Mean-Variance Hedging and Numeraire

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Abstract : We consider the mean-variance hedging problem when asset prices follow Itô processes in an incomplet market framework. The usual approach deals with self-financed portfolios with respect to the primitive assets family. By adding a numeraire as an asset to trade in, we show how self-financed portfolios may be expressed with respect to this extended assets family, without changing the set of attainable contingent claims.

We introduce the hedging numeraire and relate it to the variance-optimal martingale measure of Schweizer (1995). Using this numeraire both as a deflator and to extend the primitive assets family, we are able to transform the original mean-variance hedging problem into an equivalent and simpler one ; this transformed quadratic optimization problem corresponds to the martingale case of Föllmer and Sondermann (1986), which allows an explicit description of the optimal hedging strategy through the Kunito-Watanabe decomposition.

Couverture quadratique et numéraire

Résumé : On considère le problème de couverture quadratique lorsque les prix des actifs suivent des processus de diffusion, dans le cadre de marché incomplet. Ce problème est résolu grâce à un changement approprié de numéraire.

On introduit le numéraire de couverture et on le caractérise par l'approche dualité- martingale. Il est ensuite utilisé pour transformer le problème initial de couverture quadratique en un problème plus simple correspondant au cas martingale, pour lequel on peut appliquer le théorème de projection de Kunita-Watanabe. On obtient alors une expression explicite de la stratégie optimale de couverture. Finalement, dans le cas markovien, nous donnons une procédure en deux étapes pour déterminer la stratégie de couverture quadratique. Nous relions d'abord le numéraire de couverture à la solution d'une équation aux dérivées partielles (EDP) quasilinéaire et ensuite la stratégie optimale à la solution d'une EDP linéaire.

Keywords : Hedging, numeraire, incomplete markets, optimization
Mots clés : Couverture quadratique, numéraire, marchés incomplets, optimization.

JEL Classification : C61, G11, G12

1 Introduction

Hedging and pricing of contingent claims are two major questions in applied and theoretical finance (see the list of references). In this paper we are concerned with the hedging of some future stochastic cash-flow H, delivered at time T. The market consists of $n+1$ primitive assets, one bond of price process S^0 and n risky assets of price process S, the latter being driven by a d-dimensional Brownian motion. In such a model, incompleteness arises when n is strictly smaller than d. Indeed, when $n < d$, it is typically not possible to construct a self-financed portfolio based on the bond and the n primitive risky assets, so as to attain as terminal wealth the random variable H.

A criterion for figuring out a 'good' hedging strategy is trying to solve the *mean-variance hedging* problem, introduced by Föllmer-Sondermann (1986) :

$$\min_{\theta\in\Theta} E\left[H - V_T^{x,\theta}\right]^2 \tag{1.1}$$

where

$$V_T^{x,\theta} \;=\; S_T^0\left(x + \int_0^T \theta_t' d(S/S^0)_t\right) \tag{1.2}$$

is the terminal value of a *self-financed portfolio* in the primitive assets, with initial investment x and quantities θ invested in the risky assets.

This problem has been solved by Föllmer and Sondermann (1986) and Bouleau and Lamberton (1989) in the martingale case, i.e. $S^0 \equiv 1$ and S is a martingale under the objective probability P, thanks to a direct application of the Kunita-Watanabe projection theorem. Recently, for more general cases, Duffie and Richardson (1991), Schweizer (1992a, 1994), Monat and Stricker (1995) and Pham, Rheinländer and Schweizer (1996) have proved the existence of a solution to the optimization problem (1.1) under appropriate conditions. However, they can provide an explicit form of the optimal hedging strategy only under some restrictive assumptions (deterministic mean-variance tradeoff ...), which typically does not include the stochastic volatility models.

In the usual formulation (1.2) of self-financed portfolios, the bond is used as *deflator*. In fact, any other deflator can be used without changing the *investment opportunity set*, i.e. the set of terminal values of self-financed portfolios. This last property is known as the invariance deflator theorem (see Duffie 1992 or Geman, El Karoui and Rochet 1995). This methodology is shown to be a powerful tool for option pricing, especially in a yield curve modelling (see El Karoui and Rochet 1989, Jamshidian 1989).

In this paper, we introduce the notion of *artificial extension.* It consists in adding to the primitive assets family a *numeraire*, defined here as a positive self-financed portfolio based on the primitive assets. This numeraire is then used as deflator but also as an additional asset to be trade in, so that self-financed portfolios are expressed with respect to this *extended assets family.* This artificial extension does not change the investment opportunity set and the state price densities. In that sense, it is quite different from the fictitious completion of Karatzas, Lehoczky, Shreve and Xu (1991) and He and Pearson (1991).

Our main goal is to show how the mean-variance hedging problem can be transformed into an equivalent one much simpler thanks to a suitable choice of numeraire (the so-called *hedging numeraire*) and with the artificial extension method. Let us index this numeraire by a^*, denote by $V(a^*)$ its value process, by $X(a^*) = (S^0/V(a^*), S/V(a^*))$ the price process of the primitive assets renormalized with respect to this new numeraire. Denoting by $\phi(a^*)$ the quantities in the primitive assets of a self-financed portfolio expressed with respect to the a^*-extended assets family $\{V(a^*), X\}$, it will be seen that the initial quadratic optimization problem is equivalent to another problem :

$$\min_{\phi(a^*)\in\Phi(a^*)} E^{\tilde{P}(a^*)}\left[\frac{H}{V_T(a^*)} - x - \int_0^T \phi_t(a^*)' dX_t(a^*)\right]^2 \tag{1.3}$$

where $\tilde{P}(a^*)$ is a probability equivalent to P, uniquely defined from a^*, and such that $X(a^*)$ is a martingale under $\tilde{P}(a^*)$. We say then that $\tilde{P}(a^*)$ is an equivalent a^*-martingale measure. These properties will allow to solve (1.3) as in the martingale case.

The outline of the paper is organized as follows. In Section 2, we describe the financial market model and the evolution of the primitive assets price process $X = (S^0, S)$. In Section 3 we define a numeraire as a self-financed portfolio based on the primitive assets with unit initial investment and whose value is strictly positive at every date ; then we introduce the artificial extension method and show invariance properties of self-financed portfolios and state price densities. We define in Section 4 the hedging numeraire a^* as the numeraire which maximizes the expected quadratic utility from terminal wealth. Adapting the martingale-duality method developed by Karatzas and al (1991) and He and Pearson (1991) for a utility maximization problem and using recent results of Schweizer (1995) and Delbaen-Schachermayer (1995), we obtain the existence of the hedging numeraire a^* and relate it to the *variance-optimal martingale measure* $\tilde{P}$ recently introduced by Schweizer (1995). We define then the variance optimal a^*-martingale measure by :

$$\frac{d\tilde{P}(a^*)}{d\tilde{P}} = \frac{V_T(a^*)}{S_T^0}$$

which plays a fundamental role in the mean-variance hedging problem resolution. In this section, we also study the link between the well-known *minimal martingale measure* $\hat{P}$ of Föllmer-Schweizer (1991) and the variance optimal martingale measure $\tilde{P}$.

