INDEFINITELY SUSTAINED CONSUMPTION DESPITE EXHAUSTIBLE NATURAL RESOURCES

by

David CASS

N° 9027

INDEFINITELY SUSTAINED CONSUMPTION DESPITE EXHAUSTIBLE NATURAL RESOURCES

A B S T R A C T

This paper analyzes the feasibility of sustaining uniformly positive consumption forever -- even when flows of exhaustible resources are an indispensable input. The main result is a characterization of an economy's capability for sustaining such consumption -- under quite general maintained assumptions on technology -- in terms of a single, simple capital-resource substitution condition.

Journal of Economic Literature 111,721

Key Words : Incomplete Markets, Indeterminacy, Outside Money.

CONSOMMATION INDEFINIMENT SOUTENUE EN DEPIT DE RESSOURCES NATURELLES EPUISABLES

R E S U M E

Ce papier analyse la possibilité d'entretenir indefiniment une consommation uniformément positive - même quand les flux de ressources epuisables constituent un input indispensable. Le résultat principal est une caractérisation de la capacité d'une économie de soutenir une telle consommation - sous des hypothèses sur la technologie tout à fait générales en termes d'une seule condition de substitution entre capital et ressources naturelles.

Journal of Economic Literature 111,721

Mots clefs : Marchés incomplets, Indetermination, Monnaie externe.

INDEFINITELY SUSTAINED CONSUMPTION DESPITE EXHAUSTIBLE NATURAL RESOURCES*

by

David Cass
Department of Economics
University of Pennsylvania

and

Tapan Mitra
Department of Economics
Cornell University

Running Head: Indefinitely Sustained Consumption

Send proofs to: Professor David Cass
Department of Economics
University of Pennsylvania
Philadelphia, PA 19104

*This is a revised version of CARESS Working Paper #79-27. An even earlier preliminary paper by Mitra [1] contained the seminal idea developed more thoroughly here. Both of our research efforts were supported by the NSF, while much of Cass's participation took place during his tenure as a Sherman Fairchild Distinguished Scholar at Caltech. We thank both institutions for their support and encouragement.

Abstract: This paper analyzes the feasibility of sustaining uniformly positive consumption forever -- even when flows of exhaustible resources are an indispensable input. The main result is a characterization of an economy's capability for sustaining such consumption -- under quite general maintained assumptions on technology -- in terms of a single, simple capital-resource substitution condition.

I. Introduction and Summary

Ultimately, the only conceivable limit to continued economic progress is lack of human imagination and ingenuity. Timely accumulation of capital goods, including, in particular, stores of technical knowledge (concerning social as well as natural phenomena), can surely offset any foreseeable depletion of natural resources, no matter how shortsightedly or inflexibly specified. But is such propitious growth even within the realm of possibility?

In order to lay a solid basis for providing precise answers to this critical question, we pose the following fundamental problem: Assuming a fairly unrestricted specification of technology -- that is, a reasonably general description of an economy's feasible growth paths given initial stocks of both capital goods and exhaustible resources -- what additional technological conditions governing capital-resource substitution determine whether the economy is capable of sustaining consumption -- that is, of providing uniformly positive welfare -- forever?

Our solution to this problem yields -- under quite general maintained assumptions on technology -- a single, simple substitution condition which is both necessary and sufficient for indefinitely sustained consumption. This substitution condition is tantamount to an easily understood growth requirement: The economy must be capable, starting from arbitrary, positive stocks and growing with pure, arithmetic accumulation, of either, at best, achieving resource independence or, at worst, avoiding resource exhaustion.

Our general characterization captures the essential feature of a leading special case previously reported by both Solow [2] and Stiglitz [3,4]: When technology is specifically represented in terms of an aggregative neoclassical production function with constant elasticity of substitution, our characterization is equivalent to their condition that the elasticity of substitution be greater than one, or equal to one, provided, in addition, that the capital stock exponent exceed that of the resource flow. But our analysis also reaches far beyond this very explicit example; even within the confines of an aggregative specification of technology we permit substantial discontinuity (e.g., by admitting scale indivisibilities for capital stocks) as well as other significant nonconvexity (e.g., by admitting increasing returns to resource flows at low levels of capacity utilization). Moreover, our basic result can be readily modified to address the deeper problem of characterizing the technological possibilities for continually increasing and asymptotically unbounded consumption, and can be easily extended to include the broader situation where the specification of technology involves many different commodities, both natural (e.g., specific mineral compounds) and produced (e.g., detailed extraction techniques).

Our analysis focuses on a particularly suitable reformulation of the usual neoclassical parable. The core of the paper consists of presentation of that canonical model together with description of the fundamental problem (Section II), and statement and proof of our general characterization (Section III). A concluding discussion (Section IV) outlines some possible approaches for elaborating our analysis in order to encompass the potential for increasing and unbounded consumption as well as the presence of many different types of commodities. Finally, we also investigate the robustness of our central results to further relaxation of crucial maintained assumptions (Appendix).

II. Specification of Technology

A. A Conventional Neoclassical Model

Economic activity begins in period 0, and continues unendingly over periods $t = 0,1,2,\ldots$. In each period a (net) flow of final goods output y is produced from a flow of exhaustible resources d (for "depletion") and a previously accumulated stock of capital goods k (for "kapital") according to a (net) output production function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$. This output is then allocated between consumption goods c and (net) investment goods i. Thus, the stock of exhaustible resources available next period is simply those not utilized this period, while the stock of capital goods available next period consists of those utilized this period plus investment goods produced this period (possibly a negative quantity). The economy's intertemporal production possibilities are further constrained by its historically given initial stocks of both exhaustible resources $\tilde{r}$ and capital goods $\tilde{k}$.

These possibilities can be concisely represented by defining the economy's feasible growth paths as the solutions to the dynamical system

$$\begin{cases} (c_t,d_t,k_t) \geq 0, \quad c_t + i_t \leq y_t = f(d_t,k_t) \quad \text{and} \quad k_{t+1} = k_t + i_t \quad \text{for} \quad t \geq 0 \\ \text{and} \\ \sum\limits_{t=0}^{\infty} d_t \leq \tilde{r} \quad \text{and} \quad k_0 \leq \tilde{k}. \end{cases} \tag{1}$$

Notice that (1) presumes the availability of free disposal for both output and initial stocks. Standard specifications of the economy's static technology require that $f \in C$ be everywhere increasing in d, initially increasing in k (at least when $d > 0$), and jointly concave (and perhaps linear homogeneous) in d and k, and that it satisfy $f(0,0) = 0$ and

$f(d,k) \geq -k$. For our particular purposes, such neoclassical assumptions lead to three essentially different cases. These are exemplified in Figure 1, and will be distinguished more precisely in the following subsection.

[insert Figure 1 about here]

The model detailed below, the formulation we will actually concentrate attention on, is substantially more general than the sort of conventional parable just sketched. However, both models share several basic features which are well worth remarking at the outset:

1. Time is treated as a discrete variable. While such treatment facilitates the use of elementary analytic techniques, it also necessitates the imposition of a potentially objectionable restriction on the applicability of our central characterization. The role of this additional restriction, which essentially amounts to the technological assumption that isoquants are downward sloping, is explored in some detail in the Appendix. In particular, there we will indicate how our general theorem can be simplified conceptually -- though, in terms of the mathematical foundations required, not analytically -- by switching to a continuous time framework, and thereby avoiding any such monotonicity requirement altogether.

2. Technology is specified in terms of net rather than gross outputs. In principle, such general specification is merely a matter of notation. Thus, for instance, in describing the conventional neoclassical formulation, we could have equally well defined f in terms of a (gross) output production function $g: \mathbb{R}_+^2 \to \mathbb{R}_+$ together with a (real) capital depreciation schedule $\delta: \mathbb{R}_+^2 \to \mathbb{R}_+$ by the identity $f = g - \delta$ (requiring, in particular, that

$\delta(d,k) \le k)$.

In practice, however, our particular specification is possibly a matter of substance, since it ostensibly permits direct conversion of undepreciated capital goods (represented by the difference $k - \delta(d,k)$) into consumption goods. Thus, for instance, in (1) (net) investment goods output need only satisfy the constraint $-k_t \le i_t \le y_t$, and hence consumption goods output the constraint

$$0 \le c_t \le y_t - i_t \le y_t + k_t = \overbrace{[y_t + \delta(d_t,k_t)]}^{\text{(gross) final goods}} + \overbrace{[k_t - \delta(d_t,k_t)]}^{\text{undepreciated capital goods}}$$

for $t \ge 0$. A broad justification for this amorphous specification will be implicit in our suggested methodology for treating many different types of commodities. A more narrow justification is inherent in our present analysis itself. It turns out that, but for some exceptionally irregular technologies, all the growth paths we will consider either exhibit, or can be easily modified to exhibit the property that (net) investment goods output is always nonnegative, so that consumption goods output is always less than (net) final goods output. In other words, what on first appearance seems a substantial limitation, upon further inspection becomes just an expositional convenience.

3. <u>Exhaustible resources (e.g., fossil fuel stocks) are the only primary factors explicitly recognized</u>. As with the net-gross distinction, here again the initial impression is somewhat deceiving. For, suppose that we were equally concerned with the constraining influence of other primary factors, short-lived (e.g., labor service flows) or long-lived (e.g., land space stocks). Then, to the extent that their availability over time is constant, their restrictive properties could be incorporated directly into the

description of the economy's static technology (for instance, the specification of f). Moreover, to the extent that variation in their availability over time is endogenous, their feasible (net) output flows and input stocks could be included along with those of any other produced factors in our ultimate generalization involving many different types of commodities.

4. Allocation choices are highly aggregative. Regarding this feature, it is important to bear in mind a point we have already stressed: The central result for our canonical aggregative model largely carries over to a general disaggregative model. Such extension is more than simply generalization for its own sake. Indeed, it permits explicitly recognizing a number of basic technological considerations inherently related to questions concerning "the limits of growth." Specifically, by introducing heterogeneity of capital -- beyond the present crude division between exhaustible resources (i.e., necessarily depletable stocks) and capital goods (i.e., potentially augmentable stocks) -- we can expressly model such diverse phenomena (regarding just exhaustible resources) as (i) variability in quality, (ii) stock-flow interdependence, and (iii) recovery and recycling. Perhaps more critically, our general disaggregative model can capture significant aspects of the most fundamental long-run substitution possibility of all, the offset to resource exhaustion provided by technical progress -- at least insofar as research and development can be accurately portrayed as an activity involving the accumulation and utilization of stocks of knowledge. (Incidentally, our underlying concern with just this sort of generality in large part motivates our repeated emphasis on weakening the technological assumptions required for our analysis. Why should a growth process involving, say, the invention and utilization of fusion-electric power be subject to any of the conventionally

presumed regularities, for instance, generalized diminishing returns?)

