C.E.P.R.E.M.A.P. May, 1993

EQUILIBRIUM WITH PROFIT RATE MAXIMIZING PRODUCERS

KIRILL BORISOV[1]
St.Petersburg Institute
for Economics and Mathematics
(Russian Academy of Sciences)
and
CEPREMAP

N° 9309

Correspondence to: Kirill Borisov, St.Petersburg Institute for Economics and Mathematics, 38, Serpukhovskaya str., 198147, St.Petersburg, RUSSIA.

[1] I would like to thank C. Bidard, G.Dumenil, M.Florenzano, C.Le Van, D.Levy and C.Lucifora for help and useful discussion.

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ABSTRACT

We consider a general equilibrium model with producers maximizing their profit rates and using some fraction of profit for investments. Under some assumptions, a stationaryequilibrium exists. If at least one active producer uses some fraction of profit for dividends, not investing all his profit, an equilibrium allocation is technologically inefficient. Equilibrium prices in this model can be interpreted as modified prices of production.

We also consider a multi-regional variant of the model supposing, for simplicity, that there is only one primary factor - labor force - and study the equilibrium existence problem from another point of view. Namely, we fix wage rates in the regions and equalize labor demand and supply in every region by means of varying fractions of profit used for investments.

Key words: general equilibrium, profit rate, tehcnological inefficiency.

J.E.L. classification code : D51, D61

UN MODELE D'EQUILIBRE GENERAL AVEC MAXIMISATION DU TAUX DE PROFIT PAR LES ENTREPRISES

RESUME

Nous considérons un modèle d'équilibre général où les producteurs maximisent leurs taux de profit et utilisent une part de ces profits pour des investissements. Des conditions suffisantes sont données pour l'existence d'un equilibre stationnaire. Nous montrons que l'allocation d'équilibre est inefficace d'un point de vue technologique, dès lors qu'au moins l'un des producteurs actifs à l'équilibre ne réinvestit pas tout son profit, mais en distribue une partie sous forme de dividendes. Dans ce modèle, les prix d'équilibre peuvent etre interprétés comme des prix de production modifiés.

Nous considérons également une variante multi-régionale de ce modèle, en supposant, pour simplifier, qu'il n'existe qu'un seul facteur de production - le travail - et nous analysons le problème de l'existence de l'équilibre d'un autre point de vue : nous fixons les taux de salaire dans les régions et égalisons l'offre et la demande de travail dans chaque région, en faisant varier les fractions du profit utilisées pour l'investissement.

Mots clefs: équilibre général, taux de profit, inefficacité technologique.

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KIRILL BORISOV

1. INTRODUCTION.

Usually, in models of economic equilibrium, it is assumed that producers maximize their profit. This derives from the static framework of analysis. From a dynamic point of view, it is more natural to suppose that producers maximize their profitability.

In this paper, we consider a model of economic equilibrium with profit rate maximizing producers (model M_1). A simple version of this model was proposed in BORISOV (1988).

It should be noted that we consider stationary equilibria in a dynamic model.

Technology sets in model M_1 are cones as in models of MC KENZIE (1959, 1981), YANO (1984) and some other authors. However, in our model, both a different criterion for producers and other relations between producers and consumers are assumed. This changes the properties of the equilibrium states.

On the one hand, model M_1 is more in the style of classical than neoclassical economic thought. In particular, no intertemporal utility function is assumed: the trade-off between investments and consumption is realized by means of the fractions of profit used for investments. As for these fractions, we (implicitly) assume that they are established in the process of the competition between producers.

On the other hand, our model is not classical since rates of profit may differ. Moreover, there is a possibility of co-existence of inefficient producers with more efficient ones. If an inefficient producer wants to survive in the competition with more efficient producers he should increase the fraction of his profit which is used for investments.

Classical economists supposed that, in the long term, profit rates are equalized. (It is noteworthy that Leon Walras defined equilibrium by the following properties: i) all markets clear, ii) a uniform rate of return is established for each enterprise. But later his definition was restricted to the first property only.)

However, if we implicitly assume that there is a security market, it is possible simply to suppose that the equalization of rates of return takes place on this market. Indeed, it is natural to think that the share values of different firms with absolutely identical sets of disposal resources may differ, depending on their technologies (identical sets of resources in different firms are not identical). They may be, for example, proportional to the following ratio

sum of dividends / value of the disposal capital.

In Section 2 we propose sufficient conditions for the existence of stationary equilibria.

Section 3 is devoted to the problem of technological (in)efficiency of equilibrium allocations. We show that if at least one active producer uses some fraction of profit for dividends, not investing all his profit, equilibrium allocations are not technologically efficient. In my opinion, it is especially interesting because there are no violations of conditions of perfect competition in the statement of the model. From a mathematical point of view, this fact is of the same nature as the well-known fact of the theory of economic growth which says that in the case of discounting the level of utility at the stationary optimal state is less than in the case of no discounting. Moreover, if we considered the model with only one consumer, the equilibrium states would coincide with the stationary optimal states of a growth model with discounting.

In Section 4 we present an example with the Leontief technologies. In our model some fraction of profit is used for investments. In general position with respect to this fractions, only a small number of producers are active in the states of equilibrium (some kind of natural oligopoly). This phenomenon is well-known in models with the Leontief technologies as the non-substitution theorem (see, for example, ARROW and HAHN (1971)). Namely, if there is only one primary factor, the number of active technologies is small and does not depend on consumers, but it is determined only by technologies. In the case of the Leontief technologies this fact is also true in model $\mathbf{M}_1$.

If technologies are not of the Leontief type, equilibrium prices and the set of active technologies depend on consumers, but the number of active producers does not increase. There is no formal assertion on this topic in the paper, but it could easily be formulated.

The following question arises: does there exist an equilibrium with a large number of active producers? A variant of the positive answer is proposed in Section 6.

In this section we consider a multi-regional version of model $\mathbf{M}_1$ supposing, for simplicity, that there is only one primary factor - labor force - and study the equilibrium existence problem from another point of view. Namely, we fix wage rates in regions and equalize labor demand and supply in every region, varying the fractions of profit used for investments.

Equilibrium models with profit rate maximizing producers were considered usually in the case of no explicit restrictions on primary factors. Such a model (model $\mathbf{M}_2$) is proposed in Section 5. Equilibria which occur in model $\mathbf{M}_1$ are described as equilibria in a special case of model $\mathbf{M}_2$.

Proofs of the assertions of the paper are presented in Section 7.

We use the following conventional notations. For $x,y \in \mathbb{R}^n_+$,

$$x \le y \iff x^i \le y^i, \quad i = 1,\dots,n,$$

where x^i is the i-th coorinate of x,

$$x \ll y \iff x^i < y^i, \quad i = 1,\dots,n,$$

and by xy we denote the scalar product of x and y.

2. MODEL $\mathbf{M}_1$.

There are n_1 producible goods, n_2 primary factors (in particular, labor force), a finite set J of consumers, and a finite set R of producers.

Producers. Producer $r \in R$ is described by his technology set $K_r \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \times \mathbb{R}^{n_1}_+$ and a number $\theta_{r0} \in \,]0,1]$ showing the share of profit which is used for investments. Elements of K_r are of the form (z,h,y), where $z \in \mathbb{R}^{n_1}_+$ is an input of goods, $h \in \mathbb{R}^{n_2}_+$ is an input of primary factors, and $y \in \mathbb{R}^{n_1}_+$ is an output.

We suppose that K_r is a convex closed cone such that

$$(0,0,y) \in K_r \Rightarrow y = 0;$$

$$(z,h,y) \in K_r, \quad 0 \le \tilde{y} \le y \Rightarrow (z,h,\tilde{y}) \in K_r;$$

$$(z,0,y) \in K_r \Rightarrow y \le z.$$

Given goods prices $p \in \mathbb{R}^{n_1}_+$ and primary factors prices $w \in \mathbb{R}^{n_2}_+$, producer $r \in R$ maximizes his profitability, solving the following problem:

$$\text{maximize} \quad \frac{py - (pz + wh)}{pz + wh},$$

$$\text{s.t.} \quad (z,h,y) \in K_r.$$

More precisely, let us consider the following problem:

$$\text{maximize} \quad py - (pz + wh),$$

$$\text{s.t.} \quad (z,h,y) \in K_r, \quad pz + wh = 1.$$

Denote by $\rho_r(p,w)$ the value of this problem (the maximal rate of profit). We suppose that producer r chooses his vector of the technology activity from

$$\mathbb{K}_r(p,w) = \{(z,h,y) \in K_r \mid py = (1 + \rho_r(p,w))(pz + wh)\}.$$

By $\alpha_r(p,w)$ we denote the 'maximal rate of growth' of the r-th producer under prices (p,w), i.e.

$$\alpha_r(p,w) = 1 + \theta_{r0}\, \rho_r(p,w).$$

The distribution of profit is defined by numbers $\theta_{rj} \geq 0$, $j \in J$, $\sum_{j\in J} \theta_{rj} = 1 - \theta_{r0}$. Here θ_{rj} is the share of consumer j.

