot x_i(\lambda_n)$ $\forall i$ and $\xi = \frac{\bar{z} \cdot (x_i(\lambda)) - w_i)}{\lambda_i}$ and E is upper hemi-continuous. ■

3 Existence of equilibria

Theorem 3.1. *Assume* **H1-H6**. *Then there exists a quasi-equilibrium. Assume furthermore* $\inf_{X_i} u_i(x) < u_i(w_i) = 0$ *and* **H1 bis** *or* **H7**, *then there exists an equilibrium.*

Proof:Let $\Delta_n = \left\{\lambda \in \text{int } D, \mid \ ||\lambda|| = 1, \ \Phi(\lambda) \leq \frac{-1}{n}\right\}$ and K_n be the cone generated by Δ_n.
It follows from a generalized version of Gale-Nikaido-Debreu's lemma proven in Florenzano-Le Van

(1986) that $\forall\, n$, $\exists \lambda_n \in \Delta_n$, $\exists e_n \in E(\lambda_n)$ such that $-\lambda \cdot e_n \le 0, \forall\, \lambda \in K_n$. Let $\bar\lambda$ be a limit point of λ_n. There are three cases:

Case 1) $\bar\lambda \in \text{int } D$. Then by proposition 2.8., $e_n \to \bar e \in E(\bar\lambda)$. Thus $-\lambda \cdot \bar e \le 0, \forall\, \lambda \in D$ and hence $-\bar e \in U_\infty$. Since $\bar\lambda \in \text{int } D$, $-\bar\lambda \cdot \bar e < 0$ if $\bar e \ne 0$ which contradicts Walras-Law. Hence $\bar e = 0$. There exists $\bar z \in \partial_\varepsilon h(\bar\lambda, 0)$ such that $0 = \frac{\bar z \cdot (x_i(\bar\lambda) - w_i)}{\bar\lambda_i}$ $\forall\, i$.

$$\forall\, i,\ \forall x_i \in X_i,\ \bar\lambda_i(u_i(x_i(\bar\lambda)) - u_i(x_i)) \ge \bar z \cdot (x_i(\bar\lambda) - x_i) = \bar z \cdot (w_i - x_i).$$

Thus $\bar z \cdot x_i \le \bar z \cdot w_i$ implies $u_i(x_i) \le u_i(x_i(\bar\lambda))$. Hence $[(x_i(\bar\lambda))_{i=1}^m, \bar z]$ is an equilibrium.

Case 2) $\bar\lambda \in \partial D$ and $||e_n|| \to \infty$. Since $e_{in} \le v_i(\lambda_n), \forall i$, $e_n \in U(0), \forall n$. Hence $(\frac{e_{in}}{||e_n||}) \to \bar t \in U_\infty - \{0\}$. Since $-\lambda \cdot e_n \le 0, \forall\, \lambda \in K_n$, we get at the limit $-\lambda \cdot \bar t \le 0, \forall\, \lambda \in D$, which is impossible.

Case 3) $\bar\lambda \in \partial D$ and some subsequence of $e_n \to \bar e$. As in case 1), we have $\bar\lambda \cdot \bar e = 0$. Since $e_{in} \le v_i(\lambda_n), \forall i$, $v(\lambda_n)$ is bounded below. By proposition 1.1., it is bounded above. Without lost of generality, let us assume that $v(\lambda_n)$ converges to $\bar v$. Since

$$\sum_i \lambda_i^n v_i(\lambda_n) \ \ge \sum_i \lambda_i^n u_i, \quad \forall\, u \in U(0),$$

we have

$$\bar\lambda \cdot \bar v \ \ge\ \bar\lambda \cdot u \quad \forall\, u \in U(0),$$

hence $\infty >\ h(\bar\lambda, 0) = \bar\lambda \cdot \bar v$. Since $\bar\lambda \cdot \bar e = 0$, $h(\bar\lambda, 0) = \bar\lambda \cdot (\bar v + \bar e)$. By proposition 2.5.a, $\bar\lambda_i > 0$ implies $\bar e_i = 0$. Hence $\bar\lambda_i \bar e_i = 0$, $\forall i$.
By proposition 2.2., we still have

$$\partial_\varepsilon h(\lambda_n, 0)\varepsilon \le |h(\lambda_n, 0)| \le M,\ \forall n,\ \forall\, \varepsilon \in \mathcal{W},$$

since $h(\lambda_n, 0) \to h(\bar\lambda, 0)$. We may therefore assume that if $e_i^n = \frac{z_n \cdot (x_i(\lambda_n)) - w_i)}{\lambda_i^n}$, $z_n \in \partial_\varepsilon h(\lambda_n, 0)$, then $z_n \to \bar z$ weakly. Let $x(\bar\lambda)$ be such that $h(\bar\lambda, 0) = \sum_i \bar\lambda_i u_i(x_i(\bar\lambda))$. Then by the same proof as that of proposition 2.6., $z_n \cdot x_i(\lambda_n) \to \bar z \cdot x_i(\bar\lambda)$, $\forall\, i$. Hence

$$\forall\, i,\ \bar z \cdot (x_i(\bar\lambda) - w_i) = \lim_n e_i^n \lambda_i^n = \bar\lambda_i \cdot \bar e_i = 0.$$