B. Reformulation into Canonical Form

Because we are primarily interested in the question of the perpetual sustainability of consumption, and because its resolution ultimately depends on properties of the asymptotic substitutability of capital for resources, it is especially convenient to reformulate the representation of feasible growth paths in terms of a resource requirement function $h: H \to \mathbb{R}_+$ with $H \subset \mathbb{R} \times \mathbb{R}_+$. h describes the minimum resource depletion required to yield a given output level from a given capital stock (together with fixed primary factors), while H circumscribes the conceivable pairs of output level and capital stock. For instance, starting with the output production function f, we would have, formally,

$$h(y,k) = \text{minimum } d \text{ subject to } y \leq f(d,k) \text{ and } d \geq 0 \text{ for } (y,k) \in H$$

with

$$H = \{(y,k): k \geq 0 \text{ and there exists } d \geq 0 \text{ such that } y \leq f(d,k)\}.$$

Roughly speaking, then, h is simply a "partial inverse" of f, while H is its corresponding domain. (Of course, h as well as f could be derived directly from the primitive set of all feasible combinations of outputs and inputs, or the technology set, the more basic approach we actually adopt in treating many different types of commodities.)

Utilizing the definitions of these two fundamental constructs (and also suppressing superfluous notation for intermediate flows) it is easily verified that the dynamical system (1) is equivalent (but for permitting outright wastage of consumption goods) to the dynamical system

$$\begin{cases} c_t \geq 0 \quad \text{and} \quad (c_t + k_{t+1} - k_t, k_t) \in H \quad \text{for} \quad t \geq 0 \\ \text{and} \\ \sum_{t=0}^{\infty} h(c_t + k_{t+1} - k_t, k_t) \leq \bar{r} \quad \text{and} \quad k_0 \leq \bar{k}. \end{cases} \tag{2}$$

This particular representation of feasible growth paths will be the focus of our investigation. Before we can proceed, however, we first need to impose some structure on both H and h (now listed in the natural order for detailing their specification). One obvious way of accomplishing this task is simply to deduce the properties which they inherit from standard specifications of f. It turns out that such structure is considerably more restrictive than absolutely necessary for our purposes. Thus, while we have drawn some inspiration from examples like those shown in Figure 1, we have found it more appropriate to introduce our maintained assumptions, as it were, ab initio.

In stating these minimal assumptions, it is convenient to have at hand specific notation for cross-sections of H, as well as for certain bounds related to the possible behavior of h. So, to begin with, we define

$$H(y) = \{k: \ (y,k) \in H\} \quad \text{for} \quad y \in \mathbb{R},$$

$$H(k) = \{y: \ (y,k) \in H\} \quad \text{for} \quad k \in \mathbb{R}_+,$$

$$\bar{y} = \sup \{y: \ y \geq 0 \ \text{and} \ \inf_{k' \in H(y)} h(y,k') = 0\},$$

and

$$\underline{k}(y) = \inf \{k: k \in H(y) \ \text{and} \ \inf_{k' \in H(y), k' \leq k} h(y,k') = 0\} \quad \text{for} \quad y \in \mathbb{R}_+$$

with

$$\underline{k} = \inf_{y>0} \underline{k}(y),$$

where, by convention, $\inf \phi = - \sup \phi = \infty$. To interpret these various bounds, we ignore the (economically) implausible but (mathematically) tractable irregularity where, for some $0 < y < \bar{y}$, $\underline{k}(y) < \infty$ but either $(y,\underline{k}(y)) \notin H$ or $(y,\underline{k}(y)) \in H$ and $h(y,\underline{k}(y)) > 0$. Then, $\bar{y}$ represents the upper bound (possibly zero) on sustainable output levels, i.e., output levels which can be produced -- perhaps only from unlimited capital stocks -- without any resource depletion, while $\underline{k}$ represents the lower bound (possibly infinity) on sustaining capital stocks, i.e., minimum capital stocks from which positive output levels can be produced without any resource depletion -- themselves denoted $\underline{k}(y)$ for $y > 0$. Figure 2, which replicates the examples of Figure 1, but now labelled in terms of h rather than f, suggests the intuitive basis for these definitions. On the one hand, when

[insert Figure 2 about here]

$\bar{y} = 0$ (that is, as displayed in Figure 2a, when every isoquant lies uniformly above the k-axis) sustained consumption appears a very unlikely prospect. On the other, when $\underline{k} < \infty$ (that is, as displayed in Figure 2c, when some isoquant intersects the k-axis) it appears a very likely one. (Both conjectures are in fact correct under the neoclassical assumptions listed earlier.) While our analysis is therefore primarily directed toward the less obvious, more important intermediate case in which $\bar{y} > 0$ and $\underline{k} = \infty$ (displayed in Figure 2b), our results also cover these two polar cases.

Regarding the domain and properties of the resource requirement function itself, we assume, for H, that

H1. For every $0 < k < \underline{k}$, there exists $y > 0$ such that $(y,k) \in H$;

and

H2. If $(y,k) \in H$, $0 < y' \leq y$ and $k' \geq k$, then $(y',k') \in H$;

and, for h, that

h1. For every $k \geq 0$, h is increasing in y on $\{y: y \in H(k)$ and $y > 0\}$;

h2. If $\bar{y} > 0$, then, for every $0 < y < \bar{y}$, h is decreasing -- and hence integrable (in the sense of Riemann) -- in k on $\{k: k \in H(y)$ and $k < \underline{k}(y)\}$;

and

h3. If $\bar{y} > 0$, then there is a positive constant $\nu > 0$ with property that, for every $k \geq 0$,

$$y' \in H(k) \text{ and } 0 < y'' \leq y' \Rightarrow \frac{h(y',k)}{y'} \geq \nu \frac{h(y'',k)}{y''}.$$

Note for future reference that H2 and h1 entail that $\underline{k}(y)$ is increasing in y, so that, in particular,

$$\underline{k}(0) \leq \lim_{y \to 0^+} \underline{k}(y) = \inf_{y>0} \underline{k}(y) = \underline{k} \leq \underline{k}(y) \quad \text{for } y > 0.$$

We reemphasize the fact that these maintained assumptions, though consistent with conventional neoclassical assumptions (the "leading ease" referred to below), are very much weaker. Indeed, as should become apparent from the subsequent analysis, they are so unrestrictive as to be almost, but not quite completely unobjectionable. With this assertion clearly in mind, then, consider the following interpretive comments: (i) Both H1 and H2 are innocuous, as well as transparent, since they simply state, respectively, that some positive output level can be produced from every positive, non-sustaining capital stock, and that both output and capital can be disposed of --provided, in the background, there is sufficient accompanying depletion. (ii) h1 and h2 then sharpen the terms on which (given a feasible production point) either less output can be produced or more capital employed, since, at a minimum, they amount, respectively, to free disposal of output and (within the limits specified) free disposal of capital. (iii) Of course, h1 also embodies the obvious natural productivity hypothesis concerning increased resource utilization (and, to a somewhat lesser degree, presuming h1 obtains, h2 one obvious possible productivity hypothesis concerning increased capital intensity). That is, h1 entails that additional depletion can typically yield additional output, everything else the same (and h2, again within the limits specified, that capital goods can typically replace exhaustible resources, everything else the same). (iv) In particular, h1 and h2 would both be true in the leading case where, for every $k \geq 0$, h is increasing in y on $H(k)$, i.e., where the labelling of isoquants is increasing in the northeast direction, and where, for every $y > 0$, h is decreasing in k on $H(y)$, i.e., where isoquants are downward sloping, respectively. Recall, however, that the isoquants in question represent constant <u>net</u> final goods output, so that even as it stands, h2 is not indisputable; with capital deepening, greater depletion may be required simply to offset higher depreciation, everything else the same. (v) Finally, h3 is very mild version of the classical law of diminishing returns to a single factor. As we shall demonstrate later on, its basic purpose is to rule out the extremely implausible possibility that average returns to exhaustible resources are asymptotically unbounded. In particular, h3 would be true in the leading case where, for every $k \geq 0$, $h(0,k) = 0$ and h is convex in y on $H(k)$, i.e., where capital can be costlessly stored and depletion is in fact subject to universal diminishing returns.

Much of our lengthy analysis revolves around the two assumptions h2 and h3. Thus, it is worth emphasizing again that h2 can be replaced by a much weaker regularity requirement in continuous time (see h2′ in the Appendix), and stressing explicitly that h3 can be replaced by an even weaker curvature condition upon deeper investigation (see h3′ in the Appendix).

The basic nature of our maintained assumptions has already been exhibited by Figures 1 and 2; various aspects of their generality are now highlighted by Figure 3. Notice especially that no continuity restrictions (beyond those

[insert Figure 3 about here]

implicit in the monotonicity assumptions) are imposed on either resource utilization or capital intensity, and that, as previously remarked, only the mildest convexity restriction is imposed on resource utilization.

C. The Fundamental Existence Problem

Given the description of all feasible growth paths together with the specification of the resource requirement function, our goal is to characterize the static technologies (represented by H and h) which admit the possibility of indefinitely sustained consumption. Specifically, we seek

<u>substitution conditions on</u> h (<u>additional to the maintained assumptions h1, h2 and h3</u>) <u>which are both necessary and sufficient to guarantee that, for every positive pair of initial stocks</u> $(\tilde{r},\tilde{k}) >> 0$, <u>there exists a feasible growth path (i.e., a solution to (2))</u> $\{(c_t,k_t)\}$ <u>along which consumption goods output is uniformly positive</u> $\inf_{t\geq 0} c_t = c > 0$. Utilizing H2 and h1 in simplifying (2), this fundamental existence problem can be succinctly stated as follows: Find substitution conditions, say SC, with the property that

$$\begin{cases} \text{for every } (\tilde{r},\tilde{k}) > 0 \text{ there exists } (c,\{k_t\}) \text{ such that} \\ (c + k_{t+1} - k_t, k_t) \in H \quad \text{for } t \geq 0 \\ \text{and} \\ c > 0, \ \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t) \leq \tilde{r} \text{ and } k_0 \leq \tilde{k} \end{cases} \tag{3}$$

if and only if

h satisfies SC.

A word of caution: Our concern is with the existence <u>not</u> the magnitude of indefinitely sustained consumption. It is obvious to us -- as well as to many others who have commented on our analysis -- that an economy may well be viable in our narrow technological sense (i.e., $\inf_{t\geq 0} c_t$ may be positive) and yet not in some broad "socio-biological" sense (e.g., $\{c_t\}$ may be insufficient to forestall humanity's extinction). (All this, of course, presumes that one can conceivably define suitable units for prejudging the effects of alternative streams of consumption goods output, a point we shall return to in Section IV.) Nonetheless, we do not apologize for this seemingly damaging inadequacy in our approach. Rather, we argue that our tack is the only sensible course to pursue. We simply cannot begin to address (possibly) more germane questions without already having solved this fundamental existence problem. (See too the closely related remarks in Solow [3, especially p. 37].)

III. A Complete Characterization

Our basic theorem is a complete characterization of the circumstances under which it is feasible to sustain uniformly positive consumption forever, stated in terms of just a single capital-resource substitution condition.

Substitution Condition. For every $0 < k < \underline{k}$,

$$\lim_{y \to 0^+} 1/y \int_k^{\underline{k}(y)} h(y,u)du = 0. \tag{SC}$$

That is, formally, we will establish the following simple sustainability equivalence.

Sustainability Theorem. Suppose H satisfies H1-H2 and h satisfies h1-h3. Then (3) obtains if and only if (SC) obtains.

