REAL INDETERMINACY FROM IMPERFECT FINANCIAL MARKETS : TWO ADDENDA

by

David CASS

N° 9028

REAL INDETERMINACY FROM IMPERFECT FINANCIAL MARKETS : TWO ADDENDA

A B S T R A C T

This paper considers a competitive, pure exchange model in which households face exogenous restrictions on participating in the market for inside financial instruments. It is well-known that, except when the yields from financial instruments are denominated in commodities, such market imperfections lead to substantial indeterminacy in equilibrium allocation. So two further issues are examined : First, does the introduction of the institution of fiat or outside money reduce or eliminate this real indeterminacy ? Second, if the number of households having access to less than a complete financial market is insignificant, then is the extent of real indeterminacy also insignificant ?

Journal of Economic Literature 021,024,311

Key Words : Consumption, Exhaustible Natural Resources

INDETERMINATION REELLE A CAUSE DES MARCHES FINANCIERS IMPARFAITS : DEUX ADDENDA

R E S U M E

Ce papier considère un modèle concurrentiel d'échange pur, pour lequel les ménages sont soumis à des restrictions exogènes sur leur participation au marché des instruments financiers internes. Il est bien connu que, sauf dans le cas où les rendements des instruments financiers sont définis en biens, de telles imperfections du marché entraînent une indétermination substantielle dans les allocations d'équilibre. En conséquence, deux problèmes supplémentaires sont examinés : En premier lieu, la prise en compte de la monnaie externe réduit-elle ou supprime-t-elle cette indétermination réelle ? En second lieu, si le nombre de ménages ayant accès à des marchés financiers incomplets est insignifiant, alors l'indétermination réelle est-elle aussi insignifiante ?

Journal of Economic Literature 021,024,311

Mots Clefs : Consommation, Ressources naturelles épuisables.

Real Indeterminacy from Imperfect Financial Markets: Two Addenda*

by

David Cass
University of Pennsylvania

I. Introduction

Recently there has been a great deal of interest in examining the properties of competitive equilibrium with incomplete -- or, more generally, imperfect -- financial markets.[1/] An important branch of this research has focused on the fact that, in such economies, there is typically a large degree of price or <u>nominal</u> indeterminacy -- over and above that analogous to choosing a numeraire in the standard Walrasian model -- which also translates into allocation or <u>real</u> indeterminacy.[2/] The principal aim of this paper is to present two analyses that are basically responses to criticism of this latter development.

One sort of criticism has been based on the idea that the "reason" for indeterminacy is simply that the future "price level" is not tied down, and that, in particular, introducing the institution of fiat or <u>outside</u> money should ameliorate the problem. I will argue here that while there is some truth to this conjecture (see also Magill and Quinzii [8]), its validity depends crucially on how one conceives the operation of a monetary system and, more critically, on what one takes as variable (read "determined endogenously") in a monetary economy. My own ultimate conclusion is that extensive real indeterminacy persists despite (in the extreme case) simply imposing value to holding outside money balances, and that it will be necessary (in order to reduce or even eliminate the problem) to look much more closely at the structure of the institutions and behavior of the intermediaries that constitute the monetary -- or, more generally, financial -- sector.

A second sort of criticism has been based on the following observation. While there may be a large degree of real indeterminacy (measured, in particular, by the minimum possible dimension of the set of equilibrium allocations), this phenomenon may not be of much substantive importance (judged, for instance, against the prevalence of such indeterminacy in a comparable Walrasian environment). It is not at all obvious how one can usefully formulate this possibility in a tractable fashion. I investigate the issue by considering a sequence of economies in which relatively more and more households have access to complete financial markets. It turns out that (given my particular methodology) as the extent of market imperfection becomes insignificant, so does the substantive importance of real indeterminacy. (For a quite different approach, see also Green and Spear [7] and Zame [15]). A caution is warranted here, however. Even if (nominal or real) indeterminacy is of little substantive importance, it still presents a very difficult practical hurdle for the rational expectations hypotheses: Why should one believe households capable of concentrating their undivided attention on just one among a plethora of conceptually indistinguishable, consistent market outcomes?

In the next section I outline the basic framework for my analyses. Then, in the two subsequent sections, I consider first, the introduction of institutionalized outside money, and second, the significance of small market imperfections. Finally, in the appendix I attempt to explain -- at a fairly informal level -- what is involved in generating nominal as well as real indeterminacy when there are imperfect financial markets; this exercise can be viewed as fulfilling a secondary aim of the paper. At the outset I should emphasize that I am purposely adopting several simplifying hypotheses (which I will highlight as I go along) in order to avoid needless technical complication, and that I am purposely stressing results rather than proofs -- most of which, as the appendix also tries to indicate, are pretty simple in conception, but nonetheless pretty complicated in execution.

II. The Setting

For my purposes here it is most convenient to utilize the model with restricted participation analyzed in Balasko, Cass and Siconolfi [2]. Concerning some of its finer points (for example, the justification for the assumption A1, or the interpretation of the alternative assumptions A5′ and A5″ below) the reader is advised to consult that paper. [Note: Here I employ a more mnemonic notation than there; otherwise the models are identical.]

There are C types of physical commodities (labelled by superscript $c = 1,2,\ldots,C$, and referred to as goods), and I types of credit or financial instruments (labelled by superscript $i = 1,2,\ldots,I$, and referred to as bonds). Both goods and bonds are traded on a spot market today, while only goods will be traded on a spot market in one of S possible states of the world tomorrow (these markets are labelled by superscript $s = 0,1,\ldots,S$, so that $s = 0$ represents today and $s > 0$ the possible states tomorrow, and are referred to as spots). Thus, altogether there are $G = C(S+1)$ goods, whose quantities and (spot) prices are represented by the vectors $x = (x^0,\ldots,x^s,\ldots,x^S)$ (with $x^s = (x^{s,1},\ldots,x^{s,c},\ldots,x^{s,C})$) and $p = (p^0,\ldots,p^s,\ldots,p^S)$ (with $p^s = (p^{s,1},\ldots,p^{s,c},\ldots,p^{s,C})$), respectively.

The quantities and prices of bonds are represented by the vectors $b = (b^1,\ldots,b^i,\ldots,b^I)$ and $q = (q^1,\ldots,q^i,\ldots,q^I)$, respectively. [Note: All prices are measured in units of account, referred to as <u>dollars</u>. It will be convenient, for example, in representing dollar values of spot market transactions, to treat every price or price-like vector as a row. Otherwise I maintain the standard convention.] The typical bond, which costs q^i dollars at spot $s = 0$, promises to return a yield of $y^{s,i}$ dollars at spot $s > 0$. Let

$$Y = \begin{bmatrix} y^{1,1} & \cdots & y^{1,i} & \cdots & y^{1,I} \\ \vdots & \ddots & & & \\ y^{s,1} & & y^{s,i} & & \\ \vdots & & & \ddots & \\ y^{S,1} & & & & y^{S,I} \end{bmatrix} = \begin{bmatrix} y^1 \\ \vdots \\ y^s \\ \vdots \\ y^S \end{bmatrix}$$

$= (S \times I)$ - dimensional matrix of bond yields.

There is no loss of generality in assuming that

A1. Rank $Y = I$, <u>no redundancy</u>

which implies that $I \leq S$.

Finally, there are H households (labelled by the subscript $h = 1,2,\ldots,H$), who are described by (i) consumption sets $X_h = \mathbb{R}^G_{++}$, (ii) utility functions $u_h\colon X_h \to \mathbb{R}$, (iii) goods endowments $e_h \in X_h$ and (iv) portfolio sets $B_h \subset \mathbb{R}^I$. I assume throughout that, for $h = 1,2,\ldots,H$, A2. u_h is C^2, differentiably strictly increasing (i.e., $Du_h(x_h) >> 0$) and differentiably strictly quasi-concave (i.e., $Du_h(x_h)\Delta x = 0$ and $\Delta x \neq 0 \Rightarrow \Delta x^T D^2 u_h(x_h)\Delta x < 0$), and has indifference surfaces closed in X_h; and

A3. B_h is an I_h-dimensional linear subspace, <u>restricted participation</u> with $I_h < S$ for some h.

Let

$P = \{p \in \mathbb{R}^G_{++}\}$

= set of possible (no-free-lunch) spot goods prices,

$Q = \{q \in \mathbb{R}^I$: there is no h with $b_h \in B_h$ s.t. $\begin{bmatrix} -q \\ Y \end{bmatrix} b_h > 0\}$

= set of possible (no-financial-arbitrage) bond prices,

$\mathbf{Y} = \{Y \in \mathbb{R}^{SI}$: rank $Y = I\}$

= set of possible bond yields,

and $E = \{e = (e_1, e_2, \ldots, e_H) \in (\mathbb{R}^G_{++})^H\}$

= set of possible goods endowments (as well as allocations).

Then, given $(Y,e) \in \mathbf{Y} \times E$, $(p,q) \in P \times Q$ is a <u>financial equilibrium</u> if, when households optimize, i.e.,

given (p,q) -- and Y -- $(x_h, b_h) = (f_h(p,q,Y,e_h),\ \phi_h(p,q,Y,e_h))$ (1)

solves the problem

maximize $u_h(x_h)$

$$\text{subject to} \quad p^0(x_h^0 - e_h^0) = -qb_h,$$
$$p^s(x_h^s - e_h^s) = y^s b_h, \quad \text{for} \quad s > 0,$$
$$\text{and} \quad (x_h, b_h) \in X_h \times B_h \quad , h = 1,2,\ldots,H,$$

both spot goods and bond markets clear, i.e.,

$$\sum_h (x_h^{s,c} - e_h^{s,c}) = \sum_h (f_h^{s,c}(p,q,Y,e_h) - e_h^{s,c}) = 0, \qquad (2)$$
$$(s,c) = (0,1),(0,2),\ldots,(S,C),$$

and

$$\sum_h b_h^i = \sum_h \phi_h^i(p,q,Y,e_h) = 0, \; i = 1,2,\ldots,I. \qquad (3)$$

<u>Remarks</u> 1. Several specific aspects of this formulation greatly facilitate analyzing properties of financial equilibrium. Most notable among these are the assumptions that: (i) There are only two periods (obviously the leading case); (ii) the financial structure is exogenous (for instance, the number of bonds is given a priori), and all financial instruments are inside assets (that is, bonds are issued and redeemed by households directly) whose yields are specified in terms of dollars (which is one polar case, usually contrasted to that in which asset yields are specified in terms of bundles of goods, a kind of generalized futures contract); and (iii) the only restrictions on portfolio holdings essentially take the form of simple linear equality constraints (rather than say, more complicated bounds on borrowing which involve households' observable wealth, or even just outright prohibition of short sales in some bonds).

2. This model with restricted participation reduces to the more commonly recognized model with <u>incomplete markets</u> when $I < S$ and $B_h = \mathbb{R}^I$,

$h = 1,2,\ldots,H$. The reason I consider a more general formulation is that it easily encompasses permitting most -- but not all -- households having access to the same complete financial market when I later want to formalize the notion of small market imperfections.

The main result establishing extensive real indeterminacy in this setting requires two additional sorts of technical assumptions. The first concerns sufficiently disparate incentives for exchange of both goods and credit (in terms of numbers and also, implicitly, diversity of households), the second sufficiently flexible opportunities for exchange of credit (within the confines of restricted participation). So suppose that some group of households, say, the first H_0, faces common portfolio restrictions, say,

$$B_h = B_0,\ h = 1,2,\ldots,H_0.$$

Also, denote, for $h = 1,2,\ldots,H_0$, $I_h = I_0$, and let $D_0 = S-I_0$, an indicator of this group's <u>deficiency</u> in access to the bond market. Now assume that

A4.	$H_0 > D_0$; and $0 < I_0 < S$ (so $0 < D_0 < S$) and	<u>sufficiently numerous households</u>
A5′.	There is $b^+ \in B_0$ such that $Yb^+ \gg 0$, or	<u>possibility of positive wealth "accumulation"</u>
A5″.	Y is in general position, or	<u>possibility of maximum wealth "insurance"</u>
A5‴.	Y is variable.	<u>"endogenous" determination of bond yields</u>

Note that, under either assumption A5′ or assumption A5″, Y is taken to be exogenous or fixed. With some pair of assumptions A4 and A5$^{\bullet}$ -- given the maintained assumptions A1-A3 -- one can demonstrate a precise degree of real indeterminacy.

<u>Proposition 0</u>. Generically in endowments, the set of equilibrium allocations contains a smooth, D_0-dimensional manifold (under assumption A5′), or a smooth, (S-1)-dimensional manifold (under assumption A5″), or a smooth, $D_0 I_0$-dimensional manifold (under assumption A5‴).[3/]

III. Outside Money

In order to examine the potential role that outside money might play in reducing nominal and hence real uncertainty, I consider two of the simplest possible specifications, under what are (admittedly) the simplest possible assumptions. In the first model, outside money must be used to pay a terminal tax liability; in the second, terminal outside money balances have direct utility.

To be more specific, for $h = 1,2,\ldots,H$, let

$$e_h^m = (e_h^{m,0},\ldots,e_h^{m,s},\ldots,e_h^{m,S}) \geq 0$$

and $$x_h^m = (x_h^{m,0},\ldots,x_h^{m,s},\ldots,x_h^{m,S})$$

represent outside money endowments (at the beginning of the period) and outside money balances (at the end of the period), respectively. Also, assume that

- there is some initial outside money in the economy, i.e., $\sum_h e_h^{m,0} > 0$,

- outside money balances must be nonnegative, i.e., by fiat $x_h^m \geq 0$, $h = 1,2,\ldots,H$,

- the "price" of money is identically one (in units of account, naturally), and

- bond $i = 1$ is <u>inside</u> money, i.e., $y^{1,s} = 1$, $s = 1,2,\ldots,S$, and can be freely transacted by all households, i.e., if $b_h \in B_h$, then $b_h+(\Delta b_h^1,\ldots,0,\ldots,0) \in B_h$ for $\Delta b_h^1 \in \mathbb{R}$, $h = 1,2,\ldots,H$.

