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EDGEWORTH EQUILIBRIA, FUZZY CORE AND EQUILIBRIA OF A PRODUCTION ECONOMY WITHOUT ORDERED PREFERENCES

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EQUILIBRES D'EDGEWORTH, COEUR FLOU ET EQUILIBRES D'UNE ECONOMIE DE PRODUCTION DONT LES PREFERENCES NE SONT NI TRANSITIVES, NI COMPLETES

RESUME

Le but de ce papier est d'étendre le théorème de Debreu-Scarf sur la coïncidence, sous des hypothèses appropriées, de l'ensemble des équilibres d'Edgeworth et de l'ensemble des allocations Walrasiennes, à une économie de production, sans préférences ordonnées, définie dans un espace vectoriel topologique séparé.

Nous obtenons trois résultats :

- Des hypothèses faibles garantissent l'existence d'équilibres d'Edgeworth et le coeur flou est non vide sous une hypothèse additionnelle faible de continuité des préférences.

- Quand la dimension de l'espace des biens est finie, l'ensemble des équilibres d'Edgeworth, le coeur flou et l'ensemble des allocations Walrasiennes coïncident.

Si l'espace des biens n'est pas de dimension finie, le même théorème est démontré pour une économie hypothétique dont les équilibres d'Edgeworth p euvent être plongés dans l'ensemble des équilibres d'Edgeworth de l'économie initiale.

- Comme sous-produit, un théorème d'existence des équilibres Walrasiens étend la plupart des résultats récents d'existence de l'équilibre.

Mots clés : Théorème de Debreu-Scarf, coeur flou, préférences non transitives, théorèmes de point fixe, espace de biens de dimension infinie espaces de Riesz.

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ABSTRACT

The aim of this paper is to extend the Debreu-Scarf theorem on the coincidence, under suitable conditions, between the set of Walrasian allocations and the set of Edgeworth equilibria to production economies without ordered preferences, defined in a Hausdorff linear topological space.

We obtain three results :

- Edgeworth equilibria exist under very mild conditions. Under a weak additional continuity property of preferences, the fuzzy core is also non empty.

- In the finite dimensional case, the set of Edgeworth equilibria, the set of Walrasian allocations and the fuzzy core of a convex economy coïncide under standard assumptions.

The same is true in the infinite dimensional case for an hypothetical economy whose Edgeworth equilibria can be embedded in the Edgeworth equilibria of the original economy.

- As a by-result, an existence result for Walrasian equilibria extends most of the recent existence results.

Key words : Debreu-Scarf theorem, fuzzy core, non ordered preferences, fixed-point theorems, infinite dimensional economy, Riesz spaces.

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I - INTRODUCTION

For an economy standardly defined, an Edgeworth equilibrium is an attainable allocation whose r-fold repetition belongs to the core of the r-fold replica of the original economy, for any positive integer r.

If the definition of coalitions is enlarged in order to allow a participation of the agents with a rate belonging to the rational interval [0,1] and if the preferences are convex, an Edgeworth equilibrium can also be defined as an attainable allocation which cannot be blocked by a coalition with rational rates of participation. A fuzzy coalition is a coalition whose rates of participation can take any value in the real interval [0,1]. The fuzzy core is the set of all attainable allocations which cannot be blocked by a fuzzy coalition.

The coïncidence under suitable conditions between the set of Walrasian allocations and the set of Edgeworth equilibria for an economy with ordered preferences defined in a finite dimensional commodity space is a result by Debreu-Scarf (1963). The aim of this paper is to extend this result to production economies without ordered preferences defined in a Hausdorff linear topological space.

We obtain three results :

- Edgeworth equilibria exist under very mild conditions which are the same in the finite and in the infinite dimensional cases. Under a weak additional continuity property of preferences, the fuzzy core is also non-empty.

