INCOMPLETE FINANCIAL MARKETS AND INDETERMINACY

OF COMPETITIVE EQUILIBRIUM

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A B S T R A C T

This paper surveys and appraises recent results concerning the analysis of the problem of real indeterminacy arising from incomplete (and, even more generally, otherwise imperfect) financial markets. An appendix presents the intuitive "counting equations and variables" argument underlying the basic substantive results.

Journal of Economic Literature : 021, 024

Key Words : - Incomplete
- Financial Markets
- Indeterminacy
- Competitive Equilibrium

MARCHES FINANCIERS INCOMPLETS ET INDETERMINATION DE L'EQUILIBRE CONCURRENTIEL

R E S U M E

Cet article couvre et évalue les résultats récents concernant l'analyse du problème de l'indétermination réelle entraînée par les marchés financiers incomplets (et, plus généralement, même imparfaits par ailleurs). Un appendice présente l'argument intuitif en termes "du nombre d'équations et de variables" qui soutient les résultats intuitifs de base.

Journal of Economic Literature : 021, 024

Mots clefs : - Incomplets
- Marchés financiers
- Indétermination
- Equilibre concurrentiel

Incomplete Financial Markets and Indeterminacy of Competitive Equilibrium[#] [**]

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I. Introduction

The general conception of this line of inquiry is to broaden the canonical Walrasian or competitive equilibrium paradigm -- a la Arrow-Debreu -- to encompass (with regard to the economy's financial sector) **richer institutional structure** and **various market failures**. I believe that this is a very important undertaking for (somewhat) generalist-type theorists like myself: The Walrasian tradition is simply much too fundamental to be left to (purely) mathematical-type theorists with their excessive concern about existence in ever more abstract settings, or to (impurely?) macro- or finance-type theorists with their excessive reliance on non-robust or overly parametric examples. Be that as it may, the range of specific developments thus far has been quite modest, concentrating on inside (i.e., private) financial transactions within incomplete financial markets (using Arrow's famous reformulation of complete contingent goods markets as the benchmark), while maintaining the simplifications of perfect information, price-taking behavior, ...

As one might have predicted, this research has focused on the three classical issues in general equilibrium theory, existence, optimality and uniqueness or, better, determinacy. I will not have much to say about either existence or optimality -- the first because I view it as primarily a technical issue (and, unsurprisingly, one which has received an inordinate amount of attention), the second because I can't claim to be an expert on its intricacies. Fortunately, an excellent discussion of recent results on both problems can be found in John Geanakoplos' introduction to the special issue of the **JME** devoted to "Incomplete Markets" (Geanakoplos (1990)).

What I will expand on is the issue of determinacy, since I consider it to involve the most striking -- as well as the most troublesome -- property of these models: Incomplete markets typically lead to significant price or **nominal** indeterminacy -- that is, over and above that analogous to choosing a numeraire in the standard Walrasian model -- which also naturally translates into substantial allocation or **real** indeterminacy.

While my primary objective is to present an overview of my own and others' past work on the analysis of indeterminacy, I also have two other important objectives in writing this paper. First, specifically, in the Appendix I attempt to explain -- at a fairly informal level -- what is essential in generating nominal as well as real indeterminacy when there are incomplete financial markets. Second, more generally, throughout the paper I purposely attempt to emphasize my own considered opinions about the problem itself, which can be summarized in the following way:

- Nominal indeterminacy per se presents a severe practical hurdle for the rational expectations hypothesis. In short, is it plausible to maintain that households are capable of concentrating their beliefs (correctly) on one among a surfeit of possible market outcomes?

- Except in one very special situation -- where all yields from financial instruments depend linearly homogeneously on future spot goods prices -- competitive equilibrium with incomplete or, even more generally, otherwise imperfect financial markets exhibits pervasive real indeterminacy. Thus, in particular, this phenomenon does not depend on some very special feature of the means by which financial transactions take place.
- The degree of indeterminacy -- nominal or real -- depends on which financial parameters are treated as endogenous or **variable** (as opposed to exogenous or **fixed**). This means, taking a broad view, that the problems associated with indeterminacy will only be mitigated (and not eliminated) by elaborating the structure of the institutions and the behavior of the organizations (public or private) which constitute the financial sector.

Finally, at the outset I emphasize that -- following most of the literature in this area -- I will present the basic substantive results in the least conceptually complicated context possible (though then pointing out where important simplifications have been and should be examined further). Moreover, since this paper is in the nature of a survey, in the text itself I will concentrate on results rather than proofs (though most of these, as the Appendix tries to indicate, are pretty simple in conception, if also pretty complex in execution).

II. The Leading Case

The basic model is essentially that described in Balasko and Cass (1989) or Geanakoplos and Mas-Collel (1989). There are C types of physical commodities (labelled by the superscript $c = 1,2, \ldots ,C$, and referred to as **goods**), and I types of credit or financial instruments (labelled by the superscript $i = 1,2, \ldots ,I$, and referred to as **bonds**). Both goods and bonds are traded on a spot market today, while only goods will be traded on a spot market in one of S possible states of the world tomorrow (these markets are labelled by the superscript $s = 1,2, \ldots ,S$, so that $s = 0$ represents today and $s > 0$ the possible states tomorrow, and are referred to as **spots**). Thus, altogether there are $G = (S+1)C$ goods, whose quantities and (spot) prices are represented by the vectors

$$x = (x^0,\cdots,x^s,\cdots,x^S) \text{ (with } x^s = (x^{s,1},\cdots,x^{s,c},\cdots,x^{s,C}))$$

and $$p = (p^0,\cdots,p^s,\cdots,p^S) \text{ (with } p^s = (p^{s,1},\cdots,p^{s,c},\cdots,p^{s,C})),$$

respectively. The quantities and prices of bonds are represented by the vectors

$$b = (b^1,\cdots,b^i,\cdots,b^I)$$

and $$q = (q^1,\cdots,q^i,\cdots,q^I),$$

respectively. [Note: It will be convenient, for example, in representing dollar values of spot market transactions, to treat every price or price-like (say, for instance, marginal utility) vector as a row. Otherwise I maintain the standard convention.] The typical bond, which costs q^i dollars at spot $s = 0$, promises to return a yield of $y^{s,i}$ dollars at spot $s > 0$. Let

$$Y = \begin{bmatrix} y^{1,1} & & & & \\ & \ddots & & & \\ & & y^{s,i} & & \\ & & & \ddots & \\ & & & & y^{S,I} \end{bmatrix} = \begin{bmatrix} y^1 \\ \vdots \\ y^s \\ \vdots \\ y^S \end{bmatrix}$$

$= (S{\times}I)$-dimensional matrix of bond yields.

Since, looking ahead, households are indifferent between having access to the whole array of bonds, or just a maximally linearly independent subset, there is no loss of generality in assuming that

A1. Rank $Y = I$, **no redundancy**

which implies that $I \leq S$. What gives the model its special character is assuming that, in fact,

A2. $0 < I < S$. **incomplete markets**

It will be convenient to let $D = S - I$, the **deficiency** in the bond market.