Consumers. Under given prices $\hat{p} \in \mathbb{R}_+^{n_1}$ and his income γ_j, consumer $j \in J$ solves the following problem:

$$\text{maximize} \quad U_j(x),$$

$$\text{s.t.} \quad x \in \mathbb{R}_+^{n_1}, \quad \hat{p}x \leq \gamma_j. \qquad (2.1)$$

The utility function $U_j : \mathbb{R}_+^{n_1} \longrightarrow \mathbb{R}_+$ is supposed to be non-zero, continuous, concave and homogeneous of degree one.

Let us describe the formation of the total income for consumer j.

He has at his disposal some compact set $L_j \subset \mathbb{R}_+^{n_2}$ of available primary factors. This set can be, for example, of one of the following forms:

$$L_j = \{l \in \mathbb{R}_+^{n_2} \mid l \le \tilde{l}_j\}, \quad \text{where} \quad \tilde{l}_j \in \mathbb{R}_+^{n_2}$$

(if we mean different types of natural resources) or

$$L_j = \{l \in \mathbb{R}_+^{n_2} \mid d_j l \le \beta_j\}, \quad \text{where} \quad d_j \in \mathbb{R}_+^{n_2}, \quad \beta_j \ge 0$$

(if we mean different types of labor force).

We suppose that $\operatorname{int}\left(\sum_{j\in J} L_j\right) \ne \emptyset$.

Given prices $\hat{w} \in \mathbb{R}_+^{n_2}$ of primary factors, consumer j solves the following problem:

$$\text{maximize } \hat{w} l_j, \quad \text{s.t. } l_j \in L_j. \tag{2.2}$$

The solution to this problem represents his supply of primary factors.

We assume that there is some rate $\alpha > 1$ of Harrod-neutral technical progress in this model. For simplicity, we present this progress in the following equivalent form : if, at some moment, consumer j has at his disposal a set L_j, at the next moment he will have at his disposal the set αL_j. α is the same for all $j \in J$. This assumption is rather restrictive but as we want to consider the states of balanced growth, this assumption is inevitable.

We could consider the case $\alpha = 1$, but it would require some modification of the assertions or the statement of the model. For example, we could suppose that a firm pay dividends in some proportion to the value of its output. This modification would not change the results.

There are two sources of the total income for consumer j: payments for his primary factors and dividends.

Given $(\hat{z}_r, \hat{h}_r, \hat{y}_r) \in K_r$, $r \in R$, $\hat{l}_j \in L_j$, $j \in J$, prices

$\hat{p} \in \mathbb{R}_+^{n_1}$ and $\hat{w} \in \mathbb{R}_+^{n_2}$, γ_j is defined by

$$\gamma_j = \alpha\hat{w}\hat{l}_j + \sum_{r\in R} \theta_{rj}(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r). \tag{2.3}$$

The first term has a coefficient α since we suppose that at every moment a consumer receives his dividends for the previous period and his payments for primary factors for the following period (he sells his primary factors in advance).

Definition 2.1. A list $(\hat{p}, \hat{w}, (\hat{z}_r,\hat{h}_r,\hat{y}_r)_{r\in R}, (\hat{l}_j)_{j\in J}, (\hat{x}_j)_{j\in J})$ is called an equilibrium of model $\mathbf{M_1}$ if

A.1) $\alpha_r(\hat{p},\hat{w}) \le \alpha, \quad \forall\, r \in R;$

A.2) $\alpha_r(\hat{p},\hat{w}) < \alpha \quad \Rightarrow \quad \hat{p}\hat{y}_r = \hat{p}\hat{z}_r + \hat{w}\hat{h}_r = 0, \quad \forall\, r \in R;$

A.3) $(\hat{z}_r,\hat{h}_r,\hat{y}_r) \in \mathbb{K}_r(\hat{p},\hat{w}), \quad \forall\, r \in R;$

A.4) $\hat{x}_j$ is a solution to (2.1), where γ_j is defined by (2.3), $\forall\, j \in J;$

A.5) $\hat{l}_j$ is a solution to (2.2), $\forall\, j \in J;$

A.6) $\alpha \sum_{r\in R} \hat{z}_r + \sum_{j\in J} \hat{x}_j \le \sum_{r\in R} \hat{y}_r;$

A.7) $\sum_{r\in R} \hat{h}_r \le \sum_{j\in J} \hat{l}_j;$

A.8) $\hat{p} \sum_{r\in R} \hat{y}_r > 0.$

Denote

$$\rho_r = (\alpha-1)/\theta_{r0}, \quad r \in R,$$

and formulate one lemma which clarifies the definition and properties of equilibra of model $\mathbf{M_1}$. This lemma says that in Definition 2.1 we may replace conditions A.1) - A.4) by the following conditions:

B.1) $(\hat{z}_r,\hat{h}_r,\hat{y}_r)_r \subset \prod_{r\in R} K_r$ is a solution to the following problem:

$$\text{maximize} \quad \sum_{r \in R} \left(\frac{\hat{p} y_r}{1 + \rho_r} - \hat{p} z_r \right), \tag{2.4}$$

$$\text{s.t.} \qquad (z_r, h_r, y_r) \in K_r, \quad r \in R, \qquad \hat{w} \sum_{r \in R} h_r \le \hat{w} \sum_{j \in J} \hat{l}_j;$$

B.2) $\hat{w} \sum_{j \in J} \hat{l}_j = \sum_{r \in R} \left(\frac{\hat{p}\hat{y}_r}{1 + \rho_r} - \hat{p}\hat{z}_r \right);$

B.3) for all $j \in J$, $\hat{x}_j$ is a solution to (2.1), where

$$\gamma_j = \alpha \hat{w} \hat{l}_j + \sum_{r \in R} \frac{\theta_{rj} \rho_r}{1 + \rho_r} \hat{p} \hat{y}_r .$$

Lemma 2.1. A list $(\hat{p}, \hat{w}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r \in R}, (\hat{l}_j)_{j \in J}, (\hat{x}_j)_{j \in J})$ is an equilibrium iff B.1 - B.3 and A.5 - A.8 are satisfied.

We can now formulate the existence theorem.

Theorem 2.1. Assume that

C.1) for every $j \in J$, at least one of the following relations is satisfied:

1.1. $\text{int} \, (L_j \cap \mathbb{R}_+^{n_2}) \neq \emptyset;$

1.2. $\theta_{rj} > 0, \quad \forall \ r \in R;$

C.2) there is $(\bar{z}_r, \bar{h}_r, \bar{y}_r)_{r \in R} \subset \prod_{r \in R} K_r$ such that

$$\sum_{r \in R} \frac{\bar{y}_r}{1 + \rho_r} \gg \sum_{r \in R} \bar{z}_r, \qquad \sum_{r \in R} \bar{h}_r \in \sum_{j \in J} L_j .$$

Then there is an equilibrium in model $\mathbf{M_1}$.

3. TECHNOLOGICAL INEFFICIENCY.

Let us consider the problem of the technological (in)efficiency of equilibrium allocations in model $\mathbf{M}_1$ in the class allocations $((z_r,h_r,y_r)_{r\in R}, (l_j)_{j\in J}, (x_j)_{j\in J})$ such that

$$l_j \in L_j, \quad j \in J,$$

$$(z_r,h_r,y_r) \in K_r, \quad r \in R,$$

$$\sum_{r\in R} h_r \le \sum_{j\in J} l_j,$$

$$\alpha \sum_{r\in R} z_r + \sum_{j\in J} x_j \le \sum_{r\in R} y_r,$$

that is in the class of all attainable stationry allocations.

It is evident that if $\theta_{r0} = 1$ for all $r \in R$, any equilibrium allocation is Pareto-optimal. At the same time, if , for some r, $\theta_{r0} < 1$, that is, if some fraction of profit is used for dividends, an equilibrium allocation can be technologically inefficient and therefore, Pareto-dominated.

In our model numbers θ_{r0} substitutes, in some sense, a discount factor. In the case of one producer and one consumer, the equilibrium allocations of our model simply coincide with the stationary optimal states of the corresponding model of economic growth with the discount rate $1/(1 + \rho_r)$. Respectively, the technological inefficiency arising in model $\mathbf{M}_1$ has the same mathematical nature as the technological inefficiency of the stationary optimal states in models with discounting.

Present the technology sets in the following form:

$$K_r = \{(z,h,y) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \times \mathbb{R}_+^{n_1} \mid \phi_r(z,h,y) \le 0,$$

$$z^i = 0,\ i \in I_r^1,\ h^i = 0,\ i \in I_r^2,\ y^i = 0,\ i \in I_r^3\},$$

where ϕ_r are convex continuous functions, I_r^1, I_r^2, I_r^3 are some subsets of the sets of indices, supposing that $\phi_r(z,h,y) < 0$ for some $(z,h,y) \in K_r$.