Let us prove that $[(x_i(\bar\lambda))_{i=1}^m, \bar z]$ is a quasi-equilibrium. Since

$$\lambda_i^n[u_i(x_i(\lambda_n) - u_i(x_i)] \ge z_n \cdot (x_i(\lambda_n) - x_i),\ \forall i,$$

if $\bar\lambda_i > 0$,

$$\bar\lambda_i[u_i(x_i(\bar\lambda) - u_i(x_i)] \ge \bar z \cdot (x_i(\bar\lambda) - x_i) = \bar z \cdot (w_i - x_i).$$

Let x_i be such that $u_i(x_i) > u_i(x_i(\bar\lambda))$ (by assumption **H6** such an x_i exists). Then $\bar z \cdot x_i > \bar z \cdot w_i$. Hence $\bar z \ne 0$. If $\bar\lambda_i = 0$, then $\bar z \cdot w_i \le \bar z \cdot x_i$. Hence $[(x_i(\bar\lambda))_{i=1}^m, \bar z]$ is a quasi-equilibrium.

Assume furthermore $\inf_{X_i} u_i(x) < u_i(w_i) = 0$ and **H7** (or **H1 bis** which, with **H6** implies **H7**), let us prove that there exists an equilibrium. Let us reconsider case 3: $\lambda_n \to \bar\lambda \in \partial D$ and some subsequence of $e_n \to \bar e$. As in case 1), we have $\bar\lambda \cdot \bar e = 0$. Moreover $-\lambda \cdot \bar e \le 0, \forall\, \lambda \in D$. Since $e_{in} \le v_i(\lambda_n)$, $\forall\, i$, the sequence $(v_i(\lambda_n))_{i=1}^m$ is bounded below. By proposition 1.1., w.l.o.g. we may assume that it converges to $\bar v$. We then have: $h(\bar\lambda, 0) = \bar\lambda \cdot (\bar v + \bar e) = \bar\lambda \cdot \bar v$. Hence $\bar\lambda_i > 0$ implies $\bar e_i = 0$. Moreover if $\bar\lambda_i = 0$, $\bar e_i \le v_i < 0$ since by proposition 2.7.a $v_i = \inf_{X_i} u_i(x) < 0$. Hence

$\lambda \cdot \bar{e} \leq 0, \forall\, \lambda \in D$, which implies that $\lambda \cdot \bar{e} = 0, \forall\, \lambda \in D$, which is impossible since $\bar{e} \in U_\infty - \{0\}$ and $\text{int}D \neq \emptyset$. Hence under **H7** case 3 is impossible and only case 1 is possible. ■

Remark :From a quasi-equilibrium to an equilibrium.

It is well-known that, if $[(\bar{x}_i)_{i=1}^m, \bar{z}]$ is a quasi-equilibrium, and if $\bar{z} \cdot w_i = \bar{z} \cdot \bar{x}_i > \inf_{x_i \in X_i}\ \bar{z} \cdot x_i$, then $u_i(x_i) > u_i(\bar{x}_i)$ implies $\bar{z} \cdot x_i > \bar{z} \cdot \bar{x}_i$. Hence a sufficient condition for $[(\bar{x}_i)_{i=1}^m, \bar{z}]$ to be an equilibrium is

E1 $\quad \forall i,\ \bar{z} \cdot w_i > \inf_{x_i \in X_i}\ \bar{z} \cdot x_i,$

Let $P_i(x_i) = \{x \in X_i,\ u_i(x) > u_i(x_i)\}$. If **E1** is not satisfied, another sufficient condition is

E2 $\quad \forall i,\ P_i(\bar{x}_i)$ **is open in** F**.**

Indeed, let $x_i \in P_i(\bar{x}_i)$ and let $\mathcal{V}(0)$ be an open neighborhood of 0 such that $x_i + \mathcal{V}(0) \subset P_i(\bar{x}_i)$. Then

$$\bar{z} \cdot x_i + \bar{z} \cdot h \geq \bar{z} \cdot \bar{x}_i,\ \forall\, h \in \mathcal{V}(0),$$

which implies $\bar{z} \cdot x_i > \bar{z} \cdot \bar{x}_i$. Hence $[(\bar{x}_i)_{i=1}^m, \bar{z}]$ is an equilibrium.
As one can see in the proof of Theorem 3.1., if the quasi-equilibrium $[(x_i(\bar{\lambda}))_{i=1}^m, \bar{z}]$ is associated with some $\bar{\lambda} \in \partial D$, then there exists i such that $\bar{\lambda}_i = 0$ and $\bar{z} \cdot \bar{x}_i(\bar{\lambda}) = \inf\ \bar{z} \cdot X_i$. In other words, **E1** is not fulfilled.

