Hysteresis: what it is and what it is not

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Abstract:

The recent economic literature has made a frequent use of the concept of hysteresis, for instance in the fields of international trade and unemployment. Hysteresis has only been given vague definitions there, most of the time not equivalent to one another. These definitions refer to two relatively sketchy ideas: (i) the "dependance of the system on its past" ; (ii) the permanent effects of transitory actions". This paper stresses that beyond the different phenomenologies where hysteresis may be found, the concept refers to a set of typical formal properties, defined in terms of the response of a system to an external action. The mathematical models of hysteresis recently developed clearly enlight these properties. We propose to distinguish between a weak form of hysteresis and a strong form, according to two criteria: (i) the type of remanence the system exhibits; (ii) the type of memory the system possesses. The strong form of hysteresis involves a non-trivial transition from heterogeneous micro-elements to a macro-aggregate. On the other hand, it is shown that it is not correct to assimilate the properties of systems with zero- or unit-root dynamics to hysteresis, as it is often done in the field of unemployment theory. For an hysteretic system, two opposite shocks in a row would not take the sytem back to its initial position, contrary to what happens with zero or unit-root processes. The persistence properties of zero-root processes derive from a kind of degeneracy of the dynamics which support it.

Keywords: hysteresis, multiple equilibria, unit roots, persistence
JEL: C60

De l'hystérésis

Résumé

La littérature économique récente a fait un usage fréquent du terme "hystérésis", par exemple dans les domaines du commerce international et du chômage. L'hystérésis n'y a reçu que des définitions vagues et non équivalentes entre elles. Ces définitions renvoient à deux idées relativement vagues: (i) la dépendance du système au passé ; (ii) l'effet permanent d'actions transitoires. Ce papier souligne qu'au delà des différentes phénoménologies dans lesquelles on peut trouver l'hysteresis, le concept renvoie à un ensemble de propriétés formelles typiques, définies en termes de la réponse d'un système à une action extérieure. Les modèles mathématiques de l'hysteresis développés récemment font clairement apparaître ces propriétés. Nous proposons de distinguer une forme faible d'une forme forte de l'hysteresis, selon deux critères : (i) le type de rémanence que le système exhibe, (ii) le type de mémoire qu'il possède. La forme forte d'hystérésis implique un passage non trivial d'éléments micro hétérogènes à un agrégat macro. Par ailleurs, on montre qu'il n'est pas correct d'assimiler les propriétés des systèmes à valeur propres nulles (ou unitaires) à de l'hysteresis, comme c'est souvent fait dans la littérature. Dans un système à hysteresis, deux chocs successifs opposés ne ramèneront pas le système à sa position initiale, contrairement aux systèmes à valeurs propres nulles. Les propriétés de persistence des systèmes à racine unitaire (ou à valeurs propres nulles) semblent dériver d'une sorte de dégénérescence de la dynamique.

Mots clés : hysteresis, équilibres multiples, racine unitaire, persistance
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Hysteresis: what it is and what it is not

The term hysteresis has occurred frequently in the economic literature in the recent years, primarily in the fields of employment and unemployment (Blanchard and Summers, 1986, 1988 ; Sachs, 1985, 1987) and international trade (Baldwin and Krugman, 1989; Dixit, 1989). The precision and general nature of definitions of this concept borrowed from physics vary greatly from one author to another, and their status is uncertain. For instance, Blanchard and Summers (1986) explained that they did not intend necessarily to abide by the definition that they gave in a footnote. Still more disturbing is the fact that the definitions do not, for the most part, coincide from one text to another.

Given this context, the use of the term "hysteresis" raises two problems. To begin with, the use of a term that is ill-defined, or not defined at all, is a source of difficulty when it comes to interpretation. While an author is free to use the term that corresponds to the concept he wishes to employ, he must nevertheless be able to provide a precise definition of that term; and the way in which that term is used must correspond unambiguously to that definition.

Second, the fact that the different definitions of hysteresis, from one author to another, or from one text to another, do not coincide can cause confusion simply by virtue of the fact that the same term is used for different purposes within the same discipline, in this case economics. We wish to underscore the idea that the concept of hysteresis refers to a set of formal properties, independently of the various phenomenologies within which they are liable to be encountered (magnetism, ferro-electricity, physical mechanics, various fields of economics, etc.). The present day problem regarding the uses of the concept of hysteresis in economics is that, in addition to the legitimate diversity of economic fields within which it might find application, the formal structures employed are fundamentally heterogeneous. Thus, the models of Blanchard and Summers (1986) and Baldwin and Krugman (1989), for example, are both described as exhibiting hysteresis, whereas they differ fundamentally in both structures and formal properties.

In fact, the use of the term "hysteresis" in the recent economic literature revolves around two relatively vague ideas: on the one hand, "*dependence on the path followed*", whereby the equilibrium state of a system depends on the transition towards equilibrium; on the other, the "*permanent effects of transitory actions*" in which a system retains the traces of past external influences on it even after those influences have ceased to apply. The vagueness of these characterizations has given rise to various derivations, as a result of which the term hysteresis has been applied to phenomena that are structurally very different. Consequently, there appears to be a need for clarification. Rather than starting a sterile controversy over vocabulary, this paper aims to identify and classify, according to their relative "richness", the different kinds of processes that can be related to some extent to these properties. For that purpose, reference will be made to formal representations of hysteretic models very recently developed by mathematicians (Krasnosel'skii and Pokrovskii, 1988 ; Mayergoyz, 1986, 1991). By highlighting the formal properties of hysteresis as they originally appear in physics, they allow to show, by way of comparison, how phenomena characterized by zero root dynamics (ZRD) have weaker properties.

This comparison allows to sketch a first taxonomy where "strong hysteresis" appears as a set of rich dynamical properties. While zero root processes should rather be qualified in terms of persistence. Eventually this paper would reach its target if it helped to reveal another meaningh to hysteresis, more faithful to its origins, overlooked by economists due to the focus on ZRD and carrying a set of formal properties that could be of great help in the analysis of economic facts.

The first section presents very briefly the constructive method through which strong hysteresis can be obtained. Starting from a "weak hysteresis operator" at the micro level, an aggregation procedure makes strong hysteresis emerge at the macro level. The second section is devoted to a presentation of the dynamical properties of ZRD. The first section establishes the comparison between strong hysteresis and ZRD and states their structural differences, while arguing that persistence properties come from a degeneracy of ZRD.

I. Two forms of Hysteresis

Recent formal approaches developed by mathematicians (Krasnosel'skii and Pokrovskii, 1989 ; Mayergoyz, 1986,1991) have sought to abstract hysteresis from the particularities pertaining to the different areas of science where it may be found, and consider it as a class of phenomena. A mathematical modelling of hysteresis requires the consideration of a system subject to an external action, i.e. an *input-output system*. Hysteresis is defined as a particular type of response of the system when one modifies the value of the input: *the system exhibits some remanence when the value of the input is modified and brought back to its initial position*. In what follows, such a change in the input value will be referred to as a "*loading-unloading*". Since the value of the output is altered after such a loading-unloading, the present state of the system depends on the history of the input. The richness of this history varies with the form of hysteresis. Two forms of hysteresis, namely weak and strong, are distinguished in the following.

I.1. The weak form of hysteresis

An initial illustration of these ideas may be provided in the generic form of the fold catastrophe[1], which represents a first step towards characterizing strong forms of hysteresis. An economic model with such a structure is found in Amable, Henry, Lordon and Topol (1991) and Cross (1991). The former is based on an extension of the model of Baldwin and Krugman (1989). A firm exports on a foreign market under the following conditions: there are two fixed costs, one to enter the foreign market and another one to remain in this market. Both costs are independent of production and the entry cost is greater than the maintenance cost. The volume of exports will depend on the value of the exchange rate. The maximization of expected profits in the presence of these sunk costs leads to the particular two-branch export function depicted in figure 1.

