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INCOMPLETE MARKETS IN INFINITE HORIZON : DEBT CONSTRAINTS VERSUS NODE PRICES

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ECONOMIES EN HORIZON INFINI AVEC MARCHES INCOMPLETS : CONTRAINTES DE DETTES VERSUS COEFFICIENTS D'ACTUALISATION

Résumé

Le modèle d'équilibre général avec marchés incomplets est ici étendu au cas de l'horizon infini. Le modèle comporte un nombre fini d'agents en présence d'un marché financier composé d'actifs nominaux de maturité une période. La différence essentielle séparant le modèle infini du modèle fini est dans la nature de la contrainte d'endettement. En horizon infini, il faut explicitement une contrainte sur la dette des consommateurs, alors qu'elle est implicite dans le modèle fini. L'article établit une équivalence entre trois définitions de l'équilibre proposées dans la littérature, et contient un résultat général d'existence dans ce modèle.

Mots clés : Equilibre stochastique - Marchés incomplets - Horizon infini - Actifs à rendements nominaux - Prix de non-arbitrage.

INCOMPLETE MARKETS IN INFINITE HORIZON : DEBT CONSTRAINTS VERSUS NODE PRICES

Abstract

The general equilibrium model with incomplete markets is here extended to infinite horizon economies populated by a finite number of infinitely-lived agents. The crucial issue which divides the infinite horizon setting the finite horizon setting is in the nature of debt constraints. The paper relates three alternative definitions of equilibrium and states three equilibrium existence theorems when assets are one-period and purely financial.

Key words : Stochastic equilibrium - Incomplete markets - Infinite horizon - Purely financial, one-period securities - No-arbitrage asset pricing.

JEL : D51, D52, D90

Incomplete markets in infinite horizon : debt constraints versus node prices*

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Summary. The general equilibrium model with incomplete markets is here extended to infinite horizon economies populated by a finite number of infinitely-lived agents. The crucial issue which divides the infinite horizon setting the finite horizon setting is in the nature of debt constraints. The paper relates three alternative definitions of equilibrium and states three equilibrium existence theorems when assets are one-period and purely financial.

Journal of Economic Literature, Classification Numbers D51, D52, D90.

1. Introduction.

This paper deals with the equilibrium existence in infinite horizon economies with a finite number of agents and incomplete markets. It is in the line of a recent literature beginning with Levine (1989) and going mainly through the successive versions of papers by Hernandez and Santos (1991), Magill and Quinzii (1992), Levine and Zame (1992). In this paper, as Hernandez and Santos (1991), we restrict ourselves to the case of purely financial one-period assets. In this context, we provide three alternative definitions of equilibrium which share the property of generalizing the usual one for finite horizon economies with incomplete markets. The crucial difference between the three definitions is in the condition which prevents the agents from rolling over indefinitely their debts. As the no-arbitrage condition, such a condition is an obviously necessary condition for the existence of an equilibrium.

The first one is the notion of debt constraints developed by Levine and Zame (1992).

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The possible amount of debt hold in each date-event at the beginning of the period is bounded by a system of debt constraints in each node. As Levine and Zame, we restrict ourselves to a broad class of debt constraints : the loose and consistent debt constraints. This even allows for personalized systems of debt constraints. As asset prices, commodity prices, consumption plans and portfolio plans, these systems of debt constraints are components of the equilibrium.

The second one, adapted from Hernandez and Santos (1991) is a kind of solvency requirement. We require that the present value of the contracted debt in each date-event do not exceed the present value of future savings. Present values are computed according to some personalized present value node prices. In this case, these node prices are components of the equilibrium.

The third one is an alternative condition due to Magill and Quinzii (1992). This condition requires an asymptotic evolution of the present value of net indebtedness of the agents in each subtree of the event-tree defining the stochastic structure of the model. This definition is in the spirit of the subgame perfection; it also uses personalized present value node prices.

We establish three equilibrium existence results, comparable with the ones of Magill and Quinzii (1992) and Levine and Zame (1992) for the case of one-period numeraire assets. Actually, we prove that any asset price process that prevents arbitrage can be embedded in an equilibrium of the infinite horizon economy. In the proof of our main existence result, we construct our equilibrium as the limit for convenient topologies of each component of some particular equilibrium of the truncated economies which are classical finite horizon economies with incomplete markets. Our main result, Theorem 5 is to prove that three apparently quite distinct approaches in the definition of equilibrium give in fact under appropriate assumptions the same equilibrium allocations and prices.

The paper is organized as follows. In the next section, we present more precisely the model and we introduce different notions of equilibrium. In section 3, we state our existence result of equilibrium with debt constraints. In section 4, we compare the different definitions of equilibrium. Finally, some proofs are postponed in an appendix.

2. The model and the definitions of equilibrium.

2.1. The model.

We consider an exchange economy with countably many periods $t \in \{0, 1, \ldots\}$, and a finite number m of infinite-lived agents. The stochastic structure of the model is described by an infinite event-tree S, with a unique initial node denoted by 0 at date 0 and a finite branching number at each node of the tree. For every t, S_t denotes the set of nodes occuring at date t and S^t the set of nodes occuring before or during time period t. For any node $s \in S_t$, s^+ denotes the (finite) set of immediate successors of s at date $(t+1)$; the number of elements of s^+ is denoted by $b(s)$ and is called the branching number of the event-tree S at node s. If $s \in S_t$, $t > 0$, s^- denotes the (unique) node that immediately precedes s at date $(t-1)$. We will also use the notation $S^+(s)$ for the set of successors of s and $S(s)$ will denote the subtree starting at s. In particular, we have $S(s) = S^+(s) \cup \{s\}$ and $S = S(0)$.

At every node in the event-tree S described above, a finite number L of commodities are available and consumers trade commodities on spot markets. As in Bewley (1972), we take the commodity space to be $\ell_\infty(S \times L)$. Each consumer is classically described by a consumption set $X^i \subset \ell_\infty(S \times L)$, a preference relation $\precsim^i$ on X^i and a state-dependent initial endowment $\omega^i \in X^i$. On the other hand, at every node s, agents participate in a financial market. We assume that the set J_s of financial instruments available at s is finite and that the only available financial assets are purely financial, one-period securities. Let $r^j(\sigma)$ be the return in units of account that asset $j \in J_s$ promises to pay at $\sigma \in s^+$ and $r(\sigma) = (r^j(\sigma))_{j \in J_s}$. The matrix of returns ($b(s)$ rows, J_s columns) $R_s = (r^j(\sigma))_{\sigma \in s^+,\ j \in J_s}$ describes the financial structure at s and $R = \prod_{s \in S} R_s$ denotes the financial structure of the economy. Finally, the economy is summarized by the list of data

$$\mathcal{E} = \left(S, R, (X^i, \succsim^i, \omega^i)_{i=1}^m\right).$$

We will make on $\mathcal{E}$ the following assumptions:

For each agent i,

C1 - $X^i = \ell_\infty^+(S \times L)$.

C2 - $\omega^i \in \operatorname{int}(X^i)$.

C3 - $\succsim^i$ is a complete, convex, monotone, Mackey continuous preorder satisfying $x'^i \succ^i x^i$ and $\lambda \in [0, 1[\Rightarrow \lambda x^i + (1-\lambda)x'^i \succ^i x^i$. The preorder is monotone in the sense that for each $x^i \in X^i$ and for each $y \in \ell_\infty^+(S \times L)$, one has $x^i + y \succsim^i x^i$. Convexity and Mackey continuity mean that for all $\overline{x}^i \in X^i$, the set $\{x^i \in X^i | x^i \succsim^i \overline{x}^i\}$ is convex and closed in the Mackey topology $\tau(\ell_\infty(S\times L),\ell_1(S\times L))$ (hence closed in the weak topology $\sigma(\ell_\infty(S\times L),\ell_1(S\times L)))$ and the set $\{x^i \in X^i | x^i \succ^i \overline{x}^i\}$ is open in the same Mackey topology.

C4 - For all s, 1_s is extremely desirable for agent i in the sense that for each $x^i \in X^i$ and for each node s, one has $x^i + 1_s \succ^i x^i$. Obviously, 1_s is the direction defined by $1_s(s',\ell) = 0$ if $s' \neq s$, $1_s(s,\ell) = 1$.

Before introducing the next assumption, we will give some notations. We will denote by F, the set $\{x^i \in \ell_\infty^+(S \times L) | \; \|x^i\|_\infty \leq 2\|\sum_{i=1}^{m} \omega^i\|_\infty\}$. We emphasize that this set contains all feasible allocations. If $D \subset S$, we will also use the notation χ_D for the characteristic function of D; so if a^i is a process on $S \times L$, the element $a^i\chi_D$ is defined by $(a^i\chi_D)(s) = \begin{cases} a^i(s) & \text{si } s \in D; \\ 0 & \text{si } s \notin D. \end{cases}$ This notation will be used for both the consumption and the portfolio plans.

C5 - There exists $\beta \in [0, 1[$ such that for each agent i, for every node s and for all $x^i \in F$, $x^i\chi_{S\setminus S^+(s)} + 1_s + \beta x^i \chi_{S^+(s)} \succ^i x^i$.

We will also assume that the financial markets verify:

F - In each node, there exists a non-risky asset. There is no restriction on the space of portfolios. At node s, it is equal to the whole space $\mathbb{R}^{J_s}$.

On the consumption side, our assumptions are standard. As Bewley (1972), Magill and Quinzii (1992), Levine and Zame (1992), we restrict the consumers to positive bounded consumptions plans. Assumption C2 implies the assumption of commensurability made by Levine and Zame (1992). Together with C4, Assumption C3 implies that for each $x^i \in X^i$ and for each $y \in \ell_\infty^+(S \times L)$ satisfying $y(s,\ell) \neq 0$ for some node s and every commodity ℓ, one has $x^i + y \succ^i x^i$. This assumption of monotonicity is weaker than the strong monotonicity : for each $x^i \in X^i$ and for each $y \in \ell_\infty^+(S \times L) \setminus \{0\}$, one has $x^i + y \succ^i x^i$. Continuity in the Mackey topology $\tau(\ell_\infty(S \times L), \ell_1(S \times L))$ is the natural extension of the similar assumption made by Bewley (1972) to the case of an economy with time and uncertainty; it expresses the idea of the impatience of the agents. For a discussion of this assumption, see Bewley (1972), Brown and Lewis (1981), Mas-Colell and Zame (1991).

The assumption C5 is a condition of uniform impatience of the agents with respect to future consumption at each node. It is satisfied if the preferences of agent i are represented by an additively separable utility function :

$$u^i(x^i) = \sum_{s \in S} \rho(s) \delta_i^{t(s)} v^i(x^i(s))$$

where ρ defines a probability on each S_T, $\delta_i \in]0,1[$ is a discount factor and v^i is a continuous, increasing concave function on $\mathbb{R}_+^L$ satisfying $v^i(0) = 0$. The assumption C5 reinforces the assumptions C3 and C4. Indeed, one has the following lemma

Lemma 1. [Magill and Quinzii (1992)] Under the assumptions C3 and C4, for every node s, there exists $\beta_s \in [0,1[$ such that for each agent i, for all $x^i \in F$, $x^i \chi_{S \setminus S^+(s)} + 1_s + \beta_s x^i \chi_{S^+(s)} \succ^i x^i$.

As to the financial structure of the economy, the existence of an asset with positive returns is a classical assumption in an infinite horizon setting; it implies that the economy is "financially" connected enough. The second part of the assumption F emphasizes that there is no restriction on short-selling.

2.2. Equilibrium with debt constraints.

At every s, consumers face a commodity price $p(s) \in \mathbb{R}^L$ (in view of the above assumptions, we can assume that $p(s) \in \mathbb{R}_+^L$) and an asset price $q(s) \in \mathbb{R}^{J_s}$. Their financial constraint at s is determined by the value at $p(s)$ of their initial endowment and if $s \neq 0$ by the return paid by their preceding portfolio. Let $z_j^i(s)$ denote the number of units of the j^{th} security purchased by i at s (if $z_j^i(s) > 0$) or sold (if $z_j^i(s) < 0$) and $z^i(s) = (z_j^i(s))_{j \in J_s}$. At node s, the plan $(x^i(s), z^i(s))$ is the i^{th} agent's consumption and portfolio vector at node s and $x^i = (x^i(s))_{s \in S}$, $z^i = (z^i(s))_{s \in S}$ are the consumption and portfolio plans of i.

If an asset price process $q = (q(s))_{s \in S}$ is to be embedded in an equilibrium price process $(p,q) = (p(s), q(s))_{s \in S}$, it must preclude arbitrage. For every node s, let us denote by Φ_s the linear application $\mathbb{R}^{J_s} \to \mathbb{R}^{b(s)+1}$ defined by $\Phi_s(z(s)) = \begin{pmatrix} -q(s) \\ R_s \end{pmatrix} z(s)$. The asset price $q(s)$ is said *arbitrage-free* if there does not exist $z(s) \in \mathbb{R}^{J_s}$ such that $\Phi_s(z(s)) > 0$. The asset price process q is said *arbitrage-free* if for every s, $q(s)$ is arbitrage free. The no-arbitrage condition has several consequences. The following result should be kept in mind; its easy proof will be given in the appendix.