4 Comments on our assumptions.

1) In order to obtain support prices, Brown and Werner (1993), Dana, Le Van and Magnien (1994) assume that $\forall\, x \in A(0)$, $P_i(x_i) \neq \emptyset$, $\forall\, i$ and there exists j with int $P_j(x_j) \neq \emptyset$. In this paper, we only assume $\forall\, i\ P_i(x_i) \neq \emptyset$, $\forall\, x \in A(0)$ (**H6**) but, in order to obtain support prices, we have to assume **H3**.

2) Assumption **H4** may seem very strong; Brown and Werner, Dana, Le Van and Magnien only assume **H5**. In fact **H4** may be dropped. Indeed, consider the economy $\mathcal{E}'$ where the consumption sets X_i are replaced by:

$$X_i' = \{x \in X_i \mid u_i(x_i) \geq u_i(w_i)\}.$$

If $V(0)$ is compact, the set

$$U'(0) = \{v \in I\!\!R^m \mid v_i \leq u_i(x_i), \forall i,\ \text{for some } x \in A'(0)\}$$

where $A'(0) = \left\{ (x_1, ..., x_m) \in \prod_i X_i' \mid \sum_{i=1}^{m} x_i = w \right\}$ is obviously closed since $U'(0) = V(0) + I\!R_-^m$. Hence the economy $\mathcal{E}'$ has a quasi-equilibrium which is a quasi-equilibrium for the initial economy.

3) Assumption **H7** is fulfilled in two important cases:

i)**H1 Bis** and **H6** are fulfilled.

ii)$\forall\, i,\ X_i = F^+$ the positive cone of a Riesz space, u_i is increasing and $u_i(w_i) > u_i(0)$.

4) Let us observe that assumption **H3** is used to have:
1) w is not weakly Pareto-optimal,
2) $h(\lambda, \cdot)$ is continuous (proposition 2.1.) and the subdifferentials $\partial_\varepsilon h(\lambda_n, 0)$ are in a weakly compact set when $\lambda_n \to \lambda$ (proposition 2.2.b) . This property is crucial for the proof of theorem 3.1. For that reason, one may relax **H3** as follows:

H3 Bis **a) There exists $w' \in A(0)$ such that $u_i(w_i') > u_i(w_i), \forall\, i$**
b) There exists a neighborhood $\mathcal{W} \subseteq F$ of zero and $\beta = (\beta_1, ..., \beta_m) \in I\!R^m$ such that $\forall\, \varepsilon \in \mathcal{W}$, $\forall\, w' \in A(\varepsilon)$, $u_i(w_i') \geq \beta_i, \forall\, i$.

Indeed, it follows from **H3 Bis b** that

$$h(\lambda, \varepsilon) \geq \sum_i \lambda_i \beta_i,$$

which implies that $h(\lambda, \cdot)$ is continuous in $\mathcal{W}$. It also implies that $\forall \lambda' \in V(\lambda)$, $\partial_\varepsilon h(\lambda', 0)$ is weakly compact.

Let us also observe that under **H1 Bis**, if all u_i's are continuous and if ω is not weakly Pareto-optimal, then **H3** is true. Under the same assumptions, if ω is weakly Pareto-optimal, then there exists a price p such that $((\omega_i)_{i=1}^m,\ p)$ is an equilibrium.

5) It is well known that it is hard to verify **H4-H5**. Let us give two results concerning **H5** :

Proposition 4.1. *Assume* **H1-H2-H3a-H4**. *Then the following are equivalent:*
a)**H5** *is fulfilled,*
b) *int* $D \neq \phi$,
c) *There exists a Pareto-optimum.*

Proof: Assume first $V(0)$ bounded, hence compact. It follows from the proof of Proposition 2.3. that int $D \neq \phi$. Hence a) implies b).
It follows from proposition 2.5.that if int $D \neq \phi$, then there exists a Pareto-optimum. Hence b) implies c).
Lastly assume $V(0)$ unbounded. Then $V(0)_\infty = U(0)_\infty \bigcap I\!R_+^m \neq \{0\}$. Let $v \in V(0)_\infty$ and $x \in A(0)$

be a Pareto-optimum. Let z be such that $z_i = u_i(x_i)$, $\forall\ i$. Then $z + v \in V(0)$, hence there exists $x' \in A(0)$ such that $z_i + v_i \leq u_i(x'_i)$, $\forall\ i$. Hence $u_i(x_i) \leq u_i(x'_i)$, $\forall\ i$ a contradiction. Hence c) implies a). ■

Proposition 4.2.: H1-H2-H3-H4 bis-H7 *or* **H1bis-H2-H3-H4 bis-H6** *imply* **H5**.