Before detailing the proof of this fundamental result, it is worthwhile digressing a bit to interpret (and, indirectly, to motivate) the crucial substitution condition (SC) itself. This can be usefully accomplished on at least two distinct levels:

1. <u>The technical interpretation</u>. Note first that if $\underline{k} = 0$, then (SC) is vacuously true. So we need only be concerned with the situation in which $\underline{k} > 0$. Now, the integral in (SC) is nothing more than the area, relative to the k-axis, under the y-level isoquant from arbitrary $k < \underline{k}$ such that $k \in H(y)$, say k', to $\underline{k}(y)$. (Refer back to Figure 2.) Hence, the limit in (SC) is nothing more than the requirement that this area be finite for small positive y, and that it approach zero faster than y does.

The geometry of this interpretation is especially clear when h is linear homogeneous, i.e., when there are constant returns to scale, since then we can convert (SC) into a substitution condition involving only the unit-level isoquant. Briefly, this can be seen as follows: h linear homogeneous implies $\underline{k}(y)$ linear homogeneous implies (i) $\underline{k} > 0$ if and only if $\underline{k} = k(y) = \infty$ for $y > 0$, while (ii) $\underline{k} = 0$ if an only if $\underline{k} \leq \underline{k}(y) < \infty$ for $y > 0$. Thus, when $\underline{k} > 0$, the integral in (SC) can be rewritten (using the change of variable $v = u/y$)

$$\int_{k}^{\underline{k}(y)} h(y,u)du = y^2 \int_{k}^{\infty} h(1,u/y)du/y = y^2 \int_{k/y}^{\infty} h(1,v)dv = y^2 \int_{k/y}^{\underline{k}(1)} h(1,v)dv \qquad (4)$$

and (SC) will be verified (substituting from (4) into (SC)) if and only if

for every $0 < k < \underline{k}(1)$ such that $k \in H(1)$,

$$\int_{k}^{\underline{k}(1)} h(1,u)du < \infty, \qquad (5)$$

while, when $\underline{k} = 0$, (5) is immediately verified. In short, under this additional homogeneity property, (SC) is equivalent to (5), which is nothing more than the requirement that the area "under" the unit-level isoquant be finite. (See too the companion discussion in Mitra [2, especially pp. 15-16].)

2. <u>The economic interpretation</u>. For this interpretation we again ignore the implausible irregularity where $\underline{k}(y) < \infty$ but either $(y,\underline{k}(y)) \notin H$ or $(y,\underline{k}(y)) \in H$ and $h(y,\underline{k}(y)) > 0$. (It is, nonetheless, covered by the Sustainability Theorem.) So, consider only situations in which either $\underline{k}(y) < \infty$, $(y,\underline{k}(y)) \in H$ and $h(y,\underline{k}(y)) = 0$ or $\underline{k}(y) = \infty$. Also, consider only growth paths exhibiting <u>pure, arithmetic accumulation</u>, that is satisfying the two additional properties

$$c_t = 0 \quad \text{for} \quad t \geq 0 \tag{6}$$

and, for some $y > 0$,

$$k_0 \leq \underline{k}(y) \quad \text{and} \quad k_{t+1} = \begin{cases} k_t + y & \text{for } 0 \leq t < t_y \\ \underline{k}(y) & \text{otherwise,} \end{cases} \tag{7}$$

where $t_y = \sup\{t: t \geq 0 \text{ and } t \leq (\underline{k}(y) - k_0)/y\}$. (The descriptive label is slightly inappropriate when $\underline{k}(y) < \infty$ and therefore $t_y < \infty$.) <u>Then it can be shown that, for every positive pair of initial stocks</u> $(\bar{r},\bar{k}) >> 0$, <u>there exists a feasible growth path exhibiting pure, arithmetic accumulation (i.e., a solution to (2) satisfying (6)-(7))</u> $\{k_t\}$ <u>if and only if</u> h <u>satisfies</u> (SC). In other words, basically repeating ourselves from the Introduction, (SC) essentially amounts to the requirement that the economy must be capable, starting from arbitrary, positive stocks and growing with pure, arithmetic accumulation, of either, at best, achieving resource independence (when $\underline{k} < \infty$ and thus, typically, $t_y < \infty$) or, at worst, avoiding resource exhaustion (when $\underline{k} = \infty$ and thus, necessarily, $t_y = \infty$).

The proof of this interesting interpretive equivalence is a somewhat simplified (but nonetheless complicated) version of the proof of our main result. For this reason we will content ourselves with merely sketching the argument for the intermediate case in which we have $\bar{y} > 0$ as well as $\bar{k} = \infty$ (and hence the hypothesis in both h2 and h3 as well as $\underline{k}(y) = \infty$ in (SC)).

Sufficiency. We want to show that if h satisfies (SC), then there exists a solution to (2) satisfying (6)-(7). Toward this end, pick $k_0 = \tilde{k}$, and $0 < y < \bar{y}$ such that $(y, \tilde{k} - y) \in H$ [using H1 and H2] and

$$1/y \int_{\tilde{k}-y}^{\infty} h(y,u)du \leq \tilde{r} \tag{8}$$

[using H2 and (SC)]. Then, given these particular values of k_0 and y, the equations (6)-(7) themselves yield a solution to (2), since $(y, \tilde{k} - y) \in H$ implies $(c_t + k_{t+1} - k_t, k_t) = (y, \tilde{k} + yt) \in H$ for $t \geq 0$ [using H2], while

$$\sum_{t=0}^{\infty} h(c_t + k_{t+1} - k_t, k_t) = \sum_{t=0}^{\infty} h(y, \tilde{k} + yt)$$

$$= 1/y \sum_{t=0}^{\infty} h(y, \tilde{k} + yt)y$$

$$\leq 1/y \int_{\tilde{k}-y}^{\infty} h(y,u)du \qquad \text{[using h2]}$$

$$\leq \tilde{r} \;. \qquad \text{[using (8)]}$$

<u>Necessity</u>. We want to show that if, for every $(\tilde{r},\tilde{k}) \gg 0$, there exists a solution to (2) satisfying (6)-(7), then h satisfies (SC). Toward this end, given arbitrary $(\tilde{r},\tilde{k}) \gg 0$, suppose we have a solution to (2) satisfying (6)-(7). Then, the resource bound in (2) yields the following chain of inequalities:

$$\tilde{r} \geq \sum_{t=0}^{\infty} h(c_t + k_{t+1} - k_t, k_t)$$

$$= \sum_{t=0}^{\infty} h(y, k_0 + yt)$$

$$= \sum_{t=0}^{\infty} \frac{h(y, k_0 + yt)}{y}\, y$$

$$\geq \nu/y' \sum_{t=0}^{\infty} h(y', k_0 + yt)y \quad \text{for} \quad 0 < y' < \min\,\{y,\bar{y}\} \qquad \text{[using h3]}$$

$$\geq \nu/y' \int_{k_0}^{\infty} h(y',u)du \qquad \text{[using h2]}$$

$$\geq \nu/y' \int_{\tilde{k}}^{\infty} h(y',u)du \qquad [\text{since } k_0 \leq \tilde{k}]$$

$$> 0,$$

or, ignoring intermediate steps (and rearranging constant terms), the following inequality:

for every $0 < y' < \min\{y,\bar{y}\}$,

$$0 < 1/y' \int_{\tilde{k}}^{\infty} h(y',u)du \leq \tilde{r}/\nu. \qquad (9)$$

But, because we started with arbitrary $(\tilde{r},\tilde{k}) >> 0$, (9) is simply one equivalent restatement of (SC) in "ε-δ" form. ■

One crucial step in both sides of the foregoing argument merits explicit comment. In order to go from sums to integrals we used the elementary result for monotone functions that, under the conclusion of h2,

for every $k^i \in H(y)$ such that $k^i < \underline{k}(y)$, $i = 1,2$,

$$h(y,k^2)(k^2 - k^1) \leq \int_{k^1}^{k^2} h(y,u)du \leq h(y,k^1)(k^2 - k^1).$$

This is the characteristic role which such restriction plays in our analysis, explaining why it can be dispensed with in continuous time, where sums are naturally replaced by integrals at the outset. (The sufficiency argument for the Sustainability Theorem together with the necessity counterexamples in the Appendix will delimit the extent to which h2 can be dispensed with in discrete time.)

A final aside: It is straightforward but tedious to show that, when h corresponds to f with constant elasticity of substitution $\sigma \geq 0$, say, given arbitrary $0 < \delta < 1$ and $\rho \geq -1$,

$$f(d,k) = [(1 - \delta)d^{-\rho} + \delta k^{-\rho}]^{-1/\rho}$$

with

$$\sigma = 1/(1 + \rho),$$

h satisfies (SC) if and only if $\sigma > 1$ or $\sigma = 1$ and $\delta > 1/2$. The details of verifying this relationship (between our theorem and the Solow-Stiglitz example) are left as an exercise for the reader.

We are now ready to proceed with the

Proof of the Sustainability Theorem. Sufficiency. We will show that, under the maintained assumptions, the substitution condition (SC) can be used to construct a solution to (2) satisfying $c_t = c > 0$ for $t \geq 0$, that is, a solution to the dynamical system

$$\begin{cases} (c + k_{t+1} - k_t, k_t) \in H \quad \text{for} \quad t \geq 0 \\ \text{and} \\ c > 0, \ \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t) \leq \tilde{r} \ \text{ and } \ k_0 \leq \tilde{k}. \end{cases} \tag{10}$$

In fact, we will show that this construction can be completed without any reference to either h2 or h3.

There are two cases to consider, one $(\underline{k} = 0)$ trivial, the other $(\underline{k} > 0)$ not. In both cases the argument is slightly complicated by the possibility mentioned twice earlier, that $\underline{k}(y) < \infty$ but either $(y,\underline{k}(y)) \notin H$ or $(y,\underline{k}(y)) \in H$ and $h(y,\underline{k}(y)) > 0$. Examples of resource requirement functions exhibiting these sorts of irregularity are illustrated in Figure 4.

[insert Figure 4 about here]

Case 1. $\underline{k} = 0$.

Pick $\{\alpha_t\}$ such that $\alpha_t > 0$ for $t \geq 0$ and $\sum_{t=0}^{\infty} \alpha_t \leq \tilde{r}$ (e.g., $\alpha_t = ab^t$ with $0 < b < 1$ and $0 < a \leq (1 - b)\tilde{r}$), and using the supposition of this case, $y > 0$ such that $\underline{k}(y) \leq \tilde{k} - y/4$. Then, appealing to the definition of $\underline{k}(y)$, define $\{k_t\}$ such that

$$(y,k_t) \in H, \quad \underline{k}(y) - y/4 \leq k_t \leq \underline{k}(y) + y/4 \quad \text{and} \quad h(y,k_t) \leq \alpha_t \quad \text{for} \quad t \geq 0$$

with corresponding

$$c_t = y - (k_{t+1} - k_t) \quad \text{for} \quad t \geq 0.$$

It follows that $c_t \geq y/2 > 0$ and $(c_t + k_{t+1} - k_t, k_t) = (y,k_t) \in H$ for $t \geq 0$, while

$$\sum_{t=0}^{\infty} h(y,k_t) \leq \sum_{t=0}^{\infty} \alpha_t \leq \tilde{r}$$

and $k_0 \leq \tilde{k}$, that is, that $\{(c_t,k_t)\}$ is a solution to (2). Hence $(c,\{k_t\})$ with $c = y/2 \leq c_t$ for $t \geq 0$ is a solution to (10) [using H2 and h1].