If initially out of the market, the firm will need a sufficiently high value of the exchange rate, α, to enter the market so that its operating profits cover the entry cost. When the firm is in the market, exports are a positive function of the exchange rate as shown by the upward sloping curve. When in the market, a fall of the exchange rate below α will not drive the firm out of the market since the maintenance cost is lower than the entry cost. The firm will exit when the exchange rate passes below ß. When the firm is out of the foreign market, exports are zero, as represented by the horizontal line Oα. When the exchange rate is in the range ß,α, either schedule can apply. The important point is that there is a difference between the entry and maintenance fixed costs that leads to an asymmetry in the exit and entry exchange rates: the exit exchange rate ß is lower than the entry exchange rate α.

Such a system possesses, albeit in a weak form, the properties referred to above:

i) The history of the system matters because of the local multiplicity of equilibria. For a value of the exchange rate between ß and α, there is an indeterminacy concerning the volume of exports. It may either be zero or a positive value. It is therefore necessary to know the history of the system in order to assess its position. However, the history boils down simply to the initial value of the exchange rate and the number of times the firm has changed positions as regards its presence in the foreign market.

ii) There exists a remanence effect. Consider a change in the value of the exchange rate from E_0 to E_1 and back (Figure 1). If initially out of the market, the firm will enter when the value of the exchange rate passes α, and will stay in the market when the exchange rate decreases back to E_0. Nevertheless, it must be noted that a change in the value of the exchange rate from E_2 to E_0 or E_2 to E_1 and back will bring the system back to its initial position, via another path.

iii) The remanence, if any, is independent of the magnitude of the change in the exchange rate. If the firm is initially out of the market, a change from E_0 to E_3 and back will produce the same remanence as a change from E_0 to E_1 and back.

I.2. The strong form of hysteresis

The crux of the mathematical formalisations[2] of hysteresis such as those of Krasnosel'skii and Pokrovskii (1989) or Mayergoyz (1986,1991) is an aggregation of a large number of heterogeneous elements, or "elementary hysteretic operators" in the terminology of Mayergoyz.

Amable, Henry, Lordon and Topol (1991) took as an elementary operator the firm whose exports behaviour was described in I.1., and built a model of strong hysteresis of foreign trade based on the aggregation of heterogeneous firms exporting on a foreign market. The heterogeneity means that firms will differ according to the values of their entry and exit exchange rates α and ß.

All firms face the same demand curve with a constant elasticity of demand ϵ and have the same production costs. Hence, when in the market, all firms charge the same price and export the same quantity. If E(t) is the exchange rate at time t, the aggregate export equation is:

$$X_A(t) = N(t)\ E(t)^{\epsilon} \tag{1}$$

and N(t) is the number of firms in the market at time t.

A geometric representation may be useful. There is a one-to-one correspondence between a firm and a point in the (β,α) plane, for $\alpha \geq \beta$ since each firm is characterized by particular values for its entry and exit exchange rates, a consequence of each firm having specific values for entry and maintenance fixed costs. The exchange rate is normalized so that no firm is in the foreign market at $E = 0$ whereas all firms are in the market at $E = E^*$. A triangle T defined by the 45° line and extremum values of $\alpha = 0$ and $\alpha = E^*$ may be considered (Figure 2).

It can be shown that the representative points of the firms which are In the market belongs to a domain S^I, while those which are Out of the market are in the domain S^O (Amable, Henry, Lordon and Topol, 1991). The interface L between these two non overlapping domains is a staircase line whose vertices, have β and α coordinates that correspond respectively to the sequence of the past minima and maxima of the exchange rate. An increase in the exchange rate will have the firm whose α is below the new exchange rate value enter the market, which brings about a new horizontal line to the staircase separation line L (Figure 3). A decrease in the exchange rate makes the firms whose β is above the new exchange rate value exit from the market, which gives a new vertical line to L (Figure 4).

The history which is stored by a hysteretic system, here the aggregate exports function, consists of the sequence of the past extrema of the exchange rate. As a matter of fact, an increase in the exchange rate over the past maxima that lie under the current maximum will wipe out the part of the staircase line corresponding to those past values (Figure 5). The same wiping-out property applies when the exchange rate is decreased below some past minima. Therefore, the history is a sequence of increasing minima and decreasing maxima of the exchange rate.

Knowing what the distribution of the firms either In or Out of the market is, according to the past changes in the value of the exchange rate, it is possible to determine the volume of aggregate exports. Since the firms which are Out of the market have no exports, the aggregate exports is the sum of the exports of the firms which are In the market, i. e. whose representative points belong to the domain S^I.

At time t, the aggregate exports $X_A(t)$ reads:

$$X_A(t) = \int\int_{S^I(t)} X(E(t);\alpha,\beta)\ d\alpha\ d\beta \tag{2}$$

where $X[E(t);\alpha,\beta]$ is the exports of the individual firm characterized by its critical exchange rates (α,β). The export function of each firm only depends on the current value of the exchange rate. On the other hand, the aggregate exports function depends on the sequence of the past extrema of the exchange rate. Since every firm in the market exports the same quantity, one may write the aggregate exports as:

$$X_A(t) = E(t)^{\epsilon} \int\int_{S^I(t)} d\alpha \, d\beta \tag{3}$$

Note that an econometrician would say the exchange rate elasticity of aggregate exports is now a time varying parameter. The properties of the total volume of exports obtained by aggregation of the exports of heterogenous firms go beyond the properties of the weak form of hysteresis. In general terms, the strong form of hysteresis is characterized by the following:

i) The current output of the system, in this particular case aggregate exports, depends on a much richer history than in the weak form of hysteresis. History now consists of the sequence of the extrema of the input - here the exchange rate - that have not been wiped out. The current volume of aggregate exports depends on past values of the exchange rate.

ii) A much wider class of "loading-unloading" produces a remanence effect. Actually, every "loading-unloading" which involves an increase in the value of the exchange rate over the last local maximum or a decrease below the last local minimum will produce a remanence.

iii) By contrast to the weak form of hysteresis, the remanence depends on the magnitude of the loading-unloading.

In that case, following Mayergoyz, we may say that a system is hysteretic if "*the input-output relationship is non-linear multibranch, the transitions from branch to branch occurring each time the input reaches an extremum*".

The relationship between aggregate exports and the exchange rate is represented by the curve deduced from the interface L shown in Figure 6. The aggregate volume of exports will depend on the history of past extrema of the exchange rates, and any increase of the exchange rate over the previous maximum followed by a return to its initial value will produce a remanence: the final volume of exports will exceed the initial one.

I.3. Micro- and macro-behaviour

The strong form of hysteresis presented above involves a non-trivial transition from heterogeneous micro-elements to a macro-aggregate. Indeed, the macrobehaviour is not a mere reproduction of the microbehaviour. This claim is illustrated by the comparison between the properties of the aggregate exports, i. e. the properties of the strong form of hysteresis, and the properties of the individual firm, i. e. the weak form of hysteresis. For instance, the micro-exports curve, depicted in Figure 1, contrasts with the macro-exports curve pictured in Figure 6.

In fact, hysteretic nonlinearities such as exposed above are based on:

a) The existence of elements possessing two stable equilibria for some range of the input value.

b) The aggregation of elements that are heterogeneous with respect to the input values within which there exist two stable equilibria.

The qualitative change in the micro-macro passage is a direct consequence of the aggregation over a set of heterogeneous firms.