- In the finite dimensional case, Edgeworth equilibria belong to the fuzzy core and are Walrasian allocations of a convex economy under the classical assumptions of continuity, convexity and local non-satiation of preferences. Added to the first one, this result confirms the existence results for Walrasian equilibria of a production economy without ordered preferences which have developped in the litterature around 1975.

In the infinite dimensional case, it is well known that Walrasian equilibria may not exist under the standard assumptions. Here we use an additional assumption which unifies as well the interiority assumptions as the uniform properness assumptions which have been used since 1983 to get the existence of Walrasian equilibria. We define an hypothetical economy, whose Edgeworth equilibria can be embedded in the set of the Edgeworth equilibria of the original economy, and we prove the equivalence between the fuzzy core of this economy and the set of Walrasian equilibria of the original one.

This result contains, as a particular case, an equivalence result of Aliprantis, Brown and Burkinshaw (1987) stated in the ordered case under more restrictive uniform properness assumptions.

- A by-result of the general equivalence theorem is a general existence theorem for Walrasian equilibria in the infinite dimensional case. This general theorem can be applied in all the particular commodity spaces which have been found useful in economic applications and extends most of the recent existence results.

The paper is organized as follows.
In section II, we set the main definitions and notations. The non-emptiness theorems are proved in section III, the equivalence theorems in section IV. In the infinite dimensional case, the

equivalence theorem is obtained under an assumption previously formulated by Florenzano (1987). As in this paper, this assumption is proved in section V to be satisfied as well in the context adressed by Duffie or Jones in a locally convex topological vector commodity space as under uniform properness assumptions in a locally convex-solid topological vector lattice. Section VI is devoted to the existence of Walrasian equilibria.

II - CORE, EDGEWORTH EQUILIBRIA AND FUZZY CORE OF A PRIVATE OWNERSHIP ECONOMY

In a Hausdorff linear topological space (L,σ) as commodity space, let us consider :

$$\mathcal{E} = ((X^i, P^i, \omega^i)_{i\in M}, (Y^j)_{j\in N}, (\theta^{ij})_{i\in M, j\in N})$$

a private ownership economy with a finite set M of consumers and a finite set N of producers, standardly defined.

To each consumer i is associated a consumption set $X^i \subset L$, an initial endowment $\omega^i \in L$ and a preference correspondence $P^i : \prod_{k\in M} X^k \to X^i$. If $x = (x^k) \in \prod_{k\in M} X^K$, $P^i(x)$ is interpreted as the set of the elements of X^i which are (stictly) preferred by agent i to x^i when the consumption of each agent $k \neq i$ is equal to x^k. To each producer j is associated a production set $Y^j \subset L$. For all $i \in M$ and for all $j \in N$, $\theta^{ij} \geq 0$ is a contractual claim of the consumer i on the profit of the producer j ; the θ^{ij} are assumed to verify, for every $j \in N$, $\sum_{i\in M} \theta^{ij} = 1$.

Let $X = \prod_{i\in M} X^i$, $\omega = \sum_{i\in M} \omega^i$ and $Y = \sum_{j\in N} Y^j$. An allocation $x = (x^i) \in X$ is said attainable for economy $\mathcal{E}$ if $\sum_{i\in M} x^i \in \sum_{i\in M} \omega^i + Y$. We will denote by $\hat{X}$ the set of all attainable allocations of the economy. In the following, we will consider also for every $i \in M$ and for every $j \in N$

$\hat{X}^i = X^i \cap (\omega + Y - \sum_{i' \neq i} X^{i'})$, the <u>attainable set of consumer i,</u>

$\hat{Y}^j = Y^j \cap (\sum_{i \in M} X^i - \sum_{j' \neq i} Y^j - \{\omega\})$, the <u>attainable set of producer j,</u>

and $\hat{Y} = Y \cap (\sum_{i \in M} X^i - \{\omega\})$, the <u>attainable total production set.</u>

Now let $\mathcal{M}$ be the family of all non-empty subsets of M, ie the family of all <u>coalitions</u> of consumers. In order to define the productive power of each coalition, we assume that a coalition $B \in \mathcal{M}$ owns the technology set $\sum_{i \in B} \theta^{ij} Y^j$ at his disposal in producer j. This kind of assumption, which can be found in Rader (1964), Nikaido (1968), Hildenbrand (1970), Aliprantis et al (1987), lies on the idea that the relative shares θ^{ij} reflect consumer's stock holdings which represent proprietorships of production possibilities.