Lemma 2. If q is arbitrage-free, for every node s, for any vector $c_{min} \in \mathbb{R}^{b(s)+1}$, the set $\Gamma = \{\varphi \in \mathbb{R}^{b(s)+1} \mid \varphi \in Im\Phi_s \text{ and } \varphi \geq c_{min}\}$ is compact.

However in an infinite horizon, taking a market system of commodity and asset price processes (p,q) as given, we cannot define the *budget set* of agent i as the set $B^i(p,q) =$
$$\left\{ (x^i, z^i) \in X^i \times \prod_{s \in S} I\!R^{J_s} \,\middle|\, \begin{array}{l} p(0)\cdot x^i(0) + q(0)\cdot z^i(0) \le p(0)\cdot \omega^i(0) \\ p(s)\cdot x^i(s) + q(s)\cdot z^i(s) \le p(s)\cdot \omega^i(s) + r(s)\cdot z^i(s^-) \ \forall s \neq 0. \end{array} \right\}$$
As in the example given by Hernandez and Santos (1991), even if q is arbitrage-free, the existence of Ponzi schemes precludes the existence of any optimal point in this kind of set. So the literature has developed several additional constraints in order to prevent the consumers from rolling over their debts indefinitely.

One of them is the notion of debt constraint introduced by Levine and Zame (1992) that limits in each node the amount of debts hold by the agents. We will summarize the most important properties of the debt constraints and we refer to the article by Levine and Zame (1992) for more details.

Formally, a *system of debt constraints* for the agent i is a function $D^i : S \to]-\infty, 0]$. The portfolio $z^i(s) \in I\!R^{J_s}$ satisfies the debt constraint D^i at $\sigma \in s^+$, if $r(\sigma)\cdot z^i(s) \ge D^i(\sigma)$. If we assume for convenience that $z^i(0^-) \stackrel{\text{def}}{=} 0$, the debt constraint in node 0 will always be satisfied and the *budget set* $B^i(p,q,D^i)$ is now defined as follows :
$$\left\{ (x^i, z^i) \in X^i \times \prod_{s \in S} I\!R^{J_s} \,\middle|\, \begin{array}{ll} p(s)\cdot x^i(s) + q(s)\cdot z^i(s) \le p(s)\cdot \omega^i(s) + r(s)\cdot z^i(s^-) & \forall s, \\ r(s)\cdot z^i(s^-) \ge D^i(s) & \forall s. \end{array} \right\}$$
From this definition, it is evident that $-D^i(s)$ denotes the maximum amount of debt that i is allowed to hold at each s.

As Levine and Zame (1992), we will focus our attention on debt constraints that satisfy two conditions. The first one is a consistency condition : if the debt today is bearable, there exists a portfolio plan $z^i(s)$ that leads to a bearable debt in each state of tomorrow. At this stage, we have to notice that this notion depends on the commodity and asset price process (p,q).

Definition 1. The system of debt constraints D^i is (p,q)-consistent at node s if there exists $z^i(s) \in I\!R^{J_s}$ such that $\begin{cases} D^i(s) + p(s)\cdot\omega^i(s) - q(s)\cdot z^i(s) \ge 0, \\ \text{and for all } \sigma \in s^+,\ r(\sigma)\cdot z^i(s) \ge D^i(\sigma). \end{cases}$
The system of debt constraints D^i is said to be (p,q)-consistent if it is (p,q)-consistent at each node.

The economic interpretation of the definition is explained by the following remark.

Remark 1. For a node $s \neq 0$, an equivalent formulation of the previous definition is that for all $z^i(s^-) \in I\!R^{J_{s^-}}$ such that $r(s)\cdot z^i(s^-) \ge D^i(s)$, there exists a plan $\left(x^i(s), z^i(s)\right) \in$

$\mathbb{R}_+^L \times \mathbb{R}^{J_s}$ *such that* $\begin{cases} p(s)\cdot x^i(s) + q(s)\cdot z^i(s) \le p(s)\cdot\omega^i(s) + r(s)\cdot z^i(s^-) \\ \textit{and for all } \sigma \in s^+,\ r(\sigma)\cdot z^i(s) \ge D^i(\sigma). \end{cases}$

In other words, if the agent i arrives at node s with a debt $r(s)\cdot z^i(s^-)$ satisfying the debt constraint in node s, there exists a consumption/income transfer plan at node s, $(x^i(s), z^i(s))$, such that $z^i(s)$ finances $x^i(s)$ and $z^i(s)$ satisfies the debt constraint in every node $\sigma \in s^+$.

The second requirement is a looseness condition. If the debt today, financed by a portfolio plan $z^i(s)$ leads to a bearable debt in each state of tomorrow, then the debt today is bearable.

Definition 2. The system of debt constraints D^i is (p,q)-loose at node s if for every portfolio $z^i(s) \in \mathbb{R}^{J_s}$,
$r(\sigma)\cdot z^i(s) \ge D^i(\sigma)$, *for all* $\sigma \in s^+ \Rightarrow p(s)\cdot\omega^i(s) + D^i(s) - q(s)\cdot z^i(s) \le 0$
The system of debt constraints D^i is (p,q)-loose if it is (p,q)-loose at every node.

In particular, if D^i is (p,q)-loose at node s then $D^i(s) \le -p(s)\cdot\omega^i(s)$. The economic interpretation of the definition is explained by the following remark.

Remark 2. For a node $s \ne 0$, an equivalent formulation of the previous definition is that for every plan $(z^i(s^-), x^i(s), z^i(s)) \in \mathbb{R}^{J_{s^-}} \times \mathbb{R}_+^L \times \mathbb{R}^{J_s}$,

$$\left.\begin{array}{l} p(s)\cdot x^i(s) + q(s)\cdot z^i(s) \le p(s)\cdot\omega^i(s) + r(s)\cdot z^i(s^-) \\ \textit{and for all } \sigma \in s^+,\ r(\sigma)\cdot z^i(s) \ge D^i(\sigma). \end{array}\right\} \Rightarrow D^i(s) \le r(s)\cdot z^i(s^-).$$

In other words, if the agent i arrives at node s, with a debt $r(s)\cdot z^i(s^-)$, if he uses a consumption/income transfer plan $(x^i(s), z^i(s))$ such that $z^i(s)$ finances $x^i(s)$ and $z^i(s)$ satisfies the debt constraint in every node $\sigma \in s^+$, then the debt $r(s)\cdot z^i(s^-)$ satisfies the debt constraint at node s. It is worthwhile to remark that if the debt constraint is identically equal to zero, the system is consistent but in general not loose. In view of a better understanding of the properties of debt constraints, we recall that the definition of the budget set $B^i(p,q,D^i)$ involves both budget constraints and debt constraints. The following proposition will be proved in the appendix.

Proposition 1. Let D^i be a system of debt constraints. If q is arbitrage-free,

i) There exists $\tilde{D}^i$ (p,q)-consistent such that $B^i(p,q,\tilde{D}^i) = B^i(p,q,D^i)$ and $D^i \le \tilde{D}^i$.

ii) Moreover, if D^i is (p,q)-loose, then there exists $\tilde{D}^i$ (p,q)-loose and consistent such that $B^i(p,q,\tilde{D}^i) = B^i(p,q,D^i)$ and $D^i \le \tilde{D}^i$.

The first part of this proposition was stated by Levine and Zame (1992). The interpretation is that the consistency is to be required since a (p, q)-consistent system of debt constraints appears to be a good representative element in the set of debt constraints that give the same budget set. Looseness seems to be the most stringent requirement.

Finally, assuming that no production or intertemporal storage is possible and that assets are in zero net supply, we state the following equilibrium definition :

Definition 3. An equilibrium with debt constraints of the economy $\mathcal{E} = \left((X^i, \succsim^i, \omega^i)_{i=1}^m, S, R\right)$ *is an element* $\left((\overline{x}^i, \overline{z}^i, \overline{D}^i)_{i=1}^m, (\overline{p}, \overline{q})\right)$ *satisfying :*

i) For all i, $(\overline{x}^i, \overline{z}^i)$ *is optimal for each agent* i *in the budget set* $B^i(\overline{p}, \overline{q}, \overline{D}^i)$.

ii) For each i, $\overline{D}^i$ *is a* $(\overline{p}, \overline{q})$*-loose and consistent system of debt constraints.*

iii) $\sum_{i=1}^{m} \overline{x}^i \leq \sum_{i=1}^{m} \omega^i$.

iv) $\sum_{i=1}^{m} \overline{z}^i = 0$.

2.3. Equilibrium with present value node prices.

In the definition of the individual budget sets, a second approach adapted from Hernandez and Santos (1991) and Magill and Quinzii (1992) is to add to spot budget constraints a transversality condition on the debt contracted at each period. First let us recall a classical result on the no-arbitrage condition. For every s, if $q(s)$ is arbitrage-free, there exists a vector $\lambda_s \in I\!R_{++}^{b(s)}$, $\lambda_s = \left(\lambda_s(\sigma)\right)_{\sigma \in s+}$ such that $q(s) = \sum_{\sigma \in s+} \lambda_s(\sigma) r(\sigma)$. Here, we will use a more precise statement.

Lemma 3. [Schmachtenberg (1989)] and also [Hernandez and Santos (1991)]. If q *is a no-arbitrage asset price process for* R*, there exists* $\lambda = (\lambda_s)_{s \in S}$ *in* $I\!R_{++}^S$ *normalized by* $\lambda_0 = 1$ *such that for all* s*, we have* $\lambda_s q(s) = \sum_{\sigma \in s+} \lambda_\sigma r(\sigma)$. λ *is not necessarily unique.* λ_s *can be interpreted as the present value price of node* s.

The definition of the individual budget sets is now based on (individualized) present value node prices. Given a market system of commodity and price processes (p, q) and for each i, a system of *present value node prices* $\lambda^i = ((\lambda_s^i))$ satisfying the conditions of

Lemma 3, we define the budget set of the i^{th} agent by

$$B^i(p,q,\lambda^i) = \left\{ (x^i,z^i) \in B^i(p,q) \text{ such that } \lim_{T\to\infty} \sum_{\substack{s \in S_T \\ q(s)\cdot z^i(s) \le 0}} \lambda^i_s q(s)\cdot z^i(s) = 0. \right\}$$

Let us take λ^i as in Lemma 3; the spot constraints at node s for $s \in S^T$ imply that $\sum_{s\in S^T} \lambda^i_s p(s)\cdot x^i(s) + \sum_{s\in S_T} \lambda^i_s q(s)\cdot z^i(s) \le \sum_{s\in S^T} \lambda^i_s p(s)\cdot \omega^i(s)$. In particular, if $\pi^i \stackrel{\text{def}}{=} \left(\lambda^i_s p(s)\right)_{s\in S} \in \ell_1(S\times L)$, with a slight abuse of notations, one has $B^i(p,q,\lambda^i) \subset \{x^i \in X^i | \pi^i\cdot x^i \le \pi^i\cdot\omega^i\}$. This last set is somewhat called Arrow-Debreu budget set of agent i, for present value prices π^i.

Moreover the economic meaning of this asymptotic condition is described by the following remark.

Remark 9. [Hernandez and Santos (1991)]. Let $\pi^i \stackrel{\text{def}}{=} \left(\lambda^i_s p(s)\right)_{s\in S}$. If $\pi^i \in \ell_1(S\times L)$, the asymptotic condition is equivalent to : for every s,

$$\lambda^i_s q(s)\cdot z^i(s) \ge \sum_{\sigma \in S^+(s)} \lambda^i_\sigma p(\sigma)\cdot\left(x^i(\sigma)-\omega^i(\sigma)\right) = \pi^i\cdot\left((x^i-\omega^i)\chi_{S^+(s)}\right).$$

In other words, if $\pi^i \in \ell_1(S\times L)$ then the definition of the budget set $B^i(p,q,\lambda^i)$ involves both spot budget constraints and an infinity of additional constraints which state that, at each node, the amount of the debt contracted by consumer i should not exceed the present value of his future savings.

With such a definition of the individual budget sets, the definition of equilibrium is the following :

Definition 4. An equilibrium with present value node prices of the economy $\mathcal{E} = \left((X^i, \succsim^i, \omega^i)_{i=1}^m, S, R\right)$ is an element $\left((\overline{x}^i,\overline{z}^i,\overline{\lambda}^i)_{i=1}^m, (\overline{p},\overline{q})\right)$ satisfying :

i) For all i, $(\overline{x}^i,\overline{z}^i)$ is optimal for each agent i in the budget set $B^i(\overline{p},\overline{q},\overline{\lambda}^i)$.

ii) For each i and for every s, $\overline{\lambda}^i_s\overline{q}(s) = \sum_{\sigma\in s^+} \overline{\lambda}^i_\sigma r(\sigma)$ and $\overline{\lambda}^i_s > 0$.

iii) $\sum_{i=1}^m \overline{x}^i \le \sum_{i=1}^m \omega^i$.

iv) $\sum_{i=1}^m \overline{z}^i = 0$.