Let us consider an equilibrium $(\hat{p}, \hat{w}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}, (\hat{l}_j)_{j\in J}, (\hat{x}_j)_{j\in J})$ of model $\mathbf{M}_2$, and let R_a be the set of all active producers:

$$R_a = \{r \in R \mid (\hat{z}_r, \hat{h}_r, \hat{y}_r) \ne 0\}.$$

Theorem 3.1. Suppose that, for all $r \in R_a$, the functions ϕ_r are continuously differentiable at $(\hat{z}_r, \hat{h}_r, \hat{y}_r)$. Suppose also that, for some k_1,

$$\hat{h}_r^{k_1} > 0, \quad \forall\ r \in R_a, \quad \text{and} \quad \hat{w}^{k_1} > 0,$$

and, for some i_1, r_1 and r_2,

$$\hat{p}^{i_1} > 0, \quad \hat{z}_{r_1}^{i_1} > 0, \quad \hat{y}_{r_2}^{i_1} > 0, \quad \theta_{r_2 0} < 1.$$

Then there are $\nu > 1$ and a feasible allocation $((z_r, h_r, y_r)_{r\in R}, (l_j)_{j\in J}, (x_j)_{j\in J})$ such that

$$\sum_{r\in R} (y_r - \alpha z_r) \ge \nu \sum_{r\in R} (\hat{y}_r - \alpha \hat{z}_r).$$

4. AN EXAMPLE: THE LEONTIEF TECHNOLOGY.

In this example we suppose that there is only one primary factor, namely, labor force $(n_2 = 1)$, and that each producer produces only one product:

$$R = \bigcup_{i=1}^{n} R(i); \quad R(i_1) \cap R(i_2) = \emptyset, \quad \text{if} \quad i_1 \neq i_2,$$

where $R(i)$ is the set of the producers of i-th product.

Technology cones are defined by

$$K_r = \{(z,h,y) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+ \times \mathbb{R}_+^{n_1} \mid y^i \leq (1-\mu_r^i)z^i, \quad r \notin R(i),$$

$$y^i \leq (1-\mu_r^i)z^i + F_r(z), \quad r \in R(i)\}.$$

Here $F_r : \mathbb{R}_+^{n_1+1} \dashrightarrow \mathbb{R}_+$ is a non-zero, continuous, concave and homogeneous of degree one production function, $\mu_r^i \in [0,1]$ are coefficients of depreciation. As usually in the case of one primary factor, in this model the equilibrium prices $\hat{p}$ are determined by the production technology alone (non-substitution theorem).

Indeed, it follows from A.1) - A.3) that

$$\hat{p}^i F_r(\hat{z}_r,\hat{h}_r) + \hat{p}B_r\hat{z}_r = (1+\rho_r)(\hat{p}\hat{z}_r + \hat{w}\hat{h}_r), \quad \forall\, r \in R(i), \quad i = 1,\dots,n;$$

and

$$\hat{p}^i F_r(z,h) + \hat{p}B_r z \leq (1+\rho_r)(\hat{p}z + \hat{w}h),$$

$$\forall\, (z,h) \in \mathbb{R}_+^{n+1}, \quad \forall\, r \in R(i), \quad i = 1,\dots,n,$$

where $B_r = \operatorname{diag}\,(1-\mu_r^1,\dots,1-\mu_r^{n_1}), \quad r \in R.$

Let us define, for $r \in R$,

$$\overset{*}{F}_r(p,w) = \inf \{\, p((1+\rho_r)I - B_r)z + w(1+\rho_r)h \mid$$

$$(z,h) \in \mathbb{R}_+^{n+1}, \; F_r(z,h) \geq 1 \,\}$$

and, for $i = 1,\dots,n$,

$$\overset{*}{F}{}^{i}(p,w) = \min_{r\in R(i)} \overset{*}{F}_r(p,w).$$

If we are sure that $\hat{w} > 0$ and $\sum_{r\in R(i)} F_r(\hat{z}_r,\hat{h}_r) > 0, \quad \forall\ i = 1,\ldots,n,$ we can fix $\hat{w}$. Then $\hat{p}$ must be a solution to the following equation:

$$p = \overset{*}{\mathbb{F}}(p),$$

where the operator $\overset{*}{\mathbb{F}} : \mathbb{R}^n_+ \dashrightarrow \mathbb{R}^n_+$ is defined by

$$\overset{*}{\mathbb{F}}(p) = (\overset{*}{F}{}^{1}(p,\hat{w}),\ldots,\overset{*}{F}{}^{n}(p,\hat{w}).$$

Under some conditions $\hat{p}$ is the unique solution to this equation and

$$\hat{p} = \lim p_t,$$

where

$$p_t = \overset{*}{\mathbb{F}}(p_{t-1}), \quad t = 1,2,\ldots, \quad \text{and} \quad p_0 \gg 0.$$

We can consider the case of the purely Leontief technology. Namely, let

$$|R| = n, \quad R(i) = \{i\}, \quad \forall\ i = 1,\ldots,n, \quad B_r = \mathbb{O}, \quad \forall\ r \in R,$$

$$F_r(z) = \min \{ \min_i (z^i/a^i_r),\ z^i/\bar{l}_r \},$$

where

$$a^i_r \geq 0,\ \forall\ i,r = 1,\ldots,n; \quad \sum_{i=1}^{n} a^i_r > 0,\ \forall\ r = 1,\ldots,n;$$

$$\bar{l}_r > 0,\ \forall\ r = 1,\ldots,n.$$

In this case we have

$$\hat{p} = \hat{w}(I - A\Lambda)^{-1}\bar{L}\Lambda, \qquad (4.1)$$

where A is a $n \times n$ matrix consisting of the columns

$$A_r = \begin{pmatrix} a^1_r \\ \vdots \\ a^n_r \end{pmatrix}, \quad r = 1,\ldots,n,$$

$$\Lambda = \text{diag}\ (1+\rho_1,\dots,1+\rho_n), \quad \bar{L} = (\bar{l}_1,\dots,\bar{l}_n).$$

Equilibrium prices defined by (4.1) can be interpreted as a modification of prices of production.

5. A Von NEUMANN TYPE MODEL: MODEL $\mathbf{M}_2$.

In this section, we describe a von Neumann type model $\mathbf{M}_2$ with consumers and show that an equilibrium in model $\mathbf{M}_1$ is also an equilibrium in a special case of model $\mathbf{M}_2$.

This model is, in fact, a von Neumann-Gale type model with consumers. The notion of equilibrium in this model can be considered as a generalization of the notion of equilibrium in the von Neumann-Gale model with no consumers (see, for example, MAKAROV and RUBINOV (1977)). This model is very much in the spirit of models considered in MORISHIMA (1964) and DANA et al.(1989).

There are no exogenously restricted primary factors in model $\mathbf{M}_2$, though these factors can be assumed implicitly. The rate of growth is endogenous in this model. However, in this case, there is a possibility of inequality between the equilibrium rate of growth and the exogenous 'rate of growth' of the primary factors.

To solve this problem MORISHIMA (1964) proposed to change some parameters of his model, namely, the normative rate of consumption of workers.

In model $\mathbf{M}_1$ the primary factors are introduced explicitly and we show that in this model the definition of equilibrium can be considered as a realization of Morishima's idea in relation to model $\mathbf{M}_2$.

There are n producible goods, a finite set J of consumers, and a finite set R of producers in this model. There are no primary factors in this model.

Producers. Producer $r \in R$ is described by his technology set $K_r \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and a number $\theta_{r0} \in]0,1]$ showing the share of profit which is used for investments. Elements of K_r are of the form (z,y), where $z \in \mathbb{R}^n_+$ is an input and $y \in \mathbb{R}^n_+$ is an output. It is supposed that K_r is a convex closed cone and

$$(0,y) \in K_r \quad \Rightarrow \quad y = 0;$$

$$(z,y) \in K_r, \quad 0 \le \tilde{y} \le y \quad \Rightarrow \quad (z,\tilde{y}) \in K_r.$$

Given prices $p \in \mathbb{R}^n_+$, producer r maximizes his profit rate solving the following problem:

$$\text{maximize} \quad \frac{py - pz}{pz}, \quad \text{s.t.} \quad (z,y) \in K_r.$$

More precisely, let us consider the following problem:

$$\text{maximize} \quad py - pz, \quad \text{s.t.} \quad (z,y) \in K_r, \quad pz = 1,$$

and denote by $\rho_r(p)$ the value of this problem. We suppose that producer r chooses his vector of the technology activity from

$$\mathbb{K}_r(p) = \{(z,y) \in K_r \mid py = (1 + \rho_r(p))pz\}.$$

By $\alpha_r(p)$ we denote the 'maximal rate of growth' of r-th firm under prices p:

$$\alpha_r(p) = 1 + \theta_{r0}\, \rho_r(p).$$

The distribution of profit is determined by numbers $\theta_{rj} \ge 0$,

$j \in J$, such that $\theta_{r0} = 1 - \sum_{j\in J} \theta_{rj}$. Here θ_{rj} is the share of consumer j.