Finally, there are H households (labelled by the subscript $h = 1,2, \ldots ,H$) who are specified by (i) consumption sets $X_h = \mathbb{R}^G_{++}$, (ii) utility functions $u_h : X_h \to \mathbb{R}$ and (iii) goods endowments $e_h \in X_h$. As in most of the literature on "smooth economies", I will assume throughout that

A3. u_h is C^2 (i.e., twice continuously differentiable), differentiably strictly increasing (i.e., $Du_h(x_h) \gg 0$) and differentiably strictly quasi-concave (i.e, $\Delta x \neq 0$ and $Du_h(x_h)\Delta x = 0 \rightarrow \Delta x^T D^2u_h(x_h)\Delta x < 0$), and has indifference surfaces closed in X_h.

Let

$P = \{p \in \mathbb{R}^G_{++}\}$
= set of possible (no-free-lunch) spot goods prices,

$Q = \{q \in \mathbb{R}^I$: there is no $b \in \mathbb{R}^I$ s.t. $\begin{bmatrix} -q \\ Y \end{bmatrix} b > 0\}$
= set of possible (no-financial-arbitrage) bond prices,

$\underline{Y} = \{Y \in \mathbb{R}^{SI}$: rank $Y = I\}$
= set of possible bond yields

and $E = \{e = (e_1, \cdots, e_h, \cdots, e_H) \in (\mathbb{R}^G_{++})^H\}$
= set of possible goods endowments (as well as allocations).

Then, given $(Y,e) \in \underline{Y} \times E$, $(p,q) \in P \times Q$ is a competitive equilibrium with incomplete financial markets, referred to hereafter as a **financial equilibrium**, if, when households optimize, i.e.,

given (p,q,Y), $(x_h,b_h) = (f_h(p,q,Y,e_h), \phi_h(p,q,Y,e_h))$
solves the problem

maximize $u_h(x_h)$

$$\text{subject to} \quad p^0(x_h^0 - e_h^0) = -qb_h, \tag{1}$$

$$p^s(x_h^s - e_h^s) = y^s b_h, \quad \text{for } s > 0,$$

$$\text{and} \quad x_h \in X_h \qquad ,h = 1,2, \ldots ,H,$$

both spot goods and bond markets clear, i.e.,

$$\sum_h (x_h^{s,c} - e_h^{s,c}) = \sum_h (f_h^{s,c}(p,q,Y,e_h) - e_h^{s,c}) = 0, \quad \text{all } (s,c), \tag{2}$$

and

$$\sum_h b_h^i = \sum_h \phi_h^i(p,q,Y,e_h) = 0, \quad \text{all } i. \tag{3}$$

Remarks. 1. Several specific aspects of this formulation greatly facilitate analyzing properties of financial equilibrium (though, as we shall see later on, are not necessarily crucial to establishing that real indeterminacy is pervasive). Most notable among these are the assumptions that: (i) There are only two periods, with no production (obviously, **the leading case**); (ii) The financial structure is exogenous (for instance, the number of bonds is given a priori), and all financial instruments are inside assets (that is, are issued and redeemed by households directly); (iii) The yields on financial instruments are specified in terms of units of account (which is certainly a polar case, usually contrasted to that in which yields are specified in terms of bundles of goods, representing a kind of generalized forward contract); [Note: In the literature the former are now commonly referred to as **nominal** assets, the latter **real** assets. I prefer the more specific terminology "bonds" and "forwards", partly because the obvious prototypes are in fact a bond or a forward, respectively, but mostly because this terminology avoids suggesting an exhaustive distinction. Many (if not most) financial instruments, for example, (some types of) insurance policies, commodities futures or options and, in my opinion, corporate stocks fit neatly into neither category.] (iv) The only market imperfection takes the form of a deficiency in the number of financial instruments (as opposed, in particular, to various restrictions on financial transactions, like quantity limits on short sales).

2. Analysis of financial equilibrium amounts to analysis of the solutions to the market clearing conditions (2) and (3). By virtue of the budget constraints in (1), however, S+1 of these equations are functionally dependent on the remainder (the analogue to Walras' law). Furthermore, there is some choice about which equations to treat as redundant. In particular, it is easily verified that (under assumption A1) those concerning the bonds together with a suitable selection of those for the first type of good at just (S+1)-I spots are redundant, or, alternatively, that those concerning the first type of good at all S+1 spots are also redundant. An important implication of taking the first choice is that then variability of bond prices is not required for market clearing, and of taking the second that, again alternatively, then variability of spot prices for the first type of good is not required for market clearing. This means that, in addition to Y and e, either q (together with some household's marginal utilities of wealth at all spots) or $p^{\cdot,1}$

$= (p^{0,1},\ldots,p^{s,1},\ldots,p^{S,1})$ can be viewed as parametric. These two possible approaches dictate the form in which I present the basic substantive results concerning real indeterminacy; they also correspond, respectively, to the central approaches taken in Balasko and Cass (1989) (following Cass (1985)) and Geanakoplos and Mas-Collel (1989), where these results were first reported. (See also the paper by Werner (1986), which unifies the main results of both approaches by expanding on the first.) In the Appendix I follow the second approach, since it survives generalizing the model to encompass additional impediments on the households' abilities to utilize financial markets for providing maximal or **full** wealth insurance.

III. The Basic Results

Establishing the degree of real indeterminacy in this setting requires two further sorts of technical assumptions. The first concerns sufficiently flexible opportunities for exchange of credit (within the confines of incomplete markets), the second sufficiently disparate incentives for exchange of both goods and credit (in terms of numbers and also, implicitly, diversity of households).
So now assume that

A4'. There is $b^+ \in \mathbb{R}^I$ such that $Yb^+ \gg 0$; or **appropriately diverse yields**

A4". Y is in general position; or

A4'''. Y is variable; and

A5. H > D. **sufficiently numerous households**

Notice that, under either assumption A4' or assumption A4", Y is taken to be fixed. With various pairs of these additional assumptions A4· and A5 -- given the maintained assumptions A1-A3 --one can demonstrate the following results.

Theorem. Generically in endowments, the set of equilibrium allocations contains a smooth, D-dimensional manifold (with assumption A4'), or a smooth, (S-1)-dimensional manifold (with assumption A4"), or a smooth, DI-dimensional manifold (with assumption A4''').