Thus, in this setting,

$$Y = [1\ Y^{\backslash 1}] \quad \text{(identified with } Y^{\backslash 1} \in \mathcal{Y}^{\backslash 1} = \{Y^{\backslash 1} \in \mathbb{R}^{S(I-1)}: \text{rank } [1\ Y^{\backslash 1}] = I\}),$$

where $Y^{\backslash 1}$ is the $(S \times (I-1)$-dimensional matrix of bond yields exclusive of inside money, and the possibility of positive wealth "accumulation" is taken for granted.[4/]

With outside money, the description of financial equilibrium must be modified, depending on how the monetary system impinges on households' preferences and opportunities. Under the first specification (say, the <u>terminal tax</u> model) (1) is replaced by

given (p,q) -- and $Y^{\backslash 1}$ -- (x_h,x_h^m,b_h) solves the problem (1^{tt})

maximize $u_h(x_h)$

$$\begin{aligned}
\text{subject to}\quad & p^0(x_h^0 - e_h^0) + (x_h^{m,0} - e_h^{m,0}) = -qb_h, \\
& p^s(x_h^s - e_h^s) + (x_h^{m,s} - [x_h^{m,0} + e_h^{m,s}]) = y^s b_h, \quad \text{for} \quad s > 0, \\
& x_h^{m,s} \geq t_h, \quad \text{for} \quad s > 0, \\
\text{and}\quad & (x_h, x_h^m, b_h) \in X_h \times X_h^m \times B_h \qquad , \; h = 1,2,\ldots,H
\end{aligned}$$

where $t_h \geq 0$ represents household h's terminal tax liability, and $X_h^m = \mathbb{R}_+^{S+1}$ by fiat, while under the second specification (say, the <u>terminal utility</u> model) (1) becomes

given (p,q) -- and $Y^{\backslash 1}$ -- (x_h, x_h^m, b_h) solves $\qquad (1^{tu})$

the problem

$$\begin{aligned}
\text{maximize}\quad & u_h(x_h, x_h^m) \\
\text{subject to}\quad & p^0(x_h^0 - e_h^0) + (x_h^{m,0} - e_h^{m,0}) = -qb_h, \\
& p^s(x_h^s - e_h^s) + (x_h^{m,s} - [x_h^{m,0} + e_h^{m,s}]) = y^s b_h, \quad \text{for} \quad s > 0, \\
\text{and}\quad & (x_h, x_h^m, b_h) \in X_h \times X_h^m \times B_h \qquad , \; h = 1,2,\ldots,H
\end{aligned}$$

where $u_h: X_h \times X_h^m \to \mathbb{R}$, with $X_h^m = \mathbb{R}_+ \times \mathbb{R}_{++}^S$, is assumed to be independent of $x_h^{m,0}$ (but otherwise to exhibit the same properties described by assumption A2). For both specifications, in addition to (2) and (3), outside money markets must also clear, i.e.,

$$\sum_h (x_h^{m,0} - e_h^{m,0}) = 0 \tag{4}$$

and $\sum_h (x_h^{m,s} - [x_h^{m,0} + e_h^{m,s}]) = \sum_h (x_h^{m,s} - [e_h^{m,0} + e_h^{m,s}]) = 0, \; s = 1,2,\ldots,S.$

Proposition 0 is easily adapted to these two models.

<u>Proposition 1</u>. (The terminal tax model) Suppose that

$$e_h^{m,s} = 0, \quad \text{for} \quad s > 0,\ h = 1,2,\ldots,H, \tag{5}$$

and

$$\sum_h (e_h^{m,0} + e_h^{m,s}) = \sum_h e_h^{m,0} = \sum_h t_h. \tag{6}$$

Then, generically in endowments (<u>excluding</u> $e^{m,0} = (e_1^{m,0},\ldots,e_h^{m,0},\ldots,e_H^{m,0})$), the set of equilibrium allocations contains a smooth, D_0-dimensional manifold, or a smooth, $(S-1)$-dimensional manifold (when $[1\ Y^{\backslash 1}]$ is in general position), or a smooth, $D_0 I_0$-dimensional manifold (when $Y^{\backslash 1}$ is variable).

<u>Remarks.</u> 1. Given (5), (1^{tt}) and (4) are only consistent also given (6). Of course, if the price of money were allowed to be different than one -- which it surely should be! -- then the terminal tax model would also be consistent when (6) is replaced by

$$\sum_h e_h^{m,0} \leq \sum_h t_h; \tag{6'}$$

strict inequality would simply entail the price of money being identically zero in equilibrium. In this case the conclusions of Proposition 1 would clearly still obtain.

2. For an economy with incomplete markets, Villanacci [13] -- among other things -- considers the more general version of the terminal tax model in which (i) inside money may not be available, and (ii) terminal tax liabilities -- interpreted as being net of second period outside money endowments, or,

more simply, transfers -- may be state dependent. Since, in this more general framework he must treat $p^{0,1}$ as variable (whereas, for simplicity, I treat it as fixed; see footnote 5 below), he finds an increase in the degree of "significant" nominal and hence real indeterminacy of one dimension (under assumption A5'').

<u>Proposition 2</u>. (The terminal utility model) Suppose that

$$e_h^{m,0} + e_h^{m,s} > 0, \quad \text{for} \quad s > 0, \; h = 1,2,\ldots,H. \tag{7}$$

Then, generically in endowments (<u>including</u> $e^m = (e_h^m,\ldots,e_h^m,\ldots,e_H^m)$), equilibrium (and hence equilibrium allocation) is locally unique, or the set of equilibrium allocations contains a smooth, $D_0(I_0-1)$-dimensional manifold (when $Y^{\backslash 1}$ is variable).

<u>Proofs of Propositions 1 and 2</u>. In either model the proof basically involves elementary accounting.

To begin with, observe that, in both models, for there to be no financial arbitrage opportunities, it must be true that $q^1 \leq 1$ (since otherwise, if $q^1 > 1$, every household could immeasurably profit by selling inside money and buying outside money at spot $s = 0$), and for the initial outside money market to clear, it must be true that $q^1 \nless 1$ (since otherwise, if $q^1 < 1$, no household would be interested in buying outside money at spot $s = 0$). So $q^1 = 1$ in equilibrium.

Now consider the terminal tax model maintaining (5) and (6). Because terminal balances of outside money are only useful for meeting terminal tax liabilities, the optimal solution to (1^{tt}) must have

$$x_h^{m,s} = t_h, \quad \text{for} \quad s > 0. \tag{8}$$

Thus, by employing the notational convention that if $z = (z^1, \ldots, z^i, \ldots, z^n)$, then $z^{\backslash 1} = (z_1^2, \ldots, z^i, \ldots, z^n)$ (introduced earlier for bond yields exclusive of inside money), and reintroducing a variable $q^1 > 0$, the budget constraints in (1^{tt}) can be rewritten

$$p^{0,1}(x_h^{0,1} - [e_h^{0,1} - (t_h - e_h^{m,0})/p^{0,1}] + p^{0\backslash 1}(x_h^{0\backslash 1} - e_h^{0\backslash 1}) = -q^1(b_h^1 - [t_h - x_h^{m,0}])/q^1 - (q^1 q^{\backslash 1})b_h^{\backslash 1}/q^1$$

and $(p^s/q^1)(x_h^s - e_h^s) = (b_h^1 - [t_h - x_h^{m,0}])/q^1 + y^{s\backslash 1} b_h^{\backslash 1}/q^1$, for $s > 0$,

or, by letting $q' = (q^1, q^1 q^{\backslash 1})$, $p^{s'} = p^s/q^1$, for $s > 0$, $e_h^{0,1'} = e_h^{0,1} - (t_h - e_h^{m,0})/p^{0,1}$ and $b_h' = (b_h^1 - [t_h - x_h^{m,0}],\ b_h^{\backslash 1})/q^1$ (and then, for simplicity, suppressing "′"), just as in (1). Moreover, the initial outside money market clearing condition can be presumed satisfied (by having households -- who are indifferent between inside and outside money -- hold suitable offsetting balances of inside money in their portfolios), while the terminal outside money market clearing conditions are, by virtue of (8), automatically satisfied. But this means that the terminal tax model effectively reduces to the original model of the preceding section (with $Y = [1\ Y^{\backslash 1}]$), and, in fact, that the original version of Proposition 0 still applies.5/

The argument for the terminal utility model is even more straightforward. Here the budget constraints in (1^{tu}) can be rewritten

$$p^0(x_h^0-e_h^0) = -q^1(b_h^1+[x_h^{m,0}-e_h^{m,0}])/q^1-(q^1q^{\backslash 1})b_h^{\backslash 1}/q^1$$

and $$p^s(x_h^s-e_h^s) + (x_h^{m,s}-[e_h^{m,0}+e_h^{m,s}[) =$$
$$(b_h^1+[x_h^{m,0}-e_h^{m,0}])/q^1+y^{s\backslash 1}b^{h\backslash 1}/q^1, \quad \text{for} \quad s > 0,$$

or, by now letting $q' = (q^1,q^1q^{\backslash 1})$, $p^{s'} = p^s/q^1$, $e_h^{m,s'} = e_h^{m,0}+e_h^{m,s}$, for $s > 0$, and $b_h' = (b_h^1+[x_h^{m,0}-e_h^{m,0}],\ b_h^{\backslash 1})/q^1$ (and again, for simplicity, suppressing "′"), just as in (1) when there C types of commodities at spot $s = 0$ but $C+1$ at each spot $s > 0$, and (with appropriate relabelling of commodities at spots $s > 0$) spot goods prices are normalized so that $p^{s,1} = 1$, for $s > 0$. So this model too effectively reduces to the original model of the preceding section (but now with both $p^{\cdot 1} = 1$ and $Y = [1\ Y^{\backslash 1}]$), and an appropriate modification of Proposition 0 again applies (a result also verified in the appendix).

What do I conclude? Simply that institutionalized outside money of and by itself doesn't necessarily eliminate real indeterminacy, and may even provide a convincing rationale for a substantially greater degree than has been proposed in the bulk of the literature on this problem. It all depends on the nature of the (market clearing) equations added when characterizing the economy's monetary equilibria, as well as the variety of the (market determined) variables introduced when characterizing the economy's monetary sector. In the almost trivial, extreme models I've considered here, either the additional equations are completely redundant -- with terminal taxes -- or they are completely determinant -- with terminal utility [6/] -- <u>provided</u> that the monetary mechanism is presumed to have no influence on any nominal variables other than spot prices, for instance, on interest rates (and thereby bond yields). Of course, this last proviso runs counter to long-standing tradition in (one might even say the essence of) monetary theory, which leads me to the position that in Propositions 1 and 2 it is precisely the case where $Y^{\backslash 1}$ is variable -- and where extensive real indeterminacy is exhibited regardless of the direct consequences of institutionalized outside money -- which is by far more natural and interesting.[7/]

IV. Small Imperfections

Now returning to the original model, assume that

- the bond market is (in principle) complete, i.e., that $I = S$, and

- there are two groups of households, the <u>unrestricted</u> households $h \in H^u \subset \{1,2,\ldots,H\}$ who can freely transact on the bond market, so that

  $$B_h = \mathbb{R}^I,\ h \in H^u,$$

  and the <u>restricted</u> households $h \in H^r = \{1,2,\ldots,H\}\backslash H^u$ who are effectively constrained in their transactions on that market, so that

  B_h is an I_h-dimensional linear subspace with $0 \leq I_h < I$, $h \in H^r$.[8/]

I will compare the financial equilibria of two distinct derivative economies. In the first, say, the <u>complete market</u> economy, only the H^u unrestricted households trade, while in the second, say, the **improved participation** economy, $j \geq 1$ replicas of the H^u unrestricted households (totalling jH^u in number) together with just the original H^r restricted households trade.[9/] The basic idea is to interpret j being sufficiently large as representing the situation in which there are only small imperfections on the bond market.

Proposition 3. As j goes to infinity, the financial equilibria in the improved participation economy converge uniformly to (a subset of) those in the complete market economy.

Remark. This result explicitly pertains to equilibria, i.e., prices. To understand its ramifications for quantities, i.e., equilibrium allocations, focus on just the unrestricted households, who, for sufficiently small imperfections, carry essentially all the weight in the economy. Then applying two well-known results, we find that in the complete market economy, generically in endowments, equilibrium allocations are locally unique.[10/] Thus, Proposition 3 basically means that in the improved participation economy, "typically" equilibrium allocations to just the unrestricted households are "almost" locally unique.[11/] I don't know -- and don't see why it is especially interesting to know -- whether the "opposite" of this proposition is also true, namely, whether every equilibrium in the complete market economy can be arbitrarily closely approximated by some equilibrium in the improved participation economy (for sufficiently large j).

Proof of Proposition 3. The argument is considerably more technical than any other in the paper. It also requires, in order to distinguish between the two economies, introducing considerably more notation. So let the original model (with H replaced by H^u and $B_h = \mathbb{R}^I$, $h \in H^u$) now describe the complete market economy. Furthermore, let

$$M = \{(p,q,e) \in P \times Q \times E: \sum_{h \in H^u} (f_h(p,q,Y,e_h) - e_h, \phi_h(p,q,Y,e_h)) = 0\}$$

= equilibrium set,

$$p^{\cdot 1} = (p^{0,1}, \ldots, p^{s,1}, \ldots, p^{S,1})$$

= vector of spot goods prices for just the first type of commodity (as introduced earlier, in footnote 5),

$$P^{\cdot 1} = \{p^{\cdot 1} \in \mathbb{R}^{S+1}_{++}\}$$

and π = restriction to M of the projection of $P \times Q \times E$ onto $P^{\cdot 1} \times E$.

[Note: We know, in fact, that the set M is a smooth, $((S+1) \times H^u G)$ - dimensional manifold, and, furthermore, that the mapping π is proper and surjective (so that, in particular, equilibria exist for every $(p^{\cdot 1},e) \in P^{\cdot 1} \times E$); see Theorem 4.1 in [2]]. For the improved participation economy with j replicas of the unrestricted households, adapt the same notation by affixing "(j)" where appropriate: For example, for this economy, spot goods prices will be denoted $p(j) \in P$, while endowments and the equilibrium set will be denoted

$$e(j) = (\overbrace{(e_h)_{h \in H^u}, \ldots, (e_h)_{h \in H^u}, \ldots, (e_h)_{h \in H^u}}^{j \text{ times}}, (e_h)_{h \in H^r}) \in$$

$$E(j) = \{e(j) \in (\mathbb{R}^G_{++})^{jH^u + H^r}\} \text{ (identified with } E = \{e \in (\mathbb{R}^G_{++})^H\})$$

and $M(j) = \{(p(j),q(j),e(j)) \in P \times Q \times E(j):$

$$j \sum_{h \in H^u} (f_h(p(j),q(j),Y,e_h) - e_h,\ \phi_h(p(j),q(j),Y,e_h)) +$$

$$\sum_{h\in H^r} (f_h(p(j),q(j),Y,e_h)-e_h,\ \phi_h(p(j),q(j),Y,e_h)) = 0\},$$

[Note: Here too we know, by the same reasoning as in the proof of Theorem 4.1 in [2], that $M(j)$ is a smooth, $((S+1)\times HG)$-dimensional manifold, and that $\pi(j)$ is proper and surjective.]