Case 2. $\underline{k} > 0$.

Pick $\{\alpha_t\}$ such that $\alpha_t > 0$ for $t \geq 0$ and $\sum_{t=0}^{\infty} \alpha_t \leq \tilde{r}/2$, $0 < k \leq \tilde{k}$ such that $k < \underline{k}$ (without loss of generality, $k = \tilde{k}$), and $0 < \lambda < 1/2$ and $y > 0$ such that $(y, \tilde{k} - \lambda y) \in H$ [using H1 and H2] and

$$1/\lambda y \int_{\tilde{k}-\lambda y}^{\underline{k}(y)} h(y,u)\,du \leq \tilde{r}/2$$

[using H2 and (SC)]. Then, proceed to define $\{k_t\}$ inductively as follows: First, for $t = 0$, pick k_0 such that $\tilde{k} - \lambda y \leq k_0 < \tilde{k}$ and

$$h(y,k_0) - \alpha_0 \leq \inf_{\tilde{k}-\lambda y \leq k < \tilde{k}} h(y,k),$$

so that

$$[h(y,k_0) - \alpha_0]\lambda y \leq \int_{\tilde{k}-\lambda y}^{\tilde{k}} \left[\inf_{\tilde{k}-\lambda y \leq k < \tilde{k}} h(y,k)\right] du \leq \int_{\tilde{k}-\lambda y}^{\tilde{k}} h(y,u)\,du$$

or

$$h(y,k_0) \leq 1/\lambda y \int_{\tilde{k}-\lambda y}^{\tilde{k}} h(y,u)\,du + \alpha_0. \tag{11}$$

Then, for $t + 1 > 0$, given k_t such that $k_t + 2\lambda y \leq \underline{k}(y)$ (e.g., when $\underline{k}(y) = \infty$), pick k_{t+1} such that $k_t + \lambda y \leq k_{t+1} < k_t + 2\lambda y$ and

$$h(y,k_{t+1}) - \alpha_{t+1} \leq \inf_{k_t+\lambda y \leq k < k_t+2\lambda y} h(y,k),$$

so that

$$\lambda y \leq k_{t+1} - k_t < 2\lambda y < y \tag{12}$$

and

$$[h(y,k_{t+1}) - \alpha_{t+1}]\lambda y \leq \int_{k_t+\lambda y}^{k_t+2\lambda y} \left[\inf_{k_t+\lambda y \leq k < k_t+2\lambda y} h(y,k) \right] du \leq \int_{k_t+\lambda y}^{k_t+2\lambda y} h(y,u)\,du$$

or

$$h(y,k_{t+1}) \leq 1/\lambda y \int_{k_t+\lambda y}^{k_t+2\lambda y} h(y,u)\,du + \alpha_{t+1}. \tag{13}$$

Finally, for $t + 1 > 0$, given k_t such that $k_t < k_{t-1} + 2\lambda y \leq \underline{k}(y) < k_t + 2\lambda y$ (for convenience, defining $k_{-1} = \tilde{k} - 2\lambda y$), pick k_s for $s \geq t + 1$ such that $k_t \leq k_s < k_t + 2\lambda y$, so that

$$k_s - k_{s-1} \leq |k_s - k_{s-1}| < 2\lambda y < y, \tag{14}$$

and

$$h(y,k_s) \leq \alpha_s. \tag{15}$$

Examples of such sequences $\{k_t\}$ in the two possible cases ($\underline{k}(y) = \infty$ and $\underline{k}(y) < \infty$) are illustrated in Figure 5.

[insert Figure 5 about here]

Given our initial specifications of $\{\alpha_t\}$, λ and y, it follows from the particulars of this construction (especially those represented by the inequalities (11)-(15)) that $\{(c_t,k_t)\}$ with $c_t = y - (k_{t+1} - k_t)$ for $t \geq 0$ is a solution to (2), since $c_t = y - (k_{t+1} - k_t) > (1 - 2\lambda)y > 0$ and $(c_t + k_{t+1} - k_t,k_t) = (y,k_t) \in H$ for $t \geq 0$, while

$$\sum_{t=0}^{\infty} h(c_t + k_{t+1} - k_t,k_t) = \sum_{t=0}^{\infty} h(y,k_t)$$

$$\leq \sum_{\{t:\ t\geq 0 \text{ and } k_{t-1}+2\lambda y\leq \underline{k}(y)\}} [1/\lambda y \int_{k_{t-1}+\lambda y}^{k_{t-1}+2\lambda y} h(y,u)du + \alpha_t] + \sum_{\{t:\ t\geq 0 \text{ and } k_{t-1}+2\lambda y>\underline{k}(y)\}} [\alpha_t]$$

$$\leq 1/\lambda y \int_{\tilde{k}-\lambda y}^{\underline{k}(y)} h(y,u)du + \sum_{t=0}^{\infty} \alpha_t$$

$$\leq \bar{r}$$

and $k_0 \leq \tilde{k}$. Hence, $(c,\{k_t\})$ with $c = (1 - 2\lambda)y < c_t$ for $t \geq 0$ is a solution to (10) [using H2 and h1].

<u>Remarks</u>. We have already noted that neither h2 nor h3 plays any part whatsoever in the preceding proof. Moreover, since h1 only appears in the

final scene of that argument, it is clear that even this assumption, innocuous as it is, can be retired from the stage -- provided that the constancy requirement $c_t = c > 0$ for $t \geq 0$ is replaced by the less restrictive (but equally meaningful) uniformity requirement $\inf_{t\geq 0} c_t = c > 0$. Furthermore, in somewhat the same spirit, it turns out that the substitution condition (SC) itself can be weakened, by replacing "lim" with "lim inf" (for necessity as well as sufficiency), a development we will discuss further in the Appendix.

It is also worth noting that the foregoing construction can be easily simplified to yield a solution to (2) satisfying (6)-(7) when h does in fact exhibit the monotonicity property h2 (without the openness irregularity illustrated in Figure 4). This generalization of the interpretive sufficiency argument presented earlier is based on the readily verified observation that (SC) entails $\bar{y} > 0$, so that in defining $\{k_t\}$ above we can choose $0 < y < \bar{y}$, $\lambda = 1/2$ and -- given that h satisfies h2 -- $k_t = \min\{k_{t+1} + y, \underline{k}(y)\}$ (with $k_{-1} = \tilde{k} - y$) for $t \geq 0$.

<u>Necessity</u>. This proof naturally decomposes into two steps, the first relying on neither h2 nor h3, the second relying, in a fundamental way, on both assumptions. To underline this distinction (which is also convenient in developing the parallel argument for continuous time) we present the first as a separate result.

<u>Lemma</u>. Suppose H satisfies H2 and h satisfies h1. Given arbitrary $(\tilde{r},\tilde{k}) \gg 0$, if $(c,\{k_t\})$ is a solution to the dynamical system

$$\begin{cases} (c + k_{t+1} - k_t, k_t) \in H \quad \text{for} \quad t \geq 0 \\ \text{and} \\ c > 0, \ \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t) \leq \tilde{r} \quad \text{and} \quad k_0 \leq \tilde{k}, \end{cases} \tag{10}$$

then $c \leq \bar{y}$ and, for every $0 < y < c$, $\underline{k}(y) \leq \sup_{t \geq 0} k_t$.

<u>Proof of the Lemma</u>. Suppose $(c, \{k_t\})$ is a solution to (10). Let

$$t^+(y,\tau) = \{t\colon 0 \leq t \leq \tau \ \text{ and } \ c + k_{t+1} - k_t \geq y\}$$

and

$$\begin{aligned} t^-(y,\tau) &= \{0,1,\ldots,\tau\} - t^+(y,\tau)\} \\ &= \{t\colon 0 \leq t \leq \tau \ \text{ and } \ c + k_{t+1} - k_t < y\} \end{aligned}$$

for $y \in \mathbb{R}$, $\tau \geq 0$. In particular, the former definition reduces to

$$t^+(c,\tau) = \{t\colon 0 \leq t \leq \tau \ \text{ and } \ k_{t+1} - k_t \geq 0\}$$

for $y = c$. Also let

$$n[A] = \text{number of elements in } A$$

for $A \subset \{0,1,2,\ldots\}$. Then, there are two cases to consider.

<u>Case 1</u>. $\lim_{\tau\to\infty} n[t^+(c,\tau)] = \infty$.

To analyze this case, we use the fact that the resource bound in (10) yields the following chain of inequalities:

$$\begin{aligned}
\bar{r} &\geq \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t) \\
&\geq \sum_{t=0}^{\tau} h(c + k_{t+1} - k_t, k_t) \quad \text{for} \quad \tau \geq 0 \\
&\geq \sum_{t\in t^+(c,\tau)} h(c + k_{t+1} - k_t, k_t) \\
&\geq \sum_{t\in t^+(c,\tau)} h(y, k_t) \quad \text{for} \quad 0 < y < c \qquad \text{[using H2 and h1]} \\
&\geq \sum_{t\in t^+(c,\tau)} \inf_{k\in H(y),\, k\leq \sup_{t\geq 0} k_t} h(y,k) \\
&= n[t^+(c,\tau)] \inf_{k\in H(y),\, k\leq \sup_{t\geq 0} k_t} h(y,k).
\end{aligned}$$

By the supposition of this case, however, the final inequality can be true if and only if

$$\text{for every} \quad 0 < y < c, \qquad \inf_{k\in H(y),\, k\leq \sup_{t\geq 0} k_t} h(y,k) = 0. \tag{16}$$

But (16) implies $c \leq \bar{y}$ -- since otherwise, i.e., if $\bar{y} < c$, then by virtue of the definition of $\bar{y}$ there exists $\bar{y} < y' < c$ such that

$$0 < \inf_{k \in H(y')} h(y',k) \leq \inf_{k \in H(y'),\, k \leq \sup_{t \geq 0} k_t} h(y',k),$$

contradicting (16) -- as well as, for every $0 < y < c$, $\underline{k}(y) \leq \sup_{t \geq 0} k_t$ -- since otherwise, i.e., if there exists $0 < y' < c$ such that $\sup_{t \geq 0} k_t < \underline{k}(y')$, then by virtue of the definition of $\underline{k}(y')$

$$0 < \inf_{k \in H(y'),\, k \leq \sup_{t \geq 0} k_t} h(y',k),$$

again contradicting (16).