Proof: By proposition 4.1., it suffices to show that there exists a Pareto-optimum. It follows from **H3** that int $V(0) \neq \emptyset$. Let $z \in$int $V(0)$. The set $\{t \mid tz \in V(0)\}$ is closed (since $V(0)$ is closed), moreover it is bounded (If it was not, $U(0)$ would be equal to $I\!R^m$). Hence it is compact. Let $\bar{t} = \max t$ such that $tz \in V(0)$. Clearly $\bar{t}z$ is weakly Pareto. Hence there exists a λ such that $\lambda \cdot \bar{t}z = \sup_{u \in U(0)} \sum_i \lambda_i u_i$. Since $\bar{t}z \in V(0)$, there exists $x \in A(0)$, such that $\bar{t}z_i \leq u_i(x_i)$, $\forall\ i$. Hence $h(\lambda, 0) = \sum_i \lambda_i u_i(x_i)$. Assume that $\lambda_k = 0$ for some k and $\lambda_j > 0$. Let k_j be given by **H7** and $x' \in A(0)$ be defined by $x'_i = x_i$ if $i \neq \{k, j\}$, $x'_k = x_k - k_j$ and $x'_j = x_j + k_j$. Then $h(\lambda, 0) < \sum_i \lambda_i u_i(x'_i)$, a contradiction. Hence $\lambda_i > 0$, $\forall\ i$ and $\bar{t}z$ is Pareto-optimum. ■

As far as **H4** is concerned, it is not possible to give a general method. We shall just make a few remarks.

$$\text{For } t \in \prod_i X_i, \text{ let } A(t,0) = \left\{ x = (x_1, \ldots, x_m) \in \prod_i X_i \mid \sum_{i=1}^m x_i = \omega, u_i(x_i) \geq u_i(t_i), \forall i \right\}.$$

The following conditions are similar to those introduced by Chichilniski and Heal [1993].

G1 : F is a reflexive space and $X_i = F$, $\forall i$

G2 : For any $t \in F^m$, $A(t, 0)$ is norm bounded

G3 : $\forall i$, u_i is norm continuous and strictly concave.

Proposition 4.3. *Assume* **G1-G2-G3**, *then* **H4** *and* **H5** *are fulfilled.*

Proof: Since u_i is norm continuous for every i, for any $t \in F^m$, $A(t, 0)$ is norm closed and norm bounded, hence weakly compact, since F is reflexive. Since u_i is weakly u.s.c., $\forall\ i$, $V(0)$ is compact and **H5** is fulfilled. Lastly, for any $t \in F^m$, let

$$V(t, 0) = \{z \in I\!R^m \mid u_i(t_i) \leq z_i \leq u_i(x_i), \forall i, \text{ for some } x \in A(0)\}.$$

Similarly, for any $t \in F^m$, $V(t, 0)$ is compact. Since u_i is defined on F, $\inf_{x \in F} u_i(x) = -\infty$, $\forall\ i$. Hence $U(0) = \bigcup_{t \in F^m} V(t, 0)$ and closed, hence **H4** is fulfilled.■

It turns out that **G2** is not generally fulfilled. In fact, Cheng [1991] shows that it is not fulfilled in the case of L^p and Von Neumann-Morgenstern utilities

$$u_i(x) = \int_S U_i(x(s))\, dP(s)$$

where $U_i : I\!R \to I\!R$ fulfills the following conditions:

(i) U_i is strictly concave and strictly increasing,

(ii) U_i is C^1,

(iii) $\int U_i(x(s))\, dP(s) \in I\!R \ , \ \forall x \in L^p$,

if for at least one agent i, $\lim_{x\to-\infty} U_i'(x)$ is finite. However, as shown by Dana-Le Van [1995], **H4-H5** are fulfilled.

5 Arbitrage and Duality in finite dimension

In this section, we shall show that, in finite dimension, the demand approach which is based on the assumption of existence of a no-arbitrage price and the duality approach, which is based on the assumption of existence of a fair utility weight, are equivalent.

We shall maintain the following assumptions:

F1 $\quad X_i \subseteq I\!R^l$ **is closed and convex,** $\forall i$,

F2 $\quad \forall i,\ u_i : X_i \to I\!R$ **is strictly concave and continuous and does not have a satiation point and** $u_i(w_i) = 0$.

Remark: Clearly **F1** is equivalent to **H1** and **F2** implies **H2** and **H6**.

A vector $t \in I\!R^l$ is **useful** for i if $u_i(x+t) \geq u_i(x)$, $\forall x \in X_i$. Let W_i denote the set of useful vectors for i. It is a standard result that W_i is a closed and convex cone and that it is the asymptotic cone of the set $\{x \in X_i,\ u_i(x_i) \geq u_i(a_i),\ \forall i\}$ for any $a \in \prod_i X_i$. It follows from **F2** that $W_i \neq \{0\}$.

A vector $p \in I\!R^l$ is a **non arbitrage price** if there exists no $x \neq 0$, $x \in \sum_i W_i$ such that $p \cdot x \leq 0$ (equivalently $x \in \sum_i W_i - \{0\}$ implies $p \cdot x > 0$). Hence $p \in I\!R^l$ is a no arbitrage price if $p \in \text{int}(-\sum_i (W_i)^\circ)$.

Symmetrically, we may introduce the following definition:

Assume $U(0)$ is closed. A vector $\lambda \in I\!R^m$ is a **fair utility weight vector** if there exists no $t \neq 0$, $t \in U_\infty$ such that $\lambda \cdot t \geq 0$ (equivalently $t \in U_\infty - \{0\}$ implies $\lambda \cdot t < 0$). Hence $\lambda \in I\!R^m$ is a fair utility weight vector if $U(0)$ is closed and if $\lambda \in \text{int}\ U_\infty^\circ(0)$.