Actually, a structural break may be associated to the hysteresis phenomenon. This structural break is elementary in the weak form, where it corresponds to the jump of the system between the two stable equilibria. In the case of the model of foreign trade, a firm is either In or Out of the market. Its reactions to changes in the value of the exchange rate will thus differ according to whether it is an exporter. The break is much more sophisticated in the strong form of hysteresis:

a) The relative shares of "In" or "Out" elements are involved. In the model of foreign trade of Amable, Henry, Lordon and Topol (1991), it is the (foreign) market structure ;

b) Therefore, the exchange rate/exports relationship is continually altered by changes in the exchange rate. Since the exports are a function of the number of elements which are "In", the relationship depends on the history of the exchange rate, or more precisely the values of its extrema. The parameters of the aggregate exports equation will be unstable. The macro elasticity of exports will be conditional on past values of the exchange rate.

II. Zero and unit roots: An improper use of the concept of hysteresis

The characteristic properties of hysteresis, namely "*dependence on the history of inputs*" and "*remanence*", have given rise to more or less wide divergences from their fundamental meaning. The term "hysteresis" is often mentioned in connection with dynamic models characterized by zero-eigenvalue for continuous time (Sachs, 1985, 1987 ; Van de Klundert and Van Schaik, 1990) or by unit-roots for discrete time (Blanchard and Summers, 1986). The properties of these models bear only an approximate and distant relation to those of hysteresis. The shifts of meaning and analogical deformations are described in what follows.

II.1. Deterministic multivariate systems and "path dependence"

Some economic models lead to a problem of equilibrium indeterminacy, within a linear differential system of the type :

$$\dot{Y} = A\ Y - Z \tag{4}$$

where Y is a vector of dimension n and Z is a vector of exogenous variables. **A** is not full rank, i.e. has some zero eigenvalues.

Sachs (1985, 1987) presents a model that leads to a three-equation system of this type. The components of Y are the rate of unemployment, the rate of inflation and the NAIRU. The first equation is a Phillips curve according to which the inflation rate, denoted P, varies with the gap between the current rate of unemployment, U, and the NAIRU, U^*. The second equation models the trade-off between unemployment and inflation, derived from the loss function of the government, say with an equal weight on both objectives. Sachs adds to these two standard equation an Error Correcting adjustment of the NAIRU to the current rate of unemployment. The addition of this last equation makes **A** exhibit a zero root. The system of Sachs is similar to :

$$\dot{P} = U^* - U \tag{5}$$

$$\dot{U}^* = g\ (U - U^*) \tag{6}$$

$$\dot{U} = P - U \tag{7}$$

The general result, (see Giavazzi and Wyplosz, 1985) with such systems, is that the steady state is no longer unique, because **A** is singular. If one notes its rank r, $r < n$, the resolution of **A** $Y = Z$, presents an indeterminacy whose order $n - r$ represents the dimension of the continuum sub-space of the equilibria. As noted by Giavazzi and Wyplosz (1985), this indeterminacy can be dispelled, and one can show that the equilibrium reached, selected from within the continuum, depends on the initial conditions and the adjustment parameters. In the degenerate case where **A** possesses no eigenvalues different from zero, the economy would not move at all from any given initial position. Applied to Sachs' model, the locus of equilibria would be a line ($U^* = P = U$). The equation of this line can be derived from the transformed system in the eigenvectors base. It is the first direction associated to the zero eigenvalue. Its projection in the (NAIRU,inflation) diagram is a line ($P = U^*$), as pictured in figure 7.

From any given initial state $E_0(U_0,P_0)$, the economy, if stable[3], would converge to a point on the line of equilibria, denoted E_1^s. The final steady state is entirely determined by the initial conditions and the **A** matrix. The selected equilibrium is determined by the first intersection between the continuum of equilibria and the trajectory, which depends on the adjustment speed parameters which determine its curvature (see figure 7 for various **A** matrices). But for any given **A**, the final steady state depends only on E_0, which means only initial conditions matter. What is represented as the "dependence of equilibrium on the approach trajectory" is therefore merely a by-product of a simple property of dependence on initial conditions.

Such dependence did not exist in the case of a single equilibrium. Indeed, a single-equilibrium dissipative system "loses the memory" of the initial conditions, since the stationary state, when stable, attracts all of the trajectories from any point in the phase space (Figure 8).

Another interpretation of the multiple equilibria property might have added to the confusion between zero root dynamics and hysteresis: the "*permanent effect of transitory shock*". Consider an initial equilibrium with zero rates of inflation and unemployment (natural or current): point O on the figure 9. Any exogenous transitory shock -which means a shock applied only once- on inflation or unemployment would instantaneously displace the economy to E_0 for instance. The adjustment dynamics of the system would then lead the economy to a final non-zero inflation and unemployment steady-state. The NAIRU would have changed in the process : the steady state would be a new one, which implies a new "natural" rate of unemployment. Had the **A** matrix been non-singular, a transitory exogenous shock would have left no long-run impact on the economy, which would have come back to the zero inflation and unemployment point. The NAIRU would have stayed constant, equal to zero.

This characteristic of the zero eigenvalue dynamics allows to speak of "*persistence*". Yet, this phenomenon has little in common with the type of memory kept in hysteretic systems.

II.2 Persistence: from a deterministic to a stochastic framework

One can always compute the final steady state E^s of a multivariate linear dynamic system corresponding to a given initial position E_0, even if the transition matrix **A** possesses at least one zero root. One can take the example of a simple bivariate autonomous system with detrended income and consumption. One assumes a steady-state growth path and a unit income-elasticity of consumption. The standard Error-Correction Model[4] is:

$$\begin{pmatrix} \dot{Y} \\ \dot{C} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ g & -g \end{pmatrix} \begin{pmatrix} Y \\ C \end{pmatrix} \qquad \textbf{(8)}$$

where Y is the income and C the consumption, $0 < g < 1$. The eigenvalues of **A** are 0 and -g. The locus of equilibria (the line: C = Y) acts as an "attractor"[5]: the system would converge from an initial E_0 to a new steady state E^S.

One may take up on the previous continuous time model, use it in a discrete time specification, add stochastic disturbances, say white-noise, and obtain a simple Vectorial AutoRegressive (VAR) model between consumption and income. Note that some autocorrelation for disturbances and changes in both variables would give rise to more complex algebra, but the core of the following results would stay unchanged. Denote u^1 and u^2 the white-noise components and write the derived system, which is the well-known Error Correction representation of a level VAR between I(1) series. A more general approach (Engle and Granger, 1987) would resort to series which require even higher order of integration. Omitting time t subscript, one obtains:

$$(1 - L)\, Y = u^1 \tag{9}$$

$$(1 - L)\, C = g\, L\, Y - g\, L\, C + u^2 \tag{10}$$

where $0 < g < 0$, and L is the lag operator.

The A matrix can now be polynomial in the lag operator, denoted A(L). Compute A for L = 1, to get the so-called "long-run" transition matrix. If A(1) had a zero eigenvalue of order n, the VAR would be expressed in first-difference. The zero eigenvalue is therefore consistent with a multivariate unit-root in the system[6]. When A(1) is full-rank, all components of X are stationary and there is no unit root left. Here A(1) is not full-rank but yet is not totally degenerated either. There is one obvious "cointegrating" vector in the VAR, denoted B[7], such that B'X gives a stationary, i.e. I(0), linear combination of income and consumption. Take (9) - (10) to see that the income-consumption difference is a stationary process. Within a n-variate model, r being the rank of A(1), there would be r independent B's, as many as the independent lines of A(1)[8].