<u>Case 2</u>. $\lim_{\tau \to \infty} n[t^+(c,\tau)] < \infty$, i.e., there exists $\tau' > 0$ such that

$$t^+(c,\tau) = t^+(c,\tau') \quad \text{for} \quad \tau \geq \tau'.$$

By the supposition of this case, we know that

$$\sum_{t \in t^+(c,\tau)} (k_{t+1} - k_t) \leq \sum_{t \in t^+(c,\tau')} (k_{t+1} - k_t) = K < \infty \quad \text{for} \quad \tau \geq 0.$$

Hence, we also know that

$$0 \leq k_{\tau+1} = k_0 + \sum_{t=0}^{\tau} (k_{t+1} - k_t)$$

$$= k_0 + \sum_{t \in t^+(c,\tau)} (k_{t+1} - k_t) + \sum_{t \in t^-(c,\tau)} (k_{t+1} - k_t)$$

$$\leq k_0 + K + \sum_{t \in t^-(c,\tau)} (k_{t+1} - k_t)$$

or that (since $t \in t^-(c,\tau)$ implies $c + k_{t+1} - k_t < c$ implies $-(k_{t+1} - k_t) > 0$

$$0 \leq - \sum_{t \in t^-(c,\tau)} (k_{t+1} - k_t) \leq k_0 + K \quad \text{for} \quad \tau \geq 0.$$

It then follows that

$$0 \leq - n[t^-(y,\tau)](y - c) \leq - \sum_{t \in t^-(y,\tau)} (k_{t+1} - k_t) \leq - \sum_{t \in t^-(c,\tau)} (k_{t+1} - k_t) \leq k_0 + K$$

or that

$$0 \leq n[t^-(y,\tau)] \leq (k_0 + K)/-(y - c) \quad \text{for} \quad y < c,\ \tau \geq 0.$$

Thus, in this case, the resource bound in (10) yields the following chain of inequalities:

$$\bar{r} \geq \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t)$$

$$\geq \sum_{t=0}^{\tau} h(c + k_{t+1} - k_t, k_t) \quad \text{for} \quad \tau \geq 0$$

$$\geq \sum_{t \in t^+(y,\tau)} h(c + k_{t+1} - k_t, k_t) \quad \text{for} \quad 0 < y < c$$

$$\geq \sum_{t \in t^+(y,\tau)} h(y, k_t) \qquad \text{[using H2 and h1]}$$

$$\geq \sum_{t \in t^{+}(y,\tau)} \inf_{k \in H(y),\, k \leq \sup_{t \geq 0} k_t} h(y,k)$$

$$= n[t^{+}(y,\tau)] \inf_{k \in H(y),\, k \leq \sup_{t \geq 0} k_t} h(y,k)$$

$$= [(\tau + 1) - n[t^{-}(y,\tau)]] \inf_{k \in H(y),\, k \leq \sup_{t \geq 0} k_t} h(y,k)$$

$$\geq [(\tau + 1) - (k_0 + K)/-(y - c)] \inf_{k \in H(y),\, k \leq \sup_{t \geq 0} k_t} h(y,k).$$

As before, the final inequality entails the desired conclusion. ∎

In order to complete the proof of necessity, now suppose that $\underline{k} > 0$ and that, given arbitrary $\tilde{r} > 0$ and $0 < \tilde{k} < \underline{k}$, $(c,\{k_t\})$ is a solution to (10). Utilizing the Lemma, we will show that the existence of such a feasible growth path yields the inequality

for every $0 < y < c$,

$$0 < 1/y \int_{\tilde{k}}^{\underline{k}(y)} h(y,u)du \leq \tilde{r}/\nu \tag{17}$$

(recalling the introduction of the constant $\nu > 0$ in h3). But, as in the interpretive necessity argument presented earlier, (17) is simply one equivalent restatement of (SC) in "ε-δ" form.

So, given arbitrary $0 < y < c$, consider the two possible cases

<u>Case 1</u>. $k_t \geq \underline{k}(y)$ for some $t > 0$, i.e., there exists $t_y > 0$ such that

$$k_t \begin{Bmatrix} < \\ \geq \end{Bmatrix} \underline{k}(y) \quad \text{according as} \quad t \begin{Bmatrix} < \\ = \end{Bmatrix} t_y \tag{18}$$

and

Case 2. $k_t < \underline{k}(y)$ for every $t \geq 0$, i.e., by virtue of the second property asserted in the Lemma,

$$k_t < \underline{k}(y) \text{ for } t \geq 0 \text{ but } \sup_{t \geq 0} k_t = \underline{k}(y). \tag{19}$$

Bear in mind that in both cases y is fixed for the purposes of this analysis, and that, by virtue of the first property asserted in the Lemma, in both cases the triggering hypothesis of both h2 and h3 is satisfied.

Focus first on the possibility described by (18), and let

$$t^+(t_y) = \{t\colon 0 \leq t < t_y \text{ and } k_{t+1} - k_t \geq 0\}$$

(noticing that, by the nature of this case, $t_y - 1 \in t^+(t_y)$) and

$$\tilde{h}(y,k) = \begin{cases} h(y,\tilde{k}) & \text{for } 0 \leq k \leq \tilde{k} \\ h(y,k) & \text{for } k \geq \tilde{k} \end{cases}$$

(noticing that, again by the nature of this case, there must exist $0 \leq t' < t_y$ such that $k_{t'} \leq \tilde{k}$, $k_{t'+1} - k_{t'} \geq 0$ and, since we have $0 < y < c$, $(y + k_{t'+1} - k_{t'}, k_{t'}) \in H$ [using H2], which implies that $(y,\tilde{k}) \in H$ [using H2], which implies that h is well-defined and decreasing in k on $[0,\underline{k}(y))$ [using H2 and h2]). Then, in this case the resource bound in (10) yields the following chain of inequalities:

$$\tilde{r} \geq \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t)$$

$$\geq \sum_{t\in t^+(t_y)} h(c + k_{t+1} - k_t, k_t)$$

$$\geq \sum_{t\in t^+(t_y)} h(y + k_{t+1} - k_t, k_t) \qquad \text{[using H2 and h1]}$$

$$\geq \sum_{t\in t^+(t_y)} \frac{h(y + k_{t+1} - k_t, k_t)}{(y + k_{t+1} - k_t)} (k_{t+1} - k_t)$$

$$\geq \sum_{t\in t^+(t_y)} \nu \frac{h(y,k_t)}{y} (k_{t+1} - k_t) \qquad \text{[using h3]}$$

$$\geq \nu/y \sum_{t\in t^+(t_y)} \tilde{h}(y,k_t)(k_{t+1} - k_t) \qquad \text{[using h2]}$$

$$\geq \nu/y \sum_{t=0}^{t_y-1} \tilde{h}(y,k_t)(k_{t+1} - k_t) \qquad [\text{since } 0 \leq t < t_y \text{ but } t \notin t^+(t_y) \Rightarrow k_{t+1} - k_t < 0]$$

$$\geq \nu/y \left[\sum_{t=0}^{t_y-2} \tilde{h}(y,k_t)(k_{t+1} - k_t) + \tilde{h}(y,k_{t_y-1})(\underline{k}(y) - k_{t_y-1})\right] \qquad [\text{since } k_{t_y-1} < \underline{k}(y) \leq k_{t_y}]$$

$$\geq \nu/y \int_{\min_{0\leq t<t_y} k_t}^{\underline{k}(y)} \tilde{h}(y,u)\,du \qquad [\text{since } 0 \leq k', k'' < \underline{k}(y) \Rightarrow$$

$$\tilde{h}(y,k')(k'' - k') \geq \int_{k'}^{k''} \tilde{h}(y,u)du]$$

$$\geq \nu/y \int_{\tilde{k}}^{\bar{k}(y)} h(y,u)du \qquad [\text{since} \min_{0 \leq t < t_y} k_t \leq \tilde{k}]$$

$$> 0.$$

These in turn immediately yield the desired conclusion in (17).

For the possibility described by (19), the argument is essentially the same, but formulated in terms of the subsets of periods

$$t^+(\tau) = \{t: 0 \leq t < \tau \text{ and } k_{t+1} - k_t \geq 0\} \quad \text{for} \quad \tau \geq 0,$$

and completed by going from the chain of inequalities

$$\tilde{r} \geq \sum_{t=0}^{\infty} h(c + k_{t+1} - k_t, k_t) \geq \cdot \cdot \cdot \geq \nu/y \int_{\tilde{k}}^{\max_{0 \leq t \leq \tau} k_t} \tilde{h}(y,u)du \quad \text{for} \quad \tau \geq 0$$

to the desired conclusion in (17) by means of (19)

$$\tilde{r} \geq \nu/y \int_{\tilde{k}}^{\sup_{t \geq 0} k_t} \tilde{h}(y,u)du = \nu/y \int_{\tilde{k}}^{\bar{k}(y)} h(y,u)du > 0.$$

■

<u>Remarks</u>. The essential idea in the preceding proof is to go from the resource bound in (10) (or, more accurately, (3)) to the integral limit in (SC). The crucial steps in this logical progression involve replacing "$h(c + k_{t+1} - k_t, k_t$

by "$h(y,k_t)(k_{t+1} - k_t)/y$" and "$\Sigma$" by "$\int$", utilizing, respectively, h3 and h2. Shortly we will demonstrate that this particular procedure is not accidental, that these pivotal assumptions are in fact fundamental to the result. Less apparent is the role of h1. However, it too provides indispensable support, as we shall also soon demonstrate. Finally, while H1 plays no part whatsoever in the necessity deduction, quite obviously it was essential in the sufficiency construction, since without it, so to speak, the economy may never leave -- or, for that matter, even reach -- the starting gate. For this reason alone, we will have little more to say about this assumption -- just as, for an equally compelling reason (namely, that it is basically unimpeachable), we will also have little more to say about H2.

IV. Two Possible Generalizations

Our purpose in this final section is to outline two of the more interesting generalizations of our analysis. These involve first, the potential for increasing and unbounded consumption, and second, the presence of many different commodities. Our treatment is only intended to be suggestive, not exhaustive.

A. Increasing and Unbounded Consumption

The essential idea here is to tie consumption to the level of the capital stock, and then to characterize the growth paths exhibiting increasing and unbounded growth. For simplicity we again focus on the intermediate case in which $\bar{y} > 0$ as well as $\underline{k} = \infty$.

So now think of c_t = constant $\cdot$ $\phi(k_t)$, where $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and satisfies, say, $\phi(0) = 1$ and also $\phi(\infty) = \infty$. Then, in effec strengthening H1 and h2 to encompass ϕ as well as h, suppose that

$$\text{for every } k > 0 \text{ there is a positive constant } c > 0 \text{ such that } (c'\phi(k'),k') \in H \text{ for } k' \geq k,\ 0 < c' \leq c \tag{20}$$

and

$$h(c'\phi(k'),k')/c'\phi(k') \text{ is decreasing in } k' \text{ for } k' \geq k,\ 0 < c' \leq c. \tag{21}$$

Finally, consider feasible growth paths of the specific form

$$\begin{cases} k_{t+1} - k_t \geq 0 \text{ and } (c\phi(k_t) + k_{t+1} - k_t, k_t) \in H \text{ for } t \geq 0, \\ \lim_{t\to\infty} k_t = \infty \\ \text{and} \\ c > 0,\ \sum_{t=0}^{\infty} h(c\phi(k_t) + k_{t+1} - k_t, k_t) \leq \tilde{r} \text{ and } k_0 \leq \tilde{k}. \end{cases} \tag{22}$$

The analogue of our Sustainability Theorem obviously also requires a stronger Substitution Condition, namely, that for every $k > 0$,

$$\lim_{y\to 0^+} \int_k^{\infty} \frac{h(y\phi(u),u)}{y\phi(u)}\, du = 0. \tag{SC^s}$$

In these terms the following result is easily demonstrated.

<u>"Growth" Theorem</u>. For every $(\tilde{r},\tilde{k}) >> 0$ there exists $(c,\{k_t\})$ satisfying

(22) if and only if (SC^s) obtains.