Consumers. Under given prices $\hat{p}$ and given $(\hat{z}_r, \hat{y}_r) \in K_r$, $r \in R$, consumer $j \in J$ solves the following problem:

$$\text{maximize} \quad U_j(x), \qquad \text{s.t.} \quad x \in \mathbb{R}^n_+, \quad \hat{p}x \le \gamma_j, \tag{5.1}$$

where

$$\gamma_j = \sum_{r\in R} \theta_{rj}(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r). \tag{5.2}$$

The utility function $U_j : \mathbb{R}^n_+ \dashrightarrow \mathbb{R}_+$ is supposed to be non-zero, continuous, concave and homogeneous of degree one.

Definition 5.1. A list $(\hat{\alpha}, \hat{p}, (\hat{z}_r, \hat{y}_r)_{r\in R}, (\hat{x}_j)_{j\in J})$ is called an equilibrium of model $\mathbf{M}_2$ if

i) $\alpha_r(\hat{p}) \le \hat{\alpha}$, $\quad \forall\ r \in R$;

ii) $\alpha_r(\hat{p}) < \hat{\alpha} \quad \Rightarrow \quad \hat{p}\hat{z}_r = \hat{p}\hat{y}_r = 0, \quad \forall\ r \in R$;

iii) $(\hat{z}_r, \hat{y}_r) \in \mathbb{K}_r(\hat{p})$, $\quad \forall\ r \in R$;

iv) $\hat{x}_j$ is a solution to (5.1), where γ_j is given by (5.2);

v) $\hat{\alpha} \sum_{r\in R} \hat{z}_r + \sum_{j\in J} \hat{x}_j \le \sum_{r\in R} \hat{y}_r$;

vi) $\hat{p} \sum_{r\in R} \hat{y}_r > 0$.

Theorem 5.1. Assume that

D.1) there are $(\bar{z}_r, \bar{y}_r) \in K_r$, $\quad r \in R$, such that

$$\sum_{r\in R} \theta_{r0}(\bar{y}_r - \bar{z}_r) \gg 0;$$

D.2) $(z_r, y_r) \in K_r$, $\quad \forall\ r \in R$, $\quad \sum_{r\in R} y_r \ge \sum_{r\in R} z_r \ne 0 \quad \Rightarrow \quad \sum_{r\in R} z_r \gg 0$;

D.3) $\theta_{rj} > 0$, $\quad \forall\ r \in R$, $\quad \forall\ j \in J$.

Then there is an equilibrium in model $\mathbf{M}_2$.

Let us describe the relation between equilibria in models $\mathbf{M}_1$ and $\mathbf{M}_2$. It appears that an equilibrium in model $\mathbf{M}_1$ is also an equilibrium in a special case of model $\mathbf{M}_2$.

For simplicity, we suppose that, in model M_1,

$$J = J_1 \cup J_2 \quad \text{and} \quad J_1 \cap J_2 = \emptyset,$$

where

$$J_1 = \{j \in J \mid L_j \neq \{0\}\}, \quad J_2 = \{j \in J \mid \theta_{rj} > 0 \quad \text{for some} \quad r \in R\}.$$

Let us consider an equilibrium $(\hat{p}, \hat{w}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}, (\hat{x}_j)_{j\in J}, (\hat{l}_j)_{j\in J})$ of model $\mathbf{M}_2$ and construct a special case $\tilde{\mathbf{M}}_2$ of model $\mathbf{M}_2$ with $n = n_1 + n_2$ producible goods, the set of consumers J_2 and the set of producers $R \cup J_1$.

What were primary factors in $\mathbf{M}_1$ become producible goods in $\tilde{\mathbf{M}}_2$. We simply suppose that we can change primary factors for utilities of ex-consumers $j \in J_1$. Every ex-consumer $j \in J_1$ becomes, in some sense, a producer and his utility function plays the role of a production function.

Let us describe model $\tilde{\mathbf{M}}_2$ using tildes for marking its elements.

Producers. For $r \in R$, $\tilde{K}_r \subset \mathbb{R}^n_+ \times \mathbb{R}^n_+$ is defined by

$$\tilde{K}_r = \{(z,h,y,0) \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \times \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \mid (z,h,y) \in K_r\}.$$

For $r = j \in J_1$, $\tilde{K}_r \subset \mathbb{R}^n_+ \times \mathbb{R}^n_+$ is defined by

$$\tilde{K}_r = \{(z,0,0,g) \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \times \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \mid g \in \alpha \frac{U_r(z)}{U_r(\hat{x}_r)} L_r\}.$$

Consumers. Consumer $j \in J_2$ is the same as he was in model $\mathbf{M}_1$. His utility function $\tilde{U}_j : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ is given by

$$\tilde{U}_j(\bar{x},\bar{\bar{x}}) = U_j(\bar{x}), \quad \text{where} \quad (\bar{x},\bar{\bar{x}}) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2}.$$

As to numbers θ_{rj}, for $r \in R$ and $j \in J_2$, they are the same as they were in $\mathbf{M_1}$, and $\theta_{r0} = 1$ for $r \in J_1$.

Proposition 5.1. The set $(\alpha, (\hat{p},\hat{w}), ((\hat{z}_r,\hat{h}_r),(\hat{y}_r,\hat{g}_r))_{r \in R \cup J_1},$ $(\hat{x}_j)_{j \in J_2}),$ where

$\hat{g}_r = 0, \quad r \in R; \quad \hat{z}_r = \hat{x}_r, \quad \hat{h}_r = 0, \quad \hat{y}_r = 0, \quad \hat{g}_r = \hat{l}_r, \quad r \in J_1,$

is an equilibrium of model $\tilde{\mathbf{M}}_2$.

6. A MULTI-REGIONAL VERSION OF MODEL $\mathbf{M_1}$.

In this section we consider a version $\mathbf{M_1'}$ of model $\mathbf{M_1}$ which gives us the opportunity to look at the equilibrium existence problem from another point of view.

For simplicity, we suppose that there is only one primary factor, namely, labor force. In other words, $n_2 = 1$, and $L_j = \{l \in \mathbb{R}_+ \mid l \le \tilde{l}_j\}$, where $\tilde{l}_j \ge 0$ is the labor force of consumer j.

Let us suppose that our economy is multi-regional, and denote the set of regions by M, partitioning the sets of producers and consumers:

$$R = \bigcup_{m \in M} R_m, \quad R_{m_1} \cap R_{m_2} = \emptyset, \quad \text{if} \quad m_1 \ne m_2;$$

$$J = \bigcup_{m \in M} J_m, \quad J_{m_1} \cap J_{m_2} = \emptyset, \quad \text{if} \quad m_1 \ne m_2.$$

Definition 6.1. A list $(\hat{p}, (\hat{w}_m)_{m \in M}, (\hat{z}_r,\hat{h}_r,\hat{y}_r)_{r \in R}, (\hat{x}_j)_{j \in J})$ is called an equilibrium of model $\mathbf{M_1'}$ if

E.1) $\alpha_r(\hat{p},\hat{w}_m) \le \alpha, \quad \forall\ r \in R_m, \quad \forall\ m \in M;$

E.2) $\alpha_r(\hat{p},\hat{w}_m) < \alpha \Rightarrow \hat{p}\hat{y}_r = \hat{p}\hat{z}_r + \hat{p}\hat{h}_r = 0, \quad \forall\ r \in R_m, \quad \forall\ m \in M;$

E.3) $(\hat{z}_r,\hat{h}_r,\hat{y}_r) \in \mathbb{K}_r(\hat{p},\hat{w}_m), \quad \forall\ r \in R_m, \quad \forall\ m \in M;$

E.4) for all $j \in J_m$, $m \in M$, $\hat{x}_j$ is a solution to (2.1), where γ_j is given by

$$\gamma_j = \alpha\hat{w}_m\tilde{l}_j + \sum_{m\in M}\sum_{r\in R_m} \theta_{rj}(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}_m\hat{h}_r);$$

E.5) $\alpha \sum_{r\in R} \hat{z}_r + \sum_{j\in J} \hat{x}_j \le \sum_{r\in R} \hat{y}_r;$

E.6) $\sum_{r\in R_m} \hat{h}_r = \sum_{j\in J_m} \tilde{l}_j, \quad \forall\ m \in M;$

E.7) $\hat{p} \sum_{r\in R} \hat{y}_r > 0.$

In this definition, E.6) means that labor demand and supply are equalized in every region.

Formally, this definition can be easily reduced to Definition 1.1 and, under some conditions, it is possible to prove that in this model an equilibrium exists.