The mean-variance hedging problem is solved in Section 5 thanks to the artificial extension method. We express self-financed portfolios with respect to the hedging numeraire-extended assets family, which allows to transform the initial problem (1.1) into the equivalent simpler one (1.3), corresponding to the martingale case. We explain how a solution to the latter induces one for the former and obtain thus an explicit expression for the optimal hedging strategy solution to (1.1). The formulation (1.3) of the mean-variance hedging problem also allows to determine easily the *approximation price* for H, $x^*(H)$, i.e. the initial investment x leading to the best hedge of the cash-flow H, as the expected discounted cash-flow with respect to the variance-optimal martingale measure :

$$\begin{aligned} x^*(H) &= E^{\tilde{P}(a^*)}\left[\frac{H}{V_T(a^*)}\right] \\ &= E^{\tilde{P}}\left[\frac{H}{S_T^0}\right] \end{aligned}$$

Section 6 concludes the paper.

2 The Financial Market Model

2.1 The primitive assets family

We consider a model for the financial market as in Karatzas and al (1991). The prices are defined in current dollar, for any continuous date $t \in [0,T]$. There exist $n+1$ primitive assets of $\mathbb{R}^{n+1}$-valued price process $X = (S^0, S)$:

- one bond whose price process is given by :

$$S_t^0 = \exp \int_0^t r_s ds, \tag{2.1}$$

where r_t is the nonnegative instantaneous interest rate.

- n risky assets, indexed by $i = 1, \ldots, n$, whose $\mathbb{R}^n$-valued price process $S = (S^1, \ldots, S^n)$ satisfies the stochastic differential system[1] :

$$dS_t = diag(S_t)\left(\mu_t dt + \sigma_t dW_t\right). \tag{2.2}$$

[1]Given a $\mathbb{R}^m$-valued vector $Y = (Y^1, \ldots, Y^m)$, $diag(Y)$ denotes the diagonal $m \times m$ matrix whose i-th diagonal term is Y^i.

Here $W = (W^1, \ldots, W^d)$ is a d-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and we denote by $I\!\!F = \{\mathcal{F}_t, 0 \leq t \leq T\}$ the P-augmentation of the filtration generated by $\{W_t, 0 \leq t \leq T\}$. We assume that the instantaneous interest rate r, the $I\!\!R^n$-valued vector process μ and the $n \times d$-valued volatility matrix process σ are progressively measurable with respect to $I\!\!F$. For $1 \leq i \leq n$, we denote by σ^i the i-th row vector of the matrix σ.

In the rest of the paper we assume that, for any $t \in [0, T]$, the volatility matrix σ_t is of full rank equal to n. It implies that $d \geq n$. When $d = n$, there are as many primitive risky assets as underlying independent random factors to be hedged : the market is complete. When $d > n$, the market is *incomplete* in the Harrison-Pliska (1981) sense. Under the previous rank condition the matrix $\sigma_t \sigma_t'$ is invertible for any $t \in [0, T]$, and we can introduce the risk premium associated to the primitive assets family $\{S^0, S\}$:

$$\lambda_t \;\; := \;\; \sigma_t'(\sigma_t \sigma_t')^{-1}(\mu_t - r_t e_n), \qquad 0 \leq t \leq T$$

where $e_n = (1, \ldots, 1) \in I\!\!R^n$. The components of λ measure the price of the risks corresponding to the components of W.

We shall assume that the processes r, μ, σ and the initial conditions $S_0^i, i = 1, \ldots, n$ are such that the solutions of the system (2.1)-(2.2) are well-defined on $[0, T]$, and :

$$\int_0^T r_t dt \;\; \leq \;\; L, \qquad P \; a.s. \tag{2.3}$$

$$E\left[e^{\int_0^T \|\sigma^i\|^2 dt}\right] \;\; < \;\; +\infty, \qquad i = 1, \ldots, n, \tag{2.4}$$

$$S_T^i \;\; \in \;\; L^2(P), \qquad i = 1, \ldots, n, \tag{2.5}$$

for some given positive constant L. We also assume as in Karatzas and al (1991), the so-called boundedness condition :

$$\int_0^T \|\lambda_t\|^2 \, dt < C, \qquad P \; a.s. \tag{2.6}$$

for some constant $C > 0$.

2.2 Self-financed portfolio

Given a $\mathcal{F}_t$-adapted process V and a $I\!\!R^n$-valued $\mathcal{F}_t$-adapted process $\theta = (\theta^1, \ldots, \theta^n)$, we can define a portfolio of market value V with quantities $(V - \theta' S)/S^0, \theta^1, \ldots, \theta^n$ in the primitive assets $S^0, S^1, \ldots, S^n$. An important concept in the hedging of contingent claims is the *self-financed portfolio*.

Definition 2.1 *The portfolio (V, θ) is self-financed with respect to the primitive assets family $\{S^0, S\}$ if :*
(i) $\theta \in \Theta$, the set of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted processes satisfying :

$$\int_0^T \|\sigma_t' \, diag\left(\frac{S_t}{S_t^0}\right) \theta_t\|^2 dt < +\infty, \qquad P \ a.s. \tag{2.7}$$

(ii) the self-financing constraint is satisfied :

$$V_t = S_t^0 \left(V_0 + \int_0^t \theta_u' \, d(S/S^0)_u\right), \qquad 0 \leq t \leq T, \ P \ a.s. \tag{2.8}$$

Condition (2.7) ensures that the stochastic integral on the right hand side of (2.8) is well-defined.