Proof of the "Growth" Theorem. Sufficiency. Pick $y > 0$ such that $0 < y\phi(\tilde{k}) < \tilde{k}$, $(y\phi(k),k) \in H$ for $k \geq \tilde{k}/2$ [using (20)] and

$$\int_{\tilde{k}/2}^{\infty} \frac{h(y\phi(u),u)}{y\phi(u)} du \leq \tilde{r}/2 \tag{23}$$

[using (20) and (SC^s)], and consider the particular growth path defined by

$$\begin{cases} k_0 = \tilde{k} \\ \text{and} \\ k_{t+1} = k_t + y\phi(k_t)/2 \quad \text{and} \quad c_t = y\phi(k_t)/2 \quad \text{for} \quad t \geq 0. \end{cases} \tag{24}$$

Then

$$\begin{aligned} \sum_{t=0}^{\infty} h(c_t + k_{t+1} - k_t, k_t) &= \sum_{t=0}^{\infty} h(y\phi(k_t), k_t) \\ &= \sum_{t=0}^{\infty} \frac{h(y\phi(k_t), k_t)}{y\phi(k_t)/2} (y\phi(k_t)/2) \\ &\leq 2 \int_{\tilde{k}-y\phi(\tilde{k})/2}^{\infty} \frac{h(y\phi(u),u)}{y\phi(u)} du \qquad \text{[using (21)]} \\ &\leq 2 \int_{\tilde{k}/2}^{\infty} \frac{h(y\phi(u),u)}{y\phi(u)} du \end{aligned}$$

$$\leq \tilde{r} . \qquad \text{[using (23)]}$$

Thus, (24) yields a solution to (22) with $c = y/2$.

Necessity. Suppose that (22) has a solution for arbitrary $(\tilde{r},\tilde{k}) >> 0$. It is straightforward to show that this implies that

for every $0 < y < c$,

$$\int_{\tilde{k}}^{\infty} \frac{h(y\phi(u),u)}{y\phi(u)} du \leq \tilde{r}/\nu:$$

$$\tilde{r} \geq \sum_{t=0}^{\infty} h(c\phi(k_t) + k_{t+1} - k_t, k_t)$$

$$\geq \sum_{t=0}^{\infty} h(y\phi(k_t) + k_{t+1} - k_t, k_t) \quad \text{for} \quad 0 < y < c \qquad \text{[using H2 and h1]}$$

$$\geq \sum_{t=0}^{\infty} \frac{h(y\phi(k_t) + k_{t+1} - k_t, k_t)}{y\phi(k_t) + k_{t+1} - k_t} (k_{t+1} - k_t)$$

$$\geq \nu \sum_{t=0}^{\infty} \frac{h(y\phi(k_t), k_t)}{y\phi(k_t)} (k_{t+1} - k_t) \qquad \text{[using h3]}$$

$$\geq \nu \int_{\tilde{k}}^{\infty} \frac{h(y\phi(u),u)}{y\phi(u)} du. \qquad \text{[using (21)]}$$

■

It is worth noting -- concerning the requirement that the capital stock be increasing and unbounded -- that when, say, ϕ is uniformly Lipschitzian (i.e., there is a positive constant $\lambda > 0$ such that $|\phi(k')-\phi(k'')| < \lambda|k'-k''|$ for $k',k'' \geq 0$ and $k' \neq k''$) (22) has a solution if and only if

$$\begin{cases} (c\phi(k_t) + k_{t+1} - k_t, k_t) \in H \quad \text{for} \quad t \geq 0 \\ \text{and} \\ c > 0, \ \sum_{t=0}^{\infty} h(c\phi(k_t) + k_{t+1} - k_t, k_t) \leq \tilde{r} \quad \text{and} \quad k_0 \leq \tilde{k} \end{cases}$$

has a solution. In other words, when natural resources are actually indispensable (in the sense that $\underline{k} = \infty$), unlimited growth is equally indispensable. This result (whose proof we will not detail) also means that the preceding argument provides a simpler demonstration of our Sustainability Theorem for this intermediate case (by simply taking $\phi(k) = 1$ for $k \geq 0$).

B. Many Types of Commodities

For this kind of generalization, the crucial problem is to guarantee that the "natural" aggregation procedure provides suitable approximation. Without any pretense of presenting the most delicate analysis possible, we consider the following basic set-up.

There are $\ell \geq 1$ types of consumption goods with flows denoted $C = (c^1, c^2, \ldots, c^\ell) \in \mathbb{R}_+^\ell$, $m \geq 1$ types of exhaustible resources with (input) flows denoted $D = (d^1, d^2, \ldots, d^m) \in \mathbb{R}_+^m$ and stocks denoted $R = (r^1, r^2, \ldots, r^m) \in \mathbb{R}_+^m$, and $n \geq 1$ types of capital goods with (net output) flows denoted $Z = (z^1, z^2, \ldots, z^n) \in \mathbb{R}^n$ and (input) stocks denoted $K = (k^1, k^2, \ldots, k^n) \in \mathbb{R}_+^n$. The (final) output of economic welfare is measured by a <u>consumption index</u> $c = \psi(C)$, where $\psi: \mathbb{R}_+^\ell \to \mathbb{R}_+$ is a nondecreasing function initialized so that $\psi(0) = 0$, while the possibilities for converting inputs into (final) outputs are summarized by a technology set

$$T \subset \{(c,Z,D,K) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^n_+ : Z + K \geqq 0\}.$$

Finally, given initial stocks of exhaustible resources $\tilde{R} \gg 0$ and capital goods $\tilde{K} > 0$, in such a setting feasible growth paths are described by

$$\begin{cases} (c_t, K_{t+1} - K_t, D_t, K_t) \in T \quad \text{for} \quad t \geqq 0 \\ \text{and} \\ \sum_{t=0}^{\infty} D_t \leqq \tilde{R} \quad \text{and} \quad K_0 \leqq K. \end{cases} \tag{25}$$

To relate this model to the aggregative model, we simply identify d with $D \cdot 1$, z with $Z \cdot 1$ and k with $K \cdot 1$, where "1" always denotes a conformable vector of ones, and then define

$$H = \{(y,k) \in \mathbb{R} \times \mathbb{R}_+ : \text{ there exists } (c,Z,D,K) \in T \text{ such that } c + Z \cdot 1 \geqq y \text{ and } K \cdot 1 = k\}$$

and

$$h(y,k) = \text{infimum } D \cdot 1 \text{ on } \{D \in \mathbb{R}^m_+ : (c,Z,D,K) \in T, \; c + Z \cdot 1 \geqq y \text{ and } K \cdot 1 = k\} \text{ for } (y,k) \in H.$$

One set of assumptions about T which both entail our previous maintained assumptions and enable us to adapt our previous analysis -- and which are surely much stronger than necessary for either purpose -- are as follows:

T0. (Inactivity) $0 \in T$;

T1. (Productivity) a. If $K > 0$, then there exists $(c,Z,D) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m_+$ such that $(c,Z,D,K) \in T$, $(c,Z\cdot 1) \gg 0$ and $K + Z > 0$;

b. If $(c,Z,D,K) \in T$, $(c,Z\cdot 1) \gg 0$ and $K + Z > 0$, then there exists $(c',Z',D') \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m_+$ such that $(c',D',Z',K + Z) \in T$, $(c',Z'\cdot 1) \geq (c,Z\cdot 1)$ and $K + Z + Z' > 0$;

T2. (Disposability) There is a partition of capital goods, say, $K = (K^1,K^2) = ((k^1,k^2,\ldots,k^{n'}), (k^{n'+1},k^{n'+2},\ldots,k^n))$, such that if $(c,Z,D,K) \in T$, $0 \leq c' \leq c$, $Z' \leq Z$, $K^{1'} \geq K^1$ and $(K^{1'},K^2) + Z' \geq 0$ (resp., $K^{2'} \geq K^2$ and $(K^1,K^{2'}) + Z' \geq 0$), then $(c',Z',D,(K^{1'},K^2)) \in T)$ (resp., there exists $D' \geq D$ such that $(c',Z',D',(K^1,K^{2'})) \in T)$;

T3. (Substitutability) There are constants $0 < M < \infty$ and $1 \leq N < \infty$ such that for every $K > 0$ and associated

$$(c,z,k) \in \{(c,z,k) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ : \text{ there exists } (c',Z',D') \in \mathbb{R}_+ \times \mathbb{R}^h \times \mathbb{R}^m_+ \text{ such that } (c',Z',D',K) \in T,\ c' \geq c,\ Z'\cdot 1 \geq z,\ K + Z' > 0 \text{ and } K\cdot 1 = k$$

$$Mh(c+z,k/N) \geq \text{ infimum } D'\cdot 1 \text{ on } \{D' \in \mathbb{R}^m_+ : (c',Z',D',K) \in T,\ c' \geq c,\ Z'\cdot 1 \geq z \text{ and } K + Z' >$$

and

T4. (Convexity) T is convex.

<u>Some Brief Remarks about this Formulation</u>. 1. As we have already stressed, when $n > 1$, the notion of capital goods is very broad. Thus, for example,

these could in principle include such diverse commodities as exhaustible resources themselves (double-counted so as to permit indirect as well as direct stock-flow interaction) or cumulated technical knowledge.

2. The consumption index ψ is fairly arbitrary; it could very well, for example, incorporate subsistence level consumption requirements. More generally, there is great leeway in the choice of units for measuring the variety of commodities this model encompasses, though obviously the specification of T may be rendered more or less plausible depending on such choice. Thus, in particular, aggregation employing constant weights which are identically one is not really, in itself, especially restrictive.

3. There is no loss of generality in assuming that $\tilde{R} \gg 0$; an exhaustible resource is only relevant if it is actually available at some point.

4. In order to allow the possibility that all capital goods -- unlike consumption or investment goods -- are not freely disposable, we require the equality constraint $K \cdot 1 = k$ (rather than the inequality $K \cdot 1 \geq k$) in relating aggregates to disaggregates, for instance, in defining H and h, or in using h to provide an upper bound on minimal exhaustible resource requirements, as in T3.

5. Two simple observations clearly underscore the fact that T0-T4 provide overkill, namely, that T2 implies a strong form of h2,

$h2^s$. For every y, h is decreasing in k on H(y),

while T0, T2 and T4 imply an extremely strong form of h3,

$h3^s$. For every k, $h(0,k) = 0$ and h is convex in y on H(k).

We leave as an exercise for the reader verifying these results, as well as the two remaining requisites, that T1a implies H1, while T2 also implies H2 and h1.

Now, once again, consider feasible growth paths of the specific form

$$\begin{cases} (c, K_{t+1} - K_t, D_t, K_t) \in T \quad \text{for} \quad t \geqq 0 \\ \text{and} \\ c > 0, \ \sum_{t=0}^{\infty} D_t \leqq \tilde{R} \quad \text{and} \quad K_0 \leqq \tilde{K}. \end{cases} \tag{26}$$

<u>Generalized Sustainability Theorem</u>. Suppose T satisfies T0-T4. Then, for every $\tilde{R} \gg 0$ and $\tilde{K} > 0$ there exists $(c, \{(D_t, K_t)\})$ satisfying (26) if and only if (SC) obtains.