Now the argument per se proceeds in four steps. (i) Pick $\tilde{e}_h \in X_h$, $h \in H$. (ii) Pick $\tilde{P}^{\cdot 1} \subset P^{\cdot 1}$ compact. (iii) Define

$$\tilde{A} = \{a = (p,q) \in P \times Q\colon p^{\cdot 1} \in \tilde{P}^{\cdot 1} \ \&\ (p,q,\tilde{e}) \in M\},$$

$$\tilde{A}(j) = \{a(j) = (p(j),q(j)) \in P \times Q\colon p^{\cdot 1}(j) \in \tilde{P}^{\cdot 1} \ \&\ (p(j),q(j),\tilde{e}(j)) \in M(j)\} \quad \text{for } j \geq 1$$

and $\alpha(j) = \sup_{a(j)\in\tilde{A}(j)} \min_{a\in\tilde{A}} \|a-a(j)\|$.

Since π is proper, $\tilde{A}$ is compact, and $\min_{a\in\tilde{A}} \|a-a(j)\|$ is well-defined.

(iv) Show that

$$\limsup_{j\to\infty} \alpha(j) = \lim_{j\to\infty} \alpha(j) = 0,$$

i.e., that it is not the case that

$$\limsup_{j\to\infty} \alpha(j) > \varepsilon > 0.$$

This last step is established as follows. Suppose that the assertion were false, i.e., that, without any loss of generality, there were a sequence $a(j) \in \tilde{A}(j)$, $j \geq 1$, such that

$$\min_{a\in\tilde{A}} \|a-a(j)\| \geq \varepsilon,\ j \geq 1,$$

or

$$\|a-a(j)\| \geq \varepsilon \quad \text{for} \quad a\in\tilde{A},\ j \geq 1. \tag{9}$$

From an argument basically identical to that used to show that $\pi(j)$ is proper, or that, in particular, $\pi^{-1}(j)(\tilde{P}^{\cdot 1} \times \tilde{e}(j))$ is compact, it follows that we can pick a subsequence, without any loss of generality the original sequence itself, such that

$$\lim_{j\to\infty} a(j) = \lim_{j\to\infty} (p(j),q(j)) = a(\infty) = (p(\infty),q(\infty)) \in P \times Q \text{ (with } p^{\cdot 1}(\infty) \in \tilde{P}^{\cdot 1})$$

and

$$\lim_{j\to\infty} (f_h(p(j),q(j),Y,\tilde{e}_h),\ \phi(p(j),q(j),Y,\tilde{e})) = (f_h(p(\infty),q(\infty),Y,\tilde{e}_h),\ \phi_h(p(\infty),q(\infty),Y,\tilde{e}_h)),\ h \in H.$$

But since $(p(j),q(j),\tilde{e}(j)) \in M(j),\ j \geq 1,$ this implies that

$$\lim_{j\to\infty} \sum_{h\in H^u}(f_h(p(j),q(j),Y,\tilde{e}_h)-\tilde{e}_h,\ \phi_h(p(j),q(j),Y,\tilde{e}_h)) =$$

$$\sum_{h\in H^u}(f_h(p(\infty),q(\infty),Y,\tilde{e}_h)-\tilde{e}_h,\ \phi_h(p(\infty),q(\infty),Y,\tilde{e}_h)) =$$

$$\lim_{j\to\infty} (1/j) \sum_{h\in H^r}(f_h(p(j),q(j),Y,\tilde{e}_h)-\tilde{e}_h,\ \phi_h(p(j),q(j),Y,\tilde{e}_h)) = 0,$$

or that $(p(\infty),q(\infty),\tilde{e}) \in M$, or that $\lim_{j\to\infty} \|a(\infty)-a(j)\| = 0$ with $a(\infty) \in \tilde{A}$, which contradicts (9). ∎

Two aspects of the foregoing merit brief comment. First, the argument actually only applied to the subsets of equilibria in the improved participation economy for which $p^{\cdot 1}(j) \in \tilde{P}^{\cdot 1}$, where $\tilde{P}^{\cdot 1}$ is an arbitrary compact subset of $P^{\cdot 1}$. I simply have no idea whether there can be some sort of "perverse" behavior if $p^{\cdot 1}(j)$ goes to $p^{\cdot 1}(\infty) \in \partial \bar{P}^{\cdot 1}$ (i.e., the boundary of the closure of $P^{\cdot 1}$) as j goes to infinity. Second, while the argument was carried out for fixed or exogenous Y, it can be easily extended to cover the case of variable or endogenous Y (restricted to $\tilde{Y} \subset Y$ compact).

Appendix

The purpose of this appendix is to provide some insight into the rationale for nominal indeterminacy, as well as the logic supporting its natural translation into a corresponding degree of real indeterminacy (barring exceptional circumstances). Though the results being presented can be derived formally, my express aim here is to try and explain them in a fairly casual, intuitive manner.

__Nominal Indeterminacy__

The central idea in establishing this result is pure and simply to "count equations and unknowns". That is, the essence of the analysis involves treating the market clearing conditions (2)-(3) as a system of equations in the whole collection of variables p,q,Y and e and then -- after verifying certain crucial prerequisites (by utilizing several basic techniques from differential topology) -- employing the old workhorse of economic theory, the implicit function theorem. In following this program I intentionally give short shrift to the details of the underlying justification for treating particular price variables as being "dependent" (and the other price cum "fundamental" variables as being "independent") -- the "certain crucial prerequisites" referred to just above.

Recall what, roughly, the implicit function theorem asserts: Suppose that we are given a system of J (independent) equations in K (explicit) variables, so that necessarily $J \leq K$ (and the equations, being defined by sufficiently smooth functions of the specified variables, have Jacobian of full rank J at some particular solution). Then, locally, the system can be solved for J (distinguished) variables as continuously differentiable functions of the other K-J variables. Thus, my task basically amounts to calculating J,K and K-J for the particular system of equations at hand. In carrying out this task it is quite instructive to begin by recalling a more familiar example, that arising from the standard Walrasian model.

So now suppose that, instead of trading on many spot goods and the bond markets, households trade on a single "overall" market for current and future contingent goods. For simplicity letting the previous notation also represent prices and allocations for such an economy, then, here, given $e \in E$, $p \in P$ is a <u>Walrasian equilibrium</u> if, when households optimize (according to the usual budget-constrained, utility-maximization problem), i.e.,

$$\begin{aligned} &\text{given } p,\ x_h = g_h(p,e_h) \text{ solves the problem} \\ &\qquad \text{maximize} \quad u_h(x_h) \\ &\qquad \text{subject to} \quad p(x_h - e_h) = 0 \\ &\qquad \text{and} \quad x_h \in X_h \quad , h = 1,2,\ldots,H, \end{aligned} \tag{A1}$$

just the overall market for goods clears, i.e.,

$$\sum_h (x_h^{s,c} - e_h^{s,c}) = \sum_h (g_h^{s,c}(p,e_h) - e_h^{s,c}) = 0,\ (s,c) = (0,1),(0,2),\ldots,(S,C). \tag{A2}$$

In this setting, from the restriction imposed by the budget constraint in (A1) it follows that the market clearing conditions (A2) yield only G-1 independent equations (Walras' law), while p and e constitute the only explicit variables. Thus, obviously J = G-1 and K = G+HG, and (locally), say, the K-J = 1+HG variables $p^{0,1}$ and e uniquely determine the remaining prices p less $p^{0,1}$. In other words, there is one degree of nominal indeterminacy, the choice of the "price level", represented by $p^{0,1}$. Of course, from the linear homogeneity of the budget constraint in (A1) it also follows that such nominal indeterminacy is "insignificant", in the sense that it never engenders any real indeterminacy. For this reason it is conventional to normalize prices, for instance, by setting $p^{0,1} = 1$, and to maintain that equilibrium is locally unique (up to a "harmless" choice of numeraire), or that, say, there are no <u>significant</u> degrees of nominal indeterminacy in the Walrasian model. While I will also adopt this position in discussing the model with restricted participation, I repeat for emphasis that it is, from a practical viewpoint, quite misleading; even such "insignificant" nominal indeterminacy raises havoc for presupposing rational expectations (a very important message, but one I will hereafter take as having been delivered).

In applying similar reasoning to the model summarized by (1) - (3), it turns out that the only potential complication involves figuring out the number of significant price (or price-like) variables. So, now returning to consideration of this system, we first notice that the restrictions imposed by the budget constraints in (1) render $S+1$ of the market clearing conditions (2) - (3) redundant (for instance, those concerning just the first type of good). Since these equations number altogether $G+I$, $J = G+I-(S+1)$, while clearly, since in general p, q, Y and e are all variable, $K = G+I+SI+HG$. Thus (locally), say, the $K-J = (S+1)+SI+HG$ variables $p^{\cdot 1}, Y$ and e uniquely determine the remaining spot prices p less $p^{\cdot 1}$, and there are apparently $(S+1)+SI$ degrees of nominal indeterminacy. In order to explain why exactly 2 (when Y is exogenous or fixed) or $(S+1)+I^2$ (when Y is endogenous or variable) degrees of such nominal indeterminacy are "insignificant", it is very helpful (and indeed indispensable) to digress a moment and reformulate the budget constraints in (1). The particular reformulation I have chosen to elaborate will also be very convenient for the discussion in the succeeding subsection.

Focus on the budget constraints of a typical household h,

$$p^0(x_h^0-e_h^0) = -qb_h \tag{A3}$$
$$\text{and}\quad p^s(x_h^s-e_h^s) = y^s b_h, \quad \text{for}\quad s > 0,$$

and consider the following two-step transformation (both steps of which leave the household's consumption opportunities unaltered):

<u>Step 1</u>. Divide each of the budget constraints in (A3) by its corresponding spot price for the first type of good:

$$\bar{p}^0(x_h^0-e_h^0) = (-q/p^{0,1})b_h \tag{A4}$$
$$\text{and}\quad \bar{p}^s(x_h^s-e_h^s) = (y^s/p^{s,1})b_h, \quad \text{for}\quad s > 0,$$

where $\bar{p}^s = p^s/p^{s,1} = (1,p^{s,2}/p^{s,1},\ldots,p^{s,c}/p^{s,1},\ldots,p^{s,C}/p^{s,1}),\ s = 0,1,\ldots,S.$

<u>Step 2</u>. Assume, without loss of generality (by using assumption A1 and relabelling spots appropriately), that the last I rows of Y are linearly independent, so that we can partition

$$Y = \begin{bmatrix} \tilde{Y} \\ --- \\ \tilde{\tilde{Y}} \end{bmatrix} = \begin{bmatrix} y^1 \\ \cdot \\ \cdot \\ \cdot \\ y^D \\ --- \\ y^{D+1} \\ \cdot \\ \cdot \\ \cdot \\ y^{D+I} \end{bmatrix},$$

where $D = S-I$ and thus $\tilde{\tilde{Y}}$ is an I^2-dimensional, full-rank matrix. Then (implicitly referring to assumption A3), reduce the right-hand side of (A4) by transforming from the variables b^i to the variables $b^{i'} = (y^{D+i}/p^{D+i,1})b_h$, $i = 1,2,\ldots,I$, as follows:

$$\begin{bmatrix} -q/p^{0,1} \\ --- \\ y^1/p^{1,1} \\ \cdot \\ \cdot \\ \cdot \\ y^D/p^{D,1} \\ --- \\ y^{D+1}/p^{D+1,1} \\ \cdot \\ \cdot \\ \cdot \\ y^{D+I}/p^{D+I,1} \end{bmatrix} b_h = \begin{bmatrix} -(1/p^{0,1})q \\ \begin{pmatrix} 1/p^{1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D,1} \end{pmatrix} \tilde{Y} \\ \begin{pmatrix} 1/p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D+I,1} \end{pmatrix} \tilde{\tilde{Y}} \end{bmatrix} b_h \qquad \text{(A5)}$$

$$= \begin{bmatrix} \left(\begin{pmatrix} -(1/p^{0,1})q \\ \begin{bmatrix} 1/p^{1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D,1} \end{bmatrix} \tilde{Y} \end{pmatrix} \left(\begin{bmatrix} 1/p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D+I,1} \end{bmatrix} \tilde{\tilde{Y}} \right)^{-1} \right) \\ I \end{bmatrix} b_h'$$

$$= \begin{bmatrix} -(1/p^{0,1})q \, \tilde{\tilde{Y}}^{-1} \begin{pmatrix} p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & p^{D+I,1} \end{pmatrix} \\ \begin{pmatrix} 1/p^{1,1} & & 0 \\ & \ddots & \\ 0 & & 1/p^{D,1} \end{pmatrix} \tilde{Y} \, \tilde{\tilde{Y}}^{-1} \begin{pmatrix} p^{D+1,1} & & 0 \\ & \ddots & \\ 0 & & p^{D+I,1} \end{pmatrix} \\ I \end{bmatrix} b_h'$$

$$= \begin{bmatrix} -q' \\ \tilde{Y}' \\ I \end{bmatrix} b_h',$$

where

$$q' = q \, \tilde{\tilde{Y}}^{-1} \begin{bmatrix} p^{D+1,1}/p^{0,1} & & 0 \\ & \ddots & \\ 0 & & p^{D+I,1}/p^{0,1} \end{bmatrix}$$

and

$$\tilde{Y}' = -[\omega^{j,k}(p^{D+k,1}/p^{j,1}), j = 1,2,\ldots,D, k = 1,2,\ldots,I] \quad \text{with} \quad \Omega = -\tilde{Y}\,\tilde{\tilde{Y}}^{-1}; \tag{A6}$$

the reason for the sign change in the definition of Ω will become clear below. [Note: It is easily seen that if q are no-financial arbitrage prices for bond yields $Y = \begin{bmatrix} \tilde{Y} \\ --- \\ \tilde{\tilde{Y}} \end{bmatrix}$, then q' are also no-financial arbitrage prices for bond yields $Y' = \begin{bmatrix} \tilde{Y}' \\ --- \\ I \end{bmatrix}$. This means that, in (A5), for all practical purposes we can safely ignore the genesis of $q' \in \mathbb{R}^I$ -- but of course not at all that of $\tilde{Y}' \in \mathbb{R}^{DI}$.]