Remarks. 1.The first and third results are Theorems 5.2 and 5.3 in Balasko/Cass, the second (essentially) Theorem 1 in Geanakoplos/Mas-Collel; the first obtains even when q is fixed (and thus, a fortiori, when q is variable). I find it quite elegant, but not really essential that the degree of real indeterminacy can be precisely measured in terms of the dimension of smooth manifolds. What is essential and important is what such measures reflect, namely, that any of a number of possible variables -- or combinations of variables -- can by themselves generate a continuum of economically distinct financial equilibria. Thus, for instance, even when the particular spot goods prices $p^{s,1}$ (or, more generally, some weighted average of the spot goods prices, say, $\sum_c \alpha^{s,c}p^{s,c}$ with

$\alpha^{s,c} \geq 0$, all c, and $\sum_c \alpha^{s,c} = 1)$, all s, are fixed, variation in some or all of the bond yields $y^{s,i}$ can generate such a continuum. I will emphasize (and reemphasize) this point again, especially when I discuss the potential role of fiat or **outside** money in reducing real indeterminacy.

2. Assumptions A4' and A4" have the following interpretations (which persuade me that the first is somewhat more economic in flavor, the second somewhat more mathematic): On the one hand, since the households' financial opportunities are unaffected by replacing any particular bond with a fixed portfolio that includes that bond (a kind of mutual fund), while the choice of units of account is more or less arbitary, A4' is basically equivalent to postulating the existence of **inside** money (or, a somewhat more misleading label, a "safe" asset), say, $y^{s,i} = 1$, $s > 0$. On the other hand, A4" means specifically that every I^2-dimensionsal submatrix of Y has full rank, which translates into the practical implication that -- subject to the limitation of facing incomplete markets -- households are capable of providing "full" wealth insurance (i.e., over any given subset of I future states).

3. Consider now just the case where q is variable (the one considered in some detail in the Appendix). In this case, assumption A4' can be replaced by the weaker alternative,

A4'a. $y^s \neq 0$, $s > 0$,

i.e., for every state there is some bond which has nonzero yield in that particular state. When A4'a is adopted as a maintained assumption, a defensible move, the Theorem can be interpreted to say that D is the minimal degree of real indeterminacy, and S-1 (resp. DI) is the maximal degree of real indeterminacy with incomplete markets, when Y is fixed (resp. Y is variable). (Notice that these bounds only coincide when I = 1.) Both Geanakoplos/Mas-Collel and Werner (1990) provide fairly abstract characterizations of the intermediate possibilities (when Y is fixed), while Polemarchakis (1988) provides a more concrete characterization in terms of exchange rate variability (when Y is variable).

IV. Various Refinements and Extensions

How robust is the phenomenon of extensive real indeterminacy? In this section I sketch an answer by briefly reviewing recent work which refines and extends the model and results described in the two previous sections. While I have tried to mention all the work that I am aware of (notice just how recent most of it is!) I make no claim to being either completely comprehensive or even -- from the various authors' viewpoints -- particularly balanced.

For this discussion it is often useful to refocus attention from the separate variables bond prices q and bond yields Y to the overall variable bond **returns**, represented by $R = \begin{bmatrix} -q \\ Y \end{bmatrix}$. This manuever utilizes the fact that it is only properties of R which ultimately matter for household demand, and therefore for financial equilibrium itself as well.

Many Periods

The only significant complication in going from two to many periods is that, with more than two periods, it is plausible to incorporate retrading markets for long-lived bonds. [Note: Under this generalization the spots correspond to enumerating the nodes in the standard date-event tree, while the bonds correspond to accounting for retrades as transactions in distinct financial instruments. I am also assuming that there are no constraints on retrading; in particular, households are presumed free to short-sell either original or retraded bonds.] The implication of this extension is that, in R, original or retraded bond prices may appear as entries in every row except those corresponding to the spots on the last date. However, because the analysis in Balasko/ Cass includes the case of fixed no-arbitrage bond prices and hence arbitrary fixed R, it again establishes the deficiency in the bond markets as the minimal degree of real indeterminacy under the obvious analogue of assumption A4'. The situation with variable bond prices or yields is considerably more intricate; its analysis depends, in particular, on what sort of original bonds are available. Werner (1990) provides formulae for describing the degree of real indeterminacy in a three-period model with both (distinct) one- and two-period bonds, where the latter are originally traded today, and then retraded tomorrow.

Sunspots

I originally discovered the connection between incomplete markets and pervasive real indeterminacy while looking into the possibility of sunspots under various kinds of market failure (see Cass (1989)). My specific analysis involved the simplest leading case, i.e., where $C = 1$, $S = 2$, $I = 1$ and $H = 2$, with extrinsic rather than intrinsic uncertainty, i.e., where

$$e_h^s = e_h^1,\ s > 0, \tag{4}$$

and

$$u_h(x_h) = \sum_{s>0} \pi^s v_h(x_h^0, x_h^s),\ \text{ with } \pi^s > 0,\ s > 0, \text{ and } \sum_{s>0} \pi^s = 1. \tag{5}$$

The Theorem doesn't cover the generalization of this leading example simply because the restriction (4) is nongeneric in endowments. (The specialization (5) presents no problem.) Siconolfi (1990) has demonstrated that, in a general model of sunspots with incomplete markets, the set of equilibrium allocations contains a continuum, while Siconolfi and Villanacci (1991) have verified the minimal degree of real indeterminacy when (4) is weakened to rule out aggregate (but not individual) risk,

$$\sum_h e_h^s = \sum_h e_h^1,\ s > 0,$$

and the certainty utility functions in (5) are themselves additively separable, say,

$$v_h(x_h^0, x_h^1) = v_h^0(x_h^0) + v_h^1(x_h^1).$$

Fully extending the Theorem to cover sunspots remains a difficult open problem.

Inside Instruments other than Bonds

A common misconception is that the phenomenon of extensive real indeterminacy requires having bonds, that is, financial instruments whose yields are specified in units of account. Nothing could be farther from the truth -- a point already emphatically underlined in both Balasko/Cass and Geanakoplos/Mas-Collel.

Consider yields which in principle depend on spot goods prices and other (as yet unspecified) parameters,

$$y^{s,i} = \psi^{s,i}(p^s, \cdot). \tag{6}$$

For bonds (or so-called nominal assets), we have

$$\psi^{s,i}(p^s, \cdot) = \alpha^{s,i}, \quad \text{all} \quad (s,i), \tag{7}$$

where $\alpha^{s,i}$ is simply a number of units of account, while for forwards (or so-called real assets),

$$\psi^{s,i}(p^s, \cdot) = \sum_c \beta^{s,i,c} p^{s,c}, \quad \text{all} \quad (s,i), \tag{8}$$

where $\beta^{s,i} = (\beta^{s,i,1}, \ldots, \beta^{s,i,c}, \ldots, \beta^{s,i,C})$ is simply a vector of types of commodities. But there are obviously many other possibilities as well, for instance,

$$\psi^{s,i}(p^s,\cdot) = \alpha^{s,i} + \sum_c \beta^{s,i,c} p^{s,c}, \quad \text{all} \quad (s,i), \tag{9}$$

the general linear specification which encompasses both (7) and (8) as special cases.