Proposition 5.1. *Assume* **F1-F2**, *then the following statements are equivalent:*

1) There exists a non arbitrage price,
2) Aggregate demand exists at some price,
3) Useful vectors are positively independent: $t_i \in W_i$, $\forall i$, $\sum_i t_i = 0$ *implies* $t_i = 0$, $\forall i$,
4) There exist $\varepsilon \in \sum_i X_i$, $a \in I\!R^m$ *such that* $\{x \in A(\varepsilon) \mid a_i \leq u_i(x_i), \forall i\}$ *is compact,*
5) There exist $\varepsilon \in \sum_i X_i$, $a \in I\!R^m$ *such that* $\{v \in I\!R^m \mid a_i \leq v_i \leq u_i(x_i), \forall i, x \in A(\varepsilon)\}$ *is compact,*
6) $U(\varepsilon)$ *is closed,* $\forall\, \varepsilon \in \sum_i X_i$, , $U_\infty(\varepsilon)$ *is independent of* ε *and contains no line,*
7) $U(0)$ *is closed and there exists a fair utility weight vector,*
8) There exists a Pareto-optimal allocation when $\varepsilon = 0$.

Proof:Let us sketch these equivalences:
1) implies 2). Let p be non arbitrage price. Then $p \cdot t_i > 0$, $\forall\, i$. Hence $\forall\, i$, $\{x \in X_i \mid p \cdot x \leq p \cdot w_i,\ u_i(x_i) \geq u_i(w_i)\}$ is compact. Hence aggregate demand exists at price p.
2) implies 3). If aggregate demand exists at some price p, then p is non arbitrage price. Assume there exists $t_i \in W_i$, $\forall i$, $t_j \neq 0$, for some j, $\sum_i t_i = 0$. Then $p \cdot \sum_i t_i = 0$ while $p \cdot t_j > 0$, $p \cdot t_i \geq 0$, $\forall i \neq j$, a contradiction.
3) implies 4). Since u_i is concave and continuous $\{x \in A(\varepsilon) \mid a_i \leq u_i(x_i), \forall i, \}$ is closed and convex $\forall\, \varepsilon \in I\!R^l$, $\forall a \in I\!R^m$. If empty, it is compact. If not, its asymptotic cone $\{x \in \prod_i W_i \mid \sum_i x_i = 0\} = \{0\}$ by 2). Hence it is compact.
4) implies 5). is obvious.
5) implies 6). The fact that $U(\varepsilon)$ is closed, $\forall\, \varepsilon \in I\!R^l$ is obvious. The fact that $U_\infty(\varepsilon)$ is independent of ε and contains no line is proven in propositions 2.1.,2.3.
6) implies 7). The fact that $\lambda \in I\!R^m$ is a fair utility weight vector iff $U(0)$ is closed and $\lambda \in \text{int}\, D$ is proven in proposition 2.3.
7) implies 8). If $\lambda \in I\!R^m$ is a fair utility weight vector, then by proposition 2.5., $\lambda >> 0$ and $h(\lambda, 0) = \sum_i \lambda_i u_i(x_i)$. Clearly $(x_i)_{i=1}^m$ is Pareto-optimal.
8) implies 3). Let $(x_i)_{i=1}^m$ be a Pareto-optima. Assume there exists $t_i \in W_i$, $\forall i$, $t_j \neq 0$, for some j, $\sum_i t_i = 0$. Then for j, $u_j(x_j + t_j) > u_j(x_j)$ and $u_i(x_i + t_i) \geq u_i(x_i)$, $\forall i \neq j$. Since $(x_i + t_i)_{i=1}^m \in A(0)$, it contradicts the definition of Pareto-optimality.
3) implies 2) which is equivalent to 1). Let us first remark that condition 2) implies that $\sum_i W_i$ is closed and contains no line. If $\text{int}(-\sum_i (W_i)^\circ) = \emptyset$, then $\sum_i (W_i)^{\circ\circ} = \sum_i W_i$ contains a line, a contradiction.■

Corollary 5.2.: *Assume* **F1-F2**. *Then* **H5** *implies* **H4** .

Proof:Corollary 5.2. follows from 5) implies 6)

It also follows from Proposition 5.1. that:

Corollary 5.3.: *Assume* **F1-F2**. *Then* **H5** *is equivalent to the existence of a non arbitrage price which is equivalent to the existence of a fair utility weight.*

In finite dimension, **H3** is too strong and is not needed.

Define $\tilde{X}_i = X_i - w$, $\tilde{w}_i = w_i - w$, $\tilde{x}_i = x_i - w$, $\tilde{u}_i(\tilde{x}_i) = u_i(x_i)$, $\forall\ i$. Consider the economy $\tilde{\mathcal{E}} = [(\tilde{X}_i);\ (\tilde{u}_i);\ (\tilde{w}_i)]$.