If $X^B = \prod_{i \in B} X^i$, $x^B \in X^B$ is said to be <u>an attainable assignement for the coalition B</u> if $\sum_{i \in M} x^{iB} = \sum_{i \in B} \omega^i + \sum_{i \in B} \sum_{j \in N} \theta^{ij} Y^j$.

We will denote by $\hat{X}^B$ the set of all attainable assignements for the coalition B.

For each $B \in \mathcal{M}$, a preference correspondence $P^B: X \to X^B$ can be defined by :

$$P^B(x) = \{z^B = (z^{iB}) \in X^B / z^{iB} \in P^i(x) \ \forall i \in B\}.$$

$P^B(x)$ is interpreted as the set of the elements of X^B which are unanimously preferred to x by the members of the coalition B.

A coalition B is said to <u>block</u> an attainable allocation $x \in \hat{X}$ if there exists $z^B \in \hat{X}^B \cap P^B(x)$.

The <u>core</u> of $\mathcal{E}$ is classically defined as the set $C(\mathcal{E})$ of all attainable allocations which are blocked by no coalition.

Then let r be any positive integer. Let us consider the <u>r-fold</u>

<u>replica</u> of the economy $\mathcal{E}$, composed of r subeconomies identical to the original $\mathcal{E}$.

$$\mathcal{E}^r = ((X^{iq}, P^{iq}, \omega^{iq})_{\substack{i\in M \\ q=1,\dots,r}}, (Y^{jq'})_{\substack{j\in N \\ q'=1,\dots,r}}, (\theta^{iqjq'})_{\substack{i\in M,\ j\in N \\ q,q'=1,\dots,r}})$$

is defined as follows : for each $j\in N$, r producers of type j have the same production set $Y^{jq'} = Y^j$; for each $i \in M$, r consumers of tyupe i have the same consumption set $X^{iq} = X^i$ and the same initial endowment $\omega^{iq} = \omega^i$. For preferences and ownership of initial holdings and production possibilities, each consumer (i,q) is restricted whithin his subeconomy :

$P^{iq} : \prod_{k\in M} X^{kq} \to X^{iq}$ is defined by $P^{iq}(x) = P^i(x)$

and $\theta^{iqjq'} = \begin{cases} \theta^{ij} \text{ if } q = q' \\ 0 \text{ if } q \neq q' \end{cases}$.

If $\bar{x} \in \hat{X}$, an allocation which assignes the same consumption bundle $\bar{x}^i$ to each consumer (i, q), q = 1,..., r, belongs to the core of $\mathcal{E}^r$ if and only if there exist no $S \subset M \times \{1,\dots,r\}$, $S \neq \Phi$, and no

$x^S \in \prod_{(i,q)\in S} X^{iq}$ such that :

(1) $\sum_{(i,q)\in S} x^{iqS} \in \sum_i (\text{card } S(i))\, \omega^i + \sum_i (\text{ card } S(i)) \sum_j \theta^{ij} Y^j$

where $S(i) \overset{\text{def}}{=} \{q \in \{1,\dots, r\} / (i, q) \in S\}$ and card S(i) denotes the number of elements of S(i)

(2) $x^{iqS} \in P^i(\bar{x}) \quad \forall (i,q) \in S$

Let us denote by $C^r(\mathcal{E})$ the set of all such $\bar{x} \in \hat{X}$.