The process is indeed entirely similar to its deterministic counterpart (see above). One can then infer that the space orthogonal to B[9] is the deterministic steady-state locus, the so-called "attractor". To speak in terms of long-run equilibria, one must yet stick to cases in which "*there is an extended period with no exogenous shocks*" (Engle and Granger, 1991). This is tantamount to speaking in terms of an "impulse-response" function. The process followed by consumption and income is such that:

$$Y_t = \sum_{i \leq t} u_i^1 \qquad\qquad C_t = Y_t + \frac{u_t^1 - u_t^2}{1 - g\, L}$$

Starting from an initial situation O(0,0), a once-and-for-all shock (u_0^1, u_0^2) would yield an evolution towards a new steady-state according to:

$$Y_t - 0 = u_0^1$$

$$C_t = Y_t + g^t\, (u_0^1 - u_0^2)$$

hence:

$$Y^s = u_0^1$$

$$C^s = Y^s$$

The equilibrium locus consistent with such a shock is unique and lies on the (C = Y) line. This final steady-state might also have been affected if one had accounted for a more complex dynamics such as with the general VARMA model (I - D(L)) (1 - L) X = E(L) U where the lags operators are such that the infinite MA expression exists. Once again, the algebra would be slightly more tedious, but the final result would be left unchanged : final income depends on the initial shock (and on the adjustment parameters) but final consumption is always equal to income. Shocks would persist on income and consumption but not on the consumption-income difference.

II.3. Another history: about the "history of shocks"

It seems consequently irrelevant to speak of "path-dependence" in the case of a zero eigenvalue (or unit-root) system, as it has rather to do with some "persistence" of a once-and-for-all shock, whether in a deterministic continous or stochastic discrete framework. The latter simply boils down to its deterministic counterpart as the transient dynamics is not stochastic, but linear with constant coefficients.

Nevertheless, one can conceive an actual economic system in which at each time t, a new shock (iid to make it simpler) occurs, in order not to consider only the impulse-response function. The unit-root system is often said to keep a memory of the "history of shocks". It is possible to show that this story is not specific to unit-root systems. One has to come back to properties expressed in terms of "*persistence*" -also called "*long*" memory in the time-series jargon- to distinguish between dynamic properties. But one cannot stick to convergence in terms of the mean, the variance must also be considered, or the confidence intervals around the mean expected values.

The example of a bivariate discrete process subject to a shock iid U, gaussian with diagonal covariance matrix σI , i.e. a sequence of shocks $(u_0, u_1, \ldots, u_t, \ldots)$, is taken. Three subcases must be considered, depending on the rank of A(1). The components of the vector X at time T could be rewritten as:

i) $X_i = \sum_{t \leq T} a_i^t L^t u_i$ if **A** is full rank

ii) $(1 - L)\, X_i = \sum_{t \leq T} a_i^t L^t u_i$ if **A** has only zero roots

iii) $(1 - L) X_1 = \sum_{t \leq T} a_1^t L^t u_1$ and $X_2 = X_1 + v_2$ if A is of rank 1 and (1,-1) is a cointegrating vector (v_2 is I(0)).

For these three cases, X always depends on the "history of shocks". The discriminating point is not this dependence but whether the shocks are perfectly cumulated or not. The unit-root process possesses a zero mean but an ever increasing variance. For the above three cases, one obtains respectively:

i) a constant variance on both components of X
ii) an increasing variance in every direction
iii) an increasing variance for one of the component but a constant one for X_2 - X_1.

Figures 10 and 11 illustrate these properties. The hatched areas depicts the confidence intervals at time t, either increasingly larger or bounded as t increases.

II.4 Remanence or persistence ?

For a system possessing unit-roots, shocks would cumulate forever without progressively vanishing. Indeed, Blanchard and Summers (1986) identify their particular brand of "hysteresis" with the existence of a unit root in their univariate system. Yet, to keep such a perfect memory of shocks is not sufficient to justify the use of the term hysteresis.

There are two main reasons to this point; they both deal with the two characteristics mentioned by Mayergoyz (1991), namely the "non-linear" and "multibranch" aspects of the hysteresis dynamics.

If one wanted at all costs to establish a link between zero eigenvalue dynamics and hysteresis, one would need to take the comparison to its conclusion. Indeed a first shock followed by a second one of the same intensity in the opposite direction, takes the univariate system back to its initial level, whatever the intensity of the shock. On the contrary, a system with hysteresis would exhibit remanence subject to this kind of impulse[10]. There lies a major difference between persistence and hysteresis. The response to impulses is a linear function, which is not the case for a system exhibiting hysteresis[11]. The same remark is not restricted to univariate systems. Indeed, one may state the following proposition:

Proposition 1

Consider a dynamical sytem such as that of equation (4), where Rk A = r < n, the system being in an equilibrium position. A shock on the system followed by a second one of the same intensity in the opposite direction will bring the system back to its initial position.

Proof: see Appendix 1.a.

Proposition 2

Consider a dynamical sytem such as that of equation (4), where Rk A = r < n, the system being in an equilibrium position. A shock on the constant Z followed by a second one of the same intensity in the opposite direction will bring the system back to its initial position.

Proof: see Appendix 1.b.

Therefore, one may state that systems with zero eigenvalues or unit roots cannot exhibit remanence effects.

Actually, the theoritical gap between hysteresis and zero-root dynamics has empirical implications too. Simulations of an hysteretic system[12] prove that there is not even an observational equivalence between a unit root system and a system with strong hysteresis. In the case of a white noise input, most of the outputs have no unit root, but with a random walk input, the output does not exhibit a stable pattern in most cases. Non stationary econometrics based on cointegration would fail to capture the parameters of a strong hysteresis data generating process.

But in our view, the problem lies in the very principle of equating a shock to the system with a parametric "loading-unloading". As pointed out in part I, one can only meaningfully speak of hysteresis in the case of a system for which one can clearly distinguish a state variable, or several state variables, and an input, i.e. a control parameter, representative of an external action upon the system. Going back to the illuminating image of the potential curves suggested by Blanchard and Summers (1988), one can show - albeit contrary to the authors' intentions - that an exogenous shock to the system on the one hand and a parametric "loading-unloading" of the input on the other hand imply very different types of problems.

III. Shock to the system versus parametric loading-unloading

First and as a metaphor, it is actually practical to envisage the stationary state of a dynamics as the minimum of a certain potential[13]. The "standard" case of a linear system with a single equilibrium corresponds to a single-bottom potential curve (Figure 12). Blanchard and Summers (1988) propose that hysteretic phenomena be conceived in terms of configurations in which the potential curve presents not one but several, or even an infinity, of local minima. The first of these two cases (Figure 13) in fact corresponds to a non-linear model with a (discrete and finite) multiplicity of equilibria. The second case, which is an extreme case of the former (figure 14), corresponds to a "flat bottom" potential curve, in other words to the existence of a continuum of equilibria characteristic of a zero eigenvalue dynamics.

In the first case (Figure 13), the change of equilibrium requires a sufficiently substantial shock to allow it to breach the "potential barrier" separating the two attractors. In other words, the system must be sufficiently distanced from E to allow it to pass into the domain of attraction of F. In the case of a "flat bottom" potential curve characteristic of a zero eigenvalue dynamics (Figure 14), any shock will modify the equilibrium of the system. But in either case, the external action takes the form of a direct shock to the system, or more accurately to its state variables, whereas the very structure of the dynamic, the shape of its potential curve, is left invariant.

III.1. Hysteresis and structural change in the dynamics

A very different kind of phenomenon is involved in the weak form of hysteresis associated with the fold catastrophe for instance. Here, the action exerted from the outside consists in modifying the value of a parameter, akin to a change in the value of the input for the input/output system considered in part I, which entails a structural change in the "conflict between attractors" (Thom, 1972; Haken, 1977). If the system does move from one equilibrium to another, it is no longer as the result of an external impulse enabling it to cross a potential barrier (which may be zero in the case of a zero eigenvalue dynamics). It is because a structural modification alters the shape of the potential curve and eliminates one of its minima, leaving the system no option but to move towards the remaining attractor (Figures 15, 16, 17). Bringing the parameter (or the input) back to its initial value (Figure 18) undoubtedly produces a structural deformation that is the exact reverse of the potential curve, but the system remains the captive of attractor F (maximum delay convention) such that the "loading-unloading" may leave a remanent effect.