The previous definition adapted from Hernandez and Santos (1991) does not actually coincide with the definition of Magill and Quinzii (1992). In their definition of the individual budget sets associated with a market system of commodity and asset price processes (p, q), and for each i a system of individualized present value node prices λ^i, Magill and Quinzii require from an agent to be neither lender nor borrower at infinity. Moreover in the spirit of the subgame perfection, they require this condition on every subtree of the initial event-tree. Outside of equilibrium, it makes sense to prevent every agent from being borrower at infinity but not necessarily lender at infinity. In view of this remark, we define the alternative budget set :

$$B^i_{MQ}(p,q,\lambda^i) = \left\{ (x^i, z^i) \in B^i(p,q) \,\middle|\, \liminf_{T\to+\infty} \sum_{\sigma \in S_T \cap S(s)} \lambda^i_\sigma q(\sigma) \cdot z^i(\sigma) \geq 0 \quad \forall s \in S \right\}.$$

This budget set is related to the previous one by the following remark.

Remark 4. $B^i(p,q,\lambda^i) \subset B^i_{MQ}(p,q,\lambda^i)$; moreover if $\pi^i \stackrel{\text{def}}{=} \left(\lambda^i_s p(s)\right)_{s\in S} \in \ell_1(S \times L)$, one has also $B^i_{MQ}(p,q,\lambda^i) \subset \{x^i \in X^i | \pi^i \cdot x^i \leq \pi^i \cdot \omega^i\}$.

This definition of the budget sets leads to the corresponding concept of equilibrium.

Definition 5. A MQ-equilibrium with present value node prices of the economy $\mathcal{E} = \left((X^i, \succsim^i, \omega^i)_{i=1}^m, S, R\right)$ *is an element* $\left((\overline{x}^i, \overline{z}^i, \overline{\lambda}^i)_{i=1}^m, (\overline{p}, \overline{q})\right)$ *satisfying :*

i) For all i, $(\overline{x}^i, \overline{z}^i)$ *is optimal for each agent* i *in the budget set* $B^i_{MQ}(\overline{p}, \overline{q}, \overline{\lambda}^i)$.

ii) For each i *and for every* s, $\overline{\lambda}^i_s \overline{q}(s) = \sum_{\sigma \in s^+} \overline{\lambda}^i_\sigma r(\sigma)$ *and* $\overline{\lambda}^i_s > 0$.

iii) $\sum_{i=1}^m \overline{x}^i \leq \sum_{i=1}^m \omega^i$.

iv) $\sum_{i=1}^m \overline{z}^i = 0$.

3. Existence of equilibria with debt constraints.

In this section, we prove that every asset price process that precludes arbitrage can be embedded in an equilibrium with debt constraints of Economy $\mathcal{E}$. From now and in the whole paper, an arbitrage-free asset price process q will be assumed to be given.

Theorem 1. Under the assumptions C1 to C4 and F and if q *precludes arbitrage, there exists an equilibrium with debt constraints* $\left((\overline{x}^i, \overline{z}^i, \overline{D}^i)_{i=1}^m, \overline{p}, q\right)$ *of the economy* $\mathcal{E} = \left((X^i,\right.$

$\succsim^i, \omega^i)_{i=1}^m, S, R)$ *satisfying* $\overline{p}(s) > 0$, *for all* s.

Proof. First let us fix a λ as in Lemma 3. The main steps of our proof are as follows. If T denotes a time horizon, the first step is to obtain a particular general equilibrium with incomplete markets of the T-truncated economy $\mathcal{E}(T)$ with the same characteristics as $\mathcal{E}$, in which agents are constrained to stop trading at date T. The second step is to construct the implicit debt constraints of this T-truncated economy. The third step consists in establishing uniform bounds (in T) on the parameters of equilibria. Thus, it is possible in a fourth step to take an appropriate limit of these parameters. The fifth step is to prove that the system of debt constraints obtained at the limit is loose and consistent, and the final step shows that this limit is an equilibrium for the infinite horizon economy.

Step 1. Let us fix a time horizon T. If $p^T \in I\!R_+^{S^T \times L}$ is the commodity price process, then the budget set of agent i in the truncated economy $\mathcal{E}(T)$ is classically defined by $B^{i,T}(p^T, q) =$

$$\left\{ (x^i, z^i) \;\middle|\; \begin{array}{ll} z^i(s) = 0 & \text{for all } s \notin S^{T-1} \\ x^i(s) = \omega^i(s) & \text{for all } s \notin S^T \\ p^T(s) \cdot x^i(s) + q(s) \cdot z^i(s) \leq p^T(s) \cdot \omega^i(s) + r(s) \cdot z^i(s^-) & \text{for all } s \in S^T \end{array} \right\}$$

Even though the consumption-portfolio process of an agent be defined over the whole event-tree, a T-truncated economy is essentially a finite horizon economy with $T+1$ periods since the consumption-portfolio process of an agent is fixed after date T. As it is classical, we will not distinguish the finite dimensional vector p^T and its natural embedding in the infinite dimensional space $I\!R_+^{S \times L}$ defined by $p^T(s) = 0$ if $s \notin S^T$. The context will make it clear.

With a slight abuse of notation, we recall that

$$B^{i,T}(p^T, q) \subset \left\{ x^i \in X^i \;\middle|\; \begin{array}{ll} x^i(s) = \omega^i(s) & \text{if } s \notin S^T \\ \sum_{s \in S^T} \lambda_s p^T(s) \cdot (x^i(s) - \omega^i(s)) \leq 0 & \end{array} \right\}.$$

If we use the following notation $\pi^T = \left(\pi^T(s)\right)_{s \in S} \stackrel{\text{def}}{=} \left(\lambda_s p^T(s)\right)_{s \in S}$, the previous set can be equivalently rewritten as

$$B^{i,T}(\pi^T) \stackrel{\text{def}}{=} \left\{ x^i \in X^i \;\middle|\; \begin{array}{l} x^i(s) = \omega^i(s) \text{ if } s \notin S^T \\ \pi^T \cdot (x^i - \omega^i) \leq 0 \end{array} \right\}$$

and $B^{i,T}(p^T, q) \subset B^{i,T}(\pi^T)$.

According to Florenzano and Gourdel (1993), the finite-horizon economy $\mathcal{E}(T)$ has an equilibrium $\left((\overline{x}^{i,T}, \overline{z}^{i,T})_{i=1}^m, \overline{p}^T, q\right)$ satisfying :

i) $\overline{x}^{1,T}$ is optimal in the budget set $B^{1,T}(\overline{\pi}^T)$ where $\overline{\pi}^T(s) = \lambda_s \overline{p}^T(s)$.

ii) For all $i = 1, \dots, m$, $(\overline{x}^{i,T}, \overline{z}^{i,T})$ is optimal for agent i in the budget set $B^{i,T}(\overline{p}^T, q)$.

iii) $\sum_{i=1}^{m} \overline{x}^{i,T} = \sum_{i=1}^{m} \omega^i$.

iv) $\sum_{i=1}^{m} \overline{z}^{i,T} = 0$.

v) $\overline{p}^T \in I\!R_+^{S^T \times L}$ and for all node $s \in S^T$, $\overline{p}^T(s) \neq 0$.

With no loss of generality, we can assume that the price $\overline{p}^T$ is normalized by $\sum_{s \in S^T} \lambda_s \|\overline{p}^T(s)\|_1 = 1$. Note that this normalization requires an adaptation of the portfolios that with an abuse of notation, we will still denote by $\overline{z}^{i,T}$. So, the claims (i) to (v) are still verified for this normalized equilibrium and we can add the additional one :

vi) $\|\overline{\pi}^T\|_1 = 1$ or equivalently $\overline{\pi}^T \cdot 1 = 1$ (with $1 \in I\!R^{S \times L}$ defined by $1(s, \ell) = 1$).

Step 2. As Levine and Zame (1992) pointed out, in the finite horizon model, it is possible to associate with the above equilibrium an implicit system of debt constraints (for the finite event-tree S^T). It can be defined by

$$\overline{D}^{i,T}(s) = \inf\{\overline{p}^T(s) \cdot (x^i(s) - \omega^i(s)) + q(s) \cdot z^i(s)\}$$

where the infimum is taken among all (x^i, z^i) satisfying the spot budget constraints in all nodes of $S(s) \cap S^T$ and such that for all $s' \in S_T$, $z^i(s') = 0$. The notion of implicit debt constraint means that

$$\left\{ (x^i, z^i) \;\middle|\; \begin{array}{ll} z^i(s) = 0 & \text{for all } s \notin S^{T-1} \\ x^i(s) = \omega^i(s) & \text{for all } s \notin S^T \\ \overline{p}^T(s) \cdot x^i(s) + q(s) \cdot z^i(s) \leq \overline{p}^T(s) \cdot \omega^i(s) + r(s) \cdot z^i(s^-) & \text{for all } s \in S^T \end{array} \right\} =$$

$$\left\{ (x^i, z^i) \;\middle|\; \begin{array}{ll} r(s) \cdot z^i(s^-) \geq \overline{D}^{i,T}(s) & \text{for all } s \in S^T \\ z^i(s) = 0 & \text{for all } s \notin S^{T-1} \\ x^i(s) = \omega^i(s) & \text{for all } s \notin S^T \\ \overline{p}^T(s) \cdot x^i(s) + q(s) \cdot z^i(s) \leq \overline{p}^T(s) \cdot \omega^i(s) + r(s) \cdot z^i(s^-) & \text{for all } s \in S^T \end{array} \right\}$$

We emphasize that this construction could have been done for any price p^T, not only for the equilibrium price. As a consequence of the no-arbitrage condition on q, the main result at this step is the following proposition to be proved in the appendix :

Proposition 2. $\overline{D}^{i,T}$ is $(\overline{p}^T, q)$-loose and consistent. Moreover $\overline{D}^{i,T}$ is the unique system of $(\overline{p}^T, q)$-loose and consistent debt constraints.

Step 3. In this step, we will prove that the parameters $\Big((\overline{x}^{i,T}(s), \Phi_s(\overline{z}^{i,T}(s)),$

$\overline{D}^{i,T}(s))_{i=1}^m, \overline{p}^T(s)\Big)_{s\in S}$ of the T-horizon equilibrium are bounded in T.

• Bounds on $\overline{x}^{i,T}(s,\ell)$. Since $\overline{x}^{i,T}(s,\ell) \geq 0$ and $\sum_{i=1}^m \overline{x}^{i,T}(s,\ell) = \sum_{i=1}^m \omega^i(s,\ell) \leq M$, where $M = \|\sum_{i=1}^m \omega^i\|_\infty$. It follows that $0 \leq \overline{x}^{i,T}(s,\ell) \leq M$, for all $(s,\ell) \in S \times L$ and for each agent i.

• Bounds on $\overline{p}^T(s,\ell)$. First we recall that $\overline{p}^T(s,\ell) \geq 0$, hence our normalization $1 = \sum_{(s,\ell)\in S^T\times L} \lambda_s \overline{p}^T(s,\ell)$ leads to $0 \leq \overline{p}^T(s,\ell) \leq 1/\lambda_s$, for all $(s,\ell) \in S \times L$.

• Bounds on $\Phi_s(\overline{z}^{i,T}(s))$.

We first prove by contraposition that the first component of $\Phi_s(\overline{z}^{i,T}(s))$, which is equal to $q(s)\cdot\overline{z}^{i,T}(s)$, is bounded. If it is not true, there exists an agent i, a node s and a subsequence denoted by T_n such that $|q(s)\cdot\overline{z}^{i,T_n}(s)| \to +\infty$ when $T_n \to +\infty$. From condition (iv) in the definition of equilibrium, we can deduce the existence of an agent j such that $q(s)\cdot\overline{z}^{j,T_n}(s) \to +\infty$ when $T_n \to +\infty$. The agent j is lender at node s. But for T_n large enough, one has $\overline{p}^{T_n}(s)\cdot 1_s \leq (1-\beta_s)q(s)\cdot\overline{z}^{j,T_n}(s)$ since $\beta_s < 1$. This implies that the consumption process $(\overline{x}^{j,T_n}\chi_{S\setminus S^+(s)} + 1_s + \beta_s\overline{x}^{j,T_n}\chi_{S^+(s)})$ sustained by the portfolio process $(\overline{z}^{j,T_n}\chi_{S\setminus S(s)} + \beta_s\overline{z}^{j,T_n}\chi_{S(s)})$ is in the budget set $B^{j,T_n}(\overline{p}^{T_n}, q)$. Since by Lemma 1, it is preferred by agent j to the consumption $\overline{x}^{j,T_n}$, this contradicts the definition of equilibrium.

By local non-satiation, one has for each agent i, for all nodes s, for all $\sigma \in s^+$, $r(\sigma)\cdot\overline{z}^{i,T}(s) = q(\sigma)\cdot\overline{z}^{i,T}(\sigma) + \overline{p}^T(\sigma)\cdot(\overline{x}^{i,T}(\sigma) - \omega^i(\sigma))$. Hence, from the bounds established previously on $\overline{x}^{i,T}(\sigma)$, $\overline{p}^T(\sigma)$ and $q(\sigma)\cdot\overline{z}^{i,T}(\sigma)$, one deduces that $r(\sigma)\cdot\overline{z}^{i,T}(s)$ is also bounded. This proves that $\Phi_s(\overline{z}^{i,T}(s))$ is bounded.