Lemma 5.4.: *Assume* **F1-F2** *and* $w \in int^r(\sum_i X_i)$ *and* w *not weakly optimal. Then* **H3 Bis** *is fulfilled for* $\tilde{\mathcal{E}}$.

Proof: Since $\mathrm{int}^r(\sum_i X_i) = (\sum_i \mathrm{int}^r X_i)$ (Rockafellar, p. 49), there exists $w" = (w"_1, ..., w"_m) \in A(0)$, $w"_i \in \mathrm{int}^r X_i$, $\forall\ i$. Let U_i be a relative open neighborhood of zero such that $w"_i + U_i \subseteq X_i$. Let $Z = \sum_i U_i$. Then Z is a relative open neighborhood of zero in $\sum_i X_i - w$. Let $\varepsilon \in Z$. There exists $(\varepsilon_1, ..., \varepsilon_m) \in U_1 \times ... \times U_m$ such that $\sum_i \varepsilon_i = \varepsilon$ and $\varepsilon_i + w"_i \in X_i$, $\forall i$. Since the u_i are continuous in X_i, there exists $\beta = (\beta_1, ..., \beta_m) \in I\!R^m$ such that

$$u_i(w"_i + \varepsilon_i) \geq \inf_{\varepsilon'_i \in U_i} u_i(w"_i + \varepsilon'_i) \geq \beta_i, \forall\ i.$$

Hence **H3 Bis** is fulfilled.■

Hence, in finite dimension, an existence theorem may be given without **H3** .
Theorem 5.5. *Assume* **F1-F2-H5**. *Then there exists a quasi-equilibrium.*

Proof: Let G be the subspace generated by $\sum_i X_i - w$. We shall consider two cases:

Case 1: $w \notin \mathrm{int}^r(\sum_i X_i)$. Then there exists $p \in G - \{0\}$ such that $p \cdot w \leq p \cdot \sum_i x_i$, $\forall x \in \prod_i X_i$. Hence $p \cdot w_i \leq p \cdot x_i$, $\forall x_i \in X_i$ and $[(w_1, ...w_m), p]$ is a quasi-equilibrium.

Case 2: $w \in \mathrm{int}^r(\sum_i X_i)$. It follows from lemma 5.4. and corollary 5.2. that **H1-H2-H3 Bis-H4-H5-H6** are fulfilled for $\tilde{\mathcal{E}}$.It follows from theorem 3.1., that there exists a quasi-equilibrium $(\bar{\tilde{x}}, \bar{\tilde{p}})$ with $\bar{\tilde{p}} \in G - \{0\}$. Let $\bar{p} = (\bar{\tilde{p}}, 0)$ and $\bar{x}_i = \bar{\tilde{x}}_i + w$, $\forall i$. Then $\sum_i \bar{x}_i = w$ and $u_i(x_i) > u_i(\bar{x}_i)$ implies $\bar{p} \cdot w_i \leq \bar{p} \cdot x_i$, $\forall\ i$. Hence $[(\bar{x}, \bar{p})]$ is a quasi-equilibrium.■

Remark: Another proof of theorem 5.5. by the demand approach may be found in Nielsen (1989).

6 The differentiable Case

In this section, agents consumption spaces are the whole space and utilities are differentiable. We show that the excess utility correspondence becomes a map with good boudary behavior.

We shall maintain the following assumptions:

D1 $X_i = F$, $\forall i$, F **is a normed space;**

D2 $\forall i$, $u_i : F \to I\!R$ **is strictly concave and differentiable,** $u_i(w_i) = 0$.

Clearly **D1-D2** imply **H1bis-H2** and **H3** if w is not weakly Pareto optimal.

Examples: Assumption **D2** holds in the following cases:

1) $F = L^2(P)$, $u_i : L^2(P) \to R$ is "mean variance", in other words, there exists $a > 0$, $U_i : R\times]-a, \infty[\to R$ such that $u_i(x) = U_i(E(x), \text{var}(x))$, $\forall\, x \in L^2(P)$ where $E(x)$ and $\text{var}(x)$ denote the expectation and variance of x.

2) $F = L^p(P)$, $u_i : L^p(P) \to R$ is "Von Neumann-Morgenstern", in other words, there exists $U_i : R \to R$ such that

$$u_i(x) = \int_S U_i\,(x(s))\, dP(s)$$

where $U_i : I\!R \to I\!R$ fulfills the following conditions:

(i) U_i is strictly concave and strictly increasing,

(ii) U_i is C^1,

(iii) $\int U_i\,(x(s))\, dP(s) \in I\!R$, $\forall x \in L^p$.

The proof of differentiability of u_i, may be found in Le Van (1995).

Proposition 6.1. *Assume* **D1-D2-H4bis-H6** *and ω not weakly Paréto-optimal. Then $\forall \lambda \in$ int D, $h(\lambda, .)$. is differentiable at* 0.

