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MARKETS WITH COUNTABLY MANY PERIODS

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MARCHES SUR UNE INFINITE DENOMBRABLE DE PERIODES

Résumé

Un des premiers résultats d'existence de l'equilibre pour une économie avec une infinité de biens est celui où Peleg et Yaari (1970) considèrent une économie d'échange définie sur l'espace des suites réelles ; les consommateurs y sont supposés avoir des préférences totalement préordonnées, convexes, continues et strictement monotones, tandis que l'offre totale de ressources est supposée stictement positive.

Une abondante littérature a depuis été consacrée aux économies définies sur un espace de biens de dimension infinie. La théorie des espaces de Riesz et une hypothèse dite de "propreté uniforme" des préférences et/ou de la production se sont avérées jouer un rôle crucial dans les théorèmes d'existence.

L'objectif du papier est de trouver l'hypothèse cachée de propreté uniforme à l'oeuvre dans Peleg et Yaari et, grâce aux espaces de Riesz, de généraliser les résultats aux économies intertemporelles admettant l'incertitude à chaque période.

Mots clés : Espaces de Riesz localement convexes - Système de Riesz dual symétrique - Economie d'échange - Equilibre intertemporel - Préférences uniformément propres.

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Abstract

The existence of equilibria is proved in a stochastic market with a finite number of agents and countably many periods.

Key words : Locally convex-solid topological vector lattices - Symmetric Riesz dual systems - Exchange economy - Intertemporal equilibrium - Uniformly proper preferences.
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MARKETS WITH COUNTABLY MANY PERIODS

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The existence of equilibria is proved in a stochastic market with a finite number of agents and countably many periods.

1. Introduction

This note is intended to reconsider, at the light of some recents works, an equilibrium existence result obtained by Peleg and Yaari (1970) for an exchange economy with $\mathbb{R}^\infty$ (the space of all real sequences) as commodity space, $\mathbb{R}^\infty$ as price space and no explicit commodity-price duality.

We first restate the topological vector lattice properties of the commodity space to be considered in this model. A limited attempt in this direction is made by Besada et al. (1988) who introduce Köthe perfect spaces as commodity spaces which arise in a natural way in connection with the initial endowments of the agents. We go further and do a similar construction in the case where, at each period, the consumption choice of each agent may be stochastic. The resulting commodity-price duality is described by what is called a symmetric Riesz dual system by Aliprantis et al. (1989).

We then look at the hidden uniform properness assumptions which would establish the relevance of this vector lattice theoretic construction. As it is pointed out by Aliprantis et al. (1989), uniform properness is satisfied in a subspace of the commodity space,conveniently topologized ; in this subspace, an equilibrium existence theorem can be deduced from Mas-Colell's theorem (1986). The main result of this note is to show that under assumptions similar to the ones used by Peleg and Yaari, to this equilibrium is associated an equilibrium in the initial commodity space with prices in its dual.

2. The economic model

The notations and terminology used in this note are borrowed from Aliprantis and Burkinshaw (1978), Aliprantis et al. (1989). The index t denotes the time period. The commodity-price duality at period t is represented by a symmetric Riesz dual system $\langle E_t, E_t'\rangle$. If $x_t \in E_t$ and $p_t \in E_t'$, we write $p_t.x_t$ for $\langle x_t, p_t\rangle$.

We consider m infinitely lived agents $i = 1,\dots,m$, each of which has at its disposal an initial endowment $\omega^i = (\omega_t^i) \in \prod_{t=1}^{\infty} E_t^+$. Let $\omega = \sum_{i=1}^{m} \omega^i \in \prod_{t=1}^{\infty} E_t^+$ be the total endowment.

$$P = \{p \in \prod_{t=1}^{\infty} E_t' \mid \sum_{t=1}^{\infty} |p_t|.\omega_t < +\infty\}$$

and

$$\Lambda(P) = \{x \in \prod_{t=1}^{\infty} E_t \mid \sum_{t=1}^{\infty} |p_t|.|x_t| < +\infty \text{ for all } p \in P\}$$

will be respectively the price space and the commodity space of our model. The commodity-price duality is represented by the bilinear form : $\langle x,p\rangle = p.x = \sum_{t=1}^{\infty} p_t.x_t$. If $x \in \Lambda(P)$ and $p \in P$, $p.x$ is the value of x reckoned at the prices $p = (p_t)$.