A self-financed portfolio (V, θ) with respect to the primitive assets family $\{S^0, S\}$ with a P a.s. strictly positive value process V for each date $t \in [0, T]$ may also be characterized by the proportion of the wealth invested in the risky assets S :

$$\alpha_t = \frac{1}{V_t} diag(S_t) \, \theta_t.$$

It follows that $1 - \alpha_t' e_n$ is the proportion of wealth invested in the bond. The integrability condition (2.7) on θ is equivalent to the following integrability condition on α :

$$\int_0^T \|\alpha_u' \sigma_u\|^2 du < +\infty. \tag{2.9}$$

We denote by $\mathcal{A}$ the set of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted processes satisfying (2.9). The strictly positive value process V associated to such a strategy $\alpha \in \mathcal{A}$ may be written under an explicit exponential form :

$$V_t = V_0 S_t^0 \exp\{\int_0^t \alpha_u' \sigma_u (dW_u + \lambda_u du) - \frac{1}{2} \int_0^t \|\alpha_u' \sigma_u\|^2 du\}. \tag{2.10}$$

3 Numeraire and Artificial Extension

3.1 Definition

A *numeraire* is defined as a self-financed portfolio with respect to the primitive assets family $\{S^0, S\}$, with unit initial value, intermediate values assumed to be strictly positive P a.s., and characterized by a process $a \in \mathcal{A}$, for the proportion of the wealth invested in the primitive risky assets S. According to (2.10), the value process of the numeraire a is given by :

$$V_t(a) \quad = \quad S_t^0 \exp\{\int_0^t a'_u \sigma_u (dW_u + \lambda_u du) - \frac{1}{2}\int_0^t \|a'_u\sigma_u\|^2 du\} \tag{3.1}$$

To such a numeraire a, we can associate the $n+2$ assets consisting of this numeraire and the $n+1$ primitive assets. This assets family is called *a-extended assets family* and has price process in dollars given by :

$$(V(a), X).$$

Its price process renormalized in the new numeraire is :

$$(1, X(a)) := \left(1, \frac{X}{V(a)}\right)$$

Note that when $a \equiv 0$, $V(a)$ is the initial bond price process S^0 and $X(a) = (1, S/S^0)$.

Geman, El Karoui and Rochet (1995) defined a numeraire as a price process N_t almost surely positive for each date $t \in [0,T]$. Their change of numeraire consists in deflating the initial price process $X = (S^0, S)$ by the numeraire N to get the new price process $X/N = (S^0/N, S/N)$. Our definition of numeraire is slightly different since we impose in addition that it is a self-financed portfolio with respect to the $n+1$ primitive assets. Moreover, we use the numeraire $V(a)$ both as a deflator and as an additional asset to trade in by expressing the quantities of the portfolio with respect to the a-extended assets family $\{V(a), X\}$. This method is called *artificial extension* of the primitive assets family. In the following paragraph, we study this effect on the self-financing constraint.

3.2 Invariance of self-financed portfolios under an artificial extension

A direct application of Itô's lemma to the process $X = (S^0, S)$ given in (2.1)-(2.2) and $V(a)$ given in (3.1) provides the stochastic evolution of $X(a) = (S^0/V(a), S/V(a))$:

$$dX_t(a) \quad = \quad diag(X_t(a))\,(\beta_t(a)dt + \Sigma_t(a)dW_t)\,, \tag{3.2}$$

where $\beta(a)$ is a $I\!R^{n+1}$-valued $I\!F$-adapted process and $\Sigma(a)$ is a $n+1 \times d$-valued $I\!F$-adapted process given by :

$$\beta(a) \quad = \quad \begin{pmatrix} -a' \\ I_n - e_n a' \end{pmatrix} (\mu - re_n - \sigma\sigma' a)$$

$$\Sigma(a) \;=\; \begin{pmatrix} -a' \\ I_n - e_n a' \end{pmatrix} \sigma$$

and I_n is the $n \times n$ identity matrix.

In Definition 2.1, we have defined self-financed portfolios with respect to the primitive assets family $\{S^0, S\}$. This concept may also be applied with respect to the a-extended assets family $\{V(a), X\}$. Given a $\mathcal{F}_t$-adapted process V and a $I\!R^{n+1}$-valued $\mathcal{F}_t$-adapted process $\phi(a) = (\eta(a), \theta^1(a), \ldots, \theta^n(a))$, we can define a portfolio of market value V with quantities $(V - \phi(a)'X)/V(a), \eta(a), \theta^1(a), \ldots, \theta^n(a)$ in the $n+2$ assets $V(a), S^0, S^1, \ldots, S^n$. We have then the following definition.

Definition 3.2 *The portfolio $(V, \phi(a))$ is self-financed with respect to the a-extended assets family $\{V(a), X\}$ if :*
(i) $\phi(a) \in \Phi(a)$, the set of $I\!R^{n+1}$-valued $I\!F$-adapted processes satisfying :

$$\int_0^T \|\Sigma_t'(a)\; diag\,(X_t(a))\, \phi_t(a)\|^2 dt < +\infty, \qquad P\ a.s. \tag{3.3}$$

(ii) the self-financing constraint is satisfied :

$$V_t \;=\; V_t(a)\left(V_0 + \int_0^t \phi_u(a)'\, dX_u(a)\right), \qquad 0 \le t \le T,\ P\ a.s. \tag{3.4}$$

In contrast with the expression (2.8) of self-financed portfolios, the expression (3.4) is "symmetrical" with regard to the primitive assets family in the sense that quantities of portfolios are expressed in all the primitive assets. Artificial extension should leave invariant the investment opportunity set, as confirmed by the following result.

Proposition 3.1 *Let $a \in \mathcal{A}$ be a numeraire and $V(a)$ its value process.*
(1) If the portfolio (V, θ) is self-financed with respect to the primitive assets family $\{S^0, S\}$, then the portfolio $(V, \phi(a) = (\eta(a), \theta(a)))$ is self-financed with respect to the a-extended assets family $\{V(a), X = (S^0, S)\}$ with :

$$\eta_t(a) \;=\; \frac{V_t - \theta_t' S_t}{S_t^0} \quad and \quad \theta_t(a) \;=\; \theta_t. \tag{3.5}$$

(2) If the portfolio $(V, \phi(a) = (\eta(a), \theta(a)))$ is self-financed with respect to the a-extended assets family $\{V(a), X = (S^0, S)\}$, then the portfolio (V, θ) is self-financed with respect to the primitive assets family $\{S^0, S\}$ with :

$$\theta_t \;=\; \theta_t(a) + diag(S_t)^{-1} a_t (V_t - \phi_t(a)' X_t). \tag{3.6}$$

Proof. (1) The first assertion is straightforward from a financial viewpoint. Mathematically, it follows from the deflator invariance theorem (see Proposition 1. in Geman, El Karoui and Rochet 1995 or Duffie 1992).