Proof. Let $\lambda \in$ int D. Since $h(\lambda, .)$ is concave, it suffices to show that the set of its subgradients at 0 contains a unique element. The set of subgradients at 0 is equal to the set of multipliers. For $\lambda \in$ int D, the multiplier is unique and equals $\lambda_i u_i'(x_i(\lambda))$ $\forall\, i$.■

It follows from proposition 6.1., that in the differentiable case, the excess utility correspondence becomes a function which has the following properties:

Proposition 6.2. *Assume* **D1-D2-H4bis-H6** *and ω not weakly Paréto-optimal.*
1) For $\lambda \in$ int D, E is a continuous function, which satisfies $\lambda \cdot E(\lambda) = 0$ (Walras-law),

2) If $\lambda_n \in int\ D \to \bar{\lambda} \in \partial D$, then $||E(\lambda_n)|| \to \infty$.

Proof: The proof of 1) which is identical to that of proposition 2.9. is omitted. To prove 2) If $\lambda_n \in \text{int } D \to \bar{\lambda} \in \partial D$, then by proposition 2.8., $u_i(x_i(\lambda_n)) \to -\infty$ for some i. Since

$$E_i(\lambda_n) \leq u_i(x_i(\lambda_n)) - u_i(\omega_i),$$

$E_i(\lambda_n) \to -\infty$. ■

Theorem 6.3.: *Assume* **D1-D2-H4bis-H6**, *then there exists an equilibrium.*

Proof: One may use theorem 3.3., since **D1-D2** obviously imply **H1-H2** and **H3** if ω is not weakly Paréto-optimal. If ω is weakly Paréto-optimal, it is well known that there exists a price p such that $((w_1, ..., w_m), p)$ is an equilibrium. ■

7 Examples

7.1 C.A.P.M.

As it is well known, the C.A.P.M. played an important role in the finance literature although the problem of existence equilibrium was only discussed rather recently by Nielsen (1990.a, 1990.b) and Allingham (1991).

1.The model

There are S states of the world. A σ-field $\mathcal{S}$ models agents common information on the set S of states of the world and P is either an objective probability or agent's common subjective probability on $(S, \mathcal{S})$.

The economy E is described as follows. There is only one good taken as numeraire tradable at every state s. There are m agents. Agent i is described by a consumption space $X_i \subseteq L^2(P)$ (we do not assume here that X_i is finite dimensional), an endowment ω_i and a utility $u_i\ :\ X_i \to R$ assumed to be "mean variance", in other words, there exists $a > 0$, $U_i\ :\ R\times]-a, \infty[\to R$ such that $u_i(z) = U_i(E(z), \text{var}(z)), z \in X_i$ where $E(z)$ and var(z) denote the expectation and variance of z.

We make the following assumptions:

B1 $X_i = Z$, $\forall i$, where Z is a closed subspace of $L^2(P)$;

B2 $\omega_i \in Z$, $\forall i$; $E(\omega) = 1$ where $\omega = \sum_{i=1}^{m} \omega_i$;

B3 $\forall i$, U_i is strictly concave, C^2, $U_i(\cdot, y)$ is strictly increasing $\forall y \in R_+$, while $U_i(x, \cdot)$ is strictly decreasing $\forall x \in R$;

As Z is a closed subspace of $L^2(P)$, an asset price p being a continuous linear form is identified by Riesz representation theorem with an element of Z. We denote by $< x, y >$ the dot product of x and y in Z. Given a price p, the budget set $B_i(p)$ of agent i is defined by

$$B_i(p) = \{x_i \in Z, < p, c_i > \leq < p, \omega_i >\}.$$

Definition: An equilibrium is a pair $(\bar{x}, \bar{p}) \in Z^n \times Z$ with $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that

a) $\bar{x}_i$ maximises $u_i(x_i)$ subject to $x_i \in B_i(\bar{p})$, for every $i = 1, \ldots n$,

b) $\sum_{i=1}^{n} \bar{x}_i = \omega$.

In this example, $\mathbf{H1}, \mathbf{H2}, \mathbf{H3}$ are clearly satisfied and u_i is norm continuous for every $i = 1, \ldots, n$. In order to prove $\mathbf{H4}, \mathbf{H5}$, let us prove the following:

$$\text{For } t \in L^{2m}, \text{ let } A(t, 0) = \left\{ x = (x_1, \ldots, x_m) \in L^{2m} \mid \sum_{i=1}^{m} x_i = \omega, u_i(x_i) \geq u_i(t_i), \forall i \right\}.$$

Proposition 7.1: *For any $t \in L^{2m}$, $A(t, 0)$ is norm bounded.*

Proof: See Dana, Le Van and Magnien (1994).■

It follows from proposition 4.1., that **H4** and **H5** are fulfilled. In the case of the C.A.P.M., it turns out that the hardest hypothesis to verify is **H6**. We shall consider two cases

2.There exists a riskless asset

We assume:

B4: $1 \in Z$.

Since for every $z \in L^2, u_i(z + 1) > u_i(z)$, **H6** is verified. It follows from theorem 3.2, that there exists a quasi-equilibrium $(\bar{x}, \bar{p}) \in L^{2m} \times L^2$. Since $< \omega_i, \bar{p} > \; > \inf_{x \in L^2} < x, \bar{p} >= -\infty$, $\forall i$, the quasi-equilibrium is an equilibrium.