The construction of $\Lambda(P)$ makes it natural to consider the topology τ defined on $\Lambda(P)$ by the Riesz seminorms

$$\rho_p(x) = \sum_{t=1}^{\infty} |p_t|.|x_t| \quad , p \in P \ .$$

Similarly, τ' is the topology defined on P by the Riesz seminorms

$$\rho_x(p) = \sum_{t=1}^{\infty} |p_t|.|x_t| \quad , x \in \Lambda(P) \ .$$

As it is proved in the propositions 1 and 2 of the appendix, $\langle\Lambda(P),P\rangle$ is a dual pair ; τ (resp. τ') is a Hausdorff locally convex-solid topology on $\Lambda(P)$ (resp. P), consistent with the duality. Moreover, let $\sigma(\Lambda(P),P)$ and $\sigma(P,\Lambda(P))$ be the weak topologies associated with the duality ; every order-interval of $\Lambda(P)$ (resp. P) is $\sigma(\Lambda(P),P)$ (resp. $\sigma(P,\Lambda(P))$)-compact. In other words, $\langle\Lambda(P),P\rangle$ is a symmetric Riesz dual system.

As Peleg and Yaari (1970), Besada et al. (1988), we consider an exchange economy, to be called $\mathcal{E}$. A commodity bundle is an element of $\Lambda(P)^+$. Each agent i has an initial bundle $\omega^i > 0$ in $\Lambda(P)^+$ and a preference or indifference relation $\succsim^i$ on $\Lambda(P)^+$. The relation $\succsim^i$ is assumed to be reflexive, transitive and complete. In addition, the initial bundles and preference relations are required to satisfy the following assomptions :

1) *Strict monotonicity* : For $i = 1,\dots,m$, if $x,y \in \Lambda(P)^+$ and $x > y$, then $x >^i y$.
2) *Convexity* : For $i = 1,\dots,m$, for each $x \in \Lambda(P)^+$, the set $\{ y \in \Lambda(P)^+ \mid y \gtrsim^i x \}$ is convex.
3) *τ- continuity* : For $i = 1,\dots,m$, for each $x \in \Lambda(P)^+$, the sets $\{ y \in \Lambda(P)^+ \mid y \gtrsim^i x \}$ and $\{ z \in \Lambda(P)^+ \mid x \gtrsim^i z \}$ are both τ-closed in $\Lambda(P)^+$.

The *τ-continuity* assumption is a *myopy* assumption on preferences. A fourth assumption corresponds to the *positivity of total supply* with an additional requirement which is automatically satisfied if each E_t is an Euclidean space :

4) For every period t, ω_t is an order-unit of E_t and $E_t' = (E_t)_n^\sim$ (the order-continuous dual of E_t).

Let $\gamma^i(p) = \{ x \in \Lambda(P)^+ \mid p.x \le p.\omega^i \}$ be the budget set of i and set $\delta^i(p) = \{ x \in \Lambda(P)^+ \mid p.x < p.\omega^i \}$.

A *competitive equilibrium* (resp. a *quasi-equilibrium*) for economy $\mathcal{E}$ is an (m+1)-tuple $(\bar{x}^1,\dots,\bar{x}^m,\bar{p})$ such that $\bar{p} \in P$, $\bar{p} \ne 0$ and

a) $\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i$ (attainability of $\bar{x} = (\bar{x}^1,\dots,\bar{x}^m)$)

b) $\bar{x}^i \in \gamma^i(\bar{p})$ for each $i = 1,\dots,m$ and $y >^i \bar{x}^i$ implies $y \notin \gamma^i(p)$ (resp. $y \notin \delta^i(p)$).