For the time being, focus on just the three specifications (7), (8) and (9) (also taking, for the time being, the parameters $\alpha^{s,i}$ and $\beta^{s,i}$ as fixed). Appealing to the argument outlined in the Appendix, intuitively the typical degree of indeterminacy under each of these specifications will be the same as the degree of **significant** nominal indeterminacy, which is determined by subtracting the maximum number of permissible price normalizations, say, N, from the total number of budget constraints, S+1. Since the first budget constraint in (1) is linear homogeneous in spot goods and bond prices, it is always permissible to normalize, say, $p^{0,1} = 1$. The number of additional permissible price normalizations depends on the paricular specification. Under (7) it is also permissible to normalize, say, $p^{s,1} = 1$ at any chosen spot $s = s' > 0$ (see the Appendix), so that $N = 2$ and $(S+1)-N = S-1$; under (8) it is permissible to normalize, say, $p^{s,1} = 1$ at every $s > 1$ (since in this case each future budget constraint is linear homogeneous in its own spot goods prices), so that $N = S+1$ and $(S+1)-N = 0$; and under (9) there are no additional permissible price normalizations, so that $N = 1$ and $(S+1)-N = S$. In short, only the special case (8) entails that there is necessarily generic local uniqueness, and hence no extensive real indeterminacy; indeed, the general case (9) entails even one more degree of real indeterminacy than the special case (7) which has been the main focus of attention!

The formal analysis which validates my intuitive, "counting equations and unknowns" argument for the general case (9) can be found in Pietra (1988). Note that exactly the same sort of intuition suggests that with an arbitrary specification of yields (6) (excluding the special cases (7) and (8)), the typical degree of real indeterminacy will be S. Express support for this conjecture is contained in Krasa and Werner (1989), who analyze a model of (potentially) incomplete markets with a variety of different financial instruments, including inside money, forwards and options written on the forwards.

Two other points are worth explicit mention. First, except in the special case (7), the rank of R -- and hence the dimension of the wealth space it spans -- may vary with p, which raises a problem for existence of financial equilibrium (and undoubtedly explains some preoccupation in the literature with the special case (8)). Usually, that is, for a typical specification of (6), the property of existence will (at best) only be generic (in both endowments and the parameters specifying yields). But also, to repeat for emphasis, usually the property of indeterminacy will necessarily be generic. (Pietra and Krasa/Werner both illustrate these two points very nicely.) Second, even in the special case (8), when (some or all of) the parameters $\beta^{s,i,c}$ are treated as variables, there will be extensive real indeterminacy. More generally, suppose that the parameters in (6) reflect the characteristics of some particular array of financial instruments. Then, if various of these parameters are also taken as variable, this will usually contribute to increasing the degree of real indeterminacy beyond S. And there is just no convincing argument for taking all such parameters as fixed, since they essentially correspond -- in abstract -- to the terms on which credit is transacted between households.

Finally, I must surely note that, for the special case (8), Mas-Collel (1991) provides grounds for the assertion that, very roughly speaking, with "many" states of the world, it is not atypical that there will be commensurately "many" distinct financial equilibria. Thus, even in the most favorable circumstance, those holding to the rational expectations hypothesis may take only scant comfort from generic local uniqueness!

Outside Money

A second misconception or, perhaps better, oversimplification concerning this issue is the belief that, since "the reason" for indeterminacy is that the future "price level" is not tied down, introducing the institution of outside money per se will eliminate the problem. While there is some basis for this conjecture, its validity depends on how one conceives the operation of a monetary system and, even more critically, on what one takes as variable in a monetary economy.

Consider first what difference the particular role assigned to outside money makes, in a setting where there are both inside and outside money, and where bond yields are fixed. At one extreme, when outside money is only required in order to pay terminal taxes (Villanacci (1990)), there is -- for the same reason just previously explained -- actually one more degree of significant nominal indeterminacy, and hence real indeterminacy. At the other extreme, when outside money is instead required in order to finance spot goods consumption now (Magill and Quinzii (1988,9)), or is simply assumed by households to have value to finance spot goods consumption later (Cass (1990)), there are in fact $S-1$ less degrees of significant nominal indeterminacy, and hence no real indeterminacy. So, as conjectured, in these last two models -- but of course, not the first -- outside money does restore the generic local uniqueness associated with standard Walrasian equilibrium. [Note: The Magill-Quinzii model is somewhat cruder than most cash-in-advance models, since it amounts to imposing the ancient quantity theory of money spot-by-spot. But equally objectionable, the Cass model is just as crude as all money-in-the-utility-function models. Nonetheless, both serve the useful purpose of showing that in some monetary economy, outside money eliminates indeterminacy.]

Consider next what happens when bond yields (excluding those on inside money) are variable, especially in the two models most favorable to monetarism. It turns out (again referring to my own work on monetary models) that now there is absolutely no reduction in the degree of significant nominal indeterminacy, and hence real indeterminacy; in this situation there are $D(S-1)$ degrees of real indeterminacy. (This formula differs from that given in the Theorem since it is derived under the hypothesis that $y^{s,i} = 1$, $s > 0$, is fixed, that is, that there is inside money.)

Since I find it quite reasonable (even compelling) to believe that in a monetary economy, bond yields (as a proxy for the yields on a variety of financial instruments) as well as spot goods and bond prices are endogenous, I conclude the following (from the analysis of these extraordinarily rudimentary models): The institution of outside money may reduce the degree of real indeterminacy. After all, it is likely that there is some connection (no matter how loose) between "monetary policy" and bond yields, so that these yields are not free to vary arbitarily. However, at this time there are simply no acceptable grounds for asserting that outside money completely eliminates real indeterminacy.

Restricted Participation on Financial Markets

While there might be some disagreement over whether, in a modern, developed economy, financial markets are actually incomplete, there can hardly be any disagreement over whether at least some economic agents are variously constrained in transacting on those financial markets. Without attempting a detailed explanation of how particular constraints come about (for example, in order to resolve problems arising from moral hazard), it is still possible to extend the model of Section II to incorporate, in a very general way, their implications for financial equilibrium by adding a restriction of the form

$$b_h \in B_h \subset \mathbb{R}^I$$

to the problem (1). Here B_h represents the **portfolio set**, the possible credit transactions available to the household. Balasko, Cass and Siconolfi (1990) show that the Theorem of Section III extends to the case where the portfolio set is defined by linear homogeneous equality constraints (so that B_h is an I_h-dimensional linear subspace, with $0 \leq I_h \leq I$). Just recently Cass, Siconolfi and Villanacci (1991) have further extended these results to the case where the portfolio set is defined by smooth, quasi-concave inequality constraints. Either of these models with, say, **restricted participation** constitutes a bona fide generalization of the model with incomplete markets, but the latter potentially embodies far more interesting institutional features (and not just the flavor of restricted participation) since it permits, for instance, modeling short sales bounds or market margin requirements. Of course, in principle such constraints should themselves be determined endogenously.