• Bounds on $\overline{D}^{i,T}(s)$. First we recall that $\overline{D}^{i,T}(s) \leq 0$. Let us define on S^T the system of debt constraint D^i by :

$$D^{i,T}(s) = \frac{-\overline{\pi}^T\cdot(\omega^i\chi_{S(s)})}{\lambda_s} = -\sum_{\sigma\in S(s)}\left(\frac{\lambda_\sigma}{\lambda_s}\right)\overline{p}^T(\sigma)\cdot\omega^i(\sigma)$$

We first prove that $D^{i,T}$ is $(\overline{p}^T,q)$-loose. Consider a node $s \in S^T$. If $s \in S_T$, one has $D^{i,T}(s) = -\overline{p}^T(s)\cdot\omega^i(s)$, consequently $D^{i,T}$ is $(\overline{p}^T,q)$-loose at node s. If $s \in S^{T-1}$, let us consider a portfolio $z^i(s) \in \mathbb{R}^{J_s}$ such that for all node $\sigma \in s^+$, $r(\sigma)\cdot z^i(s) \geq D^{i,T}(\sigma)$. By definition of $D^{i,T}$, one has for all node $\sigma \in s^+$, $\lambda_\sigma r(\sigma)\cdot z^i(s) \geq -\overline{\pi}^T\cdot(\omega^i\chi_{S(\sigma)})$. This implies that $\sum_{\sigma\in s^+} \lambda_\sigma r(\sigma)\cdot z^i(s) \geq -\overline{\pi}^T \cdot \big(\sum_{\sigma\in s^+} \omega^i \chi_{S(\sigma)}\big) = -\overline{\pi}^T \cdot (\omega^i \chi_{S^+(s)})$. Since λ satisfies conditions of Lemma 3, one thus gets $\lambda_s q(s)\cdot z^i(s) \geq -\overline{\pi}^T \cdot (\omega^i \chi_{S^+(s)})$. We recall that $S(s) = S^+(s) \cup \{s\}$ and consequently $\lambda_s q(s)\cdot z^i(s) \geq -\pi^T \cdot (\omega^i \chi_{S(s)}) + \lambda_s \overline{p}^T(s)\cdot \omega^i(s)$, i.e., $D^{i,T}(s) + \overline{p}^T(s)\cdot\omega^i(s) - q(s)\cdot z^i(s) \leq 0$. Hence $D^{i,T}$ is $(\overline{p}^T, q)$-loose at node s.

Since $D^{i,T}$ is $(\overline{p}^T, q)$-loose, and since we recall that there is a unique system of $(\overline{p}^T, q)$-loose and consistent debt constraint, we can apply Proposition 1 to get the inequality $\overline{D}^{i,T} \geq D^{i,T}$. Moreover, it follows from the normalization of $\overline{\pi}^T$ that $D^{i,T}(s) \geq \dfrac{-M}{\lambda_s}$. Hence, we deduce that for all i, all nodes s, $0 \geq \overline{D}^{i,T}(s) \geq \dfrac{-M}{\lambda_s}$.

Step 4. Let $ba(S\times L) = \big(\ell_\infty(S\times L)\big)^*$ denote the norm dual of $\ell_\infty(S\times L)$ consisting of bounded finitely additive set functions on $S\times L$ and let $\|\cdot\|_{ba}$ denotes the norm of $ba(S\times L)$. For any T, the price $\overline{\pi}^T$ can be viewed as an element of $(ba(S\times L))_+$ such that $\overline{\pi}^T\cdot 1 = \|\overline{\pi}^T\|_{ba} = 1$. Let $\sigma(ba,\ell_\infty)$ denote the weak-star topology. We recall that according to Alaoglu's theorem, the unit ball of ba is $\sigma(ba,\ell_\infty)$-compact.

In view of the bounds established previously, we can apply Tychonov's Theorem to get the existence of a directed set $(\Theta,\geq)$ such that the subnet $\Big((\overline{x}^{i,T^\theta}, \big(\Phi_s(\overline{z}^{i,T^\theta}(s))\big)_{s\in S}, \overline{D}^{i,T^\theta})_{i=1}^m, \overline{p}^{T^\theta}\Big)_{\theta\in\Theta}$ converges for the product topology and $(\overline{\pi}^{T^\theta})_{\theta\in\Theta}$ converges for the $\sigma(ba,\ell_\infty)$-topology. Hence, we get the existence of an element $((\overline{x}^i, (\overline{\varphi}^i(s))_{s\in S}, \overline{D}^i)_{i=1}^m, \overline{p})$ in the set $\big(\mathbb{R}_+^{S\times L} \times \prod_{s\in S} \mathbb{R}^{b(s)+1} \times \mathbb{R}_-^S\big)^m \times \mathbb{R}_+^{S\times L}$ and an element $\overline{\pi}$ in $(ba(S\times L))_+$ such that

$$\Big((\overline{x}^{i,T^\theta}, \big(\Phi_s(\overline{z}^{i,T^\theta}(s))\big)_{s\in S}, \overline{D}^{i,T^\theta})_{i=1}^m, \overline{p}^{T^\theta}\Big) \to ((\overline{x}^i, (\overline{\varphi}^i(s))_{s\in S}, \overline{D}^i)_{i=1}^m, \overline{p})$$

for the product topology and

$$\overline{\pi}^{T^\theta} \to \overline{\pi} \text{ for the } \sigma(ba,\ell_\infty) \text{ topology with } \overline{\pi}\cdot 1 = 1.$$

Let us remark that it follows from the bounds established in Step 3 that $\overline{x}^i \in X^i$. Indeed, one has for all T and each i, $0 \leq \overline{x}^{i,T}(s,\ell) \leq M$. Moreover, it follows from the same inequality that one can assume that $\overline{x}^{i,T^\theta} \to \overline{x}^i$ in the weak topology $\sigma(\ell_\infty(S\times L), \ell_1(S\times L))$.

Since for all node s, $Im\Phi_s$ is a closed subset of $\mathbb{R}^{b(s)+1}$ and since $\Phi_s(\overline{z}^{i,T^\theta}(s)) \to \overline{\varphi}^i(s)$, one deduces that $\overline{\varphi}^i(s) \in Im\Phi_s$. I.e., for all $i = 2,\ldots,m$, there exists some $\overline{z}^i(s) \in \mathbb{R}^{J_s}$ such that $\Phi_s(\overline{z}^{i,T^\theta}(s)) \to \Phi_s(\overline{z}^i(s))$. Using the Cass' trick, if we let $\overline{z}^1 = \sum_{i=2}^m \overline{z}^i$, one deduces from the property $iv)$ of the finite horizon equilibrium that $\Phi_s(\overline{z}^{1,T^\theta}(s)) \to \Phi_s(\overline{z}^1(s))$. Note that $\overline{z}^{i,T^\theta}(s)$ does not necessarily converge to $\overline{z}^i(s)$ since redundant assets may exist, in particular the subnet $\left(\overline{z}^{i,T^\theta}(s)\right)_{\theta\in\Theta}$ may be unbounded.

Step 5. We want to prove that for each agent i, the system of debt constraint $\overline{D}^i$ is $(\overline{p}, q)$-loose and consistent.

• $\overline{D}^i$ is $(\overline{p}, q)$-loose. Let us consider an agent i, a node s and a portfolio $z^i(s) \in I\!R^{J_s}$ such that $r(\sigma)\cdot z^i(s) \geq \overline{D}^i(\sigma)$ for all $\sigma \in s^+$.

We want to prove that $\overline{D}^i(s) + \overline{p}(s)\cdot\omega^i(s) - q(s)\cdot z^i(s) \leq 0$. To this end, we will use the existence of a non risky asset at node s, denoted by $\xi(s)$. Hence for all $\varepsilon > 0$, one has $r(\sigma)\cdot(z^i(s) + \varepsilon\xi(s)) > \overline{D}^i(\sigma)$ for all $\sigma \in s^+$. Since there is a finite number of strict inequalities, there exists $\overline{\theta}$ such that for all $\theta \geq \overline{\theta}$, one has $r(\sigma)\cdot(z^i(s)+\varepsilon\xi(s)) > \overline{D}^{i,T^\theta}(\sigma)$ for all $\sigma \in s^+$. Consequently, by $(\overline{p}^{T^\theta}, q)$-looseness of the implicit debt constraint $\overline{D}^{i,T^\theta}$, one has $\overline{D}^{i,T^\theta}(s) + \overline{p}^{T^\theta}(s)\cdot\omega^i(s) - q(s)\cdot(z^i(s)+\varepsilon\xi(s)) \leq 0$. At the limit when θ goes to infinity, one gets $\overline{D}^i(s)+\overline{p}(s)\cdot\omega^i(s) - q(s)\cdot(z^i(s)+\varepsilon\xi(s)) \leq 0$. It suffices to take the limit when ε tends to 0 of the previous inequality to end the proof. □

• $\overline{D}^i$ is $(\overline{p}, q)$-consistent. Let us consider an agent i and a node s. We first recall that for all θ, the implicit debt constraint $\overline{D}^{i,T^\theta}$ is $(\overline{p}^{T^\theta}, q)$-consistent, hence there exists a portfolio $z^{i,T^\theta}(s)$ such that

$$\begin{cases} \overline{D}^{i,T^\theta}(s) + \overline{p}^{T^\theta}(s)\cdot\omega^i(s) - q(s)\cdot z^{i,T^\theta}(s) \geq 0; \\ \text{for all } \sigma \in s^+,\ r(\sigma)\cdot z^{i,T^\theta}(s) \geq \overline{D}^{i,T^\theta}(\sigma) \end{cases}.$$

Note that the portfolio $z^{i,T^\theta}(s)$ is defined without any reference to the portfolio $\overline{z}^{i,T^\theta}(s)$. The system can be equivalently rewritten as

$\Phi_s(z^{i,T^\theta}(s)) \geq \begin{pmatrix} -\overline{D}^{i,T^\theta}(s) - \overline{p}^{T^\theta}(s)\cdot\omega^i(s) \\ \left(\overline{D}^{i,T^\theta}(\sigma)\right)_{\sigma\in s^+} \end{pmatrix}$. Hence the net $\left(\Phi_s(z^{i,T^\theta}(s))\right)_{\theta\in\Theta}$ is bounded below. It follows from Lemma 2 that with no loss of generality, one can assume that $\Phi_s(z^{i,T^\theta}(s))$ tends to $\Phi_s(z^i(s))$ for some portfolio $z^i(s) \in I\!R^{J_s}$. By passing to limit in the previous inequalities, we get $\begin{cases} \overline{D}^i(s) + \overline{p}(s)\cdot\omega^i(s) - q(s)\cdot z^i(s) \geq 0; \\ \text{for all } \sigma \in s^+,\ r(\sigma)\cdot z^i(s) \geq \overline{D}^i(\sigma). \end{cases}$ This completes the proof. □

Step 6. It remains to prove that $\left((\overline{x}^i, \overline{z}^i, \overline{D}^i)_{i=1}^m, \overline{p}\right)$ is an equilibrium with debt constraints of the economy $\mathcal{E}$.

It is trivial to verify that individual consumption and income transfer plans belong to the individual budget set at each node s, and that consumption plans and income transfer plans are socially feasible.

• For each i, $\overline{x}^i \in B^i(\overline{p}, q, \overline{D}^i)$. Indeed, by passing to limit in the spot constraints, one has for each node s,

$$\begin{cases} \overline{p}(s) \cdot \overline{x}^i(s) + q(s) \cdot \overline{z}^i(s) = r(s) \cdot \overline{z}^i(s^-) + \overline{p}(s) \cdot \omega^i(s), \\ r(s) \cdot \overline{z}^i(s^-) \geq \overline{D}^i(s). \end{cases}$$

• $\displaystyle\sum_{i=1}^{m} \overline{x}^i = \sum_{i=1}^{m} \omega^i.$

• $\displaystyle\sum_{i=1}^{m} \overline{z}^i = 0.$

• We want to prove that consumption $\overline{x}^i$ is optimal in $B^i(\overline{p}, q, \overline{D}^i)$. To this end, the following claims will first prove that for all node s, $\overline{p}(s) > 0$.

Claim 1. $a^1 \succ^1 \overline{x}^1 \Rightarrow \overline{\pi} \cdot a^1 \geq \overline{\pi} \cdot \omega^1$.

We recall that $\overline{x}^{1,T^\theta} \to \overline{x}^1$ for the weak topology, and $a^1 \chi_{S^N} \to a^1$ for the Mackey topology. Since the relation $\succsim^1$ is Mackey continuous, there exists some N such that $a^1 \chi_{S^N} \succ^1 \overline{x}^1$. By the same argument, there exists some $\overline{\theta} \in \Theta$ such that for all $\theta \geq \overline{\theta}$, one has $a^1 \chi_{S^N} \succ^1 \overline{x}^{1,T^\theta}$ and we can suppose that $T^{\overline{\theta}} \geq N$. Hence by monotonicity, one obtains for all $\theta \geq \overline{\theta}$, $a^1 \chi_{S^{T^\theta}} \succ^1 \overline{x}^{1,T^\theta}$. Since $a^1 \chi_{S^{T^\theta}} + \omega^1 \chi_{S \setminus S^{T^\theta}}$ is in the finite horizon consumption set and preferred by agent 1 to the equilibrium consumption $\overline{x}^{1,T^\theta}$, it follows from the property i) of the finite horizon equilibrium that $\overline{\pi}^{T^\theta} \cdot (a^1 \chi_{S^{T^\theta}} + \omega^1 \chi_{S \setminus S^{T^\theta}}) > \overline{\pi}^{T^\theta} \cdot \overline{x}^{1,T^\theta}$. But $\overline{\pi}^{T^\theta} \cdot (a^1 \chi_{S^{T^\theta}} + \omega^1 \chi_{S \setminus S^{T^\theta}}) = \overline{\pi}^{T^\theta} \cdot a^1 \chi_{S^{T^\theta}}$ and by local non-satiation, one obtains $\overline{\pi}^{T^\theta} \cdot \overline{x}^{1,T^\theta} = \overline{\pi}^{T^\theta} \cdot \omega^1$. Hence, there exists $\overline{\theta}$ such that for all $\theta \geq \overline{\theta}$, $\overline{\pi}^{T^\theta} \cdot a^1 \geq \overline{\pi}^{T^\theta} \cdot a^1 \chi_{S^{T^\theta}} > \overline{\pi}^{T^\theta} \cdot \omega^1$. At the limit when θ goes to infinity, one gets $\overline{\pi} \cdot a^1 \geq \overline{\pi} \cdot \omega^1$. □

Since $\overline{\pi} \in ba(S \times L)$ and $\overline{\pi} \geq 0$, we can apply the Yosida-Hewitt theorem to get the existence of some $\overline{\pi}_c \in \ell_1^+(S \times L)$ such that $\overline{\pi}$ and $\overline{\pi}_c$ coincide on "finite" elements of $\ell_\infty(S \times L)$. For any $y \in \ell_\infty(S \times L)$ which has only a finite number of non-zero components, one has $\overline{\pi}_c \cdot y = \overline{\pi} \cdot y$.