In the framework of this model we can fix wage rates in different regions and the proportions between numbers θ_{rj} for all $r \in R$, and equalize labor supply and demand by means of varying numbers θ_{r0}. Since we suppose that $p \in P$, where

$$P = \{p \in \mathbb{R}_+^{n_1} \mid \sum_{i=1}^{n_1} p^i = 1\},$$

the fixation of wage rates provides some minimal level of utility per one unit of labor force.

More explicitely, let us fix $\tilde{w}_m > 0$, $m \in M$. Let us also fix numbers $\nu_{rj} \in [0,1]$, $j \in J$, $r \in R$, such that $\sum_{j\in J} \nu_{rj} = 1$, $r \in R$.

Given θ_{r0}, these numbers determine all θ_{rj} by

$$\theta_{rj} = (1 - \theta_{r0})\nu_{rj}, \quad j \in J, \quad r \in R. \tag{6.1}$$

Theorem 6.1. Assume that

F.1) there is $\varepsilon > 0$ such that, for any $p \in P$, we can find $(\bar{z}_r, \bar{h}_r, \bar{y}_r) \in \prod_{r\in R_m} K_r$ such that

$$p\Big(\sum_{r\in R_m} \bar{y}_r / \alpha - \sum_{r\in R_m} \bar{z}_r \Big) > \tilde{w}_m \sum_{j\in J_m} \tilde{l}_j + \varepsilon, \quad \sum_{r\in R_m} \bar{h}_r < \sum_{j\in J_m} \tilde{l}_j;$$

F.2) for every $j \in J_m$, $m \in M$, at least one of the following properties is true:

2.1 $\tilde{l}_j > 0$;

2.2 there is $m_1 \in M$ such that

$$\nu_{rj} > 0, \quad \forall\, r \in R_{m_1}.$$

Then there are $\theta_{r0} > 0$, $r \in R$ such that, for θ_{rj} defined by (6.1), there is an equilibrium $(\hat{p}, (\hat{w}_m)_{m\in M}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}, (\hat{x}_j)_{j\in J})$ such that

$$\hat{w}_m = \tilde{w}_m, \quad \forall\, m \in M.$$

6. PROOFS.

Proof of lemma 2.1. First, it should be noted that

$$\text{A.1} \iff \rho_r(\hat{p},\hat{w}) \le \rho_r, \quad \forall\, r \in R;$$

$$\text{A.2} \iff (\rho_r(\hat{p},\hat{w}) < \rho_r \Rightarrow \hat{p}\hat{y}_r = \hat{p}\hat{z}_r + \hat{w}\hat{h}_r = 0, \quad \forall\, r \in R);$$

$$\text{A.3} \iff 0 = \hat{p}\hat{y}_r - (1 + \rho_r(\hat{p},\hat{w}))(\hat{p}\hat{z}_r + \hat{w}\hat{h}_r) \geq$$

$$\geq \hat{p}y - (1 + \rho_r(\hat{p},\hat{w}))(\hat{p}z + \hat{w}h), \quad \forall \ (z,h,y) \in K_r, \quad \forall \ r \in R.$$

Therefore, A.1 - A.3 are equivalent to the following inequalities:

$$0 = \sum_{r \in R} \left(\frac{\hat{p}\hat{y}_r}{1 + \rho_r} - (\hat{p}\hat{z}_r + \hat{w}\hat{h}_r) \right)$$

$$\geq \sum_{r \in R} \left(\frac{\hat{p}y_r}{1 + \rho_r} - (\hat{p}z_r + \hat{w}h_r) \right), \tag{7.1}$$

$$\forall \ (z_r, h_r, y_r) \in \prod_{r \in R} K_r.$$

Let $(\hat{p}, \hat{w}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r \in R}, (\hat{l}_j)_{j \in J}, (\hat{x}_j)_{j \in J})$ be an equilibrium. Show that

$$\sum_{r \in R} \hat{w}\hat{h}_r = \sum_{j \in J} \hat{w}\hat{l}_j. \tag{7.2}$$

Since the utility functions are monotone,

$$\hat{p}\hat{x}_j = \alpha\hat{w}\hat{l}_j + \sum_{r \in R} \theta_{rj}(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r), \quad \forall \ j \in J.$$

Summing and recalling that $\sum_{j \in J} \theta_{rj} = 1 - \theta_{r0}$, $\forall \ r \in R$, we obtain

$$\sum_{j \in J} \hat{p}\hat{x}_j = \alpha \sum_{j \in J} \hat{w}\hat{l}_j + \sum_{r \in R} (1 - \theta_{r0})(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r).$$

By A.1),

$$\theta_{r0}(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r) = (\alpha - 1)(\hat{p}\hat{z}_r + \hat{w}\hat{h}_r), \quad \forall \ r \in R.$$

Therefore, by A.6),

$$0 \le \sum_{r\in R} \hat{\hat{p}}\hat{\hat{y}}_r - \left(\sum_{j\in J} \hat{\hat{p}}\hat{\hat{x}}_j + \alpha \sum_{r\in R} \hat{\hat{p}}\hat{\hat{z}}_r\right) =$$

$$\sum_{r\in R} \hat{p}\hat{y}_r - \alpha \sum_{j\in J} \hat{w}\hat{l}_j - \sum_{r\in R} (1 - \theta_{r0})(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r) - \alpha \sum_{r\in R} \hat{p}\hat{z}_r =$$

$$\sum_{r\in R} \hat{p}\hat{y}_r - \alpha \sum_{j\in J} \hat{w}\hat{l}_j - \sum_{r\in R} (\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r) +$$

$$\sum_{r\in R} (\alpha - 1)(\hat{p}\hat{z}_r + \hat{w}\hat{h}_r) - \alpha \sum_{r\in R} \hat{p}\hat{z}_r =$$

$$- \alpha \sum_{j\in J} \hat{w}\hat{l}_j + \alpha \sum_{r\in R} \hat{w}\hat{h}_r.$$

That is $\alpha \sum_{j\in J} \hat{w}\hat{l}_j \le \alpha \sum_{r\in R} \hat{w}\hat{h}_r$.

On the other hand, by A.7), $\alpha \sum_{j\in J} \hat{w}\hat{l}_j \ge \alpha \sum_{r\in R} \hat{w}\hat{h}_r$. Thus, (7.2) is true. This means, in particular, that B.2) holds.

We have

$$\sum_{r\in R} \hat{w}\hat{h}_r = \sum_{r\in R} \left(\frac{\hat{p}\hat{y}_r}{1 + \rho_r} - \hat{p}\hat{z}_r \right),$$

$$\sum_{r\in R} \hat{w}h_r \ge \sum_{r\in R} \left(\frac{\hat{p}y_r}{1 + \rho_r} - \hat{p}z_r \right), \quad \forall \quad (z_r, h_r, y_r) \in K_r, \qquad \forall\ r \in R.$$

By (7.2) these relations mean that $(\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}$ is a solution to (2.4), that is, B.1) also holds.

Taking into account A.1), we note that, for all $r \in R$,

$$\hat{p}\hat{y}_r - (\hat{p}\hat{z}_r + \hat{w}\hat{h}_r) = \hat{p}\hat{y}_r - \frac{\hat{p}\hat{y}_r}{1 + \rho_r} = \frac{\rho_r}{1 + \rho_r} \hat{p}\hat{y}_r.$$

That is

$$\alpha\hat{w}\hat{l}_j + \sum_{r\in R} \theta_{rj}(\hat{p}\hat{y}_r - \hat{p}\hat{z}_r - \hat{w}\hat{h}_r) =$$

$$\alpha\hat{w}\hat{l}_j + \sum_{r\in R} \frac{\theta_{rj}\rho_r}{1+\rho_r} \hat{p}\hat{y}_r, \qquad \forall\ j \in J. \tag{7.3}$$

This means that B.3) holds too.

Let now $(\hat{p}, \hat{w}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}, (\hat{l}_j)_{j\in J}, (\hat{x}_j)_{j\in J})$ satisfies B.1) - B.3) and A.5) - A.8).

Since $(\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}$ is a solution to (2.4), by the Kuhn-Tucker theorem (see, for example, Rockafellar(1970), Corollary 28.3), there exists $\lambda \geq 0$ such that

$$0 = \sum_{r\in R} \left(\frac{\hat{p}\hat{y}_r}{1+\rho_r} - \hat{p}\hat{z}_r \right) - \lambda \sum_{r\in R} \hat{w}\hat{h}_r \geq$$

$$\sum_{r\in R} \left(\frac{\hat{p}y_r}{1+\rho_r} - \hat{p}z_r \right) - \lambda \sum_{r\in R} \hat{w}h_r,$$

$$\forall\ (z_r, h_r, y_r) \in K_r, \qquad \forall\ r \in R.$$

If the value of (2.4) is strictly positive, then $\lambda > 0$ and by B.2), $\lambda = 1$.