This crucial difference between these formal structures should shed some light upon the gap between formalization and interpretations with respect to the idea of structural change. We would like to emphasize that the only way for structural change to make sense formally speaking is intimately linked with modifications of the parameters of the model, in this case the parameter of the dynamical system. Actually it is quite obvious that only a change in a parameter may cause a modification of the formal characteristics of a dynamics, mainly the number and stability of the equilibria. This is quite visible in the potential curve metaphor where a change in a parameter may imply the vanishing of an equilibrium. More generally hysteresis, as already seen, comes from a micro structure where the variation of a control parameter (the input) leads to the disappearance of an equilibrium and the jump towards a lower or upper one. Such an event refers exactly to what we call "a structural change in the formal characteristics of the dynamics".

On the contrary effects manifested by zero-root dynamics have nothing to do with such modifications in the formal characteristics of the dynamics. They arise because of the special nature of the equilibrium locus: a continuum. The latter remains unchanged during the shock-and-persistence phenomenon described in part II. However a confusion may come from the fact that some of the state variables of these zero-root models are sometimes interpreted as "structural" economic variables. This is particularly the case in Sachs's model where the NAIRU, taken as a "structural characteristic" of the economy, is endogenous.

There, the structural change is purely semantic and does not refer at all to the characteristics of the dynamics which are fundamentally invariant. Persistence effects thus come from a shock on the state variables (on "the economy") which affects neither the set of equilibria nor their stability properties. The economic magnitude to which pertains an interpretation in terms of "structural change" is a state variable in models with zero-root dynamics[14]. It must be an external parameter in the case of hysteresis, even in its weak form. Although Sachs's story on the NAIRU may be found economically relevant[15], it is totally neutral with respect to the formal characteristics of the dynamics. By contrast, hysteresis phenomena are crucially linked with the existence of structural change of the dynamics itself.

III.2 Zero-root versus hysteresis: global versus local structural instability

More generally and to overcome the limits of the potential curve metaphor, the difference between zero-root and hysteretic dynamics refers in last resort to the distinction between (respectively) global and local structural instability. A dynamical system is structurally unstable as soon as it possesses a stationary state in which at least one eigenvalue has a zero real part (Hirsch and Smale, 1974). It is globally structurally unstable if this property holds whatever the values of the parameters. As a matter of fact, the essential formal property behind zero-root dynamics is global structural instability. From this point of view, the model of Sachs (1985, 1987) is quite equivalent to, for instance, Goodwin's famous growth cycle (1967), which, incidentally, exhibits the same persistence effects. The model of Sachs possesses a zero-root, the model of Goodwin has two zero real part eigenvalues. The former has a linear continuum of equilibria the latter exhibits a continuous family of closed orbits in a wage share-employment rate space. The persistence effects are exactly the same : every shock on the system makes it change its orbit and select a new one in the continuum, without ever coming back to the initial one. Goodwin had discovered zero-root (pseudo) "hysteresis" without being aware (figure 19).

The common formal characteristic of both Sachs' and Goodwin's models from which the persistence property stems unfortunately leads to their mathematical rejection. Indeed, it is well known that the global structural instability of a model is quite unsatisfactory. The weakest perturbation of the functional form of its equations makes its dynamical properties drastically changed : Goodwin's closed orbits collapse into a single stationary state ; Sachs' continuum also disappears into an ordinary equilibrium. This lack of robustness is obviously a major drawback for a model. Incidentally, this crucial objection, formulated long ago against Goodwin's model, should apply to all the zero-root or unit-root models.

On the contrary, local structural instability is a much more attractive property. The eigenvalue's real part vanishes only for some particular values of the parameters[16]. Far from these critical values the dynamics is structurally stable. A modification in a parameter value is therefore quite a remarkable event when crossing such thresholds : it implies a structural change in the formal characteristics of the dynamics[17] such as, in the weak form of hysteresis, the vanishing of an equilibrium and the sudden jump toward another one.

Conclusion

The critique presented here is essentially methodological. It makes no claim to discuss the soundness of the specifically economic mechanisms proposed for the representation of unemployment. The aim of the paper was to identify and qualify the different phenomena that can be found under the label of hysteresis. The method that was suggested here, in an attempt to clarify the concept of hysteresis, consists in identifying formal properties independently of the various phenomenologies which they may be used to convey. Apart from the fact that this strategy appears generally to be the least detrimental for the purpose of effecting an interdisciplinary transfer without the risk of falling into the trap of approximate analogy, it has in this instance enabled us to identify a typology of the various forms of hysteresis, and to suggest that the zero eigenvalue dynamics (or unit root dynamics) commonly investigated by economists do not qualify for such a description. Persistence appears to come from a kind of degeneracy of the dynamics which support it. It fails to capture the idea of formal structural change and at the same time must deal with serious mathematical objections. It should be added that, in thus imprudently appropriating the concept of hysteresis, certain economists have neglected to examine all of the implications of their action, and above all to explore and appreciate its full depth. In so doing, they have overlooked a set of properties that are a good deal richer than those of simple persistence systems. Yet those properties could have broad applications in the description of economic facts.

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Appendix 1.a

Proof of proposition 1. Two shocks on the system of the same magnitude in opposite directions bring the system back to its initial position.

We consider the following dynamical system:

$$\dot{Y} = AY - Z \qquad \textbf{(A-1)}$$

Y is a (n x 1) vector of endogenous variables, Z a (n x 1) vector of exogenous variables and A is a singular (n x n) matrix, Rk A = r < n. To keep things simple, we assume that it is possible to write A as:

$$A = V \Lambda V^{-1} \qquad \textbf{(A-2)}$$

where Λ is a diagonal matrix containing the eigenvalues of A. A more general case would involve resorting to Jordan-form matrices.

The matrix is partitioned so that the d = n - r zero eigenvalues appear first.

$$\Lambda = \begin{pmatrix} 0 & | & 0 \\ 0 & | & \Lambda_r \end{pmatrix}$$

Λ_r is a (r x r) diagonal submatrix containing the (negative) non-zero eigenvalues of A.

After the transformation $X = V^{-1}Y$, the dynamical system is:

$$\dot{X} = \Lambda X - V^{-1}Z \qquad \textbf{(A-3)}$$

The steady state is thus defined as:

$$X^* = \Lambda^+ V^{-1}Z \qquad \textbf{(A-4)}$$

where Λ^+ is the generalized inverse of Λ . The existence of a solution requires that the first d elements of $V^{-1}Z$ be zero. Indeed, with A singular, a solution to $AY = Z$ exists only when Z is restricted to the subspace spanned the column-vectors of A. The diagonalized system is thus:

$$\dot{X} = \Lambda(X - X^*) \qquad \textbf{(A-5)}$$

Transforming the system back and taking a particular solution allows to express Y(t):

$$Y(t) = V\Lambda^{+}V^{-1}Z + Ve^{\Lambda t}V^{-1}(Y(0) - V\Lambda^{+}V^{-1}Z) \tag{A-6}$$