<u>Proof of the Generalized Sustainability Theorem</u>. Note that, by virtue of $h2^s$, $h(y,k) = 0$ for $k > \underline{k}(y)$, so that (SC) becomes simply, for every $k > 0$,

$$\lim_{y \to 0^+} 1/y \int_k^{\infty} h(y,u)du = 0 \tag{27}$$

<u>Sufficiency</u>. Let $\tilde{r} = \min_i \tilde{R}^i$ and $\tilde{k} = \tilde{K} \cdot 1/N$. Then pick $\{\alpha_t\}$ such that $\alpha_t > 0$ and $\sum_{t=0}^{\infty} \alpha_t \leqq \tilde{r}/2$, and $y' > 0$ such that, for every $0 < y < y'$, $(y, \tilde{k} - y/2N) \in H$ and

$$1/y \int_{\tilde{k}-y/2N}^{\infty} h(y,u)du \leq \tilde{r}(N/4M) \tag{28}$$

[using H1, H2, and (27)]. Consider, first, $\{k_t\}$ such that

$$k_t = \tilde{k} + y/2N \quad \text{for} \quad t \geq 0.$$

It follows from (28) that for <u>every</u> $0 < y < y'$

$$\sum_{t=0}^{\infty} h(y,k_t) = (2N/y) \sum_{t=0}^{\infty} h(y,\tilde{k} + (y/2N)t)(y/2N) \tag{29}$$

$$\leq (2N/y) \int_{\tilde{k}-y/2N}^{\infty} h(y,u)du \qquad \text{[using } h2^{s}\text{]}$$

$$\leq \tilde{r}/2M.$$

Now define

$$d(c,z,K) = \text{infimum } D'\cdot 1 \text{ on } \{D' \in \mathbb{R}^m_+ : (c',Z',D',K) \in T,\ c' \geq c,\ Z'\cdot 1 \geq z \text{ and } K + Z' > 0\}$$

for $(c,z,K) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n_+$, and consider, second, $\{(c_t,Z_t,D_t,K_t)\}$ such that for <u>some</u> $0 < y < y'$

$$\text{for } t = 0,\ K_0 = \tilde{K},\ (c_0,Z_0,D_0,K_0) \in T,\ c_0 = Z_0\cdot 1 = y/2,\ K_0 + Z_0 > 0 \tag{30}$$

and

$$D_0\cdot 1 - \alpha_0 \leq d(y/2,y/2,K) \qquad \text{[using T1a and T2]}$$

and

for $t > 0$, $K_t = K_{t-1} + Z_{t-1}$, $(c_t, Z_t, D_t, K_t) \in T$, $c_t = Z_t \cdot 1 = y/2$, $K_t + Z_t > 0$ (

and

$$D_t \cdot 1 - \alpha_t \leq d(y/2, y/2, K_t). \qquad \text{[using T1b and T2].}$$

Since $\{k_t\}$ and $\{K_t\}$ have been specified so that $k_t = K_t \cdot 1/N$ for $t \geq 0$, it follows from (30) and (31) that, for $i=1,2,\ldots m$,

$$\sum_{t=0}^{\infty} D_t^i \leq \sum_{t=0}^{\infty} D_t \cdot 1$$

$$\leq \sum_{t=0}^{\infty} [d(y/2, y/2, K_t) + \alpha_t]$$

$$\leq \sum_{t=0}^{\infty} [Mh(y, k_t) + \alpha_t] \qquad \text{[using T3]}$$

$$\leq \tilde{r} \qquad \text{[using (29)]}$$

$$\leq \tilde{R}^i.$$

Thus, (30) and (31) yield a solution to (26) with $c = y/2$.

Necessity. This follows directly upon noticing that every solution to (26) yields a solution to (10) with the same c, and $k_t = K_t \cdot 1$ for $t \geq 0$ as well as $(\tilde{r}, \tilde{k}) = (\tilde{R} \cdot 1, \tilde{K} \cdot 1)$. And since, given our maintained assumptions, the original proof of necessity only depended on the existence of a solution to

(10) for arbitrary $(\tilde{r},\tilde{k}) \gg 0$, it obviously applies here too, and we are finished. $\blacksquare$

Appendix

For the sake of brevity (sic!) this is omitted. A copy may be obtained by writing to either of the two authors.

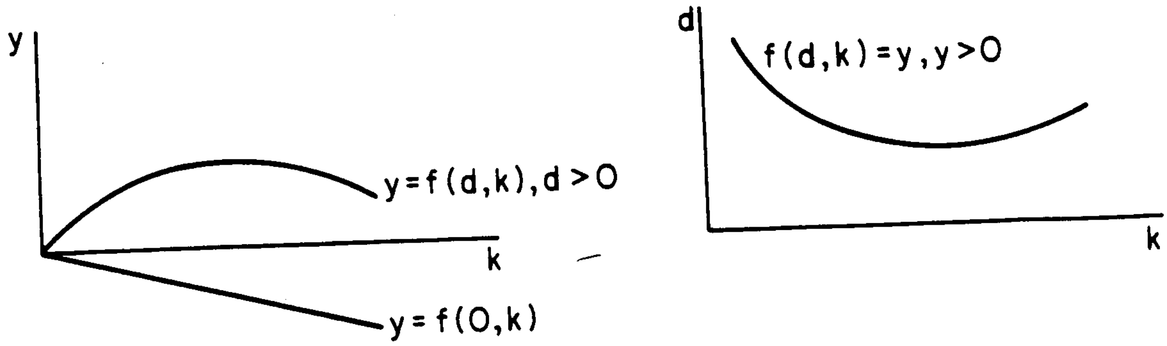
References

1. Mitra T (1978) On maintainable consumption levels and exhaustible resources. Preliminary paper, SUNY at Stony Brook.

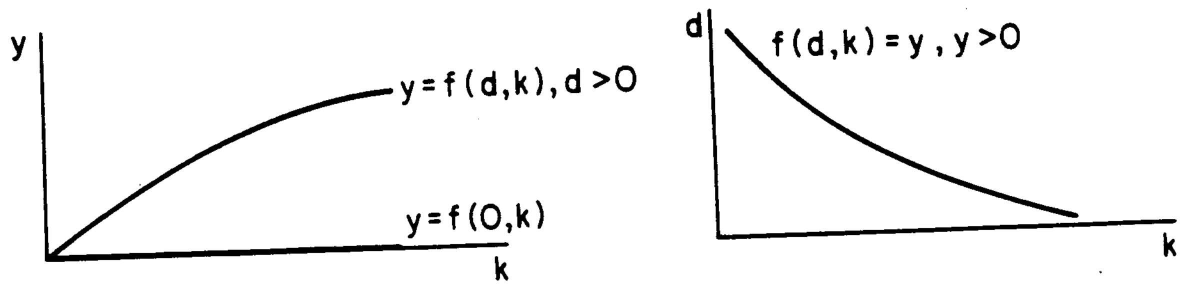
2. Solow RM (1974) Intergenerational equity and exhaustible resources. Rev Econ Stud 41: 22-49.

3. Stiglitz JE (1974) Growth with exhaustible natural resources: Efficient and optimal growth paths. Rev Econ Stud 41: 123-138.

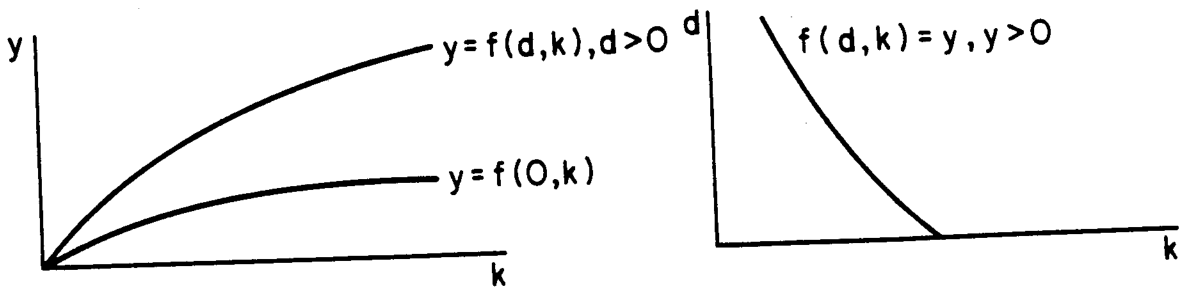
4. Stiglitz JE (1979) A Neoclassical analysis of the economics of natural resources. In Smith VK (ed.) Scarcity and grouth reconsidered. Johns Hopkins University Press, Baltimore, MD.



a. $f(0,k)<0$ for $k>0$ (and $\lim_{k\to\infty} f(d,k)<0$ for $d>0$)

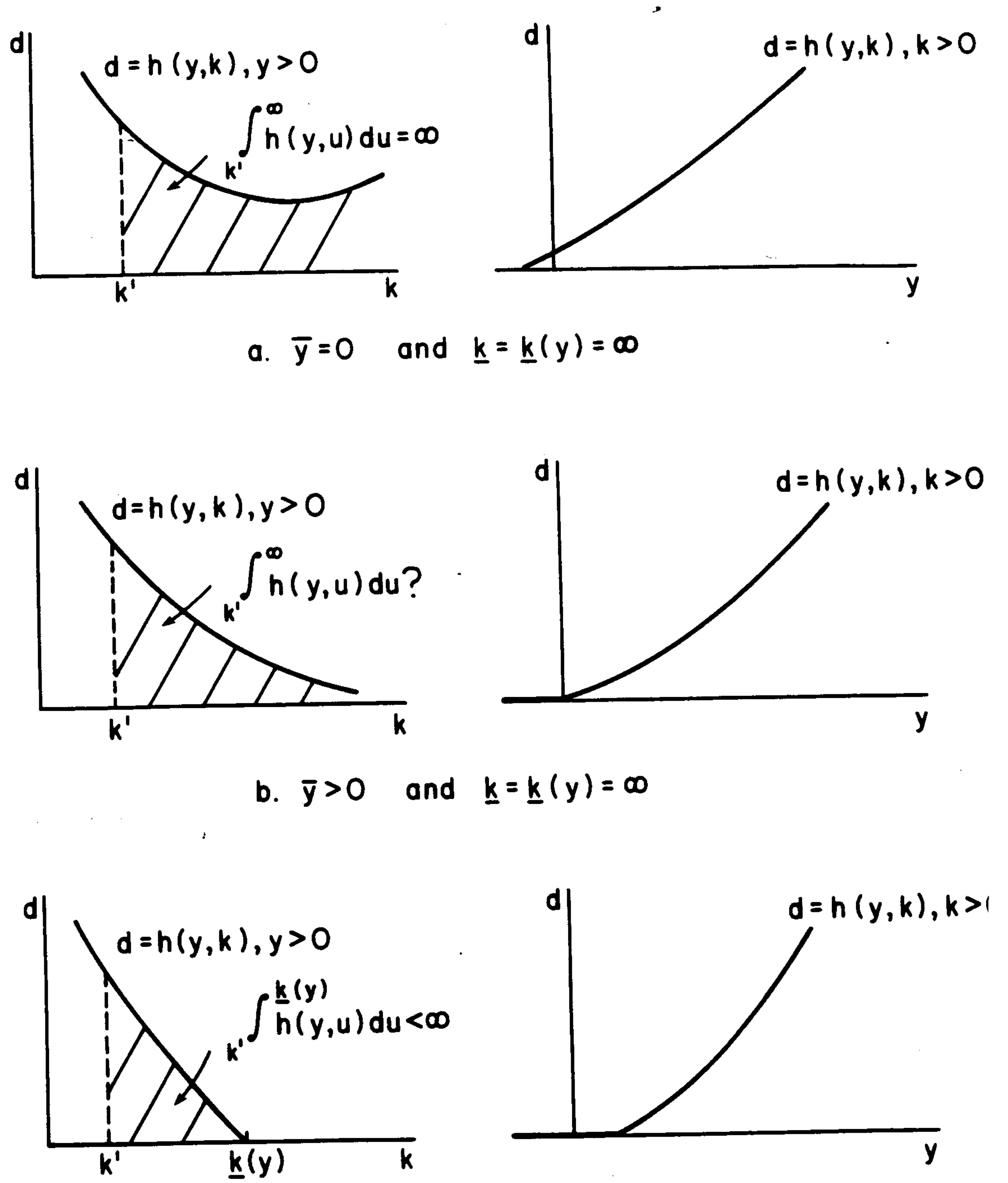


b. $f(0,k)=0$ for $k>0$ (and $\lim_{k\to\infty} f(d,k)=\infty$ for $d>0$)



c. $f(0,k)>0$ for $k>0$ (and $\lim_{k\to\infty} f(0,k)=\infty$)