Small Imperfections in Financial Markets

An intriguing question is whether a "small" departure from complete markets results in a "small" amount of indeterminacy (or, clearly related, a "small" departure from Pareto optimality). Obviously, the answer will depend on the choice of a more specific formulation of the question. Building on my joint analysis with Balasko and Siconolfi, I have provided one sort of basis for an affirmative response (Cass (1990)): Assume that there are complete markets, $I = S$, and that only some households are restricted,

$$B_h = \begin{cases} \mathbb{R}^I & , h = 1, \cdots, H' < H \\ B_h \subset \mathbb{R}^I, & \text{otherwise.} \end{cases}$$

For instance, the households $h > H'$ might only have access to a subset of all the bond markets. Now consider the "improved participation" economy consisting of M replicas of the first H' (unrestricted) households, and N replicas of the remainder, with $M/N > 1$, and let $M/N \to \infty$. Then, the set of financial equilibria converges to that of the standard Walrasian economy consisting of just the unrestricted households. So, in this precise sense, indeterminacy -- and also nonoptimality -- become (generically) insignificant as incompleteness becomes (relatively) insignificant.

Much caution is warranted here, however. Pursuing a quite different approach, Green and Spear (1989) and Zame (1988) formulate the idea of a "small" departure from complete markets by assuming that $I < S = \infty$, and then letting $I \to \infty$. They find -- modeling financial instruments as forwards, and concentrating on the issue of optimality -- that it is only under very restrictive conditions on the parametric structure of yields that equilibrium allocation converges to the set of Pareto optima. These results suggest that it is quite unlikely that under their formulation indeterminacy becomes (generically) insignificant.

Notice that even under my formulation, since the argument concerns asymptotic behavior, the rational expectations hypothesis remains highly suspect: No matter how large the economy, a continuum remains a continuum!

Concluding Comment

This review is almost as revealing for what it omitted as for what it included. For instance, I have said nothing at all about introducing firms or intermediaries, both of which create financial instruments -- and the restrictions on their trade as well -- endogenously. Obviously, I think that this undertaking, just now getting started, is an important and fascinating subject for future research. The same can be said about other possible projects that I have only alluded to earlier -- for instance, incorporating a more tenable model of the institution of outside money, or elaborating the dependence of portfolio restrictions on endogenous variables.

Appendix

The purpose of this appendix is to provide some insight into the rationale for nominal indeterminacy, as well as the logic supporting its translation into a corresponding degree of real indeterminacy (barring exceptional circumstances).

Nominal Indeterminacy

The central idea in establishing this result is pure and simply to "count equations and unknowns". That is, the essence of the analysis involves treating the market clearing conditions (2) and (3) as a system of equations in the whole collection of variables p, q, Y and e and then -- after verifying certain crucial prerequisites (by utilizing several basic techniques from differential topology) -- employing the old workhorse of economic theory, the implicit function theorem. In following this program I intentionaly give short shrift to the details of the underlying justification for treating particular price variables as being "independent" (and the other price cum "fundamental" variables as being "independent") -- the "certain crucial prerequisites" referred to just above.

Recall what, roughly, the implicit function theorem asserts: Suppose that we are given a system of J (independent) equations in K (explicit) variables, so that necessarily $J \leq K$ (and the equations, being defined by sufficiently smooth functions of the specified variables, have Jacobian of full rank J at some particular solution). Then, locally, the system can be solved for

J (distinguished) variables as continuously differentiable functions of the other K-J variables. Thus, my task basically amounts to calculating J,K and K-J for the particular system of equations at hand. In carrying out this task it is quite instructive to begin by recalling a more familiar example, that arising from the standard Walrasian model.

So now suppose that, instead of trading on many spot goods and the bond markets, households trade on a single "overall" market for current and future contingent goods. For simplicity letting the previous notation also represent prices and allocations for such an economy, then, here, given $e \in E$, $p \in P$ is a **Walrasian equilibrium** if, when households optimize (according to the usual budget-constrained, utility-maximization problem), i.e.,

$$\begin{aligned} &\text{given } p,\ x_h = g_h(p,e_h) \text{ solves the problem} \\ &\qquad \text{maximize} \quad u_h(x_h) \\ &\qquad \text{subject to} \quad p(x_h - e_h) = 0 \\ &\qquad \text{and} \quad x_h \in X_h \qquad ,h = 1,2,\cdots,H, \end{aligned} \tag{A1}$$

just the overall market for goods clears, i.e.,

$$\sum_h (x_h^{s,c} - e_h^{s,c}) = \sum_h (g_h^{s,c}(p,e_h) - e_h^{s,c}) = 0, \text{ all } (s,c). \tag{A2}$$

In this setting, from the restriction imposed by the budget constraint in (A1) it follows that the market clearing conditions (A2) yield only G-1 independent equations (Walras' law), while p and e constitute the only explicit variables. Thus, obviously $J = G-1$ and $K = G+HG$, and (locally), say, the $K-J = 1+HG$ variables $p^{0,1}$ and e uniquely determine the remaining prices p less $p^{0,1}$. In other words, there is one degree of nominal indeterminacy, the choice of the "price level" represented by $p^{0,1}$. Of course, from the linear homogeneity of the budget constraints in (A1) it also follows that such nominal indeterminacy is "insignificant", in the sense that it never engenders any real indeterminacy. For this reason it is conventional to normalize prices, for instance, by setting $p^{0,1} = 1$, and to maintain that equilibrium is locally unique (up to a "harmless" choice of numeraire), or that, say, there are no **significant** degrees of nominal indeterminacy in the Walrasian model. While I will also adopt this position in discussing the model with incomplete markets, I repeat for emphasis that it is, from a practical viewpoint, quite misleading; even such "insignificant" nominal indeterminacy raises havoc for presupposing rational expectations (a very important message, but one I will now take as having been fully delivered).

In applying similar reasoning to the model summarized by (1)-(3) it turns out that the only potential complication involves figuring out the number of significant price (or price-like) variables. So now returning to consideration of this system, we first recall that the restrictions imposed by the budget constraints in (1) render $S+1$ of the market clearing conditions (2) and (3) redundant, in particular, say, those concerning just the first type of good. Since these equations number altogether $G+I$, $J = G+I-(S+1)$, while clearly, since in general p,q,Y and e are all variable, $K = G+I+SI+HG$. Thus, (locally), say, the $K-J = (S+I)+SI+HG$ variables $p^{\cdot,1}$, Y and e uniquely determine the remaining spot prices p less $p^{\cdot,1}$ and there are apparently $(S+1)+SI$ degrees of nominal indeterminacy. In order to explain why exactly 2 (when Y is fixed) or $(S+I)+S^2$ (when Y is variable) degrees of such nominal indeterminacy are "insignificant", it is very helpful (if not indispensable)) to digress a moment and reformulate the budget constraints in (1). The particular reformulation I have chosen to elaborate will also be very convenient for the discussion in the succeeding subsection.