(2) From the self-financing constraint (3.4) with respect to the a-extended assets family $\{V(a), X\}$, we have :

$$dV_t = \left(\frac{V_t - \phi_t(a)'X_t}{V_t(a)}\right) dV_t(a) + \phi_t(a)'dX_t. \tag{3.7}$$

By definition of a numeraire characterized by a proportion of wealth a in the risky assets S, we have :

$$\frac{dV_t(a)}{V_t(a)} = (1 - a_t'e_n)\frac{dS_t^0}{S_t^0} + a_t'diag(S_t)^{-1}dS_t.$$

Substituting this last relation into (3.7), we obtain :

$$dV_t = \left(\frac{V_t - \theta_t'S_t}{S_t^0}\right) dS_t^0 + \theta_t'dS_t$$

with θ given by (3.6), which proves that (V, θ) is self-financed with respect to the primitive assets family $\{S^0, S\}$. $\square$

Let us denote by $G_T(x, \Theta)$ the set of terminal values of self-financed portfolios with respect to the primitive assets family $\{S^0, S\}$, and with initial value x :

$$G_T(x, \Theta) = \left\{S_T^0 \left(x + \int_0^T \theta_u' \, d(S/S^0)_u\right), \; \theta \in \Theta\right\},$$

and by $G_T(x, \Phi(a))$ the set of terminal values of self-financed portfolios with respect to the a-extended assets family $\{V(a), X\}$, and with initial value x :

$$G_T(x, \Phi(a)) = \left\{V_T(a) \left(x + \int_0^T \phi_u(a)' \, dX_u(a)\right), \; \phi(a) \in \Phi(a)\right\}.$$

Then, Proposition 3.1 implies that for all $x \in \mathbb{R}$ and all numeraire $a \in \mathcal{A}$, we have :

$$G_T(x, \Theta) \equiv G_T(x, \Phi(a)),$$

and relations (2.8)-(3.5) and (3.4)-(3.6) provide a one to one correspondence between these two previous sets.

3.3 Equivalent a-martingale measures

When dealing with the incomplete market $\{S^0, S\}$, the usual approach considers the set of *martingale densities*, i.e. the set of positive processes $\xi = \{\xi_t, 0 \leq t \leq T\}$ such that $\xi_0 = 1$ and $\xi, \xi S/S^0$ are local martingales under P (see Pages 1987, He and Pearson 1991, Karatzas and al 1991, Schweizer 1992b). For the primitive assets price model (2.1)-(2.2), any martingale density may be written as :

$$\xi_t^\nu = \exp\{-\int_0^t (\lambda_u + \nu_u)' dW_u - \frac{1}{2}\int_0^t (\|\lambda_u\|^2 + \|\nu_u\|^2) du\} \tag{3.8}$$

for some $\nu \in K(\sigma)$, the set of $\mathbb{R}^d$-valued $\mathbb{F}$-adapted processes such that $\int_0^T \|\nu_t\|^2 dt < +\infty$ and $\sigma_t \nu_t = 0$, for any $0 \leq t \leq T$, P *a.s.*

The previous notions and results may also be applied with respect to the a-extended assets family $\{V(a), X\}$. We may then consider the set of *a-martingale densities* as the set of positive processes $\xi(a) = \{\xi_t(a), 0 \leq t \leq T\}$ such that $\xi_0(a) = 1$ and $\xi(a), \xi(a)X(a)$ are local martingales under P. The following proposition states a one to one correspondence between martingale densities associated to different a-extended assets families. This can be interpreted as the invariance property of state price densities under an artificial extension.

Proposition 3.2 *Let $a \in \mathcal{A}$ be a numeraire and $V(a)$ its value process.*
(i) For any $\nu \in K(\sigma)$, the process :

$$\xi_t^\nu(a) := \frac{V_t(a)}{S_t^0} . \xi_t^\nu \tag{3.9}$$

is a a-martingale density. It is written as :

$$\xi_t^\nu(a) = \exp\left\{-\int_0^t (\lambda_u(a) + \nu_u)' dW_u - \frac{1}{2}\int_0^t (\|\lambda_u(a)\|^2 + \|\nu_u\|^2) du\right\} \tag{3.10}$$

where $\lambda(a) = \lambda - \sigma' a$.
(ii) Conversely, any a-martingale density may be expressed in the form (3.9).

Proof. (i) From (3.1) and (3.8), we easily check that $\xi^\nu(a)$ defined by (3.9) can be written as (3.10). Then, $\xi^\nu(a)$ is a positive local martingale under P and $\xi_0^\nu(a) = 1$. Moreover, by (3.9) we have $\xi^\nu(a)X(a) = (\xi^\nu, \xi^\nu S/S^0)$ which implies that $\xi^\nu(a)X(a)$ is a local martingale under P by definition of the martingale density ξ^ν. This shows that $\xi^\nu(a)$ is a a-martingale density.
(ii) Conversely, if $\xi(a)$ is a a-martingale density, then $\xi := \xi(a)S^0/V(a)$ is a martingale density and may be written as (3.8) for some $\nu \in K(\sigma)$. This ends the proof. □

For our purpose and since we shall apply the Kunita-Watanabe projection theorem, we need to deal with martingales instead of local martingales. Let us then define the set :

$$K_2(\sigma) \quad = \quad \{\nu \in K(\sigma),\ (\xi_t^\nu)_{0\leq t\leq T} \text{ is a square integrable } P-\text{martingale}\}$$

so that we can define for all $\nu \in K_2(\sigma)$, the probability measure equivalent to P by :

$$\frac{dP^\nu}{dP} \quad = \quad \xi_T^\nu.$$

Notice that for all $1 \leq i \leq n$, we have :

$$\frac{S_t^i}{S_t^0} \quad = \quad S_0^i exp\left(\int_0^t \sigma_u^i dW_u^\nu - \frac{1}{2}\int_0^t \parallel \sigma_u^i \parallel^2 du\right)$$

where $W^\nu := W + \int(\lambda+\nu)dt$ is a P^ν-Brownian motion, and that for all $\nu \in K_2(\sigma)$ we have from Bayes formula, Hölder inequality and condition (2.4) :

$$\begin{aligned} E^{P^\nu}\left[e^{\frac{1}{2}\int_0^T\|\sigma^i\|^2dt}\right] &= E\left[\xi_T^\nu e^{\frac{1}{2}\int_0^T\|\sigma^i\|^2dt}\right] \\ &\leq \left(E\left[\xi_T^\nu\right]^2\right)^{\frac{1}{2}}\left(E\left[e^{\int_0^T\|\sigma^i\|^2dt}\right]\right)^{\frac{1}{2}} \\ &< +\infty. \end{aligned}$$

We deduce by Novikov's criterion that S/S^0 is a P^ν-martingale. We say then that P^ν is an *equivalent martingale measure.* Note that from (3.1), we have for all $\nu \in K_2(\sigma)$ and $a \in \mathcal{A}$,

$$\frac{V_t(a)}{S_t^0} \quad = \quad \exp\{\int_0^t a_u'\sigma_u dW_u^\nu - \frac{1}{2}\int_0^t \|a_u'\sigma_u\|^2 du\}$$

so that $(V_t(a)/S_t^0)_{0\leq t\leq T}$ is a P^ν-local martingale. Let us then define for any $\nu \in K_2(\sigma)$ the set :

$$\bar{\mathcal{A}} \quad = \quad \left\{a \in \mathcal{A},\ (V_t(a)/S_t^0)_{0\leq t\leq T} \text{ is a } P^\nu-\text{martingale}, \forall \nu \in K_2(\sigma)\right\}.$$

A similar version of Proposition 3.2, with equivalent martingale measures can be stated as follows :

Corollary 3.1 *For any $a \in \bar{\mathcal{A}}$ and $\nu \in K_2(\sigma)$, the a-martingale density $\xi^\nu(a)$ defines a probability measure equivalent to P by :*

$$\frac{dP^\nu(a)}{dP} \quad := \quad \xi_T^\nu(a) \quad = \quad \frac{V_T(a)}{S_T^0}\frac{dP^\nu}{dP}. \tag{3.11}$$

Moreover, $X(a)$ is a martingale under $P^\nu(a)$. We say then that $P^\nu(a)$ is an equivalent a-martingale measure.