Theorem 7.2: *Under assumptions* $\mathbf{B1}, \mathbf{B2}, \mathbf{B3}, \mathbf{B4}$, *there exists an equilibrium.*

3.There doesn't exist a riskless asset

In this section, we assume:

B4 bis: $1 \notin Z$.

Let η denote the projection of 1 on Z. Let us first remark the following:

Proposition 7.3: *Agent's i utility has a satiation point $s_i = l_i\eta$ with $l_i > 0$.*

Proof: See Dana, Le Van and Magnien(1994).■

In the spirit of Nielsen (1990.b), we add two more assumptions in order to get **H6**:

B5: $u_i(\omega_i) > U_i(0,0)$

B6: $\max_i \left\{ -\frac{yU_{i2}(1,y)}{U_{i1}(1,y)} \right\} < \frac{1}{2}$ with $y = \frac{1-E(\eta)}{E(\eta)}$.

Proposition 7.4: *Assume* **B1, B2, B3, B4bis, B5, B6**, *then* **H6** *is fulfilled.*

Proof: See Dana, Le Van and Magnien(1994).■

We therefore have:

Theorem 7.5 -*Assume* **B1, B2 bis, B3, B4 bis, B5, B6**, *then there exists an equilibrium.*

7.2 Von Neumann-Morgenstern Utilities

As in the previous example, let $(S,\ \mathcal{S},\ P)$ be a probability space. Agent i's consumption space is assumed to be $L^p(P)$ and his utility $u_i\ :\ L^p \to R$ is assumed to be Von Neumann-Morgenstern, in other words:

$$u_i(x) = \int_S U_i\left(x(s)\right) dP(s)$$

where $U_i : I\!R \to I\!R$ fulfills the following conditions denoted by **V**:

(i) U_i is strictly concave and strictly increasing,

(ii) U_i is C^1,

(iii) $\int U_i\left(x(s)\right) dP(s) \in I\!R \ ,\ \ \forall x \in L^p$.

As we pointed out in section 6, **D2** is fulfilled. Clearly **D1, D2, H6** are true as well as **H3** if ω is not Pareto-Optimal. It follows from Dana-Le Van (1995) theorem 1 that **H4 bis** is true. It follows from proposition 3.2. that **H5** is true and from proposition 3.2. that E is a function. Existence of an equilibrium follows from theorem 6.3.

Theorem 7.6.: *Assume* **(V)**, *then there exists an equilibrium.*

8 Appendix

Lemma 2.4.: *Let* $C(\lambda) = \{u \in U(0) \mid \lambda \cdot u \geq 0\}$. *Assume* **H1-H2-H3a-H4**. *Then the correspondence* C : *int* $D \to \mathbb{R}^m_+$ *is convex, compact valued and u.h.c. Assume furthermore* **H3b**, *then it is continuous.*

Proof: Since $C(\lambda)$ is clearly closed and convex, let us show that if $\lambda \in$ int D, $C(\lambda)_\infty = \{0\}$. Let $t \in C(\lambda)_\infty$. Then $t \in U_\infty$ and $\lambda \cdot t \geq 0$, hence $\lambda \cdot t = 0$. But then $\Phi(\lambda) = 0$ which contradicts the fact that $\lambda \in$ int D. Hence C is compact valued.

Obviously C has closed graph. Let us next show that C is u.h.c. Let $V(\lambda)$ be a compact neighborhood of $\lambda \in$ int D. Let us prove that C has values in a fixed compact set. Suppose not, then there exists $\lambda_n \to \lambda$, $u^n \in C(\lambda_n)$ such that $||u^n|| \to +\infty$. Then $\frac{u^n}{||u^n||} \to \bar{u} \in U_\infty - \{0\}$ and $\lambda \cdot \bar{u} \geq 0$. Since $\lambda \in$ int D, $\lambda \cdot \bar{u} < 0$, a contradiction. Hence C has values in a fixed compact set in a neighborhood of λ and is u.h.c.

Let us lastly show that C is l.h.c. Let $u \in C(\lambda)$ and $\lambda_n \to \lambda$. If $\lambda \cdot u > 0$, then $\lambda_n \cdot u > 0$, for n large enough and C is l.h.c. at λ. If $\lambda \cdot u = 0$, let $v \in$ int $V(0)$. Then $v >> 0$ and $\lambda \cdot v > 0$, hence $\lambda_n \cdot v > 0$ for n large enough. Assume $\lambda_n \cdot u < 0$, $\forall n$. Let $\theta_n \in]0,1[$ be defined by $\lambda_n \cdot [\theta_n u + (1-\theta_n)v] = 0$. The sequence θ_n being bounded, $\theta_n \to \bar{\theta}$ and $\lambda \cdot [\bar{\theta} u + (1-\bar{\theta})v] = (1-\bar{\theta})\lambda \cdot v = 0$. Hence $\bar{\theta} = 1$ and $\theta_n u + (1-\theta_n)v \to u$ with $\theta_n u + (1-\theta_n)v \in C(\lambda_n)$ for n large enough.■

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