The function $e^{\Lambda t}$ becomes asymptotically the matrix E such that:

$$E = \begin{pmatrix} I_d & | & 0 \\ 0 & | & 0 \end{pmatrix}$$

and I_d is a (d x d) identity submatrix. The steady state solution is thus:

$$Y^{*} = V\Lambda^{+}V^{-1}Z + VEV^{-1}(Y(0) - V\Lambda^{+}V^{-1}Z) \tag{A-7}$$

which can also be written as:

$$Y^{*} = V\left[(I-E)\Lambda^{+}\right]V^{-1}Z + VEV^{-1}Y(0) \tag{A-8}$$

We suppose now that the system is initially at rest in Y_1^{*}, defined as:

$$Y_1^{*} = V\left[(I-E)\Lambda^{+}\right]V^{-1}Z + VEV^{-1}Y(0) \tag{A-9}$$

A shock is applied to the system, which now finds itself instantaneously out of equilibrium in Y_2, defined as:

$$Y_2 = Y_1^{*} + \Theta \tag{A-10}$$

where Θ is a (n x 1) vector of shocks. The new equilibrium corresponding to this new initial condition is:

$$Y_2^{*} = V\left[(I-E)\Lambda^{+}\right]V^{-1}Z + VEV^{-1}Y_2 \tag{A-11}$$

or:

$$Y_2^{*} = V\left[(I-E)\Lambda^{+}\right]V^{-1}Z + VEV^{-1}\left[V(I-E)\Lambda^{+}V^{-1}Z + VEV^{-1}Y(0) + \Theta\right]$$

since E (I - E) = 0, one has:

$$Y_2^{*} = V\left[(I-E)\Lambda^{+}\right]V^{-1}Z + VEV^{-1}Y(0) + VEV^{-1}\Theta \tag{A-12}$$

Applying the same shock in reverse displaces the system in Y_3, with:

$$Y_3 = Y_2^{*} - \Theta \tag{A-13}$$

the system will thus reach a new equilibrium:

$$Y_3^* = V\left[(I-E)\Lambda^+\right]V^{-1}Z + VEV^{-1}Y_3 \tag{A-14}$$

or:

$$Y_3^* = V\left[(I-E)\Lambda^+\right]V^{-1}Z + VEV^{-1}\left[V(I-E)\Lambda^+V^{-1}Z + VEV^{-1}Y(0) + VEV^{-1}\Theta - \Theta\right]$$

which can be written as:

$$Y_3^* = V\left[(I-E)\Lambda^+\right]V^{-1}Z + VEV^{-1}Y(0) = Y_1^* \tag{A-15}$$

Therefore, two shocks of opposite signs bring the system back to its initial position, there is no remanence.

Appendix 1.b

Proof of proposition 2. Two shocks on the constant of the same magnitude in opposite directions bring the system back to its initial position.

A shock is now applied to the (vector of) constants. We suppose that the system is at rest at its equilibrium Y_1^*:

$$Y_1^* = V\left[(I-E)\Lambda^+\right]V^{-1}Z + VEV^{-1}Y(0) \tag{A-16}$$

A shock Θ is applied to Z. The system finds itself instantaneously out of equilibrium at Y_1^*, while a new equilibrium subspace is defined, after the appropriate transformation by:

$$X^* = \Lambda^+V^{-1}Z' \tag{A-17}$$

where $Z' = Z + \Theta$. The system will reach a new steady state Y_2^*:

$$Y_2^* = V\left[(I-E)\Lambda^+\right]V^{-1}Z' + VEV^{-1}Y(0) \tag{A-18}$$

The same shock is now applied in reverse. The new equilibrium subspace is the same as before the shock. The new steady state for the system is Y_3^*:

$$Y_3^* = V\left[(I-E)\Lambda^+\right]V^{-1}Z + VEV^{-1}Y(0) = Y_1^* \tag{A-18}$$

Therefore, two shocks of the same magnitude in opposite direction applied to the constant bring the system back to its initial position. There is no remanence.

Notes

1. The fold catastrophe deals with the multiplicity of equilibria according to the value of a parameter. In what follows, it is supposed that the value of the parameter can be controlled, so that the latter is assimilated to an input.

2. Mathematical approaches aim at reproducing hysteresis loops independently of phenomenology. On the other hand, there exist physical formalisations of hysteresis based on a particular phenomenology. For instance, Maugin (1990) constructed a phenomenological model of magnetic hysteresis which is not based on the procedure of aggregation of heterogeneous elements described here.

3. Which requires negative real part eigenvalues

4. Note that Davidson, Hendry, Sbra and Yeo (1978) never thought of speaking of an "Hysteresis Model of Consumption".

5. This is also the word used in a stochastic framework, as in Engle and Granger's textbook (1991), the introduction of which is subtitled "Attractors".

6. Blanchard and Summers' (1986) univariate model is indeed a degenerate case of this type. In the case of a once-and-for-all shock, each shock gives immediately a new steady state.

7. Take the second line of A(1) and get B = (1,-1).

8. See Engle and Granger (1987) and Johansen (1988) for the general discussion.

9. Characterized by the equation B'X = 0.

10. For an hysteretic system, one must consider a loading (a variation in the input) followed by an unloading (a change of the same magnitude in the opposite direction).

11. This misinterpretation might have stemmed from the famous "hysteresis (closed) loop" graph, which is yet only a response to a very particular case of "loading-unloading".

12. The simulations, discussed in in Amable, Henry, Lordon and Topol, were performed for 100 sequences of 100 observations. We used Dickey-Fuller tests with non-zero mean for univariate results, Engle-Granger and Johansen tests - without deterministic trend- for bivariate results.

13. The idea of a potential is not merely a metaphor but absolutely rigourous for single-dimensional dynamics, which can always be considered as derived from a gradient.

14. the NAIRU in Sachs (1988).

15. Or may not be found relevant at all, but this is not the point here.

16. Local structural instability means that the critical parameters are located in a set of measure zero in the parameter space. This set is referred to as the bifurcation state.

17. The flows before and after the crossing of the bifurcation set are no longer topologically equivalent.

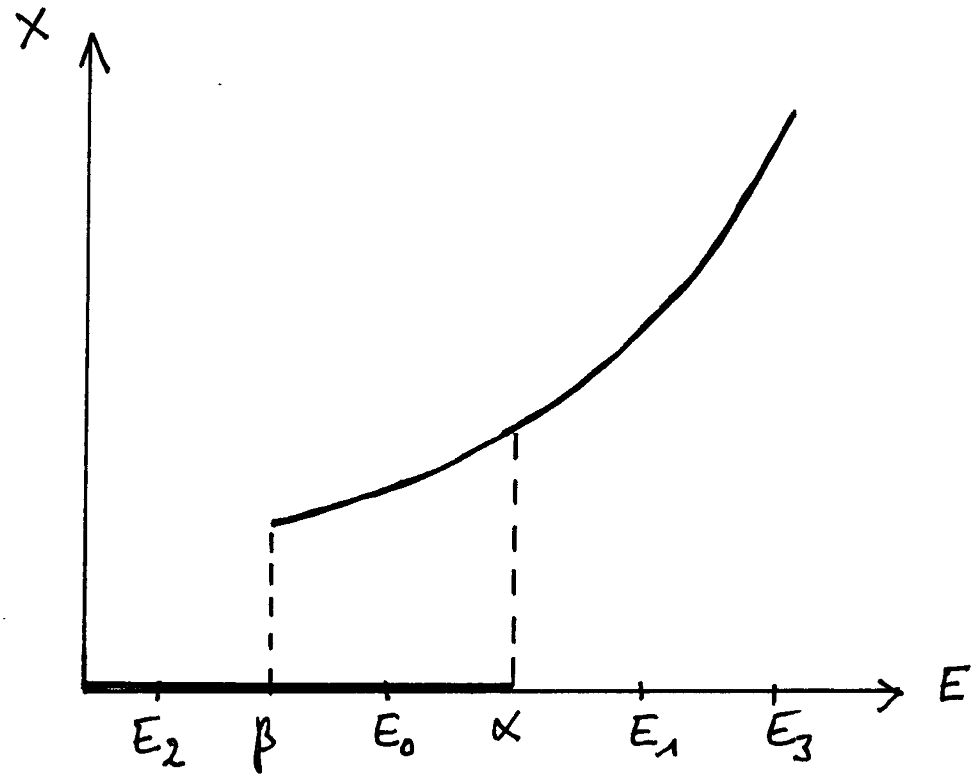
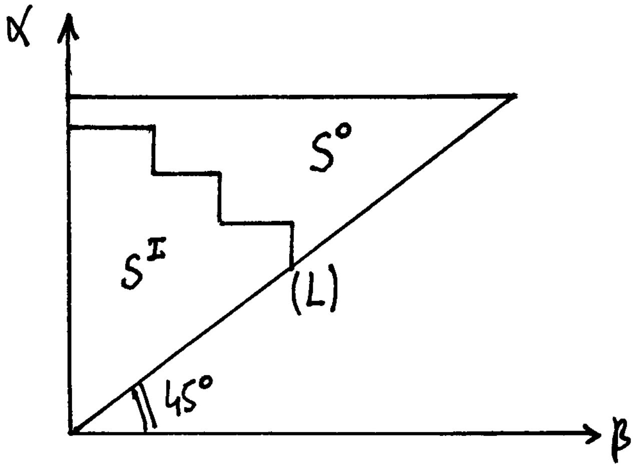