Focus on the budget constraints of a typical household h,

$$p^0(x_h^0 - e_h^0) = -qb_h$$
$$\text{and } p^s(x_h^s - e_h^s) = y^s b_h, \quad \text{for } s > 0, \tag{A3}$$

and consider the following two-step transformation (both steps of which leave the household's consumption opportunities unaltered):

<u>**Step 1**</u>. Divide each of the budget constraints in (A3) by its own spot price for the first type of good:

$$\bar{p}^0(x_h^0 - e_h^0) = (-q/p^{0,1})b_h$$
$$\text{and } \bar{p}^s(x_h^s - e_h^s) = (y^s/p^{s,1})b_h, \quad \text{for } s > 0, \tag{A4}$$

where $\bar{p}^s = p^s/p^{s,1}$, $s = 0,1,\cdots,S$.

Step 2. Assume, without loss of generality (by using assumption A1 and relabelling spots appropriately), that the last I rows of Y are linearly independent, so that we can partition

$$Y = \begin{bmatrix} \dot{Y} \\ --- \\ \ddot{Y} \end{bmatrix} = \begin{bmatrix} y^1 \\ \vdots \\ y^D \\ --- \\ y^{D+1} \\ \vdots \\ y^{D+I} \end{bmatrix},$$

where $D = S-I$ and thus $\ddot{Y}$ is an I^2-dimensional, full rank matrix. Then, reduce the right-hand side of (A4) by transforming the variables b^i to the variables $b^{i'} = (y^{D+i}/p^{D+i,1})b_h$, $i = 1,2,\cdots,I$, as follows:

$$\begin{bmatrix} -q/p^{0,1} \\ ------- \\ y^1/p^{1,1} \\ \vdots \\ y^D/p^{D,1} \\ ------- \\ y^{D+1}/p^{D+1,1} \\ \vdots \\ y^{D+I}/p^{D+I,1} \end{bmatrix} b_h = \begin{bmatrix} -(1/p^{0,1})q \\ \begin{pmatrix} 1/p^{1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D,1} \end{pmatrix} \dot{Y} \\ \begin{pmatrix} 1/p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D+I,1} \end{pmatrix} \ddot{Y} \end{bmatrix} b_h \tag{A5}$$

$$= \left[\begin{array}{c} -(1/p^{0,1})\,q \\ \begin{bmatrix} 1/p^{1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D,1} \end{bmatrix} \dot{Y} \end{array} \left(\begin{bmatrix} 1/p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D+I,1} \end{bmatrix} \ddot{Y} \right)^{-1} \\ \underline{I} \right] b_h'$$

$$= \left[\begin{array}{c} -(1/p^{0,1})\; q\; \ddot{Y}^{-1} \begin{pmatrix} p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & p^{D+I,1} \end{pmatrix} \\ \begin{pmatrix} 1/p^{1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D,1} \end{pmatrix} \dot{Y}\, \ddot{Y}^{-1} \begin{pmatrix} p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & p^{D+I,1} \end{pmatrix} \\ \underline{I} \end{array} \right] b_h'$$

$$= \begin{bmatrix} -q' \\ --- \\ \dot{Y}' \\ --- \\ \underline{I} \end{bmatrix} b_h',$$

where

$$q' = q\; \ddot{Y}^{-1} \begin{bmatrix} p^{D+1,1}/p^{0,1} & & 0 \\ & \ddots & \\ 0 & & p^{D+I,1}/p^{0,1} \end{bmatrix}$$

and

$$\dot{Y}' = \begin{bmatrix} \omega^{1,1}(p^{D+1,1}/p^{1,1}) & \cdots & \omega^{1,I}(p^{D+I,1}/p^{1,1}) \\ \ddots & & \\ \vdots & \omega^{j,k}(p^{D+k,1}/p^{j,1}) & \\ & \ddots & \\ \omega^{D,1}(p^{D+1,1}/p^{D,1}) & \cdots & \omega^{D,I}(p^{D+I,1}/p^{D,1}) \end{bmatrix} \tag{A6}$$

with $\Omega = -\dot{Y}\ddot{Y}^{-1}$;

the reason for the sign change in the definition of Ω will become clear below. [Note: It is easily seen that if q are no-financial-arbitrage prices for bond yields $Y = \begin{bmatrix} \dot{Y} \\ --- \\ \ddot{Y} \end{bmatrix}$, then q' are also no-financial-arbitrage prices for bond yields $Y' = \begin{bmatrix} \dot{Y}' \\ --- \\ \underline{I} \end{bmatrix}$.
This means that, in (A5), for all practical purposes we can safely ignore the genesis of $q' \in \mathbb{R}^I$ -- but of course, not that of $\dot{Y}' \in \mathbb{R}^{DI}$.]

Letting

$$\overline{P} = \begin{bmatrix} \overline{p}^0 & & 0 \\ & \ddots & \\ 0 & & \overline{p}^S \end{bmatrix}$$

and $$R' = \begin{bmatrix} -q' \\ \dot{Y}' \\ \underline{I} \end{bmatrix},$$

and then substituting from (A5) into (A4) (while rewriting "b_h" for "b_h'" the budget constraints in (1) can be compactly reformulated as

$$\overline{P}(x_h - e_h) = R'b_h. \tag{A7}$$

Now simply notice that, by virtue of the structure of $\dot{Y}'$ displayed in (A6), nothing significant is lost by assuming -- when Y is fixed -- that, for instance, $p^{0,1} = p^{D+1,1} = 1$ or -- when Y is variable -- that, for instance, $p^{\cdot 1} = \underline{1}$ and $\ddot{Y} = \underline{I}$ (so that, in (A6), $\dot{Y}' = -\Omega = \dot{Y}$, a (D×I)-dimensional matrix): There are only $(S+1)-2 = S-1$, or $[(S+1)+SI]-[(S+1)+I^2] = DI$ (significant) degrees of nominal indeterminacy, respectively.

Table 1 summarizes the foregoing enumeration, and should aid in digesting it.

Its Translation into Real Indeterminacy

In order to see how the conclusions of the Theorem in the text follow from these two alternative degrees of (significant) nominal indeterminacy, it is convenient to introduce an additional piece of notation that permits concentrating on just significant price or yield variation.