If the value of (2.4) is equal to zero, then $\hat{w} \sum_{j\in J} \hat{l}_j = 0$. We assumed that $\operatorname{int}\ (\sum_{j\in J} L_j) \neq \emptyset$. At the same time $\hat{l}_j$ is a solution of (2.2). Therefore $\hat{w} = 0$.

In any case (7.1) holds. Therefore, A.1) - A.3) are satisfied. It remains to note that (7.3) is also true. This means that A.4) also holds.□

Before proving Theorem 2.1 we prove

Lemma 7.1. The set $\{ y \in \mathbb{R}_+^{n_1} \mid$ there exists $(z_r, h_r, y_r) \in K_r$, $r \in R$, such that $\sum_{r\in R} h_r \in \sum_{j\in J} L_j$, $y = \sum_{r\in R} y_r \geq \alpha \sum_{r\in R} z_r \}$ is bounded.

Proof. Let us suppose that this is not true. Then there exists a sequence $(((z_r(k), h_r(k), y_r(k))_{r\in R})_{k=1}^{\infty}$ of elements of $\prod_{r\in R} K_r$ such that, for all $k = 1,2,\ldots,$

$$\sum_{r\in R} h_r(k) \in \sum_{j\in J} L_j, \qquad \sum_{r\in R} y_r(k) \geq \alpha \sum_{r\in R} z_r(k),$$

$$\lambda(k) = \Big\| \sum_{r\in R} y_r(k) \Big\| \xrightarrow[k\to\infty]{} \infty.$$

We have

$$(z_r(k)/\lambda(k), h_r(k)/\lambda(k), y_r(k)/\lambda(k)) \in K_r, \quad \forall\, r \in R, \quad \forall\, k = 1,2,\ldots,$$

$$\sum_{r\in R} y_r(k)/\lambda(k) \geq \alpha \sum_{r\in R} z_r(k)/\lambda(k), \quad \forall\, k = 1,2,\ldots,$$

$$h_r(k)/\lambda(k) \xrightarrow[k\to\infty]{} 0, \quad \forall\, r \in R.$$

Moreover, there is a subsequence $(k_i)_{i=1}^{\infty}$ such that, for all $r \in R$,

$$(z_r(k_i)/\lambda(k_i))_{i=1}^{\infty} \quad \text{and} \quad (y_r(k_i)/\lambda(k_i))_{i=1}^{\infty}$$

converge. Denote

$$\tilde{z}_r = \lim_{i\to\infty} z_r(k_i)/\lambda(k_i), \quad \tilde{y}_r = \lim_{i\to\infty} y_r(k_i)/\lambda(k_i).$$

Thus, we have $\Big\| \sum_{r\in R} \tilde{y}_r \Big\| = 1$ and

$$(\tilde{z}_r, 0, \tilde{y}_r) \in K_r, \quad \forall\, r \in R, \quad \sum_{r\in R} \tilde{y}_r \geq \alpha \sum_{r\in R} \tilde{z}_r,$$

contradicting to the following assumption about K_r:

$$(z, 0, y) \in K_r, \quad \Rightarrow \quad y \leq z. \ \square$$

Proof of Theorem 2.1. To prove this theorem we use the well-known idea (see, for example, AUBIN (1979)) of reducing our model to a constrained non-cooperative game.

Let $B_1 \subset \mathbb{R}^{n_1}$ be a closed ball such that

$$(z_r, h_r, y_r) \subset \prod_{r\in R} K_r, \quad \sum_{r\in R} h_r \in \sum_{j\in J} L_j, \quad \sum_{r\in R} y_r \geq \alpha \sum_{r\in R} z_r \quad \Rightarrow \quad \sum_{r\in R} y_r \subset \operatorname{int} B_1.$$

(This ball exists by Lemma 7.1.)

Denote

$$\nu_j = \min_r \theta_{rj}\rho_r, \qquad j \in J,$$

$$e_1 = (1,\dots,1) \in \mathbb{R}^{n_1}, \qquad e_2 = (1,\dots,1) \in \mathbb{R}^{n_2},$$

and take $\lambda > 0$, $\lambda_j > 0$, $j \in J$, $\beta > 0$ such that

$$l \in \sum_{j\in J} L_j \quad \Rightarrow \quad l \ll \lambda e_2,$$

$$\lambda_j e_2 \in L_j, \qquad \forall\ j \in J,$$

$$\sum_{r\in R} \left(\frac{\bar{y}_r}{1 + \rho_r} - \bar{z}_r \right) \geq \beta e_1.$$

We can now describe the required game. There are $2|J| + 3$ players in this game.

The first player is responsible for goods prices p. He solves the following problem:

$$\text{maximize} \quad p\Big(\alpha \sum_{r\in R} z_r + \sum_{j\in J} x_j - \sum_{r\in R} y_r \Big), \qquad \text{s.t.} \quad p \in P,$$

where $P = \{p \in \mathbb{R}_+^{n_1} \mid pe_1 = 1\}$.

The second player establishes relative prices $q \in \mathbb{R}_+^{n_2}$ of primary factors. His problem is as follows:

$$\text{maximize} \quad q\Big(\sum_{r\in R} h_r - \sum_{j\in J} l_j \Big), \qquad \text{s.t.} \quad q \in Q,$$

where $\quad Q = \{q \in \mathbb{R}_+^{n_2} \mid qe_2 = 1\}.$

The third player is connected with producers. Under given p, q, l_j, $j \in J$, his problem is as follows:

$$\text{maximize} \sum_{r\in R} \left(\frac{py_r}{1 + \rho_r} - pz_r \right),$$

$$\text{s.t.} \quad (z_r, h_r, y_r) \in K_r \cap B_1 \times B_2 \times B_1, \qquad r \in R, \tag{7.4}$$

$$q \sum_{r\in R} h_r \le q \sum_{j\in J} l_j,$$

where

$$B_2 = \{ h \in \mathbb{R}^{n_2} \mid -\lambda e_2 \le h \le \lambda e_2 \}.$$

Every consumer is represented by two players. The first one solves problem (2.2).

The second solves the following problem:

$$\text{maximize} \quad U_j(x_j) \ ,$$

$$\text{s.t.} \quad x_j \in \mathbb{R}_+^{n_1} \cap B_1, \tag{7.5}$$

$$px_j \le \gamma_j.$$

Here $\quad \gamma_j = \gamma_j(p, q, (z_r, h_r, y_r)_{r\in R}, (l_j)_{j\in J})$ is given by

$$\gamma_j = \max \{ \gamma_j^1 + \gamma_j^2, \gamma_j^3 \} > 0,$$

where

$$\gamma_j^1 = \nu_j \beta,$$

$$\gamma_j^2 = \alpha\lambda_j \beta/\lambda,$$

$$\gamma_j^3 = \alpha \frac{q l_j}{\max\{\sum_{k\in J}\lambda_k, \sum_{k\in J} q l_k\}} \times \sum_{r\in R}\left(\frac{p y_r}{1+\rho_r} - p z_r\right) + \sum_{r\in R}\frac{\theta_{rj}\rho_r}{1+\rho_r} p y_r .$$

It is noteworthy that by C.1, $\gamma_j > 0$, $\forall\ j \in J$. Therefore the correspondence

$$(p,\ q,\ (z_r,h_r,y_r)_{r\in R},\ (l_j)_{j\in J}) \longrightarrow \{ x_j \in \mathbb{R}_+^{n_1} \cap B_1 \mid p x_j \le \gamma_j \}$$

is continuous.

There is a Nash equilibrium in this game $(\hat{p},\ \hat{q},\ (\hat{z}_r,\hat{h}_r,\hat{y}_r)_{r\in R},\ (\hat{l}_j)_{j\in J},\ (\hat{x}_j)_{j\in J})$ (see, for example, AUBIN (1979), Section 9.3.2). All we need to prove the theorem is to show that the list $(\hat{p},\ \hat{w},\ (\hat{z}_r,\hat{h}_r,\hat{y}_r)_{r\in R},\ (\hat{l}_j)_{j\in J},\ (\hat{x}_j)_{j\in J})$, where

$$\hat{w} = \left(\sum_{r\in R}\left(\frac{\hat{p}\hat{y}_r}{1+\rho_r} - \hat{p}\hat{z}_r\right) \Big/ \sum_{j\in J} \hat{q}\hat{l}_j \right)\hat{q},$$

is an equilibrium of our model.