Claim 2. $a^1 \succ^1 \overline{x}^1 \Rightarrow \overline{\pi}_c \cdot a^1 \geq \overline{\pi} \cdot \omega^1$.

The proof is similar to the one of Claim 1. Since $a^1 \succ^1 \overline{x}^1$, there exists some integer N, such that $a^1\chi_{S^N} \succ^1 \overline{x}^1$. Hence, we can apply the first claim to get the following inequality : $\overline{\pi} \cdot a^1\chi_{S^N} \geq \overline{\pi} \cdot \omega^1$. By definition of $\overline{\pi}_c$, one deduces $\overline{\pi}_c \cdot a^1\chi_{S^N} \geq \overline{\pi} \cdot \omega^1$. This assertion together with the positivity of $\overline{\pi}_c$ gives us $\overline{\pi}_c \cdot a^1 \geq \overline{\pi} \cdot \omega^1$. □

Claim 3. $a^1 \succ^1 \overline{x}^1 \Rightarrow \overline{\pi}_c \cdot a^1 > \overline{\pi} \cdot \omega^1$.

First, we recall that $\omega^1 \in \mathrm{int}(\ell^+_\infty(S \times L))$, since $\overline{\pi} \cdot 1 = 1$, one deduces that $\overline{\pi} \cdot \omega^1 > 0$. Since $a^1 \succ^1 \overline{x}^1$, by continuity of $\succsim^1$, there exists some real number $\alpha < 1$ such that $\alpha a^1 \succ^1 \overline{x}^1$. It follows from Claim 2 that $\overline{\pi}_c \cdot (\alpha a^1) = \alpha \overline{\pi}_c \cdot a^1 \geq \overline{\pi} \cdot \omega^1$. Hence, one obtains $\overline{\pi}_c \cdot a^1 \geq \frac{1}{\alpha}\overline{\pi} \cdot \omega^1 > \overline{\pi} \cdot \omega^1$. □

Claim 4. $\overline{\pi}_c \cdot \overline{x}^1 \leq \overline{\pi} \cdot \omega^1$.

Let us fix an integer N, one has $\overline{\pi}^{T^\theta} \cdot (\overline{x}^{1,T^\theta}\chi_{S^N}) \leq \overline{\pi}^{T^\theta} \cdot \overline{x}^{1,T^\theta} = \overline{\pi}^{T^\theta} \cdot \omega^1$ for all θ. This can be equivalently rewritten as $\sum_{s \in S^N} \lambda_s \overline{p}^{T^\theta}(s) \cdot \overline{x}^{1,T^\theta}(s) \leq \overline{\pi}^{T^\theta} \cdot \omega^1$. Since there is only a finite number of terms on the left side, one gets at the limit in θ, $\sum_{s \in S^N} \lambda_s \overline{p}(s) \cdot \overline{x}^1(s) \leq \overline{\pi} \cdot \omega^1$. We have proved that for all N, $\overline{\pi}_c \cdot (\overline{x}^1\chi_{S^N}) \leq \overline{\pi} \cdot \omega^1$. At the limit in N, one gets $\overline{\pi}_c \cdot \overline{x}^1 \leq \overline{\pi} \cdot \omega^1$. □

Claim 5. $\overline{p}(s) > 0$ for each node s.

Finally, it follows from the assumption C3, that $\overline{x}^1 + 1_s \succ^1 \overline{x}^1$. One can apply Claim 3 to get the following inequality : $\overline{\pi}_c \cdot (\overline{x}^1 + 1_s) > \overline{\pi} \cdot \omega^1$. This assertion together with Claim 4 gives us $\overline{\pi}_c \cdot \overline{x}^1 + \overline{\pi}_c \cdot 1_s > \overline{\pi} \cdot \omega^1 \geq \overline{\pi}_c \cdot \overline{x}^1$. Consequently, one obtains : $\overline{\pi}_c \cdot 1_s = \lambda_s \|\overline{p}\|_1 > 0$ which establishes that $\overline{p}(s) > 0$. □

For each i, we will now prove by contraposition that $\overline{x}^i$ is optimal in $B^i(\overline{p}, q, \overline{D}^i)$.

Let us suppose that $a^i \succ^i \overline{x}^i$ and $(a^i, z^i) \in B^i(\overline{p}, q, \overline{D}^i)$ for some portfolio plan z^i. By Mackey continuity of the preferences, there exists some real number $\alpha < 1$ such that $\alpha a^i \succ^i \overline{x}^i$. We can use the same argument as in Claim 1 of Step 6 to get the existence of some integer N and some $\overline{\theta} \in \Theta$ such that for all $\theta \geq \overline{\theta}$, one has $(\alpha a^i \chi_{S^N}) \succ^i \overline{x}^{i,T^\theta}$. Since $(a^i, z^i) \in B^i(\overline{p}, q, \overline{D}^i)$, we have in particular the following inequalities :

$$\begin{cases} \overline{p}(s) \cdot a^i(s) + q(s) \cdot z^i(s) \leq \overline{p}(s) \cdot \omega^i(s) + r(s) \cdot z^i(s^-) & \text{for all } s \in S^N; \\ r(s) \cdot z^i(s^-) \geq \overline{D}^i(s) & \text{for all } s \in S^{N+1}. \end{cases}$$

We recall that a consequence of the $(\overline{p}, q)$-looseness of $\overline{D}^i$ is that $\overline{D}^i(s) \leq -\overline{p}(s) \cdot \omega^i(s)$ for all nodes s. This assertion together with $\overline{p}(s) > 0$ and $\omega^i(s) \gg 0$ implies that $\overline{p}(s) \cdot \omega^i(s) > 0$ and $\overline{D}^i(s) < 0$. Since $\alpha < 1$, the last remark together with the previous system of inequalities implies that

$$\begin{cases} \overline{p}(s) \cdot (\alpha a^i(s)) + q(s) \cdot (\alpha z^i(s)) < \overline{p}(s) \cdot \omega^i(s) + r(s) \cdot (\alpha z^i(s^-)) & \text{for all } s \in S^N; \\ r(s) \cdot (\alpha z^i(s^-)) > \overline{D}^i(s) & \text{for all } s \in S^{N+1}. \end{cases}$$

Since there is only a finite number of strict inequalities, one deduces the existence of an integer $\overline{T} > N$ such that for all $T \geq \overline{T}$, one has

$$\begin{cases} \overline{p}^T(s) \cdot (\alpha a^i(s)) + q(s) \cdot (\alpha z^i(s)) < \overline{p}^T(s) \cdot \omega^i(s) + r(s) \cdot (\alpha z^i(s^-)) & \text{for all } s \in S^N, \\ r(s) \cdot (\alpha z^i(s^-)) > \overline{D}^{i,T}(s) & \text{for all } s \in S^{N+1}. \end{cases}$$

With no loss of generality, we may suppose that $\overline{T} > T^{\overline{\theta}}$. But it follows from the consistency of the implicit debt constraint $\overline{D}^{i,T}$ that there exists some portfolio plan $\tilde{z}^{i,T}$ such that $\tilde{z}^{i,T}$ is equal to zero from date T and $\tilde{z}^{i,T}$ and αz^i coincide on S^N. By construction, one has $(\alpha a^i \chi_{S^N}, \tilde{z}^{i,T}) \in B^{i,T}(\overline{p}^T, q)$. Since for some T large enough, one also has $\alpha a^i \chi_{S^N} \succ^i \overline{x}^{i,T}$, this contradicts the definition of the equilibrium consumption $\overline{x}^{i,T}$. □

4. Debt constraints versus node prices.

Let us there emphasize that, as proved in Theorem 1, the existence of an equilibrium with debt constraints, compatible with the given arbitrage-free asset price process q, does not require Assumption C5. Assumption C5 was formulated by Magill and Quinzii and used in their existence result. We will prove in this section that with this additional assumption the same given arbitrage-free asset price process q can be embedded in an equilibrium with node prices. Thus the link between both approaches cannot be reduced to a simple equivalence.

Actually from (individualized) node prices, it is easy to build (individualized) debt constraints; a more precise statement will be given in Proposition 3. Under the additional assumption C5, the converse construction is only showed to be possible at an equilibrium point. Theorem 3 and 4 will make it clear. Finally, our main result, Theorem 5 is to prove that three apparently quite distinct approaches in the definition of equilibrium give in fact under the assumptions C1-5 and F the same equilibrium allocations and prices.

Proposition 3. Assume that λ^i is a system of present value node prices, satisfying the conditions of Lemma 3, such that $\pi^i \stackrel{\text{def}}{=} (\lambda^i_s p(s))_{s \in S} \in \ell_1(S \times L)$. Then there exists a (p, q)-

loose and consistent system of debt constraints $\tilde{D}^i$ such that $B^i(p,q,\lambda^i) = B^i(p,q,\tilde{D}^i)$.

Proof. Let us define the system of debt constraints D^i by : for each node s,

$$D^i(s) = \frac{-\pi^i \cdot (\omega^i \chi_{S(s)})}{\lambda^i_s} = -\sum_{\sigma \in S(s)} \left(\frac{\lambda^i_\sigma}{\lambda^i_s}\right) p(\sigma) \cdot \omega^i(\sigma)$$

Since $\pi^i \in \ell_1(S \times L)$, and since for every node s, $\lambda^i_s > 0$, D^i is well defined. We first prove that D^i is (p,q)-loose. Indeed let us consider a node s and a portfolio $z^i(s)$ such that for every node $\sigma \in s^+$, $r(\sigma) \cdot z^i(s) \geq D^i(\sigma)$. By definition of D^i, one has for every node $\sigma \in s^+$, $\lambda^i_\sigma r(\sigma) \cdot z^i(s) \geq -\pi^i \cdot (\omega^i \chi_{S(\sigma)})$. This implies that $\sum_{\sigma \in s^+} \lambda^i_\sigma r(\sigma) \cdot z^i(s) \geq -\pi^i \cdot \left(\sum_{\sigma \in s^+} \omega^i \chi_{S(\sigma)}\right) = -\pi^i \cdot (\omega^i \chi_{S^+(s)})$. Since λ^i satisfies the conditions of Lemma 3, one thus gets $\lambda^i_s q(s) \cdot z^i(s) \geq -\pi^i \cdot (\omega^i \chi_{S^+(s)})$. We recall that $S(s) = S^+(s) \cup \{s\}$ and consequently $\lambda^i_s q(s) \cdot z^i(s) \geq -\pi^i \cdot (\omega^i \chi_{S(s)}) + \lambda^i_s p(s) \cdot \omega^i(s)$, i.e., $D^i(s) + p(s) \cdot \omega^i(s) - q(s) \cdot z^i(s) \leq 0$. Hence D^i is (p,q)-loose at each node s.

Let us prove that $B^i(p,q,\lambda^i) \subset B^i(p,q,D^i)$. Indeed, let us consider (x^i, z^i) in $B^i(p,q,\lambda^i)$. It follows from the budget constraint in node s that $\lambda^i_s r(s) \cdot z^i(s^-) \geq \lambda^i_s p(s) \cdot (x^i(s) - \omega^i(s)) + \lambda^i_s q(s) \cdot z^i(s)$. This inequality together with the one of Remark 3 leads to $\lambda^i_s r(s) \cdot z^i(s^-) \geq \sum_{\sigma \in S(s)} \lambda^i_\sigma p(\sigma) \cdot (x^i(\sigma) - \omega^i(\sigma))$. Since $x^i \geq 0$, one obtains $\lambda^i_s r(s) \cdot z^i(s^-) \geq -\pi^i \cdot (\omega^i \chi_{S(s)})$. By definition of D^i, the debt constraint in node s is satisfied and $(x^i, z^i) \in B^i(p,q,D^i)$.