Letting

$$\bar{P} = \begin{bmatrix} \bar{p}^0 & & 0 \\ & \ddots & \\ 0 & & \bar{p}^S \end{bmatrix}$$

and $R' = \begin{bmatrix} -q' \\ \tilde{Y}' \\ I \end{bmatrix}$,

and then substituting from (A5) into (A4) (while rewriting "b_h" for "b_h'"), the budget constraints in (1) can be compactly reformulated as

$$\bar{P}(x_h - e_h) = R'b_h. \tag{A7}$$

Now simply notice that, by virtue of the structure of $\tilde{Y}'$ displayed in (A6), nothing significant is lost by assuming -- when Y is fixed -- that, for instance, $p^{0,1} = p^{D+1,1} = 1$, or -- when Y is variable -- that, for instance, $p^{\bullet 1} = 1$ and $\tilde{\tilde{Y}} = I$ (so that, in (A6), $\tilde{Y}' = -\Omega = \tilde{Y}$, a $(D \times I)$-dimensional matrix): There are only $(S+1)-2 = S-1$, or $[(S+1)+SI]-[(S+1)+I^2] = DI$ (significant) degrees of nominal indeterminacy, respectively.

Finally, to show that, with inside money (where $Y = [1\ Y^{\backslash 1}]$ by definition, so that only $Y^{\backslash 1}$ and not all of Y is potentially variable), nominal indeterminacy is unaffected in the terminal tax model, and reduced by just D degrees in the terminal utility model (where, in demonstrating its relation to the original model, we also introduced the restriction $p^{\bullet 1} = 1$ by definition), I make two observations. First (referring to the terminal tax model), the immediately preceding argument -- when Y is variable -- can be directly recast in terms of fixing $p^{0,1} = 1$ and, for instance, $y^{1,s} = 1$, for $s > 0$, rather than $p^{\bullet 1} = 1$ (since here the force of the argument is unaffected by having restricted Y to vary in $\{Y \in \mathcal{Y}: y^{1,s} > 0,$ for $s > 0\}$, an open subset of $\mathcal{Y}$). Second (now referring to the terminal utility model), notice that, given $p^{\bullet 1} = 1$, there is a

$$Y^{\backslash 1} = \begin{bmatrix} \tilde{Y}^{\backslash 1} \\ - - - \\ \tilde{\tilde{Y}}^{\backslash 1} \end{bmatrix} \in \{Y^{\backslash 1} \in \mathcal{Y}^{\backslash 1}:\ \text{rank}\ [\ 1\ \tilde{\tilde{Y}}^{\backslash 1}] = I\}$$

(an open subset of $\mathcal{Y}^{\backslash 1}$) such that

$$\tilde{Y}' = -\Omega = [1\ \tilde{Y}^{\backslash 1}]\ [1\ \tilde{\tilde{Y}}^{\backslash 1}]^{-1}$$

if and only if

$$\Omega \in \boldsymbol{\Omega} = \{\Omega \in \mathbb{R}^{DI}: \sum_k \omega^{j,k} = -1,\ j = 1,2,\ldots,D\},$$

so that -- again referring to the structure of $\tilde{Y}'$ displayed in (A6) -- here nothing significant is lost by parameterizing the effects, given $p^{\cdot 1} = 1$, of varying $Y^{\backslash 1}$ within $\mathbf{Y}^{\backslash 1}$ (whose dimension is $S(I-1)$) in terms of varying Ω within $\boldsymbol{\Omega}$ (whose dimension is also $S(I-1)$). Hence, in this model, there are only $D(I-1)$ degrees of (significant) nominal indeterminacy.

Table 1 summarizes the foregoing enumeration, and should aid in digesting it.

<u>Its Translation into Real Indeterminacy</u>

In order to see how the conclusions of Proposition 0 follow from the existence of these two possible degrees of (significant) nominal indeterminacy, it is convenient to introduce a simplifying change in notation that reduces the analysis to that for the model with incomplete markets, as well as an additional piece of notation that permits concentrating on just significant price or yield variation.

- Consider the households $h = 1,2,\ldots,H_0$ distinguished by the common portfolio restrictions $B_h = B_0$ with $0 < \dim B_0 = I_0 < S$. By virtue of assumption (A3), there is no loss of generality in assuming,

Nominal Indeterminacy

	Walrasian Equilibrium	Financial Equilibrium: Y fixed	$Y^{\backslash 1}$ variable (with $p^{\cdot 1}=1$)	Y variable
Equations	$\sum_h (g_h^{s,c}(p,e_h)-e_h^{s,c}) = 0$, all (s,c)		$\sum_h (f_h^{s,c}(p,q,Y,e_h)-e_h^{s,c})=0$, all (s,c) and $\sum_h \phi_h^i(p,q,Y,e_h)=0$, all i	
no. of equations	G		G+I	
interdependencies (no. of budget constraints)	1		S+1	
J = no. of independent equations	G-1		G+I-(S+1)	
Variables	p,e	p,q,e	p less $p^{\cdot 1}$,q,$Y^{\backslash 1}$,e	p,q,Y,e
no. of variables	G+HG	G+I+HG	[G-(S+1)]+I+S(I-1)+HG	G+I+SI+HG
insignificancies (no. of price "normalizations")	1	2	I(I-1)	$(S+1)+I^2$
K = no. of significant variables	(G-1)+HG	(G-2)+I+HG	[G-(S+1)]+I+D(I-1)+HG	[G-(S+1)]+I+DI+HG
K-J = no. of "independent" and significant variables	HG, say, e	(S-1)+HG, say, $p^{\cdot 1}$(s.t. $p^{0,1}=p^{1,1}=1$) & e	D(I-1)+HG say, Ω(s.t. $\Omega 1=-1$) & e	DI+HG, say, Ω & e
K-J-HG = degree of (significant) nominal indeterminacy	0	S-1	D(I-1)	DI

Table 1

in fact, that

$$B_0 = \{b \in \mathbb{R}^I : b^i = 0,\ i = I_0+1, I_0+2, \ldots, I\}.$$

Since I will be almost exclusively concerned with just this group of households, hereafter I simply let unsubscripted notation (in particular, I, $D = S-I$ and $B = \mathbb{R}^I$) represent their pertinent characteristics -- in effect dropping, from their own viewpoint, irrelevant financial opportunities.

- Consider the, transformed representation of bond yields in (A6), the matrix $\tilde{Y}'$. Since I will only be concerned with perturbing $p^{\cdot 1}$ (with $p^{0,1} = p^{D+1,1} = 1$, for Y fixed) or Y (with $p^{\cdot 1} = 1$ and $\tilde{\tilde{Y}} = I$, for Y variable), let

$$\omega = (p^{1,1}, \ldots, p^{D,1}, 1, p^{D+2,1}, \ldots, p^{D+I,1})$$

as well as

$$\Omega = - \tilde{Y}\, \tilde{\tilde{Y}}^{-1}.$$

Then, for simplicity suppressing the superfluous "$'$" (since q' depends only indirectly on ω and Ω under either hypothesis), we can rewrite $\tilde{Y}'$ and R' as

$$\tilde{Y} = \tilde{Y}(\omega,\Omega) = -\begin{bmatrix} \omega^{1,1}(1/\omega^1) & \ldots & \omega^{1,k}(\omega^{D+k}/\omega^1) & \ldots & \omega^{1,I}(\omega^{D+I}/\omega^1) \\ \vdots & \ddots & & & \\ \omega^{j,1}(1/\omega^j) & & \omega^{j,k}(\omega^{D+k}/\omega^j) & \ldots & \omega^{j,I}(\omega^{D+I}/\omega^j) \\ \vdots & & & \ddots & \\ \omega^{D,1}(1/\omega^D) & & & & \omega^{D,I}(\omega^{D+I}/\omega^D) \end{bmatrix} \quad \text{(A8}$$

$$\text{and} \quad R = R(q,\omega,\Omega) = \begin{bmatrix} -q \\ \tilde{Y}(\omega,\Omega) \\ I \end{bmatrix},$$

respectively.

The particular approach I prefer basically involves analyzing the overall implications of the households' personalized no-financial arbitrage conditions (which derive from the Lagrangean characterization of the optimal solution to (1) after the budget constraints have been reformulated according to (A7); cf., again, [1]). [Note: Alternatively, another approach basically involves analyzing the overall implications of the budget constraints themselves; cf. Geanakoplos and Mas-Colell [6]. In my opinion this second approach is not nearly as efficient (or powerful) for drawing conclusions about properties of the mapping, say, for Y fixed, $f:M \to E$ such that $(p,q,e) \to (x_1,\ldots,x_h,\ldots x_H) = f(p,q,Y,e) = (f_1(p,q,Y,e_1),\ldots,f_h(p,q,Y,e_h),\ldots,f_H(p,q,Y,e_H))$, where $M \subset P \times Q \times E$ again represents the equilibrium set, so that $f(M)$ represents the corresponding allocation set.] Associating the Lagrange multipliers

$\lambda_h = (\lambda_h^0, \ldots, \lambda_h^s, \ldots, \lambda_h^S) \in \mathbb{R}_{++}^{S+1}$ with the constraints (A7) -- and incorporating the hypothesis that $B_h = B$ -- the first-order conditions for (1) become

$$Du_h(x_h) = \lambda_h \bar{P} \tag{A9}$$

and

$$\lambda_h R = \tilde{\lambda}_h \begin{bmatrix} -q \\ \tilde{Y} \end{bmatrix} + \tilde{\tilde{\lambda}}_h = 0$$

or

$$\lambda_h = \tilde{\lambda}_h \begin{bmatrix} I & \begin{pmatrix} q \\ -\tilde{Y} \end{pmatrix} \end{bmatrix}, \tag{A10}$$

where, as before, I partition $\lambda_h = (\tilde{\lambda}_h, \tilde{\tilde{\lambda}}_h) = ((\lambda_h^0, \ldots, \lambda_h^D), (\lambda_h^{D+1}, \ldots, \lambda_h^{D+I}))$. Substituting from (A10) into (A9) yields the fundamental construct for verifying the generic existence of precise degrees of real indeterminacy,

$$Du_h(x_h) = \tilde{\lambda}_h \begin{bmatrix} I & \begin{pmatrix} q \\ -\tilde{Y} \end{pmatrix} \end{bmatrix} \bar{P}, \; h=1,2,\ldots,H_0. \tag{A11}$$

The key mechanism for generating real from nominal indeterminacy is, in principle, quite simple. Perturbations of the S-1 spot prices ω (typically) or the, say, reduced form bond yields Ω (generally) alter the linear subspace orthogonal to that spanned by the columns of R (alternatively, and equivalently, the latter subspace itself). But this in turn (typically) changes the equilibrium allocations consistent with R

according -- in particular -- to (A11) (alternatively, to (A7)). In order for this chain process to actually work out it must be the case, first, that R is sufficiently sensitive to price or yield variation, and second, that as a whole, households are sufficiently sensitive to their altered financial opportunities. Assumption A5 is designed to guarantee the former, and assumption (A4) (together with enough variety of endowments, given preferences) the latter. Before describing how these two assumptions operate, it is quite illuminating to look at several examples in which, though (significant) nominal indeterminacy is pervasive, it doesn't necessarily induce any real indeterminacy. For simplicity, I always suppose that $H = H_0$.

Example 1. Fully Complete Markets: Suppose that $I=S$. Then (adapting previous usage in the natural way), $Y = \tilde{\tilde{Y}}$ and

$$R = \begin{bmatrix} -q \\ I \end{bmatrix}$$

and clearly, whether Y is fixed or variable, since R is essentially independent of both ω and Ω, there is purely nominal indeterminacy. I must reemphasize here, however, that even in such an idyllic situation, households would still surely be in a real quandary about what prices they could reasonably expect in the future (as they certainly are in any actual economy!).

Example 2. Fully Incomplete or Restricted Markets: Suppose that $I=0$ or, to the same end, that $0 < I \leq S$ but $Y = 0$. Then (again adapting previous usage in the natural way), necessarily $q = 0$, so that

$$R = 0,$$

and we have precisely the same outcome as in the opposite case where there are fully complete markets.

Example 3. Incomplete Markets with Arrow Securities (for a Subset of Future Spots): Suppose that $0 < I < S$ and

$$y^{s,i} = \begin{cases} 1, & \text{for } s=D+i,\ i=1,2,\ldots,I \\ 0, & \text{otherwise.} \end{cases}$$

Then, $Y = \begin{bmatrix} 0 \\ I \end{bmatrix}$, so $\Omega = 0$ and

$$R = \begin{bmatrix} -q \\ 0 \\ I \end{bmatrix},$$

and again, clearly, for Y fixed, since R is independent of ω, there is purely nominal indeterminacy.

<u>Example 4</u>. <u>Incomplete Markets with Inside Money plus a Subset of Arrow Securities</u>: Suppose, slightly modifying the previous example, that bond 1 is inside money (rather than the Arrow security paying off at spot D+1). Then,

$$Y = \begin{bmatrix} & 0 \\ 1 & \\ & I \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ --- \\ \tilde{\tilde{Y}} \end{bmatrix} = \begin{bmatrix} (1 & & 0) \\ - & - & - \\ \begin{pmatrix} (1,0,\ldots,0) \\ 1 \qquad I \end{pmatrix} \end{bmatrix}$$

with

$$\tilde{\tilde{Y}}^{-1} = \begin{bmatrix} (1,0,\ldots,0) \\ -1 \qquad I \end{bmatrix},$$

so

$$\Omega = -[1 \quad 0] \qquad \text{and} \qquad R = \begin{bmatrix} -q \\ \begin{pmatrix} 1/\omega^1 & \\ 1/\omega^2 & \\ \vdots & 0 \\ 1/\omega^D & \end{pmatrix} \\ I \end{bmatrix}.$$

In this example, for Y fixed, only perturbations of the first D elements of ω, say, $\tilde{\omega} = (\omega^1, \omega^2, \ldots, \omega^D)$, affect R and therefore possibly generate real indeterminacy; Y satisfies assumption A5′ but not A5″. [Note:

Generally, these two assumptions are not nested, so either can be satisfied when the other is not.]

Example 5. Pareto Optimality: Suppose that e is a Pareto optimal allocation (which is always true when $H=1$). Then, clearly, since the only equilibrium allocation is autarky, households' equilibrium behavior is independent of R, and there is, once again, purely nominal indeterminacy.