Since $\hat{l}_j$ is, for all $j \in J$, a solution to (2.2), by the choice of λ_j,

$$\sum_{j\in J} \hat{q}_j \hat{l}_j > \sum_{j\in J} \lambda_j .$$

At the same time, since $(\hat{z}_r,\hat{h}_r,\hat{y}_r)_{r\in R}$ is a solution to (7.4) under $p = \hat{p}$, $w = \hat{w}$, $l_j = \hat{l}_j$, $j \in J$, by the choice of β,

$$\sum_{r\in R}\left(\frac{\hat{p}\hat{y}_r}{1+\rho_r} - \hat{p}\hat{z}_r\right) \ge \beta .$$

Hence, for every $j \in J$,

$$\alpha \frac{\hat{q}\hat{l}_j}{\sum_{k\in J}\hat{q}\hat{l}_k} \sum_{r\in R}\left(\frac{\hat{p}\hat{y}_r}{1+\rho_r} - \hat{p}\hat{z}_r\right) \ge \alpha \frac{\lambda_j q e_2}{\lambda q e_2}\beta = \gamma_j^2 ,$$

$$\sum_{r\in R} \frac{\theta_{rj}\rho_r}{1+\rho_r}\, \hat{p}\hat{y}_r \geq \nu_j \sum_{r\in R} \frac{\hat{p}\hat{y}_r}{1+\rho_r} \geq$$

$$\nu_j \sum_{r\in R} \left(\frac{\hat{p}\hat{y}_r}{1+\rho_r} - \hat{p}\hat{z}_r \right) \geq \nu_j\beta \geq \gamma_j^1 .$$

These inequalities yield that, for all $j \in J$, $\hat{x}_j$ is a solution to (7.5), where

$$\gamma_j = \hat{\gamma}_j = \alpha \frac{\hat{q}\hat{l}_j}{\sum_{k\in J} \hat{q}\hat{l}_k} \sum_{r\in R} \left(\frac{\hat{p}\hat{y}_r}{1+\rho_r} - \hat{p}\hat{z}_r \right) + \sum_{r\in R} \frac{\theta_{rj}\rho_r}{1+\rho_r}\, \hat{p}\hat{y}_r ,$$

and

$$\hat{p} \sum_{j\in J} \hat{x}_j \leq \sum_{j\in J} \hat{\gamma}_j = \sum_{r\in R} \left(\frac{\alpha}{1+\rho_r} + \frac{(1-\theta_{r0})\rho_r}{1+\rho_r} \right) \hat{p}\hat{y}_r - \alpha \sum_{r\in R} \hat{p}\hat{z}_r .$$

We have $\theta_{r0}\rho_r = 1 - \alpha$. Hence,

$$\frac{\alpha}{1+\rho_r} + \frac{(1-\theta_{r0})\rho_r}{1+\rho_r} = 1$$

and, therefore,

$$\hat{p} \sum_{j\in J} \hat{x}_j + \alpha\hat{p} \sum_{r\in R} \hat{z}_r \leq \hat{p} \sum_{r\in R} \hat{y}_r . \qquad (7.6)$$

Remind that $\hat{p}$ is a solution to the following problem:

$$\text{maximize} \quad p\Big(\sum_{j\in J} \hat{x}_j + \alpha \sum_{r\in R} \hat{z}_r - \sum_{r\in R} \hat{y}_r \Big),$$

$$\text{s.t,} \quad p \in P.$$

Thus, (7.6) yields the following inequality:

$$\sum_{j\in J} \hat{x}_j + \alpha \sum_{r\in R} \hat{z}_r \leq \sum_{r\in R} \hat{y}_r .$$

This inequality means, in particular, that

$$(\hat{z}_r, \hat{h}_r, \hat{y}_r) \in \text{int } (B_1 \times B_2 \times B_1), \quad \forall\ r \in R,$$

and

$$\hat{x}_j \in \text{int } B_1, \quad \forall\ j \in J.$$

Hence, the constraints

$$(z_r, h_r, y_r) \in B_1 \times B_2 \times B_1, \qquad r \in R,$$

and

$$x_j \in B_1, \quad j \in J,$$

in problems (7.4) and (7.5), respectively, are not essential.

It remains to refer to Lemma 2.1. □

Proof of Theorem 3.1. We know that, for $r = r_1, r_2$, $(\hat{z}_r, \hat{h}_r, \hat{y}_r)$ is a solution to the following problem:

$$\text{maximize} \quad \frac{\hat{p} y_r}{1 + \rho_r} - \hat{p} z_r - \hat{w} h_r,$$

$$\text{s.t.} \quad (z_r, h_r, y_r) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \times \mathbb{R}_+^{n_1}, \qquad \phi_r(z_r, h_r, y_r) \le 0,$$

$$z_r^i = 0, \quad i \in I_r^1, \qquad h_r^i = 0, \quad i \in I_r^2, \qquad y^i = 0, \quad i \in I_r^3.$$

Therefore, for some $\lambda_r > 0$, $r = r_1, r_2$,

$$\frac{\partial \phi_{r_1}}{\partial z^{i_1}} \left(\hat{z}_{r_1}, \hat{h}_{r_1}, \hat{y}_{r_1} \right) = - \lambda_{r_1} \hat{p}^{i_1},$$

$$\frac{\partial \phi_{r_2}}{\partial y^{i_1}} \left(\hat{z}_{r_2}, \hat{h}_{r_2}, \hat{y}_{r_2} \right) = \lambda_{r_2} \frac{\hat{p}^{i_1}}{1 + \rho_{r_2}},$$

$$\frac{\partial \phi_{r_1}}{\partial h^{k_1}} \left(\hat{z}_{r_1}, \hat{h}_{r_1}, \hat{y}_{r_1} \right) = - \lambda_{r_1} \hat{w}^{k_1},$$

$$\frac{\partial \phi_{r_2}}{\partial h^{k_2}} \left(\hat{z}_{r_2}, \hat{h}_{r_2}, \hat{y}_{r_2} \right) = - \lambda_{r_2} \hat{w}^{k_1}.$$

Remind that $\alpha < 1 + \rho_{r_2}$, take δ^1 and δ^2 such that

$$\frac{\hat{p}^{i_1}}{(1+\rho_{r_2})\hat{w}^{k_1}} < \delta^2 < \delta^1 < \frac{\hat{p}^{i_1}}{\alpha\hat{w}^{k_1}},$$

and define vectors $\Delta \in \mathbb{R}^{n_1}$ and $\Delta_1, \Delta_2 \in \mathbb{R}^{n_2}$ by

$$\Delta^i = 0, \quad i \neq i_1, \quad \Delta^{i_1} = 1,$$

$$\Delta_1^k = 0, \quad k \neq k_1, \quad \Delta_1^{k_1} = \delta^1,$$

$$\Delta_2^k = 0, \quad k \neq k_1, \quad \Delta_2^{k_1} = \delta^2.$$

We have

$$\operatorname{grad} \phi_{r_1}\left(\hat{z}_{r_1}, \hat{h}_{r_1}, \hat{y}_{r_1} \right) \times (\Delta/\alpha, -\Delta_1, 0) < 0,$$

$$\operatorname{grad} \phi_{r_2}\left(\hat{z}_{r_2}, \hat{h}_{r_2}, \hat{y}_{r_2} \right) \times (0, \Delta_2, \Delta) < 0,$$

Therefore, for some $\hat{\mu} > 0$ and for

$$\tilde{z}_{r_1} = \hat{z}_{r_1} + \hat{\mu}\Delta/\alpha, \quad \tilde{h}_{r_1} = \hat{h}_{r_1} - \hat{\mu}\Delta_1, \quad \tilde{y}_{r_1} = \hat{y}_{r_1},$$

$$\tilde{z}_{r_2} = \hat{z}_{r_2}, \quad \tilde{h}_{r_2} = \hat{h}_{r_2} + \hat{\mu}\Delta_2, \quad \tilde{y}_{r_2} = \hat{y}_{r_2} + \hat{\mu}\Delta,$$

we derive

$$\phi_r(\tilde{z}_r, \tilde{h}_r, \tilde{y}_r) < 0, \quad r = r_1, r_2.$$

Moreover, what is important,

$$\left(\tilde{y}_{r_1} + \tilde{y}_{r_2} \right) - \alpha\left(\tilde{z}_{r_1} + \tilde{z}_{r_2} \right) = \left(\hat{y}_{r_1} + \hat{y}_{r_2} \right) - \alpha\left(\hat{z}_{r_1} + \hat{z}_{r_2} \right),$$

$$\tilde{h}_{r_1} + \tilde{h}_{r_2} \leq \hat{h}_{r_1} + \hat{h}_{r_2},$$

and

$$\sigma = \left(\hat{h}_{r_1}^{k_1} + \hat{h}_{r_2}^{k_1} \right) - \left(\tilde{h}_{r_1}^{k_1} + \tilde{h}_{r_2}^{k_1} \right) > 0.$$

Thus, if we define $\tilde{h}_r$ for $r \in R_a$, $r \neq r_1, r_2$, by

$$\tilde{h}_r^k = \hat{h}_r^k, \quad k \neq k_1, \quad \tilde{h}_r^{k_1} = \hat{h}_r^{k_1} + \sigma/|R_a|,$$

we shall derive

$$\phi_r(\tilde{z}_r, \tilde{h}_r, \tilde{y}_r) < 0, \quad \forall\ r \in R_a,$$

$$\sum_{r \in R_a} \tilde{y}_r - \alpha \sum_{r \in R_a} \tilde{z}_r = \sum_{r \in R_a} \hat{y}_r - \alpha \sum_{r \in R_a} \hat{z}_r,$$

$$\sum_{r \in R_a} \tilde{h}_r \le \sum_{r \in R_a} \hat{h}_r.$$

It remains to note that, for some $\lambda > 1$ and for

$$z_r = \lambda \tilde{z}_r, \quad h_r = \tilde{h}_r, \quad y_r = \lambda \tilde{y}_r, \quad r \in R_a,$$

we have

$$\phi_r(z_r, h_r, y_r) \le 0, \quad \forall\ r \in R_a. \ \square$$

Proof of Theorem 5.1 is similar to the proof of Theorem 2.1.□

Proof of Proposition 5.1. It follows from the definitions.