Consider the transformed representation of bond yields in (A6), the matrix $\dot{Y}'$. Since I will only be concerned with perturbing $p^{\cdot 1}$ (with $p^{0,1} = p^{D+1,1} = 1$, for Y fixed) or Y (with $p^{\cdot 1} = \underline{1}$ and $\ddot{Y} = \underline{I}$, for Y variable), let

Table 1. Nominal Indeterminacy

	Walrasian Equilibrium	Financial Equilibrium: Y fixed	Financial Equilibrium: Y variable
Equations	$\sum_h (g_h^{s,c}(p,e_h) - e_h^{s,c}) = 0$, all (s,c)	$\sum_h (f_h^{s,c}(p,q,Y,e_h) - e_h^{s,c}) = 0$, all (s,c) and $\sum_h \phi_h^i(p,q,Y,e_h) = 0$, all i	
no. of equations	G	G+I	
interdependencies (no. of budget constraints)	1	S+1	
J = **no. of independent equations**	G−1	G+I−(S+1)	
Variables	p,e	p,q,e	p,q,Y,e
no. of variables	G+HG	G+I+HG	G+I+SI+HG
insignificancies (no. of price "normalizations")	1	2	$(S+I)+I^2$
K = **no. of significant variables**	(G−1)+HG	(G−2)+HG	[G−(S+1)]+I+DI+HG
K−J = **no. of "independent" and significant variables**	HG, say, e	(S−1)+HG say, $p^{s,1}$, $s > 1$, and e	DI+HG say, Ω and e
K−J−HG = **degree of (significant) nominal indeterminacy**	0	S−1	DI

$$\omega = (p^{1,1},\cdots,p^{D,1},1,p^{D+2,1},\cdots,p^{D+I,1})$$

as well as

$$\Omega = -\dot{Y}\ddot{Y}^{-1}.$$

Then, for simplicity suppressing the superfluous "'" (since q' depends only indirectly on ω and Ω under either hypothesis) we can rewrite $\dot{Y}'$ and R' as

$$\dot{Y} = \dot{Y}(\omega,\Omega) = -\begin{bmatrix} \omega^{1,1}(1/\omega^{1}) & \cdots & \omega^{1,I}(\omega^{D+I}/\omega^{1}) \\ \ddots & & \\ \vdots & \omega^{j,k}(\omega^{D+k}/\omega^{j}) & \\ & \ddots & \\ \omega^{D,1}(1/\omega^{D}) & \cdots & \omega^{D,I}(\omega^{D+I}/\omega^{D}) \end{bmatrix} \quad \text{(A8)}$$

and $R = R(q,\omega,\Omega) = \begin{bmatrix} -q \\ \dot{Y}(\omega,\Omega) \\ \underline{I} \end{bmatrix}$,

respectively (with a corresponding simplification of (A7)).

At this point, the particular approach I prefer basically involves analyzing the overall implications of the households' personalized no-financial-arbitrage conditions (which derive from the Lagrangean characterization of the optimal solution to (1) after the budget constraints have been reformulated according to (A7); cf, again, Balasko/Cass.) [Note: An alternative approach involves analyzing the overall implications of the budget constraints themselves; cf, again, Geanakoplos/Mas-Collel. In my opinion this second approach is not nearly as efficient (or powerful) for drawing conclusions about properties of the mapping, say, for Y fixed, $f: M \to E$ such that

$$
\begin{aligned}
(p,q,e) \mapsto (x_1,\cdots,x_h,\cdots,x_H) & \\
&= f(p,q,Y,e) \\
&= (f_1(p,q,Y,e_1),\cdots,f_h(p,q,Y,e_h),\cdots,f_H(p,q,Y,e_H)),
\end{aligned}
$$

where $M \subset P \times Q \times E$ represents the equilibrium set, and thus $f(M) \subset E$ represents the corresponding allocation set.] Associating the Lagrange multipliers $\lambda_h = (\lambda_h^0,\cdots,\lambda_h^s,\cdots,\lambda_h^S) \in R_{++}^{S+1}$ with the constraints (A7), the first-order conditions for (1) become

$$Du_h(x_h) = \lambda_h \overline{P} \tag{A9}$$

and

$$\lambda_h R = \acute{\lambda}_h \begin{bmatrix} -q \\ \dot{Y} \end{bmatrix} + \ddot{\lambda}_h = \underline{0}$$

or

$$\lambda_h = \acute{\lambda}_h \left[\; \underline{I} \; \begin{pmatrix} q \\ -\dot{Y} \end{pmatrix} \right], \tag{A10}$$

where, as before, I partition $\lambda_h = (\acute{\lambda}_h, \ddot{\lambda}_h) = ((\lambda_h^0,\cdots,\lambda_h^D),(\lambda_h^{D+1},\cdots,\lambda_h^{D+I}))$. Substituting from (A10) into (A9) yields the fundamental construct for verifying generic existence of precise degrees of indeterminacy,

$$Du_h(x_h) = \lambda_h \left[\underline{I} \begin{pmatrix} q \\ -\dot{Y} \end{pmatrix} \right] \overline{P}, \ h = 1,2,\cdots,H. \tag{A11}$$

The key mechanism for generating real from nominal indeterminacy is, in principle, quite simple. Perturbations of the S-1 spot prices ω (typically) or the, say, reduced form bond yields Ω (generally) alter the linear subspace orthogonal to that spanned by the columns of R (alternatively, and equivalently, the latter subspace itself). But this in turn (typically) changes the set of equilibrium allocations consistent with R according -- in particular -- to (A11) (alternatively, to (A7)). In order for this chain process to actually work out it must be the case, first, that R is sufficiently sensitive to price or yield variation, and second, that as a whole, households are sufficiently sensitive to their altered financial opportunities. Assumption A4· is designed to guarantee the former, and assumption A5 (together with enough variety of endowments, given preferences) the latter. Before describing how these two assumptions operate, it is quite illuminating to look at several examples in which, though (significant) nominal indeterminacy is pervasive, it doesn't necessarily induce any real indeterminacy.

Example 1. Fully Complete Markets: Suppose that $I = S$. Then (adapting previous usage in the natural way), $Y = \ddot{Y}$ and

$$R = \begin{bmatrix} -q \\ \underline{I} \end{bmatrix}$$

and clearly, whether Y is fixed or variable, since R is essentially independent of both ω and Ω there is purely nominal indeterminacy. I must reemphasize, however, that even in such an idyllic situation, households would still surely be in a real quandary about what prices they could reasonable expect in the future (as they surely are in any actual economic environment!).

Example 2. Fully Incomplete Markets: Suppose that $I = 0$ or, to the same end, that $0 < I \leq S$ but $Y = \underline{0}$. Then (again adapting previous usage in the natural way), necessarily $q = 0$, so that

$$R = \underline{0} \ ,$$

and we have exactly the same outcome as in the opposite case where there there are fully complete markets.

Example 3. Incomplete Markets with Arrow Securities (for a Subset of Future Spots): Suppose that $0 < I < S$ and

$$y^{s,i} = \begin{matrix} 1, & \text{for } s = D+i,\ 1,2,\cdots,I \\ 0, & \text{otherwise.} \end{matrix}$$

Then, $Y = \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix}$, so $\Omega = \underline{0}$ and

$$R = \begin{bmatrix} -q \\ \underline{0} \\ \underline{I} \end{bmatrix},$$

and again, clearly, for Y fixed, since R is independent of ω, there is purely nominal indeterminacy.