To see what can be learned from the first four examples -- and understand why perturbations of ω or Ω alter the column span of R', say, for simplicity, **span** R, and thereby its orthogonal complement as well, that is, for short, why such perturbations are **effective** -- it is important to bear in mind that, for this analysis, when Y is fixed, then $\Omega = -\tilde{Y}\tilde{\tilde{Y}}^{-1}$ is also fixed, and only ω is perturbed, while when Y is variable, then $\omega=1$ itself is fixed, and only Ω is perturbed.

From examination of the examples it is apparent that in each of the first three the difficulty is simply that no permissible perturbation is effective, while in the fourth, that only certain permissible perturbations (namely, of the subvector $\tilde{\omega}$) are effective. More generally, and equally apparent from examination of R as displayed in (A8), is why assumptions A5' and A5'' guarantee that perturbations of $\tilde{\omega}$ and ω, respectively, are effective. In the first instance, $Yb^{+} \gg 0$ is equivalent to $(\tilde{Y}\,\tilde{\tilde{Y}}^{-1})b^{+\prime} \gg 0$, where $b^{+\prime} = \tilde{\tilde{Y}}b^{+}$, and assumption A5' is tantamount to assuming that at least one element in each row of Ω is nonzero. Hence, for $j=1,2,\ldots,D$,

$$\omega^{j,k} \neq 0 \;\&\; \omega^{j\prime} \neq \omega^{j\prime\prime} \quad (\text{with } \omega^{s\prime} = \omega^{s\prime\prime} \text{ for } s=D+k, k=1,2,\ldots,I) \Rightarrow$$

$R(q',\omega',\Omega)b \notin \text{span } R(q'',\omega'',\Omega)$, for $b^i \neq 0$, $i = k$, $= 0$, otherwise.

In the second instance, assumption A5″ implies that every element of Ω is nonzero. Hence, for $j=1,2,\ldots,D, k=2,3,\ldots,I$,

$\omega^{j,k} \neq 0$ (resp. $\omega^{j,1} \neq 0$), $\omega^{j'} = \omega^{j''}$ & $\omega^{D+k'} \neq \omega^{D+k''}$ (resp. $\omega^{j'} \neq \omega^{j''}$) $\Rightarrow R(q',\omega',\Omega)b \notin \text{span } R(q'',\omega'',\Omega)$, for $b^i \neq 0$, $i = k$ (resp. $i = j$), $= 0$, otherwise.

[Note: To say that "Y is in 'general position'" means precisely that every I^2-dimensional submatrix of Y has full rank. This condition is violated if, for some (j,k), $\omega^{j,k} = 0$, because then replacing the k^{th} row in $\tilde{\tilde{Y}}$ with the j^{th} row in $\tilde{Y}$ yields an I^2-dimensional submatrix with rank I-1.]

It should now be more or less obvious why assumption A5‴ works equally well:

$\omega^{j,k'} \neq \omega^{j,k''} \Rightarrow R(q',\omega,\Omega')b \notin \text{span } R(q'',\omega,\Omega'')$ for $b^i \neq 0$, $i = k$, $= 0$, otherwise.

Finally, it is also worthwhile mentioning explicitly that while assumption A5′ (but not A5″) has economic content -- and thus permits arguing for real indeterminacy without mathematical artifice -- it also entails a weaker result than assumption A5″, since it only provides a lower bound on the degree of real indeterminacy. (For further analysis of various specific alternative assumptions about the structure of bond yields, see, especially, Werner [14].)

Once it has been determined that suitable perturbations of spot prices or bond yields are effective, the rest is easy (at least in conception). With enough households, as specified by assumption A4, the property that

$$\text{rank } [Du_h(x_h), h=1,2,\ldots,H_0] = \text{rank } [\tilde{\lambda}_h, h=1,2,\ldots,H_0] = D+1, \qquad \text{(A12)}$$

which is its maximal value, is -- like the property that $p^{\cdot 1}$, Y and e can be taken as "independent" variables -- generic in endowments. That is, this "rank" property obtains on an open, dense subset of E. [Note: of course, Example 5 illustrates the difficulty when (A12) fails, since, if x is a Pareto optimal allocation, then

$$\text{rank } [Du_h(x_h),\ h=1,2,\ldots,H_0] = \text{rank } [Du_h(x_h),\ h=1,2,\ldots,H] = 1. \qquad \text{(A13)}$$

This last result therefore also ratifies the intuition that, in the presence of restricted participation, the coordination required by (A13) is most unlikely.] So, locally, variation in ω or Ω (by means of perturbing the overall returns exhibited in (A8)) must typically map diffeomorphically into variation in x (by virtue of satisfying the gradient restrictions exhibited in (A11)). As in the preceding subsection, I give short shrift to the argument supporting this last step -- which again amounts to utilizing basic techniques from differential topology. For a more detailed account, the interested reader is once more referred to [1].

Concluding Comment

The particular sort of financial instruments considered in this paper, for which underlying asset yields are specified in terms of units of account, is the most convenient for demonstrating a precise degree of real indeterminacy. However, it is only in the contrasting polar case, for which underlying asset yields are specified in terms of bundles of goods (which are also taken as exogenous or fixed) -- so that the asset yields themselves are linear homogeneous in corresponding spot prices -- that all nominal indeterminacy is necessarily "insignificant". Some interesting analysis of real indeterminacy in various intermediate cases can be found in [1, pp. 148-9], [6, pp. 36-8] and, especially (the most general treatment available), Pietra [10].

Footnotes

* This paper, intended to be a modest contribution to the theory of general competitive equilibrium, is dedicated to Lionel McKenzie. The importance of Lionel's seminal influence on the modern development of this fundamental discipline almost requires no elaboration (and is, anyway, very amply detailed elsewhere in this volume). More personally, Lionel has been an incomparable role model for me, as well as my friend and benefactor.

Research support from the NSF under grant #SES 88-08524 is gratefully acknowledged, as are useful comments by numerous (supportive and yet critical) colleagues.

1. Two somewhat different kinds of surveys of this literature can be found in Cass [4] and Geanakoplos [5]. Both contain extensive references.

2. This branch developed from an example I constructed that emphasized the possible sunspot interpretation of this phenomenon (see Cass [3]). Generalization and refinement (as of this date) can be found in Balasko and Cass [1], Balasko, Cass and Siconolfi [2], Geanakoplos and Mas-Colell [6], Mas-Colell [9], Pietra [10], Polemarchakis [11], Siconolfi and Villanacci [12] and Werner [14]; the list is not exhaustive.

3. This is essentially Theorem 5.1 in [2]. The very last assertion, however, requires straightforward modification of the proof of Theorem 5.3 in Balasko and Cass [1]; it will be central to the argument in the subsequent section.

4. Notice that postulating the existence of inside money is equivalent to assumption A5′ (for an appropriate choice of units of account), since the households' financial opportunities are unaffected by replacing any

particular bond with a fixed portfolio that includes that bond (a sort of mutual fund). As far as I can see, this is a pretty innocuous assumption, taken by itself. Much less innocuous is the further assumption that all households can freely transact on the market for inside money.

5. Here two minor technical points need checking: First, in the transformed model, $p^{0,1}$ must be large enough to guarantee (implicitly) that $e_h^{0,1} > 0$ (i.e., in the terminal tax model itself, to guarantee that $e_h^{0,1}-(t_h-e_h^{m,0})/p^{0,1} > 0$). This interiority restriction can be satisfied by utilizing the fact that, in the original model of the preceding section, $p^{\cdot 1} = (p^{0,1},\ldots,p^{s,1},\ldots,p^{S,1})$ as well as e can be treated as parameters (see [2], Section 4.3). Second, in the transformed model, appropriate perturbations of $p^{\cdot 1}$ and $Y^{\setminus 1}$ must still yield D_0I_0 "significant" degrees of nominal indeterminacy. This result, involving routine linear algebra, is verified in the appendix.

6. An even more direct derivation of this seemingly strong conclusion follows upon just appending spot-by-spot price normalization to the original model -- the crudest version of the venerable quantity theory of money -- for instance, in the form of the additional equations

$$\sum_c \alpha^{s,c} p^{s,c} = \beta^s,$$

where $\alpha^s = (\alpha^{s,1},\ldots,\alpha^{s,c},\ldots,\alpha^{S,C}) > 0$ and $\beta^s > 0$, $s = 0,1,\ldots,S$. This is, for all practical purposes, the route followed by Magill and Quinzii [8].

7. I recognize that it makes sense to consider variation in $\begin{bmatrix} -q \\ Y \end{bmatrix}$ as being limited by specific functional characteristics of the financial sector. Nonetheless, the fundamental conclusion, that endogenous bond yields -- or, more generally, financial instruments -- contribute to extensive real indeterminacy seems to me to be quite robust. My position receives some support from Proposition 2 itself: To be concrete, suppose that $S = 3$, $I = 2$, $H > 1$ and the market imperfection is purely incompleteness. When bond 1 is inside money, so that $y^{1,s} = 1$, $s = 1,2,3$, but bond 2 a variable rate loan, so that $y^{2,s} = 1+r^s > 1$, $s = 1,2,3$, the set of equilibrium allocations will still typically contain a continuum.

8. For simplicity, I will also denote the numbers of unrestricted and restricted households by H^u and H^r, respectively (and hereafter, for symmetry, also the set of all households by $H = H^u \cup H^r$). No confusion should result.

9. Equivalently, in this second type of economy there are $J \geq 1$ replicas of the H^u unrestricted households and $K \leq J$ replicas of the H^r restricted households, and $j = J/K \geq 1$.

10. The two results required are that (i) equilibrium allocations in the complete market economy are identical to those in the corresponding Walrasian economy with the same fundamentals (i.e., preferences and endowments), and (ii) in a smooth Walrasian economy, generically in endowments, equilibria (up to price normalization) and hence equilibrium allocations are locally unique. [Note: Here and in the text, translating from prices to allocations only utilizes the underlying result that demand functions are smooth].

11. "Typically" also refers to just endowments of the unrestricted households. I doubt that much can be inferred regarding the asymptotic behavior of equilibrium allocations to the restricted households. There is simply no reason to expect their goods demands to be systematically delimited on the sets of (financial) equilibrium prices

$$\{(p,q) \in P \times Q: p = (p^{0\prime}, p^{1\prime}/\lambda^1, \ldots, p^{s\prime}/\lambda^s, \ldots, p^{S\prime}/\lambda^S) \text{ with } \lambda = qY^{-1} \gg 0\}$$

associated with equilibrium prices p' in the corresponding Walrasian economy -- as they certainly are for the unrestricted households (precisely the content of the first result mentioned in the preceding footnote).

References

1. Balasko Y, Cass D (1989) The structure of financial equilibrium with exogenous yields: The case of incomplete markets. Econometrica 57: 135-162.

2. Balasko Y, Cass D, Siconolfi P (1990) The structure of financial equilibrium with exogenous yields: The case of restricted participation. J Math Econ 19: 195-216.

3. Cass D (1989) Sunspots and incomplete financial markets: The leading example. In Feiwel G (ed) The economics of imperfect competition: Joan Robinson and beyond. MacMillan, London.

4. Cass D (1990) Perfect equilibrium with incomplete financial markets: An elementary exposition. In McKenzie LW, Zamagni S (eds) "Value and Capital" fifty years later. MacMillan, London.

5. Geanakoplos J (1990) An introduction to general equilibrium with incomplete asset markets. J Math Econ 19: 1-38.

6. Geanakoplos J, Mas-Colell A (1989) Real indeterminacy with financial assets. J Econ Theory 47: 22-38.

7. Green RC, Spear SE (1989) Equilibria in large commodity spaces with incomplete financial markets. Preliminary paper, Carnegie-Mellon University.

8. Magill M, Quinzii M (1988) Real effects of money in general equilibrium. MRG Working Paper, University of Southern California.

9. Mas-Colell A (1989) Indeterminacy in incomplete market economies. Preliminary paper, Harvard University.

10. Pietra T (1988) Indeterminacy in general equilibrium models with incomplete financial markets: Mixed asset returns. Preliminary paper, Rutgers University.

11. Polemarchakis HM (1988) Portfolio choice, exchange rates and indeterminacy. J Econ Theory 46: 414-421.

12. Siconolfi P, Villanacci A (1989) Real indeterminacy in incomplete financial market economies without aggregate risk. CARESS Working Paper, University of Pennsylvania.

13. Villanacci A (1990) Indeterminacy of equilibria, taxes and outside money in exchange economies with incomplete financial markets. CARESS Working Paper, University of Pennsylvania.

14. Werner J (1990) Structure of financial markets and real indeterminacy of equilibria. J Math Econ 19: 217-232.

15. Zame WR (1988) Asymptotic behavior of asset markets, I: Asymptotic inefficiency, Preliminary paper, SUNY at Buffalo (Department of Mathematics).

Appendix to CARESS Working Paper #90-13, **"Indefinitely Sustained Consumption Despite Exhaustible Natural Resources"** by David Cass and Tapan Mitra (to be published in <u>Economic Theory</u>)

We have clearly demonstrated that sufficiency of the substitution condition (SC) basically rests on just H1 and H2. In contrast, we have merely asserted that necessity critically depends on h1, h2, and h3 (as well as H2). The purpose of this appendix is to substantiate this latter claim. We will demonstrate by example that, without any one of these three assumptions, (3) may be satisfied even when (SC) is not. Along the way we will also establish that h2 can be replaced by a weaker regularity requirement when time is modeled as being continuous rather than discrete, and that h3 can be replaced by a weaker curvature condition provided that (SC) itself is reformulated in terms of "lim inf" rather than "lim".