Proof of Theorem 6.1. As in Theorem 2.1, we construct a non-cooperative game.

It follows from F.1) that, for every $m \in M$, we can find $\bar{\theta}_m < 1$ such that, for any $p \in P$, there is $(\bar{z}_r, \bar{h}_r, \bar{y}_r)_{r \in R_m} \in \prod_{r \in R_m} K_r$ such that

$$p\left(\sum_{r \in R_m} \frac{\bar{\theta}_m}{\alpha - 1 + \bar{\theta}_m} \bar{y}_r - \bar{z}_r \right) > \tilde{w}_m \sum_{j \in J_m} \tilde{l}_j,$$

$$\sum_{r \in R_m} \bar{h}_r < \sum_{j \in J_m} \tilde{l}_j.$$

Denote

$$\bar{\theta}_{rj} = \nu_{rj}(1 - \bar{\theta}_m), \quad r \in R_m, \quad m \in M, \quad j \in J,$$

$$\mu_j^m = \min_{r \in R_m} \bar{\theta}_{rj}(\alpha - 1) \,/\, \bar{\theta}_m, \quad m \in M, \quad j \in J$$

(note that $\max_{m\in M} \mu_j^m > 0, \quad \forall\ j \in J$).

We can now describe the mentioned game.

The first player is responsible for goods prices $p \in P$. He is the same as in the proof of Theorem 2.1.

Two players are connected with every $m \in M$. The first of them solves (uder given p and $\theta_m \in [0,1]$) the following problem:

$$\text{maximize} \quad \sum_{r\in R_m} \left(\frac{\theta_m}{\alpha - 1 + \theta_m} py_r - pz_r \right),$$

$$\text{s.t.} \quad \sum_{r\in R_m} h_r \le \sum_{j\in J_m} \tilde{l}_j,$$

$$(z_r, h_r, y_r) \in K_r \cap B_1 \times [0, L_m+\varepsilon] \times B_1, \quad r \in R_m,$$

where B_1 is the same as in the proof of Theorem 2.1, $L_m = \sum_{j\in J_m} \tilde{l}_j$.

The second player is responsible for θ_m. His problem is as follows:

$$\text{maximize} \quad \theta_m \left[\tilde{w}_m \sum_{j\in J_m} \tilde{l}_j - p \sum_{r\in R_m} \left(\frac{\theta_m}{\alpha - 1 + \theta_m} py_r - pz_r \right) \right],$$

$$\text{s.t.} \quad \theta_m \in [0,1].$$

Every consumer $j \in J$ is represented by a player. He solves problem (7.5) where $\gamma_j = \gamma_j(p, (z_r, h_r, y_r)_{r\in R}, (\theta_m)_{m\in M})$ is given by

$$\gamma_j = \max \{ \gamma_j^1 + \gamma_j^2, \ \gamma_j^3 \}.$$

Here, for $j \in J_m$,

$$\gamma_j^1 = \alpha \tilde{w}_m \tilde{l}_j,$$

$$\gamma_j^2 = \sum_{m\in M} \mu_j^m \tilde{w}_m \sum_{k\in J_m} \tilde{l}_k,$$

$$\gamma_j^3 = \alpha \frac{\tilde{l}_j}{\sum_{k\in J_m} \tilde{l}_k} \sum_{r\in R_m} \left(\frac{\theta_m}{\alpha - 1 + \theta_m} py_r - pz_r \right) + \sum_{r\in R} \frac{\theta_{rj}(\alpha - 1)}{\theta_{r0} + \alpha - 1} py_r .$$

Note that $\gamma_j > 0, \quad \forall\, j \in J_m, \quad \forall\, m \in M$. Therefore the correspondences

$$(p, (z_r, h_r, y_r)_{r\in R}, (\theta_m)_{m\in M}) \dashrightarrow \{ x_j \in \mathbb{R}_+^{n_1} \cap B_1 \mid px_j \leq \gamma_j \}$$

is continuous.

There is a Nash equilibrium in this game $(\hat{p}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R_m},$ $m \in M, \ (\hat{\theta}_m)_{m\in M}, \ (\hat{x}_j)_{j\in J})$ (see, for example, AUBIN (1979), Section 9.3.2), and all we need is to show that, under

$$\theta_{r0} = \hat{\theta}_m, \quad r \in R_m, \quad m \in M,$$

the set $(\hat{p}, (\tilde{w}_m)_{m\in M}, (\hat{z}_r, \hat{h}_r, \hat{y}_r)_{r\in R}, (\hat{x}_j)_{j\in J})$ is an equilibrium of our model.

Let us show that, for all $m \in M$,

$$\tilde{w}_m \sum_{j\in J_m} \tilde{l}_j = \hat{p} \sum_{r\in R_m} \left(\frac{\hat{\theta}_m}{\alpha - 1 + \hat{\theta}_m} \hat{p}\hat{y}_r - \hat{p}\hat{z}_r \right). \qquad (7.7)$$

Indeed, if the right-hand-side were bigger than the left-hand-side, we would have $\hat{\theta}_m = 0$ (see the problem of the player who establish θ_m), which is impossible. If the left-hand-side were bigger than the right-hand-side, we would have $\hat{\theta}_m = 1$, which is impossible too. Note that $\hat{\theta}_m < \bar{\theta}_m, \quad \forall\, m \in M$.

Denote $\rho_r = (\alpha - 1)/\theta_{r0}, \quad r \in R,$ and prove that, for $j \in J_m$,

$$\alpha \frac{\tilde{l}_j}{\sum_{k\in J_m} \tilde{l}_k} \sum_{r\in R_m} \left(\frac{\hat{p}\hat{y}_r}{1 + \rho_r} - \hat{p}\hat{z}_r \right) + \sum_{r\in R} \frac{\theta_{rj}\, \rho_r}{1 + \rho_r} \hat{p}\hat{y}_r =$$

$$\alpha \frac{\tilde{l}_j}{\sum_{k\in J_m} \tilde{l}_k} \sum_{r\in R_m} \left(\frac{\theta_{r0}}{\alpha - 1 + \theta_{r0}} \hat{p}\hat{y}_r - \hat{p}\hat{z}_r \right) + \sum_{r\in R} \frac{\theta_{rj}(\alpha - 1)}{\theta_{r0} + \alpha - 1} \hat{p}\hat{y}_r \geq$$

$$\geq \gamma_j^1 + \gamma_j^2.$$

First, note that (7.7) yields

$$\alpha \frac{\tilde{l}_j}{\sum_{k \in J_m} \tilde{l}_k} \sum_{r \in R_m} \left(\frac{\hat{\hat{py}}_r}{1 + \rho_r} - \hat{\hat{pz}}_r \right) = \gamma_j^1. \qquad (7.8)$$

Let us show that

$$\sum_{r \in R} \frac{\theta_{rj}\,\rho_r}{1 + \rho_r} \hat{\hat{py}}_r \geq \gamma_j^2. \qquad (7.9)$$

We have, for $r \in R_m$,

$$\theta_{rj}\rho_r = \theta_{rj}(\alpha - 1)/\theta_{r0} \geq \bar{\theta}_{rj}(\alpha - 1)/\bar{\theta}_m = \mu_j^m.$$

Therefore,

$$\sum_{r \in R} \frac{\theta_{rj}\,\rho_r}{1 + \rho_r} \hat{\hat{py}}_r \geq \sum_{m \in M} \mu_j^m \sum_{r \in R_m} \frac{\hat{\hat{py}}_r}{1 + \rho_r} \geq$$

$$\sum_{m \in M} \mu_j^m \sum_{r \in R_m} \left(\frac{\hat{\hat{py}}_r}{1 + \rho_r} - \hat{\hat{pz}}_r \right) = \sum_{m \in M} \mu_j^m \tilde{w}_m \sum_{k \in J_m} \tilde{l}_k = \gamma_j^2.$$

Adding up (7.8) and (7.9) we derive the required inequality.

We can now repeat the argument used in the proof of Theorem 2.1. □

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