3. Existence theorem

$\Lambda(P)$ is Dedekind-complete. Let A_ω be the principal ideal of $\Lambda(P)$ generated by ω,

$$A_\omega = \{ x \in \Lambda(P) \mid \textit{there exists } \lambda > 0 \textit{ with } |x| \le \lambda\omega \}.$$

Recall that under the Riesz norm

$$\|x\|_\infty = \mathit{inf}\{ \lambda > 0 \mid |x| \le \lambda\omega \}$$

A_ω is an AM-space with unit ω. The norm dual of $(A_\omega, \|.\|_\infty)$ will be denoted by A_ω'.

According to Mas-Colell (1986), an economy is said to satisfy the *closedness condition* whenever for every sequence of attainable allocations $(x^{1,\nu},\dots,x^{m,\nu})$ which satisfies $x^{i,\nu+1} \gtrsim^i x^{i,\nu}$ for all ν and all $i=1,\dots,m$, there exists another attainable allocation $(x^1,\dots,x^m)$ satisfying $x^i \gtrsim x^{i,\nu}$ for all ν and all $i=1,\dots m$. If v^i and V^i are respectively a vector and a 0-neighborhood of the commodity space L, $\gtrsim^i$ is said to be (v^i, V^i)-*uniformly proper* if for all $x \in L_+$, $x + \lambda v^i + z \in L_+$, $\lambda > 0$ and $z \in \lambda V^i$ imply $x + \lambda v^i + z >^i x$. ω is said *desirable* for every $i = 1,\dots,m$, if $x \in L^+$

implies $x + \alpha\omega >^i x$ for all $\alpha > 0$. In a topological vector lattice, if an exchange economy with convex, monotone, continuous, uniformly proper preferences and a desirable total endowment satisfies the closedness condition, then it has a quasi-equilibrium with prices in the topological dual of the commodity space.

In order to apply Mas- Colell's existence result, we will first consider $\mathcal{E}_\omega$, the same economy than $\mathcal{E}$, but restricted to the commodity space A_ω. It is worth noticing that $\mathcal{E}$ and $\mathcal{E}_\omega$ have the same attainable allocations.

Proposition 1. Under the assumptions 1), 2) and 3), $\mathcal{E}_\omega$ has a quasi-equilibrium $(\bar{x}, \bar{p}) \in (A^+_\omega)^m \times A'^+_\omega$ with $\bar{p}(\omega) = 1$.

Proof. It follows from assumption 1) that ω is desirable for every $i=1,\ldots,m$ .It is also well known that, under assumption 1), every point of $int(A^+_\omega)$ (the $\|.\|$-interior of A^+_ω), in particular $\frac{1}{m}\,\omega$, is a properness vector for every $\gtrsim^i$ in $\mathcal{E}_\omega$. As it is pointed out by Aliprantis et al. (1989), the $\|.\|_\infty$- convergence implies the order-convergence ; since τ is a Lebesgue topology, the order-convergence implies the τ-convergence. So the $\|.\|_\infty$-continuity of each $\gtrsim^i$ in $\mathcal{E}_\omega$ follows from assumption 3). Finally, $\mathcal{E}_\omega$ satisfies the closedness condition as it follows easily from the assumptions 2) and 3) and the $(\sigma(\Lambda(P), P))^m$-compactness of the set of all attainable allocations. The conclusion follows from Mas-Colell's theorem.

□

Corollary. Let $\bar{\bar{p}}$ be the normal component of $\bar{p}$. $(\bar{x}, \bar{\bar{p}})$ is a quasi-equilibrium of $\mathcal{E}_\omega$ and $\bar{\bar{p}}(\omega) > 0$.