Following Aliprantis et al (1987), we will say that $\bar{x} \in \hat{X}$ is an <u>Edgeworth equilibrium</u> if $\bar{x} \in \bigcap_{r\geqslant 1} C^r(\mathcal{E})$ and we will denote by $C^e(\mathcal{E}) = \bigcap_{r\geqslant 1} C^r(\mathcal{E})$ the set of all Edgeworth equilibria.

Now let us replace (2) by

(2') $x^{iqS} \in \text{co } P^i(\bar{x})$ (the convex hull of $P^i(\bar{x})$) $\forall (i,q) \in S$

and denote by $C'^{r}(\mathcal{E})$ the set of all $\bar{x} \in \hat{X}$ such that there exist no $S \subset M \times \{1,\dots,\}$, $S \neq \Phi$ and no $x^S \in \prod_{(i,q)\in S} X^{iq}$ satisfying (1) and (2'). Let $C'^{e}(\mathcal{E}) = \bigcap_{r\geq 1} C'^{r}(\mathcal{E})$.

If we assume (this assumption will be made later) that every X^i is convex and if we define $t^i = \dfrac{\text{card } S(i)}{r}$, $t = (t^i)_{i\in M}$ and for each i such that $t^i > 0$, $x^{it} = \dfrac{1}{\text{card } S(i)} \sum_{q\in S(i)} x^{iqS}$,we can replace (1) and (2') by :

$$(3) \qquad \sum_{t^i>0} t^i x^{it} \in \sum_{i\in M} t^i \omega^i + \sum_{i\in M} t^i \sum_{j\in N} \theta^{ij} Y^j$$

$$(4') \qquad x^{it} \in \text{co } P^i(\bar{x}) \qquad \forall i : t^i > 0$$

while (4) denotes the relation

$$(4) \qquad x^{it} \in P^i(\bar{x}) \qquad \forall i : t^i > 0.$$

Here t^i is a rational number in [0,1] ($t^i \in [0,1] \cap Q$) which can be understood as the rate of participation of i to the coalition S while x^{it} is the mean consumption that i achieves by participating to the coalition.

Let $T = [0,1]^M \setminus \{0\}$ and $T_Q = T \cap Q^M$. Obviously $\bar{x} \in \hat{X}$ belongs to $C'^{e}(\mathcal{E})$ if and only if there exists no $t = (t^i) \in T_Q$ and no $x^t \in \prod_{t^i>0} X^i$ satisfying (3) and (4').

Allowing, as Aubin (1979), that the rates of participation take all values in the real interval [0,1], we will say that $\bar{x} \in \hat{X}$ belongs to $C^f(\mathcal{E})$, the fuzzy core of $\mathcal{E}$, (resp. to $C'^{f}(\mathcal{E})$) if there exists no $t = (t^i) \in T$ and no $x^t \in \prod_{t^i>0} X^i$ satisfying (3) and (4) (resp. (3) (4')).

Between all the core concepts defined in this section, we have the following relations :

$$C'^{f}(\mathcal{E}) \subset C'^{e}(\mathcal{E}) = \bigcap_{r\geq 1} C'^{r}(\mathcal{E}) \subset C^{e}(\mathcal{E}) = \bigcap_{r\geq 1} C^{r}(\mathcal{E}) \subset C(\mathcal{E})$$

$$C'^{f}(\mathcal{E}) \subset C^{f}(\mathcal{E}).$$

In the next section we will prove the non-emptiness of $C'^{e}(\mathcal{E})$ under the following assumptions on economy $\mathcal{E}$ (provided that X is endowed with the topology induced by the product topology on L^M) :

A_1- $\forall i \in M$, X^i is convex and $\omega^i \in X^i$

$\forall x \in X$, $x^i \notin co\, P^i(x)$

P^i has σ^m -open lower sections (i.e for every $z^i \in X^i$, the set $(P^i)^{-1}(z^i) = \{x \in X / z^i \in P^i(x)\}$ is σ^m -open in X)

A_2- $\forall j \in N$, $0 \in Y^j$

A_3- Y is convex and $\hat{X}$ is σ^m-compact.