FIGURE 1



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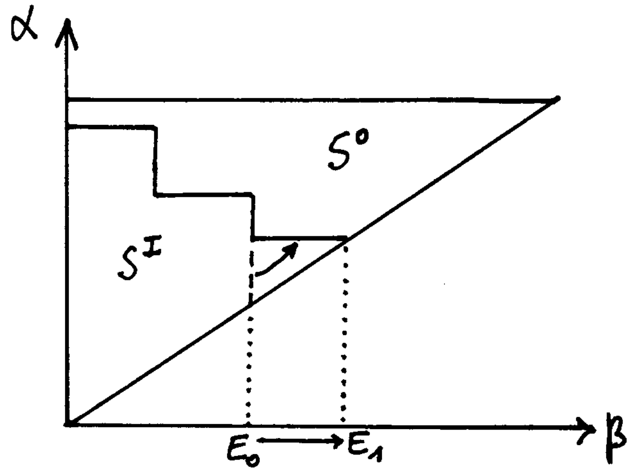

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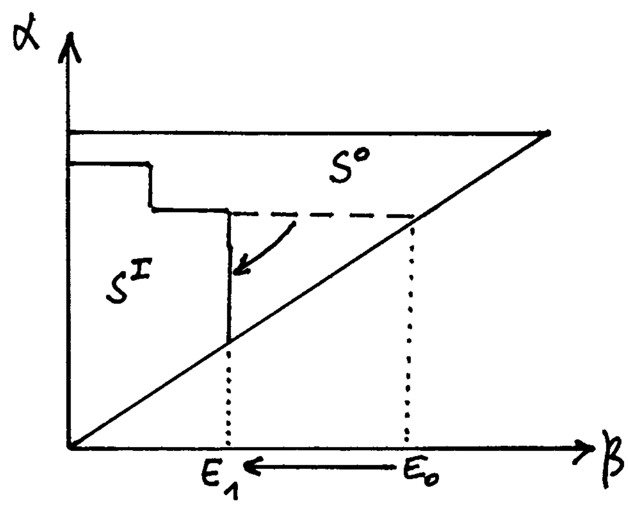

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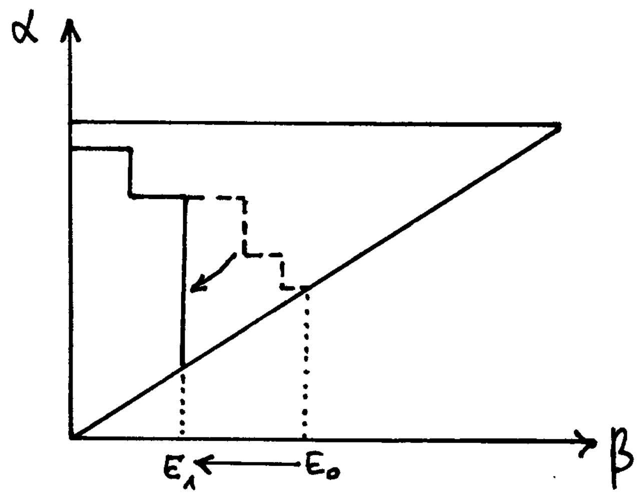

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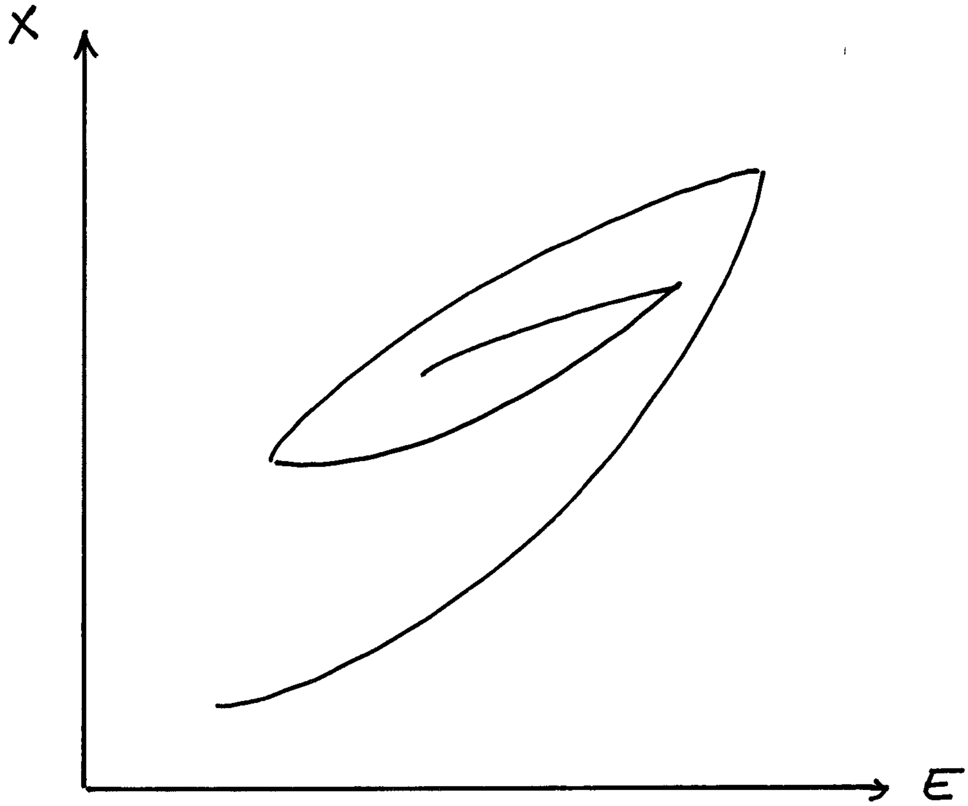

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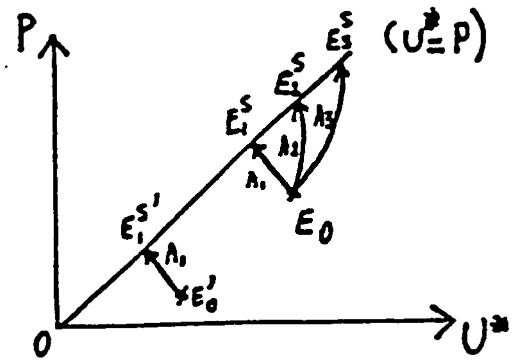

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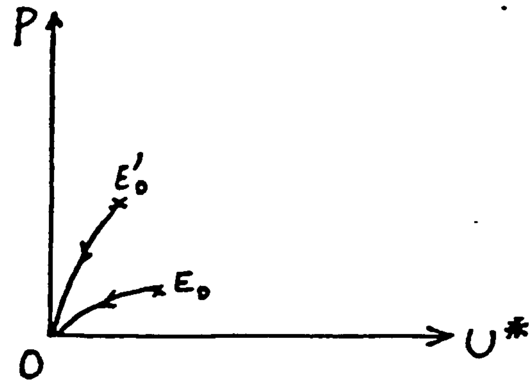