Proof. It follows from (3.9) that for any $a \in \bar{\mathcal{A}}$ and $\nu \in K_2(\sigma)$, $E[\xi_T^\nu(a)] = E^{P^\nu}[V_T(a)/S_T^0] = 1$, so that $P^\nu(a)$ defined by (3.11) is a probability measure equivalent to P. Moreover, from definition (3.11), we have $E^{P^\nu(a)}[S_T^0(a)] = 1$ and $E^{P^\nu(a)}[S_T^i(a)] = E^{P^\nu}[S_T^i/S_T^0] = S_0^i = S_0^i(a)$, so that $X(a)$ is not only a $P^\nu(a)$ local (positive) martingale but a 'true' $P^\nu(a)$-martingale. □

Remarks.
3.1. If $a \in \mathcal{A}$ is such that $E[e^{\int_0^T \|a'\sigma\|^2 dt}] < +\infty$, then by Hölder inequality we have for any $\nu \in K_2(\sigma)$: $E^{P^\nu}[e^{\int_0^T \frac{1}{2}\|a'\sigma\|^2 dt}] < +\infty$. This implies by Novikov's criterion that $(V(a)/S^0)$ is a P^ν-martingale and so $a \in \bar{\mathcal{A}}$.
3.2. Note that the bond numeraire $a = 0$ is obviously in $\bar{\mathcal{A}}$ and thus $P^\nu(0) = P^\nu$ for any $\nu \in K_2(\sigma)$.
3.3. From condition (2.6), $\nu = 0 \in K_2(\sigma)$ and the equivalent martingale measure $\hat{P} := P^0$ is the *minimal martingale measure* defined by Föllmer and Schweizer (1991). If $a \in \bar{\mathcal{A}}$, the probability measure

$$\hat{P}(a) \;:=\; P^0(a) \;=\; \frac{V_T(a)}{S_T^0}.\hat{P} \tag{3.12}$$

is called minimal a-martingale measure.

4 Variance optimal martingale measure and hedging numeraire

Let us consider the optimization problem :

$$(\mathcal{P}) \qquad \min_{a \in \bar{\mathcal{A}}} E\left[V_T(a)\right]^2.$$

This problem is a mean-variance hedging problem corresponding to a zero cash-flow $H = 0$ and subject to the constraint of positive portfolio value. It has been introduced in a discrete time framework by Gouriéroux and Laurent (1995). Problem $(\mathcal{P})$ may also be seen as a problem of maximizing the quadratic expected utility from terminal wealth starting with a unit investment and with a constraint of strictly positive wealth. Such a problem is considered for classical utility functions in Karatzas et al (1991) and He-Pearson (1991), and is solved by a martingale duality approach. However, their assumptions on utility functions are not satisfied for a quadratic utility function. Then, we shall adapt the martingale duality method to our quadratic optimization problem.

Since $E^{P^\nu}[V_T(a)/S_T^0] = 1$ for any $a \in \bar{\mathcal{A}}$ and $\nu \in K_2(\sigma)$, we have by Hölder inequality :

$$\begin{aligned} 1 &= \left(E\left[\frac{1}{S_T^0}\frac{dP^\nu}{dP}V_T(a)\right]\right)^2 \\ &\leq E\left[\frac{1}{S_T^0}\frac{dP^\nu}{dP}\right]^2 E[V_T(a)]^2. \end{aligned} \tag{4.1}$$

Defining then the $\mathcal{F}_T$-measurable random variable :

$$V_T^\nu := \frac{1}{E\left[\frac{1}{S_T^0}\frac{dP^\nu}{dP}\right]^2} \cdot \frac{1}{S_T^0}\frac{dP^\nu}{dP}$$

inequality (4.1) implies that for all $\nu \in K_2(\sigma)$:

$$E[V_T^\nu]^2 \leq \min_{a \in \bar{\mathcal{A}}} E\left[V_T(a)\right]^2. \tag{4.2}$$

Let us then consider the dual quadratic problem of $(\mathcal{P})$:

$$(\mathcal{D}) \qquad \max_{\nu \in K_2(\sigma)} E\left[V_T^\nu\right]^2$$

or equivalently :

$$(\mathcal{D}) \qquad \min_{\nu \in K_2(\sigma)} E\left[\frac{1}{S_T^0}\frac{dP^\nu}{dP}\right]^2.$$

Problem $(\mathcal{D})$ is closely related to the *variance-optimal martingale measure* problem considered by Schweizer (1995) and Delbaen-Schachermayer (1995). Indeed, we have the following result.

Proposition 4.1 *There exists a unique solution $\nu^* \in K_2(\sigma)$ to the problem $(\mathcal{D})$ and P^{ν^*} is the variance-optimal martingale measure.*

Proof. Let us denote by $\mathbb{P}^s$ the set of signed martingale measures, i.e. the set of signed measures Q on $(\Omega, \mathcal{F})$ such that $Q(\Omega) = 1$, $Q << P$ with $\frac{1}{S_T^0}\frac{dQ}{dP} \in L^2(P)$ and

$$E\left[\frac{dQ}{dP}\left(\frac{S_t}{S_t^0} - \frac{S_s}{S_s^0}\right)|\mathcal{F}_s\right] = 0, \quad P \text{ a.s.} \, \forall \, 0 \leq s \leq t \leq T.$$