Moreover, if τ is a vector space topology on L, not necessarily identical to the initial topology of L, we will prove that $C'^{f}(\mathcal{E})$ is non-empty, under the following additional assumption :

A_4 - $\forall i \in M$, $\forall x \in X$, $P^i(x)$ is τ-open in X^i.

III - NON-EMPTINESS THEOREMS

For any $t \in T$, let us define :

$$Y^t = \sum_{i \in M} t^i \sum_{j \in N} \theta^{ij}\, Y^j \quad ; \quad X^t = \prod_{t^i > 0} X^i$$

$$\hat{X}^t = \{x^t \in X^t \ / \sum_{t^i>0} t^i x^{it} \in \sum_{i \in M} t^i \omega^i + Y^t\}$$

$\overline{co}\ \hat{X}^t$, the closed convex hull of $\hat{X}^t$, and, if $x \in X$

$$P^t(x) = \{z^t \in X^t \ / \ z^{it} \in P^i(x) \quad \forall i : t^i > 0\}$$

$$Q^t(x) = \{z^t \in X^t \ / \ z^{it} \in co\, P^i(x)\ \forall i : t^i > 0\} .$$

Y^t, $\hat{X}^t$, $P^t : X \to X^t$ may be respectively interpreted as the <u>production set</u>, the <u>attainable set</u> and the <u>preference correspondence</u> of the <u>fuzzy coalition</u> t.

$$\text{If } \Delta^{T} = \left\{ \lambda = (\lambda_t) \in \mathbb{R}^{T} \;\middle|\; \begin{array}{l} \lambda_t \geqslant 0 \quad \forall\, t \in T \\ \lambda_t = 0 \text{ for almost all indices } t \\ \sum_{t \in T} \lambda_t\, t^i = 1 \quad \forall i \in M \end{array} \right\}$$

and if Y is convex, we first observe that economy $\mathcal{E}$ satisfies the following balancedness condition :

$$\lambda \in \Delta^{T} \Rightarrow \sum_{t \in T} \lambda_t Y^t \subset Y.$$

We will first prove (proposition 1 and proposition 2) that for any $r \geqslant 1$, $C'^{r}(\mathcal{E}) \neq \Phi$. When L is $\mathbb{R}^{l}$, the l-dimensional Euclidian space, in view of the balancedness property of $\mathcal{E}$, the argument is strongly related to the fixed-point argument used in Florenzano (1987) to prove that, under similar assumptions, the core of a balanced coalitional production economy is non-empty. By considering traces of economy $\mathcal{E}$ on finite dimensional subspaces of the commodity space, the result is extended to the infinite dimensional case. Then the non-emptiness of $C'^{e}(\mathcal{E})$ (proposition 3) and $C'^{f}(\mathcal{E})$ (proposition 4) are quite straight-forward.

Let $Y' \subset L$ be such that $Y \subset Y'$ and define:

$\hat{X}' = \{x = (x^i)_{i \in M} \in X \;/\; \sum_{i \in M} x^i \in \sum_{i \in M} \omega^i + Y'\}$.

In proposition 1, we replace the assumption A_3 by :

A_3' - Y' is convex and $\hat{X}'$ is compact.

Proposition 1. Assume A_1, A_2, A_3' and that $L = \mathbb{R}^{l}$. Then if r is any positive integer and if

$$T_r = \{t = (t^i)_{i \in M} \in T \;/\; rt^i \in \{0,1,\dots,r\} \quad \forall\, i \in M\},$$

there exists $\bar{x} \in \hat{X}'$ such that $\hat{X}^{t} \cap Q^{t}(\bar{x}) = \Phi \quad \forall\, t \in T_r$.