Proof. $\bar{p} \in (A'_\omega)^+ \subset (A^\sim_\omega)^+$,so by the Riesz decomposition theorem, there is a unique decomposition of $\bar{p}$, $\bar{\bar{p}} \in (A_\omega)^\sim_n$ and $q \in ((A_\omega)^\sim_n)^d$, the disjoint complement of $(A_\omega)^\sim_n$, $\bar{\bar{p}} \geq 0$, $q \geq 0$, such that $\bar{p} = \bar{\bar{p}} + q$. We first remark that $(A_\omega)^\sim_n$ separates the points of A_ω ; indeed, as it is shown immediately after the Proposition 2 of the appendix, $P \subset (\Lambda(P))^\sim_n \subset (A_\omega)^\sim_n$ and P separates the points of A_ω . Then let $N_q = \{ x \in A_\omega \mid |q|\ (x) = 0 \}$ be the null ideal of q in A_ω.It follows from Theorem 2.2 in Aliprantis et al. (1989) that N_q is order dense.

Then suppose that for some i, $x^i >^i \bar{x}^i$ in $\mathcal{E}_\omega$. There exists a net $(x^{i\alpha}) \subset N_q$, $x^{i\alpha} \xrightarrow{o} x^i$. For large enough α, $x^{i\alpha} >^i \bar{x}^i$ and $\bar{\bar{p}}(x^{i\alpha}) = \bar{p}(x^{i\alpha}) \geq (\bar{\bar{p}} + q)(\omega^i) \geq \bar{\bar{p}}(\omega^i)$. By passing to limit, $\bar{\bar{p}}(x^i) \geq \bar{\bar{p}}(\omega^i)$. Since there exists i such that $(\bar{\bar{p}} + q)(\omega^i) > 0$, it is easily deduced from the strict monotonicity of preferences that $\bar{\bar{p}} \neq 0$. From the strict monotonicity of preferences, we get that for each i, $\bar{\bar{p}}(\bar{x}^i) \geq \bar{\bar{p}}(\omega^i)$. Finally, since $\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i$, $\bar{\bar{p}}(\bar{x}^i) = \bar{\bar{p}}(\omega^i)$ holds for all i.

Recall now that A_ω is a Banach lattice and thus that A'_ω is a band of $A^{\sim}_\omega$ (see Theorem 6.4 in Aliprantis and Burkinshaw (1978)). It follows that $\bar{\bar{p}} \in A'_\omega$; since ω is in the $\|.\|$- interior of A^{+}_ω, $\bar{\bar{p}}(\omega) > 0$ and $\bar{\bar{p}}$ can be re-normalized so that $\bar{\bar{p}}(\omega) = 1$.

□

Proposition 1 is nothing else than Theorem 5.6 in Aliprantis et al. (1989), reproved there in order to keep this note self-contained . The construction used in the proof of the corollary is classical since Bewley (1972). To go further, we need to exploit Assumption 4), through the proposition 3 of the appendix.

Proposition 2. Under the assumptions 1), 2), 3) and 4), $(\bar{x}, \bar{\bar{p}})$ is an equilibrium of $\mathcal{E}$ and $\bar{\bar{p}}$ is a strictly positive element of P.

Proof. From Proposition 3 in the appendix, we know that A_ω is order dense and thus τ- dense in $\Lambda(P)$ and that $P = (A_\omega)^{\sim}_{n}$. Then if $(\bar{x}, \bar{\bar{p}})$ is a quasi-equilibrium of $\mathcal{E}_\omega$ as in the corollary of Proposition 1, $\bar{\bar{p}} \in P$.

Suppose now that for some i, $x^i >^i \bar{x}^i$ in $\mathcal{E}$. There exists a net $(x^{i\alpha}) \subset A_\omega$, $x^{i\alpha} \xrightarrow{\tau} x^i$. For large enough α, $x^{i\alpha} >^i \bar{x}^i$ and $\bar{\bar{p}}(x^{i\alpha}) = \bar{\bar{p}}\,.x^{i\alpha} \geq \bar{\bar{p}}(\omega^i) = \bar{\bar{p}}.\omega^i$. By passing to limit, $\bar{\bar{p}}.x^i \geq \bar{\bar{p}}.\omega^i$, which shows that $(\bar{x}, \bar{\bar{p}})$ is a quasi equilibrium of $\mathcal{E}$.