Then $\mathbb{D}^s := \{\frac{1}{S_T^0}\frac{dQ}{dP}, Q \in \mathbb{P}^s\}$ is a nonempty $(\frac{1}{S_T^0}\frac{d\hat{P}}{dP} \in \mathbb{D}^s)$ closed convex set of $L^2(P)$ and the problem

$$(\mathcal{D}') \qquad \min_{Q \in \mathbb{P}^s} E\left[\frac{1}{S_T^0}\frac{dQ}{dP}\right]^2$$

introduced by Schweizer (1995) has a unique solution Q^* called the variance-optimal martingale measure. Now, since S is a continuous process and there exists an equivalent martingale measure with square integrable density (in fact $\hat{P}$), it has been proved by Delbaen-Schachermayer (1995 Theorem 1.3.) that Q^* is actually a probability measure equivalent to P. From the characterization of equivalent martingale measures with square integrable density in our diffusion model, it follows that there exists $\nu^* \in K_2(\sigma)$ such that $Q^* = P^{\nu^*}$. Since the set $\{\frac{1}{S_T^0}\frac{dP^\nu}{dP},\ \nu \in K_2(\sigma)\}$ is obviously contained in $I\!D^s$, this implies that problems $(\mathcal{D})$ and $(\mathcal{D}')$ are equivalent and so ν^* is the unique solution of $(\mathcal{D})$. □

We can now state the following remarkable property of the variance-optimal martingale measure in relation with the primal problem $(\mathcal{P})$. This basic fact was already observed in Schweizer (1995) and Delbaen-Schachermayer (1995).

Theorem 4.1 *There exists a process $a^* \in \bar{\mathcal{A}}$ such that :*

$$\frac{1}{S_T^0}\frac{dP^{\nu^*}}{dP} \;=\; E\left[\frac{1}{S_T^0}\frac{dP^{\nu^*}}{dP}\right]^2 .V_T(a^*) \tag{4.3}$$

and a^* is solution to the mean-variance hedging problem :

$$(\mathcal{P}) \qquad\qquad \min_{a \in \bar{\mathcal{A}}} E\left[V_T(a)\right]^2 .$$

a^* is called hedging numeraire.

Proof. (4.3) was proved by Delbaen-Schachermayer (1995 Lemma 2.2.) when $S^0 = 1$. The proof is easily adapted when $r \neq 0$, and is omitted here. Noting that (4.3) means that :

$$V_T(a^*) \;=\; V_T^{\nu^*},$$

we immediately deduce from (4.2) that a^* is solution to $(\mathcal{P})$. □

Remarks.

4.1. From (4.3), we easily note that for all $a \in \bar{\mathcal{A}}$:

$$E[V_T(a^*)]^2 \;=\; E[V_T(a^*)V_T(a)] \;=\; \frac{1}{E\left[\frac{1}{S_T^0}\frac{dP^{\nu^*}}{dP}\right]^2}.$$

The first equality may also be deduced from the fact that by the optimality of $V_T(a^*)$, the random variable $V_T(a^*) - V_T(a)$ is orthogonal to $V_T(a^*)$ in $L^2(P)$.

4.2. From the P^{ν^*}-martingale property of $(V(a^*)/S^0)$, relation (4.3) and Bayes formula, we deduce that the hedging numeraire portfolio process is given by :

$$V_t(a^*) \;=\; \frac{S_t^0}{\xi_t^{\nu^*}} \frac{E_t\left[\frac{1}{S_T^0}\xi_T^{\nu^*}\right]^2}{E\left[\frac{1}{S_T^0}\xi_T^{\nu^*}\right]^2} \tag{4.4}$$

where E_t denotes the conditional expectation given $\mathcal{F}_t$.

4.3. Because of the martingale duality method used to determine the hedging numeraire, we shall also call the variance-optimal martingale measure as the quadratic minimax martingale measure following the terminology of Karatzas and al (1991) and He-Pearson (1991). Moreover, to alleviate notations, we denote $\tilde{P} := P^{\nu^*}$, which is then defined from the hedging numeraire by :

$$\frac{d\tilde{P}}{dP} \;=\; \frac{V_T(a^*)S_T^0}{E[V_T(a^*)]^2}. \tag{4.5}$$

Next, we study the relation between the variance-optimal martingale measure $\tilde{P}$ and the minimal martingale measure $\hat{P}$. Let us consider the *numeraire portfolio*, introduced by Long (1990) and Bajeux and Portait (1994). It is defined as the numeraire a^0 (say), which maximizes the expected terminal logarithmic utility function :

$$\max_{a\in\mathcal{A}} E\left[Log V_T(a)\right].$$

Recall that a^0 is given by :

$$a_t^0 \;=\; (\sigma_t\sigma_t')^{-1}(\mu_t - r_t e_n), \qquad 0 \le t \le T, \; P \; a.s.$$

Note that $\sigma' a^0 = \lambda$ and then from condition (2.6) and Remark 3.1., $a^0 \in \bar{\mathcal{A}}$. Since $\lambda(a^0) = \lambda - \sigma' a^0 = 0$, we deduce that the minimal a^0-martingale coincides with the objective probability :

$$\hat{P}(a^0) \;=\; P, \tag{4.6}$$

which is another characterization of a^0.

The following proposition relates the variance-optimal martingale measure to the minimal martingale measure.

Proposition 4.2 *$\tilde{P}$ and $\hat{P}$ are related by :*

$$\frac{d\tilde{P}}{d\hat{P}} = \frac{V_T(a^*)V_T(a^0)}{E[V_T(a^*)]^2}.$$

In particular, $\tilde{P} = \hat{P}$ if and only if there exists a constant $C > 0$ such that :

$$V_T(a^*) = \frac{C}{V_T(a^0)}.$$

Proof. From (3.12) and (4.6), we have :

$$\frac{d\hat{P}}{dP} = \frac{S_T^0}{V_T(a^0)}$$

which gives the result by comparing with (4.5). □

Hipp's assumption corresponds precisely to the case $\tilde{P} = \hat{P}$, so that the value of the hedging numeraire portfolio is equal to the inverse, up to a constant, of the value of the numeraire portfolio. Pham, Rheinländer and Schweizer (1996) give several examples where this assumption is or is not satisfied. We study here a particular example for which one can also explicitly determine the hedging numeraire.