AI. <u>Counterexample to Dropping Assumption h1</u>

Suppose that

$$h(y,k) = \begin{cases} g(k) - y & \text{for } y \leq g(k) \\ a & \text{otherwise} \end{cases} \tag{A1}$$

with $H = \mathbb{R} \times \mathbb{R}_+$, where $g: \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies $g(0) > g(\infty) = \lim_{k\to\infty} g(k) > 0$, and $a > 0$. Such a resource requirement function effectively embodies the antithesis of h1, since, up to some limit (determined solely by the capital stock), more output always requires less depletion. The salient features of (A1) are depicted in Figure A1. For an economy enjoying this peculiar static technology, it is fairly obvious that

[insert Figure A1 about here]

(3) is satisfied, since, for instance, given arbitrary $(\tilde{r},\tilde{k}) >> 0$, the growth path described by

$$c_t = g(\infty) \quad \text{for} \quad t \geq 0$$

and

$$k_0 = \tilde{k} \quad \text{and} \quad k_{t+1} = k_t + [g(k_t) - g(\infty)] \quad \text{for} \quad t \geq 0$$

is a solution to (10). Equally obvious is the fact that (SC) is not satisfied, since $\underline{k}(y) = \underline{k} = \infty$ for $0 < y \leq g(\infty)]$ but, for every $0 < k < \underline{k}$,

$$\lim_{y \to 0^+} 1/y \int_k^{\underline{k}(y)} [g(u) - y]du = \left[1/y \int_k^{\infty} [g(u) - y]du \right]_{0<y<g(\infty)}$$

$$= \left[1/y \int_k^{\infty} [g(\infty) - y]du \right]_{0<y<g(\infty)}$$

$$= \infty.$$

Two further points about this particular example are worth making explicitly. (i) The argument is very direct. Though much more complicated in detail, the basic logic for each of the succeeding examples in this appendix is absolutely identical: First, describe a particular resource requirement function. Second, exhibit a solution to (10) (given arbitrary $(\tilde{r},\tilde{k}) >> 0$). Third, and finally, verify violation of (SC). (ii) (A1) fails to satisfy h3 as well as h1. This follows immediately from the observation that

$$\frac{h(y,k)}{y} = \frac{g(k)}{y} - 1 \begin{Bmatrix} > \\ = \end{Bmatrix} 0 \quad \text{according as} \quad y \begin{Bmatrix} < \\ = \end{Bmatrix} g(k),$$

so that, though $\bar{y} = g(0)$, for every $\nu > 0$ and every $k \geq 0$, if $y' = g(k)$ and $0 < y'' < y'$, then

$$0 = \frac{g(k)}{y'} - 1 = \frac{h(y',k)}{y'} < \nu \frac{h(y'',k)}{y''} = \nu \left[\frac{g(k)}{y''} - 1 \right] .$$

Thus, one could legitimately puzzle over the appropriate interpretation of the example. However, because (A1) does satisfy the still weaker version of diminishing returns presented below, it seems quite properly viewed as basically a counterexample to dropping h1 rather than to dropping h3.

AII. <u>Counterexamples to Dropping Assumption h2</u>

These two examples exhibit quite distinct kinds of nonmonotonicity. Though either one by itself would suffice for our present purpose, we include both in order to emphasize the wide variety of situations which can be covered in continuous, but not discrete time. (Refer to the analysis in the following subsection.) Our elaboration of each example is intentionally abbreviated, since, as noted above, it is logically straightforward. In particular, among other more minor details, we leave to the reader explicit verification that, in both examples, h1 and h3 are satisfied.

<u>Example A2a</u>. Suppose that

$$h(y,k) = f(y)g(k) \tag{A2}$$

with $H = \mathbb{R} \times \mathbb{R}_+$, where $f\colon \mathbb{R} \to \mathbb{R}_+$ is increasing and convex, and satisfies $f(y) \begin{Bmatrix} > \\ = \end{Bmatrix} 0$ according as $y \begin{Bmatrix} > \\ = \end{Bmatrix} 0$ and $\lim_{y \to 0^+} f(y)/y = 0$, while if (i) $g_1\colon [0,1] \to \mathbb{R}_+$ is continuous and satisfies $g_1(x) \begin{matrix} > \\ = \end{matrix} 0$ according as $x(1-x) \begin{Bmatrix} > \\ = \end{Bmatrix} 0$, and (ii) $g_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies $0 < \int_0^\infty g_2(u)du < \infty$, then

$$g(k) = \begin{cases} g_1(k - n) + g_2(k) & \text{for } n \leq k \leq n + 1, \quad n = 0,2,4,\ldots \\ g_2(k) & \text{otherwise.} \end{cases}$$

In a manner of speaking, the isoquants suffer chronic (as well as uniform) "bumpiness," as exhibited in Figure A2a.

[insert Figure A2 about here]

To see that (10) has a solution, let

$$y_{nm} = (n - \tilde{k})/m \quad \text{for } n \geq \tilde{k} \text{ and } m \geq 2(n - \tilde{k}) \text{ both integral,}$$

and consider the growth paths described by

$$c_t = y_{nm} \quad \text{for} \quad t \geq 0$$

and

$$k_0 = \tilde{k} \quad \text{and} \quad k_{t+1} = \begin{cases} k_t + y_{nm} & \text{for} \quad 0 \leq t \leq m-1 \\ k_t + 1 & \text{otherwise.} \end{cases}$$

Then all that needs checking is that, for some n and m, the resource bound will be satisfied. Direct calculation yields

$$\begin{aligned} \sum_{t=0}^{\infty} h(c_t+k_{t+1}-k_t,k_t) &= \sum_{t=0}^{m-1} h(2y_{nm},\tilde{k}+y_{nm}t) + \sum_{t=m}^{\infty} h(y_{nm}+1,\tilde{k}+y_{nm}m+(t-m)) \qquad \text{(A3)} \\ &= (f(2y_{nm})/y_{nm}) \sum_{t=0}^{m-1} g(\tilde{k}+y_{nm}t)y_{nm} + f(y_{nm}+1) \sum_{t=m}^{\infty} g_2(n+(t-m)) \\ &\leq (f(2y_{nm})/y_{nm}) \sum_{t=0}^{m-1} g(\tilde{k}+y_{nm}t)y_{nm} + f(y_{nm}+1) \int_{n-1}^{\infty} g_2(u)du. \end{aligned}$$

So just pick n large enough so that

$$f(y_{nm}+1) \int_{n-1}^{\infty} g_2(u)du \leq f(2) \int_{n-1}^{\infty} g_2(u)du \leq \tilde{r}/2, \qquad \text{(A4)}$$

and m large enough (with corresponding y_{nm} small enough) so that

$$\begin{cases} \sum_{t=0}^{m-1} g(\tilde{k}+y_{nm}t)y_{nm} \le 2 \int_{\tilde{k}}^{n} g(u)du \\ \text{and} \\ 4(f(2y_{nm})/2y_{nm}) \int_{\tilde{k}}^{n} g(u)du \le \tilde{r}/2, \end{cases} \tag{A5}$$

and the desired conclusion follows immediately upon using (A4)-(A5) to extend the chain of inequalities in (A3).

Finally, to see formally that (SC) is not satisfied (informally, this is obvious from Figure (A2a), let

$$n(k) = \min\{n: n \text{ is even and } n \ge k\} \quad \text{for} \quad k \ge 0.$$

Then, because $\underline{k} = \infty$, for every $0 < k < \tilde{k}$,

$$\int_k^{\underline{k}(y)} h(y,u)du = f(y) \left[\sum_{i=0}^{\infty} \int_{n(k)+2i}^{n(k)+2i+1} g_1(u - (n(k) + 2i))du + \int_k^{\infty} g_2(u)du\right]$$

or

$$\lim_{y\to 0^+} 1/y \int_k^{\underline{k}(y)} h(y,u)du = \left[(f(y)/y) \left[\sum_{i=0}^{\infty} \int_{n(k)+2i}^{n(k)+2i+1} g_1(u - (n(k) + 2i))du + \int_k^{\infty} g_2(u)du\right]\right]_{y>0} = \infty.$$

Example A2b. Suppose that

$$h(y,k) = \begin{cases} h_1(y,k) & \text{for} \quad (y,k) \in H_1 \\ h_1(y,k)+h_2(y) & \text{otherwise} \end{cases} \tag{A6}$$

with $H = \mathbb{R} \times \mathbb{R}_+$ and

$$H_1 = \{(y,k) \colon 0 \le y \le 1/2^{n-1} \text{ and } k = 1 - i/2^n \text{ for } i = 0,1,\ldots,2^n,\ n = 1,2,3\ldots\},$$

where $h_1\colon [0,1]^2 \to \mathbb{R}_+$ is increasing in y, decreasing in k and convex in y, and satisfies $\lim_{y\to 0^+} h_1(y,k)/y = 0$ and $h_1(y,k) \begin{Bmatrix} > \\ = \end{Bmatrix} 0$ according as $k \begin{Bmatrix} < \\ = \end{Bmatrix} 1$ for $0 < y \le 1$, while $h_2\colon \mathbb{R} \to \mathbb{R}_+$ is increasing and convex, and satisfies $h_2(0) = 0$ and $\lim_{y\to 0^+} h_2(y)/y > 0$. This example is hard to describe in words, but not in pictures, as Figure A2b attests.

Now let n be a positive integer such that $1/2^n \le \bar{k}$, and consider the growth paths described by

$$c_t = 1/2^n \quad \text{for} \quad t \ge 0$$

and

$$k_0 = 1/2^n \quad \text{and} \quad k_{t+1} = \begin{cases} k_t + 1/2^n & \text{for} \quad 0 \le t \le 2^n - 2 \\ 1 & \text{otherwise.} \end{cases}$$

Then, in this instance direct calculation yields

$$\sum_{t=m}^{\infty} h(c_t+k_{t+1}-k_t,k_t) = \sum_{t=0}^{2^n-2} h(1/2^{n-1},(t+1)/2^n)$$

$$= \sum_{t=0}^{2^n-2} h_1(1/2^{n-1},(t+1)2^n)$$

$$= 1/(1/2^n) \sum_{t=0}^{2^n-2} h_1(1/2^{n-1}, (t+1/2^n)(1/2^n)$$

$$\leq 2/(1/2^{n-1}) \int_0^{1-1/2^n} h_1(1/2^{n-1}, u)du$$

$$\leq 2/(1/2^{n-1}) \int_0^{1} h_1(1/2^{n-1}, u)du$$

$$\leq \tilde{r}$$

for n large enough.

Finally, observe that, for every $0 < y \leq 1$, h is integrable in k on $[0,1]$, and has closed form, for every $0 \leq k^1 \leq k^2 \leq 1$,

$$\int_{k^1}^{k^2} h(y,u)du = \int_{k^1}^{k^2} h_2(y)du = h_2(y)(k^2-k^1)$$

(since, if $1/2^n < y \leq 1/2^{n-1}$, then there are only a finite number of points $k_i = 1 - i/2^n$ for $i = 0,1,\ldots,2^n$). Hence, because $\underline{k}(y) = \underline{k} = 1$ for $0 < y \leq 1$, for every $0 < k < \underline{k}$,

$$\lim_{y\to 0^+} 1/y \int_k^{\underline{k}(y)} h(y,u) = \lim_{y\to 0^+} 1/y \int_k^1 h_2(y)du = \left[\lim_{y\to 0^+} h_2(y)/y \right](1-k) > 0.$$

AIII. **Modeling with Continuous Time: Weakening Assumption h2**

For simplicity we again illustrate the main point by focusing on the intermediate case in which $\bar{y} > 0$ as well as $\underline{k} = \infty$. A similar modification of the analysis (but with additional -- and purely technical -- complication) applies in the general case.

In this setting a feasible growth path is a solution to the dynamical system

$$\begin{cases} c(t) \geq 0 \quad \text{and} \quad (c(t) + (\dot{k}(t), k(t)) \in H \quad \text{for} \quad t \geq 0 \\ \text{and} \\ \displaystyle\int_0^\infty h(c(s) + \dot{k}(s), k(s))ds \leq \tilde{r} \quad \text{and} \quad k(0) \leq \bar{k}, \end{cases} \tag{A7}$$

where $t \in [0,\infty]$ and $k(t) = k(0) + \int_0^t \dot{k}(s)ds$ (so it is understood that, by hypothesis, all requisite integrals are well-defined, say, in the sense of Riemann). Now replace h2 by the weaker regularity requirement

h2′. If $\bar{y} > 0$, then, for every $0 < y < \bar{y}$, h is integrable in k on $\{k\colon k \in H(y) \text{ and } k < \underline{k}(y)\}$,

and consider feasible growth paths of the specific form

$$\begin{cases} \dot{k}(t) \geq 0 \quad \text{and} \quad (c + \dot{k}(t),\ k(t)) \in H \quad \text{for} \quad t \geq 0, \\ \lim_{t \to \infty} k(t) = \infty \\ \text{and} \\ c > 0,\ \int_0^\infty h(c + \dot{k}(s), k(s))ds \leq \tilde{r} \quad \text{and} \quad k(0) \leq \tilde{k}. \end{cases} \tag{A8}$$

In the same spirit as the argument underlying the Growth Theorem in Section IVA, the following result is also easily demonstrated.

<u>Sustainability Theorem in Continuous Time</u>. For every $(\tilde{r},\tilde{k}) >> 0$ there exists $(c, \{k(t)\})$ satisfying (A8) if and only if (SC) obtains.

<u>Proof of the Sustainability Theorem in Continuous Time</u>. <u>Sufficiency</u>. Pick $y > 0$ such that $(y,\tilde{k}) \in H$ [using H1] and

$$1/y \int_{\tilde{k}}^\infty h(y,u)du \leq \tilde{r}/2 \tag{A9}$$

[using (SC)], and consider the particular growth path defined by

$$\begin{cases} k(0) = \tilde{k} \\ \text{and} \\ c(t) = y/2 \quad \text{and} \quad \dot{k}(t) = y/2 \quad \text{for} \quad t \geq 0. \end{cases} \tag{A10}$$

Then

$$\int_0^\infty h(c(s) + \dot{k}(s), k(s))ds = \int_0^\infty h(y, k(0) + (y/2)s)ds$$

$$= 2/y \int_{\tilde{k}}^\infty h(y,u)du \qquad \text{[changing variables from "s" to "u = k(s)"]}$$

$$\leq \tilde{r}. \qquad \text{[using (A9)]}$$

Thus, (A10) yields a solution to (A8) with $c = y/2$.

Necessity. Suppose that (A8) has a solution for arbitrary $(\tilde{r},\tilde{k}) >> 0$. Again it is straightforward to show that this implies that

for every $0 < y < c$,

$$1/y \int_{\tilde{k}}^\infty h(y,u)du \leq \tilde{r}/\nu:$$

$$\tilde{r} \geq \int_0^\infty h(c + \dot{k}(s), k(s))ds$$

$$\geq \int_0^\infty h(y + \dot{k}(s), k(s))ds \quad \text{for} \quad 0 < y < c \qquad \text{[using H1 and h1]}$$

$$\geq \int_0^\infty \frac{h(y)+\dot{k}(s), k(s))}{y+\dot{k}(s)} \dot{k}(s)ds$$

$$\geq \nu \int_0^{\infty} \frac{h(y,k(s))}{y} \dot{k}(s)ds$$ [using h3]

$$= \nu/y \int_{\tilde{k}}^{\infty} h(y,u)du.$$ [changing variables from "s" to "u = k(s)"]

AIV. <u>Counterexamples to Dropping Assumption h3</u>

As in the examples concerning the necessity of h2, these two examples differ markedly from one another. However, unlike there, here the difference has substantial significance. On the one hand, the first example -- involving average returns to exhaustible resources which become "too large" at high levels of output -- represents situations where a substitution condition like (SC) is simply inappropriate; sustained consumption typically requires geometric accumulation. (These are, in our view, extremely implausible situations anyway). On the other hand, the second example -- involving average returns to exhaustible resources which behave "too erratically" at low levels of output -- represents situations where, with suitable relaxation of h3, a parallel relaxation of (SC) will still apply, as we demonstrate in the following section. (These are, in our view, distinctly plausible situations, especially when translated into the disaggregative terms we proposed in Section IVB.) Once again, our argument is purposely terse.