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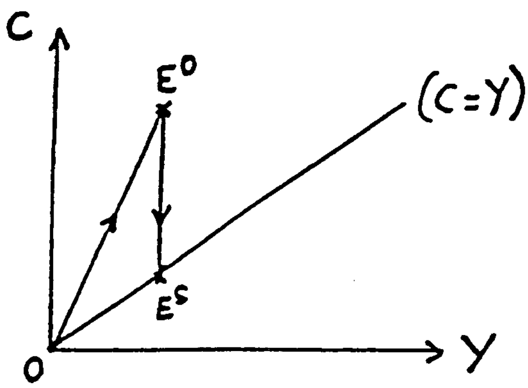

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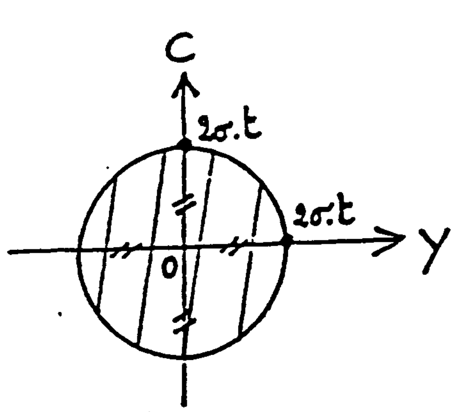
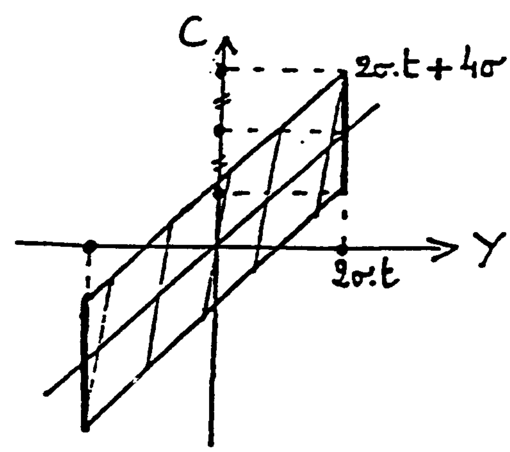


FIGURE 10



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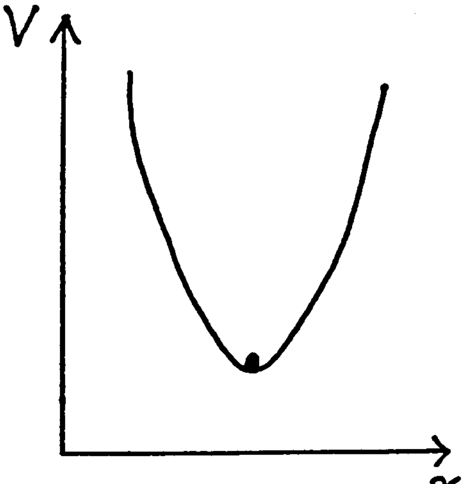


FIGURE 12

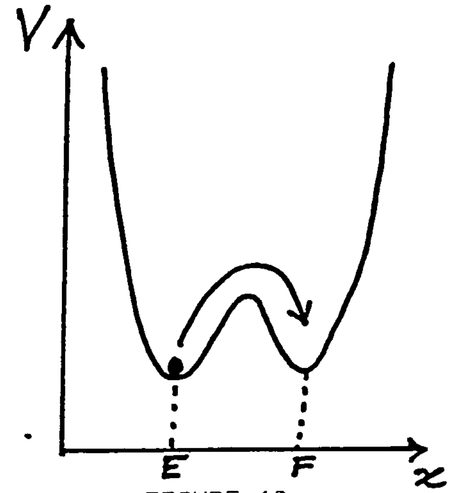
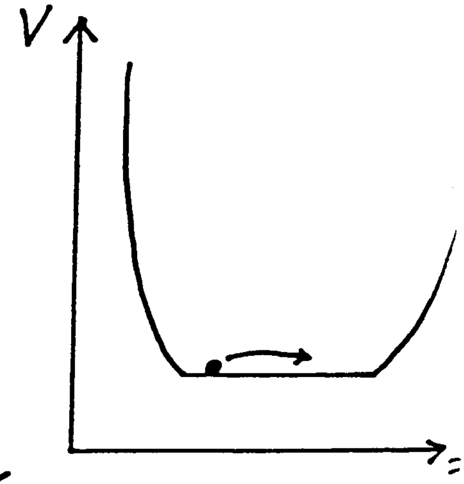


FIGURE 13



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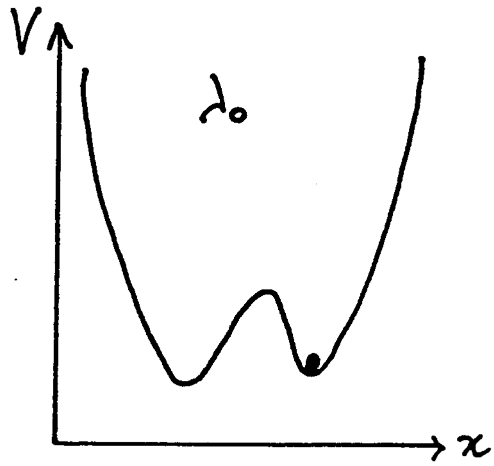

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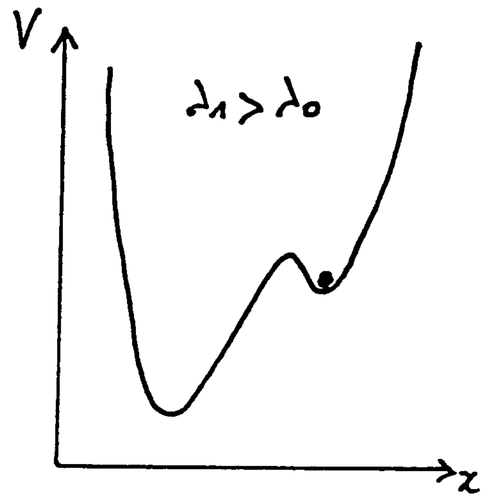

FIGURE 16

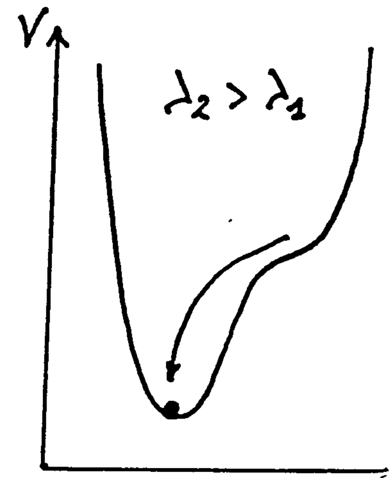

FIGURE 17

FIGURE 18

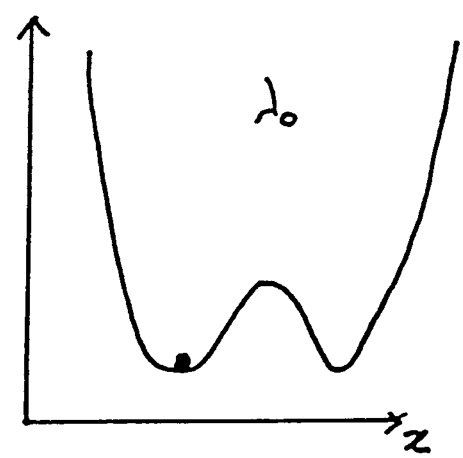

FIGURE 19