Example

Suppose that $S^0 \equiv 1$ and $\hat{K}_T := \int_0^T \parallel \lambda_t \parallel^2 dt$ is deterministic. It is showed in Schweizer (1995) that $\tilde{P} = \hat{P}$. Moreover, if we denote $\hat{Z}_t = E_t^{\hat{P}}[d\hat{P}/dP]$, we easily check as in Pham, Rheinländer and Schweizer (1996) that :

$$\hat{Z}_t = e^{\hat{K}_T}\left(1 - \int_0^t \hat{Z}_u \hat{\lambda}_u' dS_u\right) \tag{4.7}$$

with

$$\hat{\lambda} = diag(S)^{-1}(\sigma\sigma')^{-1}\mu. \tag{4.8}$$

Since $V(a^*)$ is a $\hat{P}$-martingale, we obtain from (4.5) that :

$$V_t(a^*) = E[V_T(a^*)]^2 \hat{Z}_t. \tag{4.9}$$

Now by definition of the proportion of wealth invested in the risky assets, we have :

$$V_t(a^*) = 1 + \int_0^t V_u(a^*)(a_u^*)' diag(S_u)^{-1} dS_u$$

which implies from (4.7)-(4.9) that :

$$a^* = -diag(S)\hat{\lambda} = -(\sigma\sigma')^{-1}\mu.$$

5 The Mean-Variance Hedging Problem

5.1 Formulation of the problem

Given a contingent claim delivering a stochastic cash-flow $H \in L^2(P)$ at time T, we are looking for a self-financed portfolio with respect to the primitive assets family, with initial investment x, that minimizes the expected square of the hedging residual. Let us define the set of strategies :

$$\tilde{\Theta} \;=\; \left\{ \theta \in \Theta,\ \int \theta' d(S/S^0) \text{ is a } \tilde{P} - \text{martingale and } E\left[\int_0^T \theta'_t d(S/S^0)_t\right]^2 \;<\; +\infty \right\}.$$

We are then interested in the following problem :

$$(\mathcal{H}(x)) \qquad \min_{\theta \in \tilde{\Theta}} E\left[H - S^0_T\left(x + \int_0^T \theta'_t d(S/S^0)_t\right)\right]^2$$

and we denote by $J(x, H)$ the associated *minimal quadratic risk.* It gives the hedging accuracy corresponding to the initial investment x.

Remark.
Schweizer (1994) considers a slightly different hedging problem since its quadratic optimization problem is over the set Ψ defined by :

$$\Psi \;=\; \left\{ \theta \in L(S/S^0),\ \int \theta' d(S/S^0) \;\in\; S^2(P) \right\}.$$

We refer to Schweizer (1994) for the above notations. It can be easily checked that $\Psi \subset \tilde{\Theta}$.

A solution to problem $(\mathcal{H}(x))$, if it exists, will be denoted by $\theta^*(x, H)$ and called *optimal hedging strategy.* We also define its associated value process

$$V^*_t(x, H) \;=\; S^0_t\left(x + \int_0^t \theta'_u \; d(S/S^0)_u\right)$$

so that $(V^*(x, H), \theta^*(x, H))$ is a self-financed portfolio with respect to the primitive assets family, called *optimal hedging portfolio.*

In a second step it is also interesting to consider the effect of the initial hedging investment and to solve the problem :

$$(\mathcal{H}) \qquad \min_{x \in \boldsymbol{R}} J(x, H)$$

whose solution $x^*(H)$ will provide the "limit initial investment" to reach the best hedging accuracy. $x^*(H)$ is also called *approximation price* for H, following the terminology of Schweizer (1995). If H is attainable in the usual sense that it can be written as $H = S_T^0(H_0 + \int_0^T (\theta_t^H)' d(S/S^0)_t)$ for some $(H_0, \theta^H) \in I\!R \times \tilde{\Theta}$, then $x^*(H) = H_0$ and is the usual arbitrage-free price of H.

As usual the choice of the pure quadratic criterion function may be questioned. But it is known that problem (5.1) is the basis for solving a class of quadratic optimization problems including general quadratic utility problems, but also mean-variance hedging problems (see Duffie and Richardson (1991), Schweizer (1992a)).

5.2 Determination of the solution of problems $(\mathcal{H}(x))$ and $(\mathcal{H})$

The idea for solving problem $(\mathcal{H}(x))$ is to transform it into a simpler one corresponding to the martingale case thanks to the artificial extension method. Let us consider the hedging numeraire $a^* \in \bar{\mathcal{A}}$ and the associated a^*-extended assets family. We can then define the equivalent a^*-martingale measure $\tilde{P}(a^*) := P^{\nu^*}(a^*)$, which is given according to Proposition 3.1 and relation (4.5) by :

$$\frac{d\tilde{P}(a^*)}{dP} = \frac{V_T(a^*)^2}{E[V_T(a^*)]^2}. \tag{5.1}$$

$\tilde{P}(a^*)$ is called variance-optimal a^*-martingale measure or quadratic minimax a^*-martingale measure. Let us then define the set of strategies :

$$\tilde{\Phi}(a^*) = \left\{\phi(a^*) \in \Phi(a^*),\ \int \phi(a^*)' dX(a^*) \text{ is a square integrable } \tilde{P}(a^*) - \text{martingale}\right\}$$

and the quadratic optimization problem :

$$\left(\mathcal{H}^{a^*}(x)\right) \qquad \min_{\phi(a^*) \in \tilde{\Phi}(a^*)} E^{\tilde{P}(a^*)} \left[\frac{H}{V_T(a^*)} - x - \int_0^T \phi_t(a^*)' dX_t(a^*)\right]^2 .$$

We also denote by $J^{a^*}(x, H)$ the associated minimal quadratic risk.

The following proposition shows equivalence between this last problem and the initial mean-variance hedging problem and explains how their solutions are related.

Proposition 5.1 *Problems* $(\mathcal{H}(x))$ *and* $(\mathcal{H}^{a^*}(x))$ *are equivalent : if* θ *is solution to* $(\mathcal{H}(x))$, *then* $\phi(a^*)$ *given by* (2.8)-(3.5) *is a solution to* $(\mathcal{H}^{a^*}(x))$. *Conversely, if* $\phi(a^*)$ *is a solution* *to* $(\mathcal{H}^{a^*}(x))$*, then* θ *given by* (3.4)-(3.6) *(for* $a = a^*$*) is a solution to* $(\mathcal{H}(x))$*. Moreover, their minimal quadratic risks are related by :*

$$J(x, H) \quad = \quad E[V_T(a^*)]^2 J^{a^*}(x, H). \tag{5.2}$$

Proof. First, let us notice that by definition (5.1) of $\tilde{P}(a^*)$, problem $(\mathcal{H}^{a^*}(x))$ may be written equivalently as

$$\min_{\phi(a^*) \in \tilde{\Phi}(a^*)} E\left[H - V_T(a^*)\left(x + \int_0^T \phi_t(a^*)' dX_t(a^*)\right)\right]^2.$$

This is a mean-variance hedging problem for H where self-financed portfolios are expressed with respect to the a^*-extended assets family. Therefore, according to the invariance property of the investment opportunity set under an artificial extension (Proposition 3.1), we only have to check that integrability conditions on $\theta \in \tilde{\Theta}$ and $\phi(a^*) \in \tilde{\Phi}(a^*)$ related by (2.8)-(3.5) and (3.4)-(3.6) are equivalent. Indeed, by noting that $V_t(a^*)/S_t^0 = E^{\tilde{P}}[d\tilde{P}(a^*)/d\tilde{P}|\mathcal{F}_t]$, we have :

$$E^{\tilde{P}}\left[\frac{d\tilde{P}(a^*)}{d\tilde{P}} \,|\, \mathcal{F}_t\right]\left(x + \int_0^t \phi_u(a^*)' dX_u(a^*)\right) \;=\; x + \int_0^t \theta_u' d(S/S^0)_u.$$