<u>Example A3a</u>. Suppose, as earlier, that

$$h(y,k) = f(y)g(k) \tag{A2}$$

with $H = \mathbb{R} \times \mathbb{R}_+$, where now $f: \mathbb{R} \to \mathbb{R}_+$ is increasing (and strictly increasing for $y \geq 0$) and satisfies $\lim_{y \to 0^+} f(y)/y = 0$ and $0 < f(\infty) = \lim_{y \to 0} f(y) < \infty$ (so that $\lim_{y \to \infty} f(y)/y = 0$), while $g: \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies, for every $k > 0$,

$$\int_k^\infty g(u)du = \infty \quad \text{and} \quad \int_k^\infty \frac{g(u)}{u}\, du < \infty$$

(for instance, $g(k) = k^{-1/2}$). As Figure A3a suggests, h3 isn't satisfied, since obviously $\bar{y} = \infty$ (noting that $\lim_{k \to \infty} g(k) = 0$), but for every $\nu > 0$ and every $k \geq 0$, if $0 < y'' < y'$, then

$$\frac{h(y',k)}{y'} = \frac{f(y')}{y'}\, g(k) < \nu\, \frac{f(y'')}{y''}\, g(k) = \nu\, \frac{h(y'',k)}{y''}$$

for y' large enough (noting that $g(k) > 0$).

[insert Figure A3 about here]

Let $0 < y \leq \bar{k}/2$, $0 < \lambda < 1$ and n be a positive integer, and consider the growth paths described by

$$c_t = y \quad \text{for} \quad t \geq 0$$

and

$$k_0 = \bar{k} \quad \text{and} \quad k_{t+1} = \begin{cases} k_t + y & \text{for } 0 \leq t \leq n - 1 \\ (1 + \lambda)k_t & \text{otherwise.} \end{cases}$$

Then direct calculation yields

$$\sum_{t=0}^{\infty} h(c_t+k_{t+1}-k_t,k_t) = \sum_{t=0}^{n-1} f(2y)g(\tilde{k}+yt) + \sum_{t=0}^{\infty} f(y+\lambda k_t)g(k_t) \tag{A11}$$

$$\leq (f(2y)/y) \sum_{t=0}^{n-1} g(\tilde{k}+yt)y + f(\infty) \sum_{t=n}^{\infty} \frac{g(k_t)}{\lambda k_t} (k_{t+1}-k_t)$$

$$\leq 2(f(2y)/2y) \int_{\tilde{k}-y}^{\tilde{k}+(n-1)y} g(u)du + (f(\infty)/\lambda) \int_{(1-\lambda)(\tilde{k}+ny)}^{\infty} \frac{g(u)}{u} du.$$

So just pick ny large enough so that

$$(f(\infty)/\lambda) \int_{(1-\lambda)(\tilde{k}+ny)}^{\infty} \frac{g(u)}{u} du \leq \bar{r}/2, \tag{A12}$$

and (fixing ny) y small enough so that

$$2(f(2y)/2y) \int_{\tilde{k}-y}^{\tilde{k}+(n-1)y} g(u)du \leq 2(f(2y)/2y) \int_{\tilde{k}/2}^{\tilde{k}+ny} g(u)du \leq \bar{r}/2, \tag{A13}$$

and, as before, the desired conclusion follows immediately upon using (A12)-(A13) to extend the chain of inequalities in (A11).

Finally, because $\underline{k} = \infty$, for every $0 < k < \underline{k}$,

$$\lim_{y\to 0^+} 1/y \int_k^{k(y)} h(y,u)du = \left[1/y \int_k^{\infty} f(y)g(u)du\right]_{y>0} = \left[(f(y)/y) \int_k^{\infty} g(u)du\right]_{y>0} = \infty.$$

Example A3b. Suppose, once more, that

$$h(y,k) = f(y)g(k) \tag{A2}$$

with $H = \mathbb{R} \times \mathbb{R}_+$, where now $f\colon \mathbb{R} \to \mathbb{R}_+$ is as depicted in Figure A3b, while $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies, for every $k > 0$, $0 < \int_k^{\infty} g(u)du < \infty$.

Explicitly, f is defined by the formula

$$f(y) = \begin{cases} (1+\alpha)y_1 + (y-y_1) & \text{for } y \geq y_1 \\ y_i/i + (y-y_i)/(i+1) & \text{for } y_{i+1} \leq y < y_i,\ i = 2,3,\ldots \\ 0 & \text{for } y \leq 0, \end{cases}$$

where $\alpha > 0$, $y_1 > 0$ and $y_{i+1} = \min\{y\colon y_i/i + (y-y_i)/(i+1) \leq (1+\alpha)y\}$ for $i > 1$. For this example, h3 isn't satisfied since again $\bar{y} = \infty$, but now, for every $\nu > 0$ and every $k \geq 0$, if $0 < y'' = y_{i+1} < y' < y_i$, then

$$\lim_{y'\to y_i^-} \frac{h(y',k)}{y'} = \frac{1}{i}\, g(k) < \nu(1+\alpha)g(k) = \nu\, \frac{h(y'',k)}{y''}$$

for i large enough.

In fact, the critical feature of this example is precisely that

$$\liminf_{y\to0^+} \frac{f(y)}{y} = \lim_{i\to\infty} \left[\lim_{y\to y_i^-} \frac{f(y)}{y} \right] = \lim_{i\to\infty} \frac{1}{i} = 0 <$$

$$(1+\alpha) = \lim_{i\to\infty} \frac{f(y_i)}{y_i} = \lim_{i\to\infty} \left[\lim_{y\to y_i^+} \frac{f(y)}{y} \right] = \limsup_{y\to0^+} \frac{f(y)}{y} .$$

Hence, on the one side, the demonstration that (10) has a solution follows from the observations that $\underline{k} = \infty$ and that, for every $0 < k < \underline{k}$,

$$\liminf_{y\to0^+} 1/y \int_k^{\underline{k}(y)} h(y,u)du = \left[\liminf_{y\to0^+} \frac{f(y)}{y}\right] \int_k^{\infty} g(u)du = 0. \qquad \text{(A14)}$$

(As we show in the following section, (A14) completely captures the only implication of (SC) we actually needed for the sufficiency argument in the proof of the Sustainability Theorem.) On the other side, the verification that (SC) itself is not satisfied follows from the observations that $\underline{k} = \infty$ and that, for every $0 < k < \underline{k}$,

$$\limsup_{y\to0^+} 1/y \int_k^{\underline{k}(y)} h(y,u)du = \left[\limsup_{y\to0^+} \frac{f(y)}{y}\right] \int_k^{\infty} g(u)du = (1+\alpha) \int_k^{\infty} g(u)du > 0.$$

AV. <u>Erratically Behaved Average Returns: Weakening Assumption h3</u>

Consider replacing h3 by the weaker assumption h3'. If $\bar{y} > 0$, $\underline{k} = \infty$ and, for every $k \geq 0$,

$$\liminf_{y \to 0^+} \frac{h(y,k)}{y} = 0,$$

then there exist a positive constant $\nu > 0$ and, for every $c > 0$, a smaller, positive output level $0 < y < c$ $\nu > 0$ with the property that, for every $k \geq 0$,

$$y' \in H(k),\ y'' = y \text{ and } y'' \leq y' \Rightarrow \frac{h(y',k)}{y'} \geq \nu \frac{h(y'',k)}{y''},$$

and (SC) by the weaker condition

for every $0 < k < \underline{k}$,

$$\liminf_{y \to 0^+} 1/y \int_0^{\underline{k}(y)} h(y,u)du = 0. \qquad \text{(SC')}$$

h3' is a significant relaxation in the sense that it permits including the additional cases shown in Figure A4.

[insert Figure A4 about here]

These changes require some minor adjustments in the previous proof of the Sustainability Theorem.

Sufficiency. Clearly (SC′) suffices for our earlier construction, since it too permits picking $0 < \lambda < 1/2$ and $y > 0$ such that $(y, \tilde{k} - \lambda y) \in H$ and

$$1/\lambda y \int_{\tilde{k}-\lambda y}^{\underline{k}(y)} h(y,u)\,du \leq \tilde{r}/2$$

(cf. top p. 22).

Necessity. We introduce an additional lemma, and then suitable corresponding modification of the concluding argument (currently on pp. 30-33), as follows:

Lemma′ (given the Lemma on p. 25). If (10) has a solution for every $(\tilde{r}, \tilde{k}) \gg 0$, then

$$\text{for every } 0 < k < \underline{k},$$

$$\liminf_{y \to 0^+} \frac{h(y,k)}{y} = 0. \qquad \text{(A15)}$$

Proof of the Lemma′. Given $\tilde{r} > 0$ and $0 < \tilde{k} < \underline{k}$, suppose that $(c, \{k_t\})$ is a solution to (10). Then, by virtue of the Lemma, $0 < c \leq \bar{y}$ -- so that the conclusion of h2 obtains -- while, for every $0 < y < c$, $\underline{k}(y) \leq \sup_{t \geq 0} k_t$ -- so that (recalling that $k_0 \leq \tilde{k} < \underline{k} \leq k(y)$ for $y \geq 0$), for every $\tilde{k} < k' < \bar{k}$, there exists $0 < t' < \infty$ such that

$$k_t \begin{Bmatrix} < \\ \geq \end{Bmatrix} k' \text{ according as } 0 \leq t \begin{Bmatrix} < \\ = \end{Bmatrix} t'.$$

So let

$$t^+(t') = \{t\colon 0 \leq t < t' \quad \text{and} \quad k_{t+1} - k_t > 0\}$$

(noting that $t'-1 \in t^+(t')$) and

$$k'_{t+1} = \min\{k', k_{t+1}\} \quad \text{for} \quad t \in t^+(t')$$

(noting that $k'_{t+1} = k'$, for $t = t'-1$, $= k_{t+1}$, for $t \in t^+(t')$, $t < t'-1$). Then the resource bound in (10) yields the following chain of inequalities:

$$\tilde{r} \geq \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t)$$

$$= \sum_{t \in t^+(t')} h(c + k_{t+1} - k_t, k_t)$$

$$\geq \sum_{t \in t^+(t')} h(c + k'_{t+1} - k_t, k_t) \qquad \text{[using H2 and h1]}$$

$$\geq \sum_{t \in t^+(t')} h(k'_{t+1} - k_t, k') \qquad \text{[using H2, h1 and h2]}$$

$$= \sum_{t \in t^+(t')} \frac{h(k'_{t+1} - k_t, k')}{k'_{t+1} - k_t} (k'_{t+1} - k_t)$$

$$\geq \sum_{t \in t^+(t')} \left[\inf_{y \in \{y\colon\, y \in H(k') \text{ and } y \leq k'\}} \frac{h(y,k')}{y} \right] (k'_{t+1} - k_t)$$

$$\geq \left[\inf_{y \in \{y:\ y \in H(k')\ \text{ and }\ y \leq k'\}} \frac{h(y,k')}{y} \right] (k' - \tilde{k}).$$

But since $\tilde{r} > 0$ and $0 < \tilde{k} < k' < \underline{k}$ are arbitrary, the last inequality obtains only if

for every $0 < k < \underline{k}$,

$$\inf_{y \in \{y:\ y \in H(k)\ \text{ and }\ y \leq k\}} \frac{h(y,k)}{y} = 0$$

a property which itself obtains only if (A15) does. ■

The balance of the proof distinguishes the two cases $\underline{k} < \infty$ and $\underline{k} = \infty$.

<u>Case 1</u>. $\underline{k} < \infty$.

Take $0 < k < \underline{k}$. Then, by virtue of H1 and h2, there is y' such that

for every $0 < y < y'$,

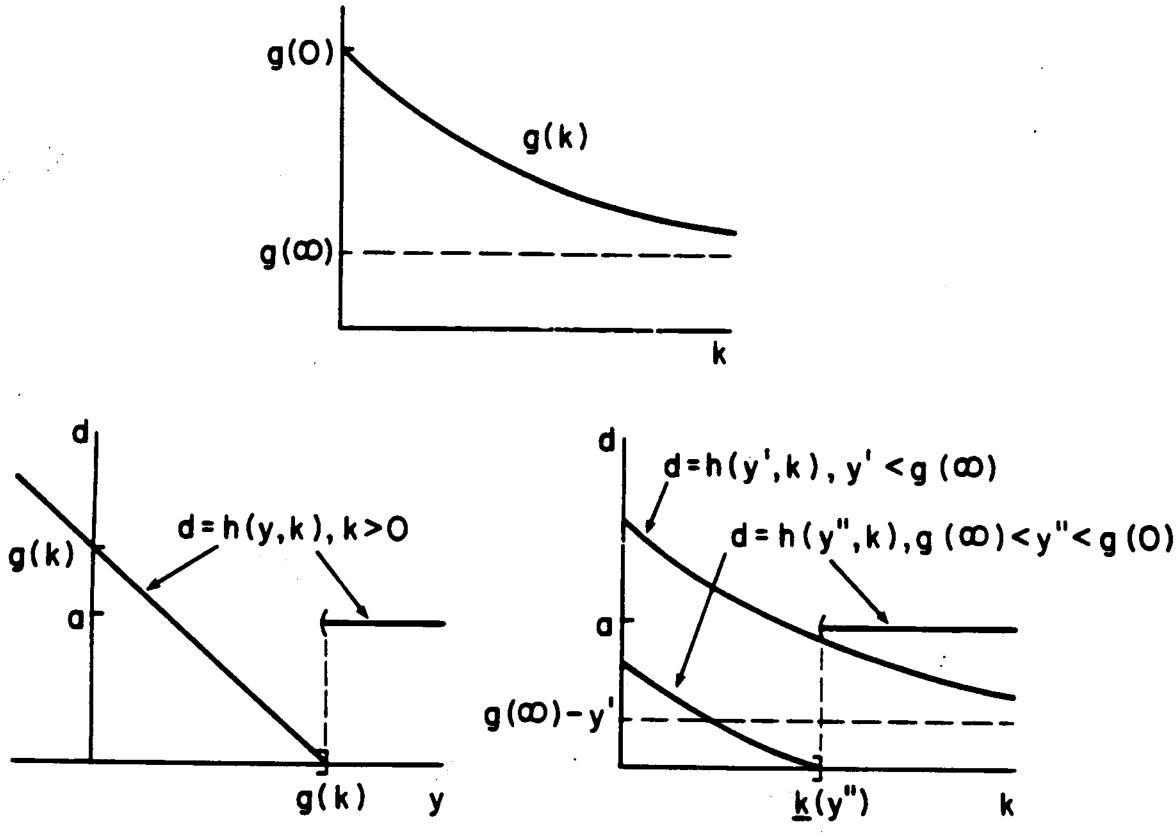
$$(y,k) \in H \quad \text{and} \quad 0 < 1/y \int_{k}^{\underline{k}(y)} h(y,u)du \leq \frac{h(y,k)}{y} (\underline{k}(y) - k).$$

Hence, by virtue of the Lemma' together with $\lim_{y \to 0^+} \underline{k}(y) = \underline{k} > k$,

$$\liminf_{y \to 0^+} 1/y \int_{k}^{\underline{k}(y)} h(y,u)\,du = 0.$$

Case 2. $\underline{k} = \infty$.