Fig. 1 Typical Examples of the Standard (Net) Output Production Function

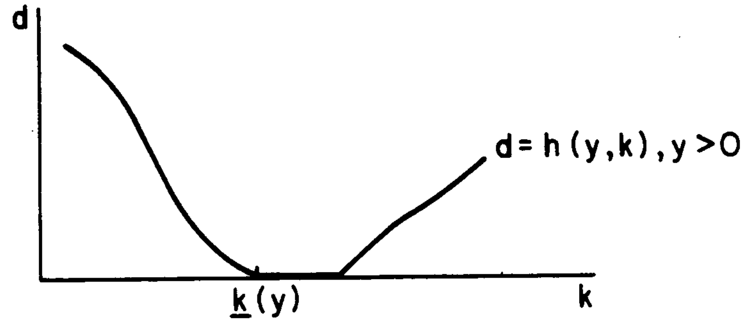


a. $\bar{y} = 0$ and $\underline{k} = \underline{k}(y) = \infty$

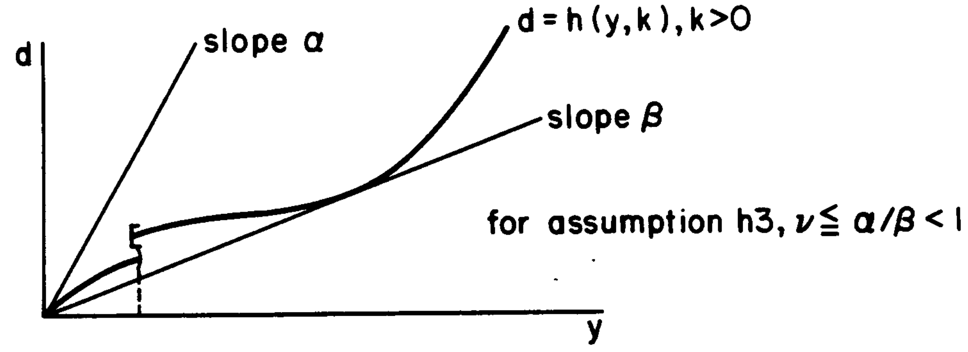
b. $\bar{y} > 0$ and $\underline{k} = \underline{k}(y) = \infty$

c. $\bar{y} = \infty$ and $0 = \underline{k} \leq k(y) < \infty$

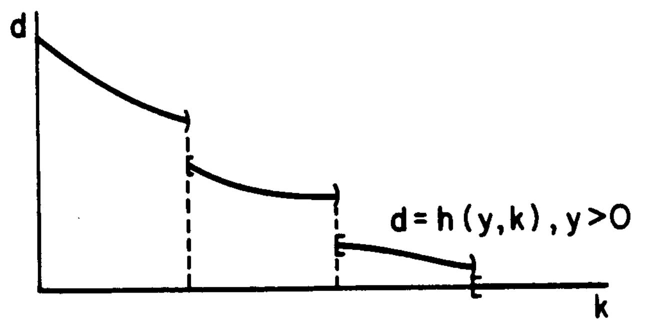
Fig. 2 Corresponding Examples of our Canonical Resource Requirement Function



a. Substantial depletion required to offset depreciation with large stocks of capital goods

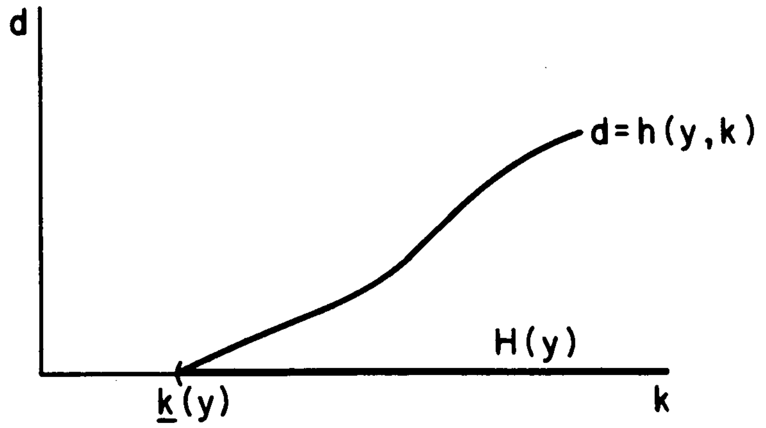


b. Increasing returns to resources at low levels of capacity utilization

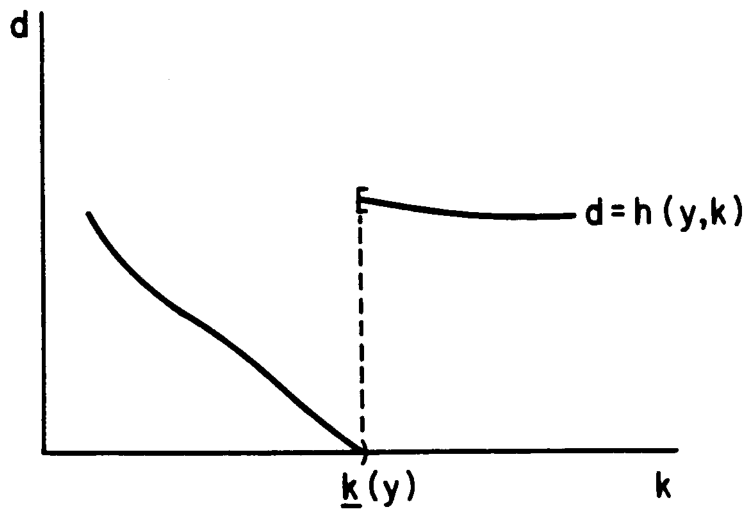


c. Significant indivisibilities in substituting capital for resources

Fig. 3 Various Productivity Phenomena Encompassed by the Technological Assumptions



a. $\underline{k}(y) < \infty$ but $(y, \underline{k}(y)) \notin H$



b. $\underline{k}(y) < \infty$ but $(y, \underline{k}(y)) \in H$ and $h(y, \underline{k}(y)) > 0$

Fig. 4 Illustration of the Irregularity Involved in the Proof of Sufficiency

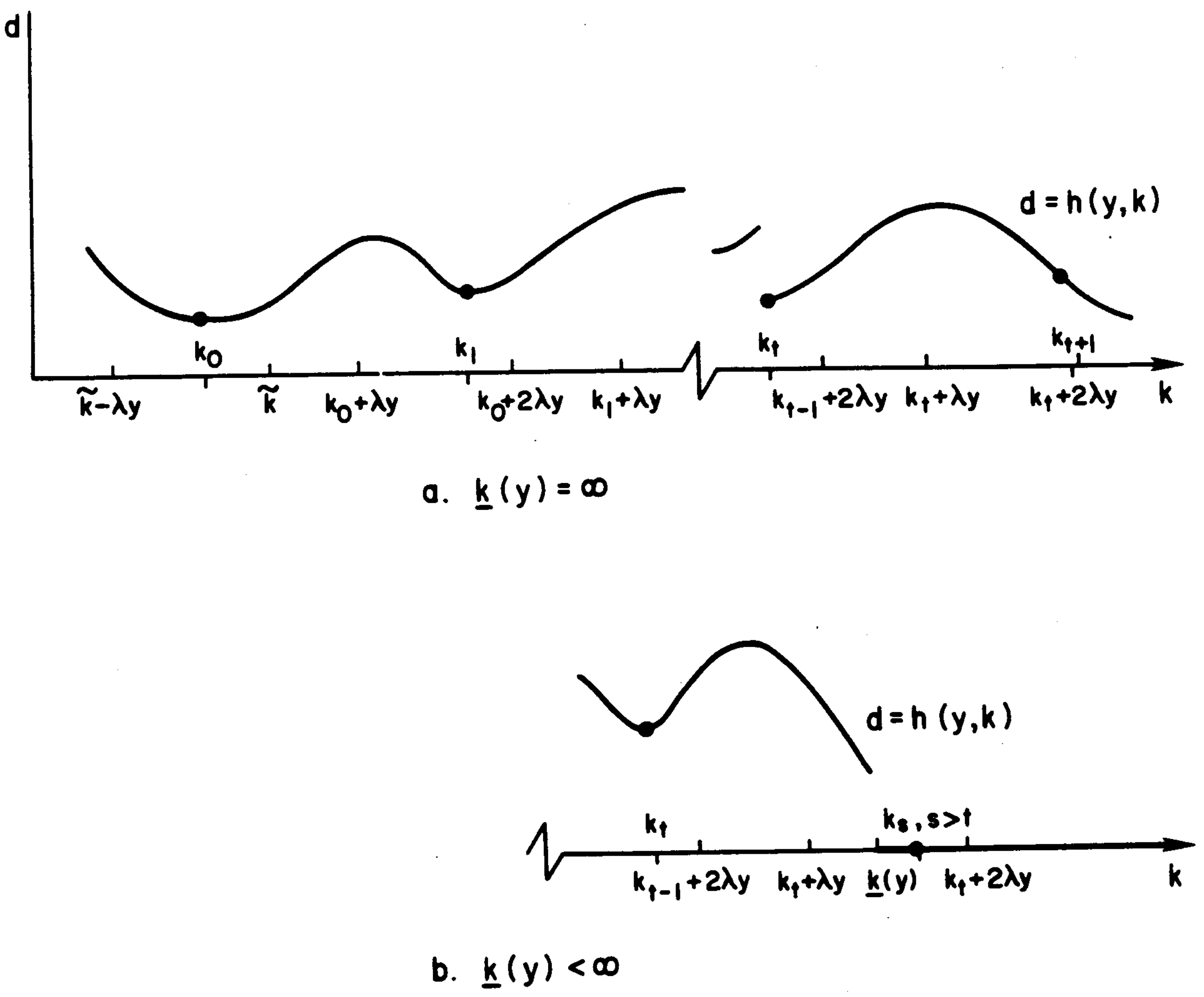


Fig. 5 Illustration of the Construction Employed in the Proof of Sufficiency