Let us prove that conversely $B^i(p,q,D^i) \subset B^i(p,q,\lambda^i)$. Indeed, let us consider (x^i, z^i) in $B^i(p,q,D^i)$. Since D^i is (p,q)-loose, one has for all node s, $D^i(s) + p(s) \cdot \omega^i(s) - q(s) \cdot z^i(s) \leq 0$. This implies that $\lambda^i_s q(s) \cdot z^i(s) \geq -\pi^i \cdot (\omega^i \chi_{S^+(s)})$ and

$$0 \geq \sum_{\substack{s \in S_T \\ q(s) \cdot z^i(s) \leq 0}} \lambda^i_s q(s) \cdot z^i(s) \geq -\pi^i \cdot (\omega^i \chi_{S \setminus S^T}).$$

Since the term on the right side of the previous inequality tends to zero in the limit in T, one obtains (x^i, z^i) in $B^i(p,q,\lambda^i)$.

Finally, we can apply Proposition 1 to get the existence of a (p,q)-loose and consistent system of debt constraint $\tilde{D}^i$ such that $B^i(p,q,D^i) = B^i(p,q,\tilde{D}^i)$. □

The previous construction can be achieved for any price system (p,q). If we apply Proposition 3 to the equilibrium prices, we obtain the following result.

Theorem 2. Let $\left((\overline{x}^i,\overline{z}^i,\overline{\lambda}^i)_{i=1}^m,\overline{p},q\right)$ *be an equilibrium with present value node prices of the economy* $\mathcal{E}=\left((X^i,\succsim^i,\omega^i)_{i=1}^m,S,R\right)$. *If for each* i, $\overline{\pi}^i \stackrel{\text{def}}{=} \left(\overline{\lambda}_s^i\overline{p}(s)\right)_{s\in S}\in \ell_1(S\times L)$, *then there exists some debt constraints* $\overline{D}^i$ *such that* $\left((\overline{x}^i,\overline{z}^i,\overline{D}^i)_{i=1}^m,\overline{p},q\right)$ *is an equilibrium with debt constraints of the economy* $\mathcal{E}$.

Under the additional assumption C5, conversely one also has :

Theorem 3. Let $\left((\overline{x}^i,\overline{z}^i,\overline{D}^i)_{i=1}^m,\overline{p},q\right)$ *be an equilibrium with debt constraints of the economy* $\mathcal{E}=\left((X^i,\succsim^i,\omega^i)_{i=1}^m,S,R\right)$. *Under the assumptions C1-C5 and F, there exists some present value node prices* $(\overline{\lambda}^i)_{i=1}^m$ *such that* $\left((\overline{x}^i,\overline{z}^i,\overline{\lambda}^i)_{i=1}^m,\overline{p},q\right)$ *is an equilibrium with present value node prices of the economy* $\mathcal{E}$. *Moreover* $\left((\overline{x}^i,\overline{z}^i,\overline{\lambda}^i)_{i=1}^m,\overline{p},q\right)$ *is also a MQ-equilibrium with present value node prices of the economy* $\mathcal{E}$.

Proof of Theorem 3. Let us consider for any agent i, and for any time horizon T, the set $B_T^i(\overline{p},q)$ defined by

$$\left\{ x^i\in I\!R^{S^T\times L} \;\middle|\; \exists z^i\in\prod_{s\in S^T} I\!R^{J_s} \;\middle|\; \begin{array}{l} \overline{p}(s)\cdot x^i(s)+q(s)\cdot z^i(s)\le \overline{p}(s)\cdot\omega^i(s)+r(s)\cdot z^i(s^-) \\ \hfill \forall s\in S^T; \\ -r(s)\cdot z^i(s^-)\le 0\ \forall s\in S_{T+1}. \end{array}\right\}.$$

As it is classical, we will not distinguish the finite dimensional set $B_T^i(\overline{p},q)$ and its natural embedding in the infinite dimensional space $\ell_\infty(S\times L)\times\prod_{s\in S} I\!R^{J_s}$ defined by $\left(x(s),z(s)\right)=(\omega^i(s),0)$ if $s\notin S^T$. The context will make it clear. Note that the sets $B_T^i(\overline{p},q)$ are nonempty and convex and that $B_T^i(\overline{p},q)\subset B_{T+1}^i(\overline{p},q)$, for all $T\ge 1$. We can now define the two convex subsets of $\ell_\infty(S\times L)$ defined by

$$U^i=\{x^i\in\ell_\infty^+(S\times L) | x^i\succ\overline{x}^i\};$$

$$A^i=\operatorname{co}\left\{\{\overline{x}^i\}\cup\bigcup_{T\ge 1}B_T^i(\overline{p},q).\right\}$$

This set can be equivalently rewritten as

$$A^i=\left\{x^i\in\ell_\infty(S\times L) \;\middle|\; \exists(\mu,\tilde{x}^i)\in[0,1]\times\bigcup_{T\ge1}B_T^i(\overline{p},q) \text{ such that } x^i=\mu\overline{x}^i+(1-\mu)\tilde{x}^i\right\}.$$

Claim 1. $U^i\cap A^i=\emptyset$.

Let us consider a point x^i in the intersection. First, we remark that it follows from the definition of U^i that x^i is non-negative. Since $x^i\in A^i$, there exists some integer

T, a point $(\tilde{x}^i, \tilde{z}^i) \in B^i_T(\overline{p}, q)$ and a real number of $[0,1]$ such that $x^i = \mu\overline{x}^i + (1-\mu)\tilde{x}^i$. We can associate the portfolio plan $z^i = \mu\overline{z}^i + (1-\mu)\tilde{z}^i$. We want to prove that $(x^i, z^i) \in B^i(\overline{p}, q, \overline{D}^i)$. Clearly, the spot constraints are satisfied in every node. Moreover (x^i, z^i) satisfies the debt constraints $\overline{D}^i$ in all nodes $s \notin S^T$. Since x^i is non-negative, it follows from the $(\overline{p}, q)$-looseness of the debt constraint $\overline{D}^i$ that (x^i, z^i) satisfies the debt constraints in all nodes. Hence $(x^i, z^i) \in B^i(\overline{p}, q, \overline{D}^i) \cap U^i$, which contradicts the definition of equilibrium with debt constraints. □

The assumption C4 implies that $\overline{x}^i + 1 \in \text{int}(U^i)$. Hence, we can apply the Hahn-Banach theorem to get the existence of some non-zero element P^i in $(\ell_\infty(S \times L))'_+ = (ba(S \times L))_+$ such that for all $(x^i, y^i) \in A^i \times U^i$, one has $P^i \cdot x^i \leq P^i \cdot y^i$. Since $\overline{x}^i \in A^i$, and since by local non-satiation $\overline{x}^i \in \text{cl}(U^i)$, one deduces that for all $(x^i, y^i) \in A^i \times U^i$, one has $P^i \cdot x^i \leq P^i \cdot \overline{x}^i \leq P^i \cdot y^i$. Since P^i is in $ba(S \times L)_+$, we can apply the Yosida-Hewitt theorem to get the existence of some $\overline{\pi}^i \in \ell_1^+(S \times L)$ such that P^i and π^i coincide on "finite" elements of $\ell_\infty(S \times L)$.

We first claim that $a^i \succ \overline{x}^i \Rightarrow \pi^i \cdot a^i \geq P^i \cdot \overline{x}^i$. Indeed, we recall that $a^i \chi_{S^N} \to a^i$ for the Mackey topology and consequently, there exists some integer N such that $a^i \chi_{S^N} \succ^i \overline{x}^i$. It follows that $P^i \cdot a^i \chi_{S^N} \geq P^i \cdot \overline{x}^i$. By definition of π^i, one deduces that $\pi^i \cdot a^i \chi_{S^N} \geq P^i \cdot \overline{x}^i$. This assertion together with the positivity of π^i leads to $\pi^i \cdot a^i \geq P^i \cdot \overline{x}^i$.

By local non satiation of the preferences, we deduce from the last claim that $\pi^i \cdot \overline{x}^i \geq P^i \cdot \overline{x}^i$. This implies that $\pi^i \cdot \overline{x}^i = P^i \cdot \overline{x}^i$ by definition of π^i.

Claim 2. $\overline{x}^i$ is optimal for agent i in the set $\{x^i \in X^i | \pi^i \cdot x^i \leq \pi^i \cdot \overline{x}^i\}$.

We first prove that $\pi^i \cdot \overline{x}^i > 0$. Since $(\omega^i, 0) \in A^i$, one deduces that $P^i \cdot \omega^i \leq \pi^i \cdot \overline{x}^i$. Recalling that P^i is a non-zero element of $(ba(S \times L))_+$ and $\omega^i \in \text{int}(\ell_\infty^+(S \times L))$, it follows that $P^i \cdot \omega^i > 0$ and consequently $\pi^i \cdot \overline{x}^i > 0$. Let us now consider $x^i \in X^i$ such that $x^i \succ^i \overline{x}^i$. It follows from the continuity of the preferences that there exists some real number $\alpha < 1$ such that $\alpha x^i \succ^i \overline{x}^i$. Hence we deduce from the previous results that $\pi^i \cdot (\alpha x^i) \geq \pi^i \cdot \overline{x}^i$. One thus gets $\pi^i \cdot x^i \geq \frac{1}{\alpha}\pi^i \cdot \overline{x}^i > \pi^i \cdot \overline{x}^i$. □

In particular, we deduce from Assumption C4 and Claim 2 that $\pi^i(s) > 0$, for every node s.

Let us now construct the node prices $\overline{\lambda}^i$ corresponding to the present value prices π^i. For any time horizon T, for all $x^i \in B^i_T(\overline{p}, q)$ one has $P^i \cdot x^i \leq \pi^i \cdot \overline{x}^i$. This implies that $\pi^i \cdot x^i \chi_{S^T} + P^i \cdot \omega^i \chi_{S \setminus S^T} \leq \pi^i \cdot \overline{x}^i$, and consequently $\pi^i \cdot x^i \chi_{S^T} \leq \pi^i \cdot \overline{x}^i$. We have proved that the obviously consistent system of inequalities defining $B^i_T(\overline{p}, q)$ implies that $\sum_{s \in S^T} \pi^i(s) \cdot x^i(s) \leq \pi^i \cdot \overline{x}^i$. Hence we can apply the non-homogeneous Farkas' lemma (cf. Rockafellar (1970), Theorem 22.3) to get the existence of some $\left((\lambda^{i,T}_s)_{s \in S^T}, (\mu^{i,T}_s)_{s \in S_{T+1}}\right) \in \mathbb{R}^{S^T}_+ \times \mathbb{R}^{S_{T+1}}_+$ such that :

$$\pi^i(s) = \lambda^{i,T}_s \overline{p}(s) \text{ for all } s \text{ in } S^T; \tag{1}$$

$$\lambda^{i,T}_s q(s) = \sum_{\sigma \in s+} \lambda^{i,T}_\sigma r(\sigma) \text{ for all } s \text{ in } S^{T-1}; \tag{2}$$

$$\lambda^{i,T}_s q(s) = \sum_{\sigma \in s+} \mu^{i,T}_\sigma r(\sigma) \text{ for all } s \text{ in } S_T;$$

$$\text{and } \pi^i \cdot \overline{x}^i \geq \sum_{s \in S^T} \lambda^{i,T}_s \overline{p}(s) \cdot \omega^i(s).$$

Recalling that $\pi^i(s) > 0$ for all $s \in S$, we deduce from (1) that the real number $\lambda^{i,T}_s$ is positive and independent of T. This allow the construction of some $\overline{\lambda}^i$ in $\mathbb{R}^S_{++}$ satisfying $\pi^i(s) = \overline{\lambda}^i_s \overline{p}(s)$ for all s in S; moreover using (2), one also has $\overline{\lambda}^i_s q(s) = \sum_{\sigma \in s+} \overline{\lambda}^i_\sigma r(\sigma)$ for all s in S. Since $\pi^i \cdot \omega^i \leq \pi^i \cdot \overline{x}^i$, one deduces that $B^i(\overline{p}, q, \overline{\lambda}^i) \subset \{x^i \in X^i | \pi^i \cdot x^i \leq \pi^i \cdot \omega^i\} \subset \{x^i \in X^i | \pi^i \cdot x^i \leq \pi^i \cdot \overline{x}^i\}$. Recalling that $\overline{x}^i$ is optimal in the last set, it suffices to prove that $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \overline{\lambda}^i)$ to get the optimality of $(\overline{x}^i, \overline{z}^i)$ in $B^i(\overline{p}, q, \overline{\lambda}^i)$.

Claim 3. $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \overline{\lambda}^i)$.