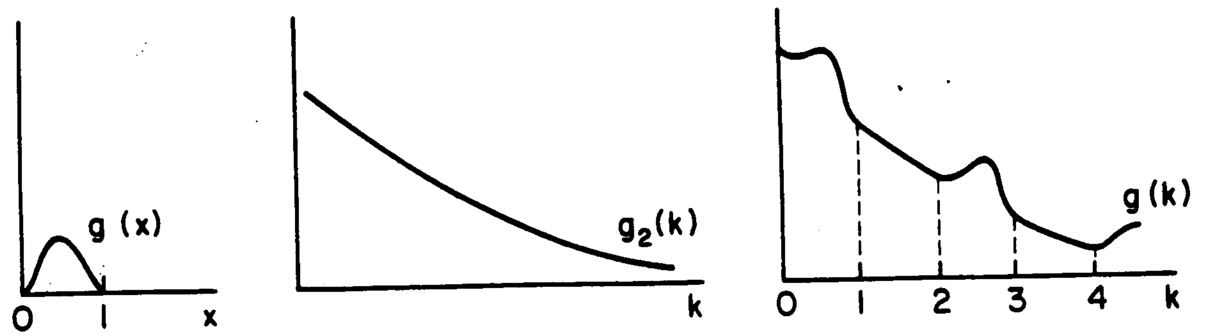
Now, by virtue of the Lemma', all the triggering hypotheses of h3' are satisfied, so that our previous argument is virtually unchanged (by picking suitable $0 < y < c$). ■



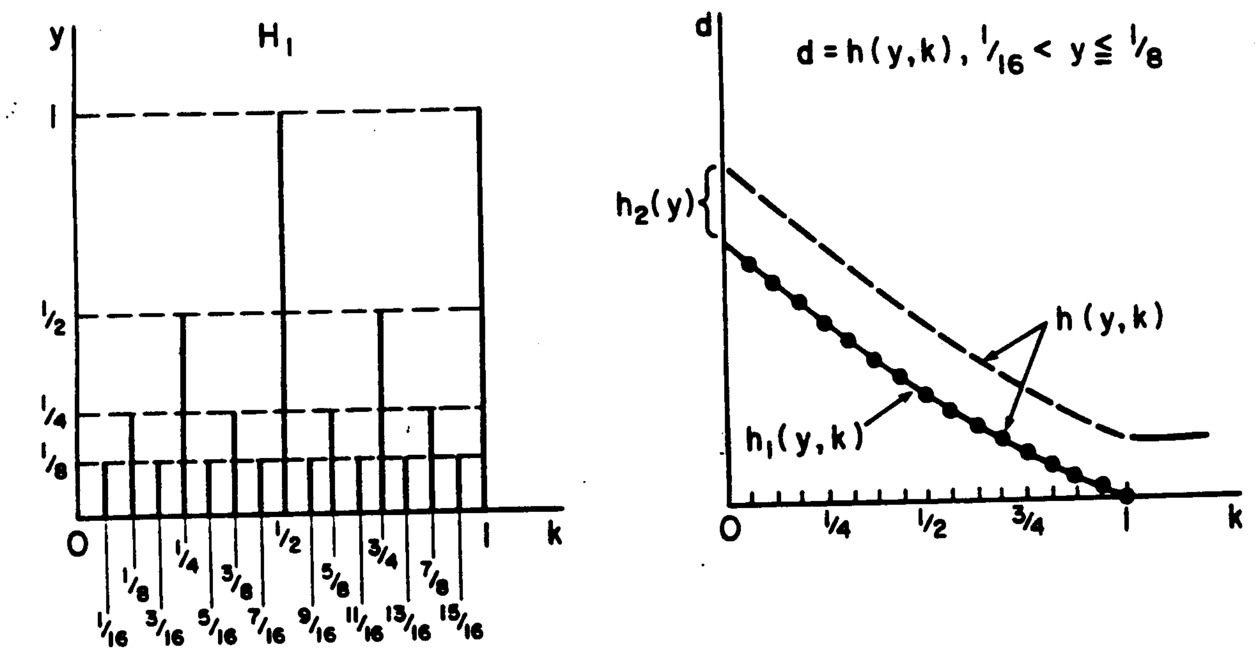
$$h(y,k) = \begin{cases} g(k) - y & \text{for } y \leqq g(k) \\ a & \text{otherwise} \end{cases}$$ with $H = \mathbb{R} \times \mathbb{R}_+$, where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$

is decreasing and satisfies $g(0) > g(\infty) > 0$, and $a > 0$

Fig. A1 Critical Features of the Example Showing that without Assumption h1, Condition (SC) is not Necessary



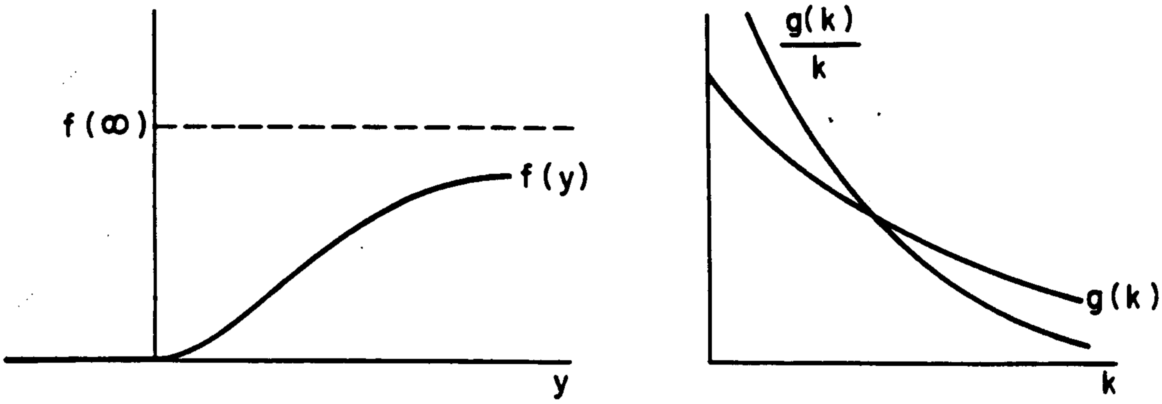
a. Example where $g(k) = \begin{cases} g_1(k-n) + g_2(k) & \text{for } n \leqq k \leqq n+1,\ n = 0, 2, 4, \cdots \\ g_2(k) & \text{otherwise} \end{cases}$



b. Example where $h(y,k) = \begin{cases} h_1(y,k) & \text{for } (y,k) \in H_1 \quad \text{with} \\ h_1(y,k) + h_2(y) & \text{otherwise} \end{cases}$

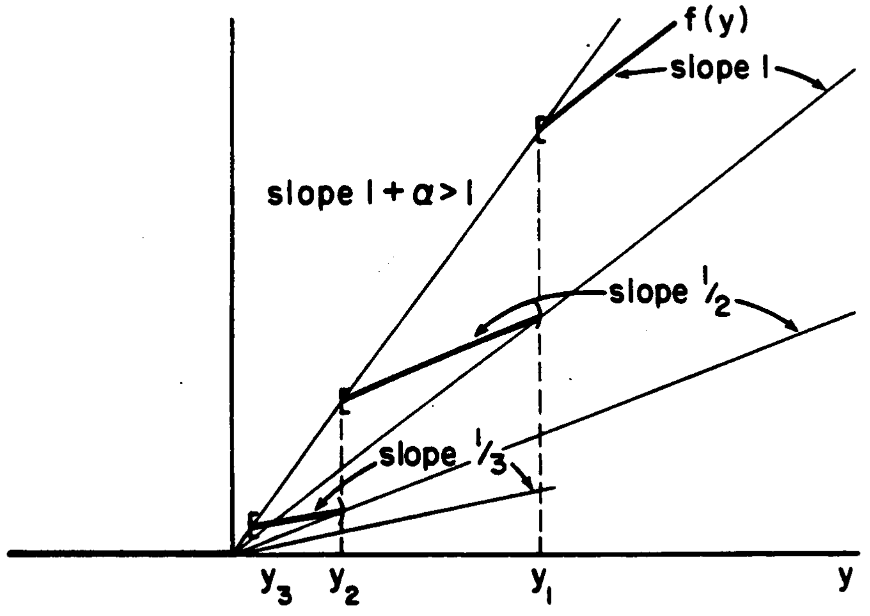
$$H = \left\{ (y,k) : 0 \leqq y \leqq 1/2^{n-1} \text{ and } k = 1 - i/2^n \text{ for } i = 0, 1, \cdots, 2^n,\ n = 1, 2, 3, \cdots \right\}$$

Fig. A2 Critical Features of the Two Examples Showing that without Assumption h2, Condition (SC) is not Necessary



a. Example where $\lim_{y \to \infty} f(y) = f(\infty) < \infty$ (so that $\lim_{y \to \infty} \frac{f(y)}{y} = 0$), while

$$\int_k^{\infty} g(u)\,du = \infty \quad \text{but} \quad \int_k^{\infty} \frac{g(u)}{u}\,du < \infty \quad \text{for } k > 0$$



b. Example where $\liminf_{y \to 0^+} \frac{f(y)}{y} = 0 < \limsup_{y \to 0^+} \frac{f(y)}{y}$

Fig. A3 Critical Features of the Two Examples Showing that without Assumption h3, Condition (SC) is not Necessary

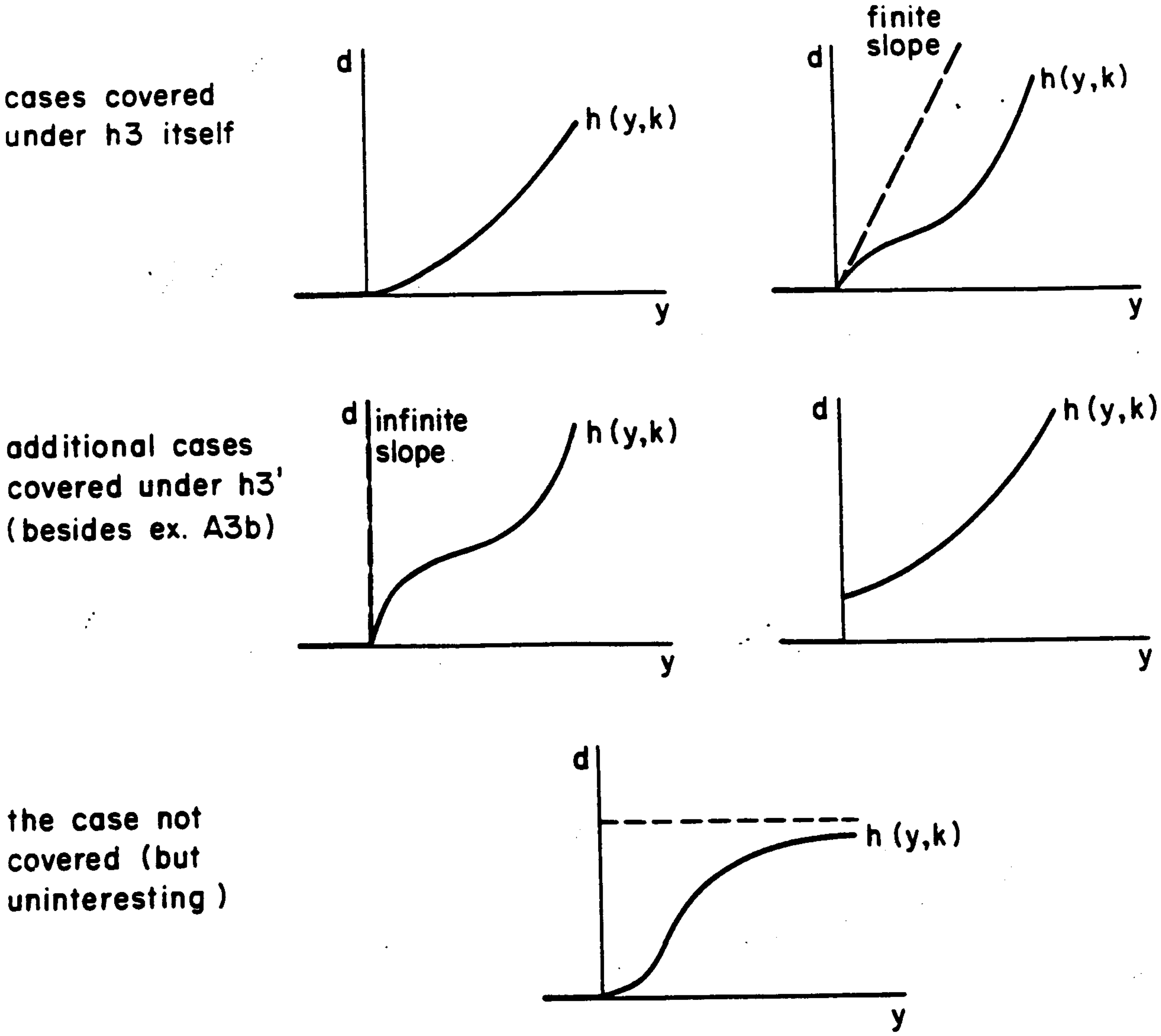


Fig. A4 Relaxation of Assumption h3