Since $\bar{\bar{p}}\,.\,\omega > 0$, there exists i such that $\bar{\bar{p}}\,.\,\omega^i > 0$. For all $z > 0$ in $\Lambda(P)$, by the strict monotonicity of preferences, $\bar{x}^i + z >^i \bar{x}^i$ and by the τ-continuity of preferences, $\bar{\bar{p}}\,.\,(\bar{x}^i + z) > \bar{\bar{p}}\,.\,\omega^i = \bar{\bar{p}}\,.\,\bar{x}^i$. We get $\bar{\bar{p}}^i.z > 0$, which shows that $\bar{\bar{p}}$ is strictly positive. Hence $\bar{\bar{p}}\,.\,\omega^i > 0$ for all i and $(\bar{x}, \bar{\bar{p}})$ is an equilibrium.

□

Proposition 2 is a straight generalization (as to the commodity space) of the existence statements in Peleg and Yaari (1970) or Besada et al. (1988). Assumptions 1-4 on the economic model are exactly the assumptions made in these two papers. By adding an explicit desirability assumption for the total endowment ω, it would be easy to weaken the monotonicity assumption on preferences. The quasi-equilibrium price would not need be strictly positive and an irreducibility assumption would be necessary to guarantee the existence of an equilibrium.

The construction of the commodity space $\Lambda(P)$ together with the assumption 4) allow to consider at each period a commodity-price duality $\langle L_\infty, L_1\rangle$ which captures the idea of a stochastic choice at each period. A simpler application is the case where at each period, the commodity space is an Euclidean space, of which the dimension may depend on time.

Appendix

Let $\langle\Lambda(P),P\rangle$, τ and τ' be defined as in the beginning of Section 2.

Proposition 1. $\langle\Lambda(P),P\rangle$ is a dual pair. τ (resp. τ') is a Hausdorff locally convex-solid topology consistent with this duality.

Proof. Let Φ (resp. Φ') denote the set of all elements of $\prod_{t=1}^{\infty} E_t$ (resp. $\prod_{t=1}^{\infty} E_t'$) with only finitely many non-zero coordinates. Obviously $\Lambda(P)$ and P are Riesz spaces, more precisely ideals of $\prod_{t=1}^{\infty} E_t$ and $\prod_{t=1}^{\infty} E_t'$ containing Φ et Φ'. The separation properties of the bilinear form $\langle x, p\rangle = p.x = \sum_{t=1}^{\infty} p_t.x_t$ follow from the last remark.

Each seminorm $\rho_p(x)$ (resp. $\rho_x(p)$) $= \sum_{t=1}^{\infty} |p_t|.|x_t|$ is obviously monotone, i.e. a Riesz seminorm : $|x| \leq |y| \Rightarrow \rho_p(x) \leq \rho_p(y)$ (resp. $|p| \leq |q| \Rightarrow \rho_x(p) \leq \rho_x(q)$) ; thus τ (resp. τ') is a locally convex-solid topology on $\Lambda(P)$ (resp. P).