Let us consider a node s and an agent i. Let $(\tilde{x}^i_s, \tilde{z}^i_s)$ be the consumption/portfolio plan defined by $(\overline{x}^i \chi_{S \setminus S+(s)} + 1_s + \beta \overline{x}^i \chi_{S+(s)}, \overline{z}^i \chi_{S \setminus S(s)} + \beta \overline{z}^i \chi_{S(s)})$. Since by assumption C5, $\tilde{x}^i_s \succ^i \overline{x}^i$, one deduces that $(\tilde{x}^i_s, \tilde{z}^i_s) \notin B^i(\overline{p}, q, \overline{D}^i)$. Since $(\tilde{x}^i_s, \tilde{z}^i_s)$ obviously satisfies the debt constraint in every node and the spot constraint in every node $s' \neq s$, it follows that the spot constraint in node s is not satisfied. This implies that $\overline{p}(s) \cdot (x^i(s) + 1_s) + q(s) \cdot \beta \overline{z}^i(s) > \overline{p}(s) \cdot \omega^i(s) + r(s) \cdot \overline{z}^i(s^-)$. This inequality together with the spot constraint satisfied by $(\overline{x}^i, \overline{z}^i)$ at node s leads to $\overline{p}(s) \cdot 1_s + q(s) \cdot (\beta \overline{z}^i(s)) > q(s) \cdot \overline{z}^i(s)$. One thus gets $q(s) \cdot \overline{z}^i(s) < \dfrac{\|\overline{p}(s)\|_1}{1 - \beta}$. Since this inequality holds for all i, and recalling that $\sum_{i=1}^{m} \overline{z}^i(s) = 0$, it follows that $q(s) \cdot \overline{z}^i(s) > -\dfrac{m-1}{1-\beta} \|\overline{p}(s)\|_1$ for all nodes s, and all agents i. These inequalities lead to

$$\sum_{\substack{s \in S_T \\ q(s)\cdot \overline{z}^i(s) \le 0}} \overline{\lambda}^i_s q(s)\cdot \overline{z}^i(s) > -\frac{m-1}{1-\beta} \sum_{\substack{s \in S_T \\ q(s)\cdot \overline{z}^i(s) \le 0}} \overline{\lambda}^i_s \|\overline{p}(s)\|_1.$$

Hence we deduce that

$$0 \ge \sum_{\substack{s \in S_T \\ q(s)\cdot \overline{z}^i(s) \le 0}} \overline{\lambda}^i_s q(s)\cdot \overline{z}^i(s) > -\frac{m-1}{1-\beta} \sum_{s \in S_T} \overline{\lambda}^i_s \|\overline{p}(s)\|_1 = -\frac{m-1}{1-\beta}\pi^i \cdot 1\chi_{S_T}.$$

Since π^i is in $\ell_1(S \times L)$, the right term tends to zero, when T goes to infinity. Hence one has $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \overline{\lambda}^i)$. □

Moreover $(\overline{x}^i, \overline{z}^i)$ is optimal in $B^i_{MQ}(\overline{p}, q, \overline{\lambda}^i)$. Indeed, it follows from Remark 4 that $B^i(\overline{p}, q, \overline{\lambda}^i) \subset B^i_{MQ}(\overline{p}, q, \overline{\lambda}^i) \subset \{x^i \in X^i | \pi^i \cdot x^i \le \pi^i \cdot \omega^i\} \subset \{x^i \in X^i | \pi^i \cdot x^i \le \pi^i \cdot \overline{x}^i\}$. Recalling that $\overline{x}^i$ is optimal in the largest set and that $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \overline{\lambda}^i)$, one gets the conclusion of the last assertion of Theorem 3. □

Theorem 4. Let $((\overline{x}^i, \overline{z}^i, \overline{\lambda}^i)_{i=1}^m, \overline{p}, q)$ *be a MQ-equilibrium with present value node prices of the economy* $\mathcal{E} = ((X^i, \succsim^i, \omega^i)_{i=1}^m, S, R)$. *Under the assumptions C1-C5 and F, there exists some present value node prices* $(\tilde{\lambda}^i)_{i=1}^m$ *such that* $((\overline{x}^i, \overline{z}^i, \tilde{\lambda}^i)_{i=1}^m, \overline{p}, q)$ *is an equilibrium with present value node prices of the economy* $\mathcal{E}$, *satisfying for each* i, $\tilde{\pi}^i \stackrel{\text{def}}{=} (\tilde{\lambda}^i_s \overline{p}(s))_{s \in S} \in \ell_1(S \times L)$.

Proof of Theorem 4. The proof of Theorem 4 is similar to the one of Theorem 3. Let us consider for any agent i, the two convex subsets of $\ell_\infty(S \times L)$ defined by

$$A^i = \left\{ x^i \in \ell_\infty(S \times L) \;\middle|\; \exists z^i \in \prod_{s\in S} \mathbb{R}^{J_s} \;\middle|\; \begin{array}{ll} \overline{p}(s)\cdot (x^i(s) - \omega^i(s)) + q(s)\cdot z^i(s) \le r(s)\cdot z^i(s^-) & \\ & \forall s \in S; \\ \liminf_{T\to+\infty} \sum_{\sigma \in S_T \cap S(s)} \overline{\lambda}^i_\sigma q(\sigma)\cdot z^i(\sigma) \ge 0 & \forall s \in S. \end{array} \right\}$$

Since $\overline{x}^i$ is optimal in the budget set $B^i_{MQ}(\overline{p}, q, \overline{\lambda}^i)$, it follows that $U^i \cap A^i = \emptyset$; moreover the assumption C4 implies that $\overline{x}^i + 1 \in \mathrm{int}(U^i)$. As previously, we can apply the Hahn-Banach theorem to get the existence of some non-zero element P^i in $(\ell_\infty(S \times L))'_+ = (ba(S \times L))_+$ such that for all $(x^i, y^i) \in A^i \times U^i$, one has $P^i \cdot x^i \le P^i \cdot y^i$. Since $\overline{x}^i \in B^i_{MQ}(\overline{p}, q, \overline{\lambda}^i) \subset A^i$, and since by local non-satiation $\overline{x}^i \in \mathrm{cl}(U^i)$, one deduces that for all $(x^i, y^i) \in A^i \times U^i$, one has $P^i \cdot x^i \le P^i \cdot \overline{x}^i \le P^i \cdot y^i$. Since P^i is in $ba(S \times L)_+$, we can apply the Yosida-Hewitt theorem to get the existence of some $\tilde{\pi}^i \in \ell_1^+(S \times L)$ such that P^i and π^i coincide on "finite" elements of $\ell_\infty(S \times L)$.

As previously, we prove that $\overline{x}^i$ is optimal for agent i in the set $\{x^i \in X^i | \tilde{\pi}^i \cdot x^i \leq \tilde{\pi}^i \cdot \overline{x}^i\}$. In particular, we deduce from Assumption C4 that $\tilde{\pi}^i(s) > 0$, for every node s.

Let us now construct the node prices $\tilde{\lambda}^i$ corresponding to the present value prices $\tilde{\pi}^i$. For any time horizon T, let us consider the set $B_T^i(\overline{p}, q)$ defined as in the proof of Theorem 3. Since $B_T^i(\overline{p}, q)$ is clearly a subset of A^i, it follows that $\tilde{\pi}^i \cdot x^i \chi_{S^T} \leq \tilde{\pi}^i \cdot \overline{x}^i$ for all $x^i \in B_T^i(\overline{p}, q)$. We have proved that the obviously consistent system of inequalities defining $B_T^i(\overline{p}, q)$ implies that $\sum_{s \in S^T} \tilde{\pi}^i(s) \cdot x^i(s) \leq \tilde{\pi}^i \cdot \overline{x}^i$. As previously, there exists $((\tilde{\lambda}_s^{i,T})_{s \in S^T}, (\mu_s^{i,T})_{s \in S_{T+1}}) \in \mathbb{R}_+^{S^T} \times \mathbb{R}_+^{S_{T+1}}$ such that :

$$\tilde{\pi}^i(s) = \tilde{\lambda}_s^{i,T} \overline{p}(s) \text{ for all } s \text{ in } S^T; \tag{1}$$

$$\tilde{\lambda}_s^{i,T} q(s) = \sum_{\sigma \in s+} \tilde{\lambda}_\sigma^{i,T} r(\sigma) \text{ for all } s \text{ in } S^{T-1}; \tag{2}$$

$$\tilde{\lambda}_s^{i,T} q(s) = \sum_{\sigma \in s+} \mu_\sigma^{i,T} r(\sigma) \text{ for all } s \text{ in } S_T;$$

$$\text{and } \tilde{\pi}^i \cdot \overline{x}^i \geq \sum_{s \in S^T} \tilde{\lambda}_s^{i,T} \overline{p}(s) \cdot \omega^i(s).$$

Recalling that $\tilde{\pi}^i(s) > 0$ for all $s \in S$, we deduce from (1) that the real number $\tilde{\lambda}_s^{i,T}$ is positive and independent of T. This allow the construction of some $\tilde{\lambda}^i$ in $\mathbb{R}_{++}^S$ satisfying $\tilde{\pi}^i(s) = \tilde{\lambda}_s^i \overline{p}(s)$ for all s in S; moreover using (2), one also has $\tilde{\lambda}_s^i q(s) = \sum_{\sigma \in s+} \tilde{\lambda}_\sigma^i r(\sigma)$ for all s in S. Since $\tilde{\pi}^i \cdot \omega^i \leq \tilde{\pi}^i \cdot \overline{x}^i$, one deduces that $B^i(\overline{p}, q, \tilde{\lambda}^i) \subset \{x^i \in X^i | \tilde{\pi}^i \cdot x^i \leq \tilde{\pi}^i \cdot \omega^i\} \subset \{x^i \in X^i | \tilde{\pi}^i \cdot x^i \leq \tilde{\pi}^i \cdot \overline{x}^i\}$. Recalling that $\overline{x}^i$ is optimal in the last set, it suffices to prove that $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \tilde{\lambda}^i)$ to get the optimality of $(\overline{x}^i, \overline{z}^i)$ in $B^i(\overline{p}, q, \tilde{\lambda}^i)$.

Claim 2. $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \tilde{\lambda}^i)$.

Let us consider a node s and an agent i. Let $(\tilde{x}_s^i, \tilde{z}_s^i)$ be the consumption/portfolio plan defined by $(\overline{x}^i \chi_{S \setminus S+(s)} + 1_s + \beta \overline{x}^i \chi_{S+(s)}, \overline{z}^i \chi_{S \setminus S(s)} + \beta \overline{z}^i \chi_{S(s)})$. Since by assumption C5, $\tilde{x}_s^i \succ^i \overline{x}^i$, one deduces that $(\tilde{x}_s^i, \tilde{z}_s^i) \notin B_{MQ}^i(\overline{p}, q, \overline{\lambda}^i)$. Since $(\tilde{x}_s^i, \tilde{z}_s^i)$ obviously satisfies the Magill-Quinzii transversality condition in every node and the spot constraint in every node $s' \neq s$, it follows that the spot constraint in node s is not satisfied. As previously, one gets

$$\sum_{\substack{s \in S_T \\ q(s)\cdot \overline{z}^i(s) \leq 0}} \tilde{\lambda}^i_s q(s) \cdot \overline{z}^i(s) > -\frac{m-1}{1-\beta} \sum_{\substack{s \in S_T \\ q(s)\cdot \overline{z}^i(s) \leq 0}} \tilde{\lambda}^i_s \|\overline{p}(s)\|_1.$$

Hence we deduce that

$$0 \geq \sum_{\substack{s \in S_T \\ q(s)\cdot \overline{z}^i(s) \leq 0}} \tilde{\lambda}^i_s q(s) \cdot \overline{z}^i(s) > -\frac{m-1}{1-\beta} \sum_{s \in S_T} \tilde{\lambda}^i_s \|\overline{p}(s)\|_1 = -\frac{m-1}{1-\beta} \tilde{\pi}^i \cdot 1\chi_{S_T}.$$

Since $\tilde{\pi}^i$ is in $\ell_1(S \times L)$, the right term tends to zero, when T goes to infinity. Hence one has $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, q, \tilde{\lambda}^i)$. □

We emphasize that in Theorem 4, the "initial" present value price $\pi^i \stackrel{\text{def}}{=} \left(\overline{\lambda}^i_s \overline{p}(s)\right)_{s \in S}$ does not necessarily belong to $\ell_1(S \times L)$. Using the same argument as in the proof of Theorem 4, we obtain the following remark.

Remark 5. Let $\left((\overline{x}^i, \overline{z}^i, \overline{\lambda}^i)_{i=1}^m, \overline{p}, q\right)$ be an equilibrium with present value node prices of the economy $\mathcal{E} = \left((X^i, \succsim^i, \omega^i)_{i=1}^m, S, R\right)$. There exists some present value node prices $\tilde{\lambda}^i$ such that $\left((\overline{x}^i, \overline{z}^i, \tilde{\lambda}^i)_{i=1}^m, \overline{p}, q\right)$ is an equilibrium with present value node prices of the economy $\mathcal{E}$, satisfying for each i, $\tilde{\pi}^i \stackrel{\text{def}}{=} \left(\tilde{\lambda}^i_s \overline{p}(s)\right)_{s \in S} \in \ell_1(S \times L)$.

Remark 5 allows us to summarize Theorems 2,3 and 4 under the additional assumption C5. Actually, we state an obvious equivalence result.

Theorem 5. Assume that the economy $\mathcal{E} = \left((X^i, \succsim^i, \omega^i)_{i=1}^m, S, R\right)$ satisfies Assumptions C1-C5 and F; then for any consumption/portfolio plan $(\overline{x}^i, \overline{z}^i)_{i=1}^m$, and for any price $\overline{p}$, the following assertions are equivalent.

- *There exists some present value node prices $\overline{\lambda}^i$ such that $\left((\overline{x}^i, \overline{z}^i, \overline{\lambda}^i)_{i=1}^m, \overline{p}, q\right)$ is an equilibrium with present value node prices of the economy $\mathcal{E}$.*
- *There exists some present value node prices $\tilde{\lambda}^i$ such that $\left((\overline{x}^i, \overline{z}^i, \tilde{\lambda}^i)_{i=1}^m, \overline{p}, q\right)$ is a MQ-equilibrium with present value node prices of the economy $\mathcal{E}$.*
- *There exists some system of debt constraints $\overline{D}^i$ such that $\left((\overline{x}^i, \overline{z}^i, \overline{D}^i)_{i=1}^m, \overline{p}, q\right)$ is an equilibrium with debt constraints of the economy $\mathcal{E}$.*

Appendix

Proof of Lemma 2.