<u>**Example 4. Incomplete Markets with Inside Money plus a Subset of Arrow Securities**</u>: Suppose, slightly modifying the previous example, that bond 1 is inside money (rather than the Arrow security paying off at spot D+1). Then,

$$Y = \begin{bmatrix} & \underline{0} \\ \underline{1} & \\ & \underline{I} \end{bmatrix} = \begin{bmatrix} \dot{Y} \\ --- \\ \ddot{Y} \end{bmatrix} = \begin{bmatrix} (\underline{1} \quad \underline{0}) \\ --------- \\ \begin{pmatrix} (1,0,\cdots,0) \\ \underline{1} \quad\quad \underline{I} \end{pmatrix} \end{bmatrix}$$

with

$$\ddot{Y}^{-1} = \begin{bmatrix} (1,0,\cdots,0) \\ -\underline{1} \quad\quad \underline{I} \end{bmatrix},$$

so $\Omega = -[\underline{1} \quad \underline{0}]$ and

$$R = \begin{bmatrix} -q \\ \begin{pmatrix} 1/\omega^1 & \\ \vdots & \underline{0} \\ 1/\omega^D & \end{pmatrix} \\ \underline{I} \end{bmatrix}.$$

In this example, for Y fixed, only perturbations of the first D

elements of ω, say, $\dot{\omega} = (\omega^1, \omega^2, \cdots, \omega^p)$, affect R and therefore possibly generate real indeterminacy; Y satisfies assumption A4' but not A4". [Note: Generally, these two assumptions are not nested, so either can be satisfied when the other is not.]

Example 5. Pareto Optimality: Suppose that e is a Pareto optimal allocation (which is always true when $H = 1$). Then, clearly, since the only equilibrium allocation is autarky, the households' equilibrium behavior is independent of R, and there is, once again, purely nominal indeterminacy.

To see what can be learned form the first four examples -- and understand why perturbations of ω and Ω alter the column span of R, say, for simplicity, **span** R, and thereby its orthogonal complement as well, that is, for short, why such perturbations are **effective** -- it is important to bear in mind that, for this analysis, when Y is fixed, then $\Omega = -\dot{Y}\ddot{Y}^{-1}$ is also fixed, and only ω is perturbed, while when Y is variable, then $\omega = 1$ itself is fixed, and only Ω is perturbed.

From examination of the examples it is apparent that in each of the first three the difficulty is simply that no permissible pertubation is effective, while in the fourth, that only certain permissible perturbations (namely, of the subvector $\dot{\omega}$) are effective. More generally, and equally apparent from examination of R as displayed in (A8), is why assumptions A4' and A4" guarantee that perturbations of $\dot{\omega}$ and ω, respectively, are effective. In the first instance, $Yb^+ > 0$ is equivalent to $(\dot{Y}\ddot{Y}^{-1})b^{+'} > 0$, where $b^{+'} = \ddot{Y}b^+$, and assumption A4' is tantamount to assuming that at least one element in each row of Ω is nonzero (which is, of course, why why A4' can be replaced by A4'a). Hence, for $j = 1,2,...,D$,

$$\omega^{j,k} \neq 0 \ \& \ \omega^{j'} \neq \omega^{j''} \ (\text{with } \omega^{s'} = \omega^{s''} \text{ for } s = D+k,\ k = 1,2,\cdots,I) \rightarrow$$
$$R(q',\omega',\Omega)b \notin \text{span } R(q'',\omega'',\Omega), \text{ for } b^i \neq 0,\ i = k,\ = 0, \text{ otherwise}$$

In the second instance, assumption A4" implies that every element of Ω is nonzero. [Note: To say that "Y is in general position" means precisely that every I^2-dimensional submatrix of Y has full rank. This condition is violated if, for some (j,k), $\omega^{j,k} = 0$, because then replacing the k^{th} row in $\ddot{Y}$ with the j^{th} row in $\dot{Y}$ yields an I^2-dimensional submatrix with rank equal I-1.] Hence, for $j = 1,2,...,D$, $k = 2,3,...,I$

$$\omega^{j,k} \neq 0 \ (\text{resp. } \omega^{j,1} \neq 0),\ \omega^{j'} = \omega^{j''} \ \& \ \omega^{D+k'} \neq \omega^{D+k''} (\text{resp. } \omega^{j'} \neq \omega^{j''}) \rightarrow$$
$$R(q',\omega',\Omega)b \notin \text{span } R(q'',\omega'',\Omega), \text{ for } b^i \neq 0,\ i = k$$
$$(\text{resp. } i = j),\ = 0, \text{ otherwise.}$$

It should now be more or less obvious why assumption A4''' works equally well:

$\omega^{j,k'} \neq \omega^{j,k''} \Rightarrow$

$R(q',\omega,\Omega')b \notin \text{span}\, R(q'',\omega,\Omega'')$ for $b^i \neq 0$, $i = k$, $= 0$, otherwise

Finally, it is worthwhile mentioning again that while assumption A4' (but not A4") has some economic content -- and thus permits arguing for real indeterminacy without mathematical artifice -- it also entails a weaker result than assumption A4", since it only provides a lower bound on the degree of real indeterminacy.

Once it has been determined that suitable perturbations of spot prices or bond yields are effective, the rest is easy (at least in conception). With enough households, as specified by assumption A5, the property that

$$\text{rank}\,[Du_h(x_h),\ h = 1,2,\cdots,H] = \text{rank}\,[\lambda_h,\ h = 1,2,\cdots,H] = D+1, \qquad \text{(A12)}$$

which is its maximal value, is -- like the property that $p^{\cdot,1}$, Y and e can be taken as "independent" variables -- generic in endowments. That is, this "rank" property obtains (for some financial equilibrium) on an open, dense subset of E. [Note: Of course, Example 5 illustrates the difficulty when (A12) fails, since, if x is a Pareto optimal allocation, then

$$\text{rank}\,[Du_h(x_h),\ h = 1,2,\cdots,H] = 1. \qquad \text{(A13)}$$

This last result therefore also basically ratifies the intuition that, in the presence of incomplete markets, the coordination required by (A13) is most unlikely.] So, locally, variation in ω or Ω (by means of perturbing the overall returns exhibited in (A8)) must typically map diffeomorphically into variation in x (by virtue of satisfying the gradient restrictions exhibited in (A11)). As in the preceding subsection, I give short shrift to the argument supporting this last step -- which again amounts to utilizing basic techniques from differential topology. For a detailed account, the interested reader is once more referred to Balasko/Cass.

Footnotes

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Symbols*

$=$ = "equals"

$>$, $\geq$, $\gg$ = "greater than", "greater than or equal", "much greater than"

$\mathbb{R}$ = "real number"

$\rightarrow$ = "right arrow"

ϵ = "belongs to"

Δ (e.g., Δx) = "uc delta"

$\times$ = "times"

ϕ = "phi"

$\sum$ = "sum"

α = "alpha"

π = "pi"

ψ = "psi"

β = "beta"

$\dot{Y}$ = "uc yi dot"

$\ddot{Y}$ = "uc yi double dot"

ω = "omega"

Ω = "uc omega"

λ = "lambda"

$\Rightarrow$ = "double right arrow"

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*In order of appearance in the paper (text, then appendix).