It should be clear that $P \subset (\Lambda(P),\tau)'$, the topological dual of $\Lambda(P)$. To prove the converse inclusion, let us remark that on each factor E_t, τ coincides with the absolute weak topology $|\sigma|(E_t,E_t')$ generated by the seminorms : $x_t \to |p_t|.|x_t|$. Hence, if $f \in (\Lambda(P),\tau)'$, for each t there exists $p_t \in E_t'$ such that $f(0,\ldots,0,x_t,0,\ldots) = p_t.x_t$. Since for all x in $\Lambda(P)$, $(x_1,x_2,\ldots,x_t,0,\ldots)$ τ-converges to x, $f(x) = \sum_{t=1}^{\infty} p_t.x_t$. It remains to prove that $p = (p_t) \in P$. $|f|(0,\ldots,0,x_t,0,\ldots) = |p_t|.x_t$, so that $|f|(\omega) = \sum_{t=1}^{\infty} |p_t|.\omega_t < +\infty$. From a similar proof, it follows that $\Lambda(P) = (P,\tau')'$. Finally, if $\sigma(\Lambda(P),P)$ and $\sigma(P,\Lambda(p))$ are the weak topologies associated with the duality $\langle\Lambda(P),P\rangle$, remark that $\sigma(\Lambda(P),P) \subset \tau$ and $\sigma(P,\Lambda(P)) \subset \tau'$, which proves that τ and τ' are Hausdorff.

□

Proposition 2. Each order-interval of $\Lambda(P)$ *(resp.* P*) is* $\sigma(\Lambda(P),P)$ *(resp.* $\sigma(P,\Lambda(P))$*-compact.*

Proof. If $u > 0$ in $\Lambda(P)$, let $[-u,+u]$ be an order-interval of $\Lambda(P)$. For any $\varepsilon > 0$, $\rho_u(p) < \varepsilon$ and $x \in [-u,+u]$ imply

$$\left| \sum_{t=1}^{\infty} p_t . x_t \right| \leq \sum_{t=1}^{\infty} |p_t . x_t| \leq \sum_{t=1}^{\infty} |p_t| . |x_t| \leq \rho_u(p) < \varepsilon .$$

This shows the τ'- equicontinuity of $[-u,+u]$. The $\sigma(\Lambda(P),P)$-relative compactness of $[-u,+u]$ follows from the Alaoglu-Bourbaki theorem. Recall now that each $\langle E_t,E_t'\rangle$ is a symmetric Riesz dual system. Each $[-u_t,+u_t]$ is $\sigma(E_t,E_t')$-compact. It follows easily that $[-u,+u]$ is $\sigma(\Lambda(P),P)$-closed.

The $\sigma(P,\Lambda(p))$-compactness of order-intervals of P is proved in a similar way.

□

Applying Theorem 22.1 of Aliprantis and Burkinshaw(1978), we see that $\Lambda(P)$ (resp.P) is Dedekind-complete and τ (resp. τ') is a Lebesgue topology. If $(\Lambda(P))_n^{\sim}$ (resp. $P_n^{\sim}$) denotes the collection of all order-continuous, order-bounded linear forms on $\Lambda(P)$ (resp. P), $P \subset (\Lambda(P))_n^{\sim}$ (resp. $\Lambda(P) \subset P_n^{\sim}$).

Assume now, as in assumption 4), that for every t, ω_t is an order-unit of E_t and that $E_t' = (E_t)_n^{\sim}$.Recall that A_ω is the principal ideal generated by ω in $\Lambda(P)$.

Proposition 3. Under assumptions 4), A_ω *is order dense in* $\Lambda(P)$ *and*

$$(A_\omega)_n^{\sim} \subset P \subset (\Lambda(P))_n^{\sim} \subset (A_\omega)_n^{\sim}.$$

Proof. From the first part of the assumption, it follows that each factor E_t can be considered as a Riesz subspace of A_ω. For all x in $\Lambda(P)$, $(x_1,x_2,\ldots,x_t,0,\ldots) \uparrow x$. This proves the first part of the statement.

To prove the first inclusion, let f be any element of $(A_\omega)_n^{\sim}$. From the second part of the assumption, one deduces that there exists $p_t \in (E_t)_n^{\sim} = E_t'$ such that $f(x) = \sum_{t=1}^{\infty} p_t . x_t$ and that $p = (p_t) \in$ P.

The second inclusion was proved above; the third one is obvious.

□

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