It follows that $\int \phi(a^*)' dX(a^*)$ is a $\tilde{P}(a^*)$-martingale if and only if $\int \theta' d(S/S^0)$ is a $\tilde{P}$-martingale. Moreover, we have from (5.1) :

$$E^{\tilde{P}(a^*)}\left[x + \int_0^T \phi_u(a^*)' dX_u(a^*)\right]^2 \;=\; \frac{1}{E[V_T(a^*)]^2} E\left[S_T^0\left(x + \int_0^T \theta_u' d(S/S^0)_u\right)\right]^2.$$

which proves, by using also (2.3), the equivalence of the additional square integrability conditions. Finally, relation (5.2) is immediately deduced from the following equality derived again from (5.1) :

$$E^{\tilde{P}(a^*)}\left[\frac{H}{V_T(a^*)} - x - \int_0^T \phi_u(a^*)' dX_u(a^*)\right]^2 = \frac{1}{E[V_T(a^*)]^2} E\left[H - S_T^0\left(x + \int_0^T \theta_u' d(S/S^0)_u\right)\right]^2.$$

$\square$

Let us now turn out to the resolution of the simpler quadratic optimization problem $(\mathcal{H}^{a^*}(x))$. From conditions (2.3) and (2.5), X_T is square integrable with respect to P. We deduce then immediately from (5.1) and Bayes formula that $X_T(a^*)$ is square integrable with respect to $\tilde{P}(a^*)$. It follows that the process $\{X_t(a^*), 0 \leq t \leq T\}$ is a square integrable $\tilde{P}(a^*)$-martingale. Moreover, the square integrability of H under P also implies from (5.1) the square integrability of $H/V_T(a^*)$ under $\tilde{P}(a^*)$. Hence, problem $(\mathcal{H}^{a^*}(x))$ is a mean-variance hedging problem corresponding to the martingale case of Föllmer and Sondermann (1986). We can then apply the Kunita-Watanabe projection theorem and obtain the existence of a process $\phi^H(a^*) \in \tilde{\Phi}(a^*)$ and a square integrable $\tilde{P}(a^*)$-martingale $\tilde{R}(a^*)$, orthogonal to $X(a^*)$, such that :

$$\frac{H}{V_T(a^*)} = E^{\tilde{P}(a^*)}\left[\frac{H}{V_T(a^*)}\right] + \int_0^T \phi_t^H(a^*)' dX_t(a^*) + \tilde{R}_T(a^*). \tag{5.3}$$

We directly deduce from the decomposition (5.3) that the solution of problem $\left(\mathcal{H}^{a^*}(x)\right)$ is $\phi^H(a^*)$. Morover, its associated minimal quadratic risk is given by :

$$J^{a^*}(x,H) = \left(E^{\tilde{P}(a^*)}\left[\frac{H}{V_T(a^*)}\right] - x\right)^2 + E^{\tilde{P}(a^*)}\left[\tilde{R}_T(a^*)\right]^2. \tag{5.4}$$

Hence, the Kunita-Watanabe decomposition (5.3) gives the optimal allocation $\phi^H(a^*) = (\eta^H(a^*), \theta^H(a^*))$ in the primitive assets S^0, S of the optimal hedging portfolio expressed with respect to the hedging numeraire-extended assets family. Therefore, by applying Proposition 5.1, we can provide the solution to the quadratic problem $(\mathcal{H}(x))$, i.e. the optimal hedging strategy in the primitive risky assets of the optimal hedging portfolio expressed with respect to the primitive assets family.

Theorem 5.1 *There exists a solution $\theta^*(x,H)$ to the mean-variance hedging problem $(\mathcal{H}(x))$. It is explicitly given by :*

$$\theta_t^*(x,H) = \theta_t^H(a^*) + diag(S_t)^{-1} a_t^* \left(V_t^*(x,H) - \phi_t^H(a^*)' X_t\right) \tag{5.5}$$

where $V^*(x,H)$ is given by :

$$V_t^*(x,H) = V_t(a^*)\left(x + \int_0^t \phi_u^H(a^*)' dX_u(a^*)\right). \tag{5.6}$$

Proof. The existence of a solution to problem $(\mathcal{H}(x))$ follows from the existence of the solution $\phi^H(a^*)$ to problem $(\mathcal{H}^{a^*}(x))$ and from the equivalence between those two quadratic problems. Relations (5.5) and (5.6) are deduced from the correspondence between the solutions via (2.8)-(3.5) and (3.4)-(3.6). □

In summary the determination of the optimal hedging portfolio requires two steps :
- firstly the determination of the hedging numeraire a^*, which is independent of the cash-flow H to be hedged,

- secondly, cash-flow by cash-flow, the determination of the Kunita-Watanabe decomposition (5.3).

Remark.

It would be interesting to know whether or not the solution $\theta^*(x,H)$ defined in (5.5) lies in Ψ so that it would be in fact the solution of the mean-variance hedging problem of Schweizer (1994).

Finally, from the expression (5.4) of the minimal quadratic risk $J^{a^*}(x,H)$ and its relation (5.2) with the minimal quadratic risk $J(x,H)$, we immediately deduce that the approximation price for H, i.e. the solution of problem $(\mathcal{H})$, is given by :

$$x^*(H) \;=\; E^{\tilde{P}(a^*)}\left[\frac{H}{V_T(a^*)}\right].$$

Now, from expression (3.11), we can express the approximation price for H as the expected discounted random cash-flow H under the variance-optimal martingale measure. Let us point us that this fact was already observed by Schweizer (1995) thanks to a different argument.

Theorem 5.2 *The approximation price for H is given by :*

$$x^*(H) \;=\; E^{\tilde{P}}\left[\frac{H}{S_T^0}\right].$$

The last theorem shows that the variance-optimal martingale measure can be interpreted as a viable price system (following the terminology of Harrison and Kreps 1979) corresponding to a quadratic cost criterion. This generalizes the familiar pricing concept from a complete to an incomplete market.

6 Concluding Remarks

In this paper, we have explained how to easily solve the mean-variance hedging problem thanks to the artificial extension method and by introducing the so-called hedging numeraire, indexed by a^*. The main point is to express self-financed portfolios with respect to the a^*-extended assets family and then to transform the initial mean-variance hedging problem into the simpler one quadratic optimization problem under the variance-optimal a^*-martingale measure. We obtain thus an explicit expression of the optimal hedging strategy and of the approximation price.

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