It follows from the closedness of $Im\Phi_s$ that Γ is closed. If Γ is unbounded, there exists some non-zero vector $\overline{\varphi}$ in the asymptotic cone of Γ. Since this set is bounded below, one has $\overline{\varphi} \geq 0$. Moreover, $\overline{\varphi}$ is also in the asymptotic cone of $Im\Phi_s$ which is a linear space. Finally, we have proved the existence of an element $\overline{\varphi} > 0$ in the set $Im\Phi_s$, which contradicts the no-arbitrage condition. □

Proof of Proposition 1.

To prove Proposition 1, we will first state a lemma.

Lemma 4. Let D be a system of debt constraints. If q is arbitrage-free,

i) If D is not (p,q)-consistent, there exists some system of debt constraints D' such that $D' > D$ and $B^i(p,q,D') = B^i(p,q,D)$.

ii) If D is (p,q)-loose and not (p,q)-consistent, there exists some system of debt constraints D' such that $D' > D$, $B^i(p,q,D') = B^i(p,q,D)$ and D' is (p,q)-loose.

Proof of Lemma 4.

i) Since D is not (p,q)-consistent, let s be a node such that D is not (p,q)-consistent at node s. Hence, for all portfolio $z(s) \in \mathbb{R}^{J_s}$, one has

$r(\sigma) \cdot z(s) \geq D(\sigma)$ for all $\sigma \in s^+ \Rightarrow D(s) + p(s) \cdot \omega^i(s) - q(s) \cdot z(s) < 0$.

Let us define $D'(s) = \inf \Delta$ where Δ is the set

$$\left\{ d \in \mathbb{R}^- \text{ such that } \exists z(s) \in \mathbb{R}^{J_s} \text{ satisfying } \left| \begin{array}{l} d + p(s) \cdot \omega^i(s) - q(s) \cdot z(s) \geq 0; \\ r(\sigma) \cdot z(s) \geq D(\sigma) \text{ for all } \sigma \in s^+. \end{array} \right. \right\}$$

First, we remark that this set is non-empty since we can take $d = 0$ associated with $z(s) = 0$. It follows that if $D'(s)$ exists, then $D'(s) \leq 0$. An equivalent formulation for Δ is $\Delta = \{d \in \mathbb{R}^- | \Gamma(d) \neq \emptyset\}$ with $\Gamma(d) = \left\{ \varphi \in I_m\Phi_s \middle| \varphi \geq \begin{pmatrix} -d - p(s) \cdot \omega^i(s) \\ (D(\sigma))_{\sigma \in s^+} \end{pmatrix} \right\} \subset \left\{ \varphi \in I_m\Phi_s \middle| \varphi \geq \begin{pmatrix} -p(s) \cdot \omega^i(s) \\ (D(\sigma))_{\sigma \in s^+} \end{pmatrix} \right\}$. The second set is then non-empty and it follows from Lemma 2 that it is compact. Hence we deduce that Δ is bounded below, and this proves the existence of $D'(s)$. Moreover, Δ is closed. Indeed, let $d^\nu \to d$; for every ν, there exists $\varphi^\nu \in I_m\Phi_s$ and $\varphi^\nu \geq \begin{pmatrix} -d^\nu - p(s) \cdot \omega^i(s) \\ (D(\sigma))_{\sigma \in s^+} \end{pmatrix}$. With no loss of generality, one can assume

$\varphi^\nu \to \varphi \in Im\Phi_s$ and $\varphi \geq \begin{pmatrix} -d - p(s)\cdot\omega^i(s) \\ (D(\sigma))_{\sigma\in s+} \end{pmatrix}$, so that $d \in \Delta$.

Finally $D'(s) = \min\Delta$ and for some $z(s) \in \mathbb{R}^{J_s}$, consequently one has

$$D'(s) + p(s)\cdot\omega^i(s) - q(s)\cdot z(s) \geq 0 > D(s) + p(s)\cdot\omega^i(s) - q(s)\cdot z(s)$$

which shows that $D'(s) > D(s)$. We can now define $D'(s') = D(s')$ for all $s' \neq s$. By definition of D', D' is a system of debt constraints satisfying $D < D'$.

Thus it only remains to show that the two systems of debt constraints give the same budget set. First we remark that it follows from the inequality between D and D' that $B^i(p,q,D') \subset B^i(p,q,D)$. Conversely, let us take any consumption plan x in the budget set $B^i(p,q,D)$ sustained by the portfolio plan z. We want to prove that for each node s', one has $r(s')\cdot z(s'^-) \geq D'(s')$. Since D and D' differ in node s only, it suffices to prove that $r(s)\cdot z(s^-) \geq D'(s)$. But it follows from the budget/debt constraints that

$$\begin{cases} r(s)\cdot z(s^-) + p(s)\cdot(\omega^i(s) - x(s)) - q(s)\cdot z(s) \geq 0, \\ \text{for all } \sigma \in s^+,\ r(\sigma)\cdot z(s) \geq D(\sigma). \end{cases}$$

One deduces that $\begin{cases} r(s)\cdot z(s^-) + p(s)\cdot\omega^i(s) - q(s)\cdot z(s) \geq 0, \\ \text{for all } \sigma \in s^+,\ r(\sigma)\cdot z(s) \geq D(\sigma). \end{cases}$

Hence $r(s)\cdot z(s^-) \in \Delta$, and by definition of $D'(s)$, one obtains $r(s)\cdot z(s^-) \geq D'(s)$.

ii) Let us define D' as above. Clearly, one has $D' > D$ and $B^i(p,q,D') = B^i(p,q,D)$. Since D and D' differ on node s only, D' is also (p,q)-loose in each node of $D \setminus \{s, s^-\}$. By construction, D' is (p,q)-loose at s. If $s \neq 0$, let us take $z(s^-) \in \mathbb{R}^{J_{s^-}}$ satisfying $r(\sigma)\cdot z(s^-) \geq D'(\sigma)$ for all $\sigma \in (s^-)^+$. Since $D' > D$, one deduces that $r(\sigma)\cdot z(s^-) \geq D(\sigma)$ for all $\sigma \in (s^-)^+$. This assertion together with the (p,q)-looseness of D at node s^- leads to $D(s^-) + p(s^-)\cdot\omega^i(s^-) - q(s^-)\cdot z(s^-) \leq 0$. Since D and D' coincide at node s^-, D' is also (p,q)-loose at node s^- and consequently D' is (p,q)-loose. □

a) Proof of assertion i) of Proposition 1.

Recall that the initial system of debt constraints is denoted by D^i.

Let X be the set of systems of debt constraints defined by $X = \{D$ such that $B^i(p,q,D) = B^i(p,q,D^i)$ and $D \geq D^i\}$. We will consider the classical ordering on X, $D \leq D'$ if and only if for all s in S, one has $D(s) \leq D'(s)$. We now claim that every totally ordered subset of X has an upper bound.

Indeed, let us consider a totally ordered subset of X, $(D^{\theta})_{\theta\in\Theta}$. Since for every node s, for all θ, $D^{\theta}(s) \leq 0$, we can define $\overline{D}(s)$ by $\overline{D}(s) = \sup\{D^{\theta}(s) \mid \theta \in \Theta\}$. Clearly $\overline{D}$ is a system of debt constraints satisfying for all θ, $\overline{D} \geq D^{\theta} \geq D^{i}$. Consequently, one obtains $B^{i}(p,q,\overline{D}) \subset B^{i}(p,q,D^{i})$. Conversely, let us take any consumption plan x in the budget set $B^{i}(p,q,D^{i})$ sustained by the portfolio plan z. We want to prove that $r(s)\cdot z(s^{-}) \geq \overline{D}(s)$ for every node s. Since for all θ, one has $B^{i}(p,q,D^{\theta}) = B^{i}(p,q,D^{i})$, one obtains that $r(s)\cdot z(s^{-}) \geq D^{\theta}(s)$ for every node s. By taking the supremum in $\theta \in \Theta$, one gets the required inequality. Hence, $\overline{D} \in X$, and the set $(D^{\theta})_{\theta\in\Theta}$ has an upper bound.

We can apply Zorn's lemma to get the existence of a maximal element $\tilde{D}^{i}$ in the set X. To end the proof, since $\tilde{D}^{i}$ is in X, it remains to prove that $\tilde{D}^{i}$ is (p,q)-consistent. Indeed, if $\tilde{D}^{i}$, is not (p,q)-consistent, we can apply lemma 4 to get the existence of a system of debt constraints D' in X such that $D' > \tilde{D}^{i}$. This contradicts the definition of $\tilde{D}^{i}$. □

b) Proof of assertion ii)

The proof of assertion ii) is similar to the one of assertion i). We replace the set X by the set $X' = \{D \text{ such that } B^{i}(p,q,D) = B^{i}(p,q,D^{i}),\ D \geq D^{i} \text{ and D is } (p,q)-\text{loose}\}$. As before, one checks that every totally ordered subset of X' has an upper bound and that a maximal element of X' satisfies the condition of assertion ii). □

Proof of Proposition 2.

Let us first remark that the definition of $(\overline{p}^{T},q)$- consistency and looseness at a terminal node of the truncated economy $\mathcal{E}(T)$ requires the obvious adaptations : If D^{i} is a system of debt constraints defined on S^{T}, D^{i} is $(\overline{p}^{T},q)$-consistent at $s \in S_{T}$ if $D^{i}(s)+\overline{p}^{T}\cdot\omega^{i}(s) \geq 0$; D^{i} is $(\overline{p}^{T},q)$-loose at $s \in S_{T}$ if for every $x^{i}(s) \in \mathbb{R}^{L}_{+}$, $\overline{p}^{T}\cdot x^{i}(s) \leq \overline{p}^{T}\cdot\omega^{i}(s) + r(s)\cdot z^{i}(s^{-}) \Rightarrow r(s)\cdot z^{i}(s^{-}) \geq D^{i}(s)$. Obviously, D^{i} is $(\overline{p}^{T},q)$-loose and consistent at $s \in S_{T}$ if and only if $D^{i}(s) = -\overline{p}^{T}(s)\cdot\omega^{i}(s)$.

Let us now recall the definition of $\overline{D}^{i,T}$;

$$\overline{D}^{i,T}(s) = \inf\{\overline{p}^{T}(s)\cdot\left(x^{i}(s)-\omega^{i}(s)\right) + q(s)\cdot z^{i}(s)\}$$

where the infimum is taken among all (x^{i},z^{i}) satisfying the spot budget constraints in all nodes of $S(s)\cap S^{T}$ and such that for all $s' \in S_{T}$, $z^{i}(s') = 0$. Exactly as in the proof of Proposition 1, one deduces from the no-arbitrage condition on q that $\overline{D}^{i,T}(s)$ is actually a minimum. Then it is obvious that $\overline{D}^{i,T}$ is $(\overline{p}^{T},q)$-loose and consistent.

Finally, let D^{i} be $(\overline{p}^{T},q)$-loose and consistent. We prove by decreasing induction on t $(0 \leq t \leq T)$ that D^{i} and $\overline{D}^{i,T}$ are equal on S_{t}. It follows from the initial remark that

$D^i(s) = \overline{D}^{i,T}(s)$ for every $s \in S_T$. Let us assume that D^i and $\overline{D}^{i,T}$ are equal on S_t for some $t \in \{1, \ldots, T\}$. Consider a node $s \in S_{t-1}$. Since D^i is $(\overline{p}^T, q)$-consistent in node s, there exists a portfolio $z^i(s) \in \mathbb{R}^{J_s}$ such that $\begin{cases} D^i(s) + \overline{p}^T(s) \cdot \omega^i(s) - q(s) \cdot z^i(s) \geq 0; \\ \text{for all } \sigma \in s^+,\ r(\sigma) \cdot z^i(s) \geq D^i(\sigma). \end{cases}$

Since $s^+ \subset S_t$, it follows from the induction assumption that $\overline{D}^{i,T}(\sigma) = D^i(\sigma)$ for all $\sigma \in s^+$. This implies that for all $\sigma \in S^+$, $r(\sigma) \cdot z^i(s) \geq \overline{D}^{i,T}(\sigma)$. Since $\overline{D}^{i,T}$ is $(\overline{p}^T, q)$-loose at node s, one gets $\overline{D}^{i,T}(s) + \overline{p}^T(s) \cdot \omega^i(s) - q(s) \cdot z^i(s) \leq 0$. This inequality together with $D^i(s) + \overline{p}^T(s) \cdot \omega^i(s) - q(s) \cdot z^i(s) \geq 0$ leads to $D^i(s) \geq \overline{D}^{i,T}(s)$. To end the proof, it suffices to remark that the reverse inequality can be obtained by the same argument applied to the portfolio $\tilde{z}^i(s)$ defined by the $(\overline{p}^T, q)$-consistency at node s of $\overline{D}^{i,T}$. Hence, one has $D^i(s) = \overline{D}^{i,T}(s)$ for all $s \in S_{t-1}$. □

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