Proof

Let $\Delta^{\mathcal{T}r} = \{\lambda = (\lambda_t) \in \mathbb{R}^{\mathcal{T}r} / \lambda_t \geqslant 0 \;\; \forall\, t \in \mathcal{T}_r \text{ and } \sum_{t \in \mathcal{T}} \lambda_t t^i = 1 \;\; \forall i \in M\}$

For each $(x, z, \lambda) \in \hat{X}' \times \prod_{t \in \mathcal{T}_r} \overline{co}\, \hat{X}^t \times \Delta^{\mathcal{T}r}$, let us define :

$$I(x) = \{t \in \mathcal{T}_r \;/\; \hat{X}^t \cap Q^t(x) \neq \Phi\}$$

and - $\theta(z,\lambda) = (x'^i)_{i \in M}$ with for each $i \in M, x'^i = \sum_{t \in \mathcal{T}_r} \lambda_t\, t^i\, z^{it}$

- $\varphi(x) = (\varphi^t(x))_{t \in \mathcal{T}_r}$ with for each $t \in \mathcal{T}_r$, $\varphi^t(x) = \overline{co}\, \hat{X}^t \cap Q^t(x)$

$$- \psi(x,\lambda) = \begin{cases} \bigcap_{I(x)} \{\mu \in \Delta^{\mathcal{T}r} \;/\; \mu_t > \lambda_t\} & \text{if } I(x) \neq \Phi \\ \Phi \quad \text{if } I(x) = \Phi & \end{cases}$$

It is easily seen that $\Delta^{\mathcal{T}r}$ is a non-empty, convex and compact subset of $\mathbb{R}^{\mathcal{T}r}$, that $\hat{X}'$ is non-empty and convex and that each $\hat{X}^t$ is non-empty and relatively compact. Hence for each $t \in \mathcal{T}_r$, $\overline{co}\, \hat{X}^t$ is compact and $\hat{X}' \times \prod_{t \in \mathcal{T}_r} \overline{co}\, \hat{X}^t \times \Delta^{\mathcal{T}r}$ is a non-empty, convex and compact subset of some finite dimensional Euclidian space. It follows from the convexity of X^i for each $i \in M$, the convexity of Y' and the balancedness condition that $\theta(\Delta^{\mathcal{T}r} \times \prod_{t \in \mathcal{T}r} \overline{co}\, \hat{X}^t) \subset \hat{X}'$. Since $\hat{X}'$ is compact and θ is continuous, $\theta(\Delta^{\mathcal{T}r} \times \prod_{t \in \mathcal{T}_r} \overline{co}\, \hat{X}^t) \subset \hat{X}'$. It can be shown, exactly as in Florenzano (1987), that there exists $(\bar{x}, \bar{z}, \bar{\lambda}) \in \hat{X}' \times \prod_{t \in \mathcal{T}_r} \overline{co}\, \hat{X}^t \times \Delta^{\mathcal{T}r}$ such that

(1) $\bar{x} = \theta(\bar{z}, \bar{\lambda})$

(2) $\forall t \in \mathcal{T}_r$, $\bar{z}^t \in \overline{co}\, \hat{X}^t \cap Q^t(\bar{x})$ or $\overline{co}\, \hat{X}^t \cap Q^t(\bar{x}) = \Phi$

(3) $\psi(\bar{x}, \bar{\lambda}) = \Phi$

To complete the proof, we show by contraposition that $I(\bar{x}) = \Phi$.
If not, by a classical separation argument, it follows from (3) that there exists $\bar{p} = (\bar{p}_t) \in \mathbb{R}_+^{\mathcal{T}r} \setminus \{0\}$, $\bar{p}_t = 0 \;\; \forall\, t \notin I(\bar{x})$, such that $\bar{\lambda}$ is a solution of the linear programming problem :

$$\max \sum_{t\in T_r} \bar{p}_t \;\; \mu_t$$

$\sum_{t\in T_r} \mu_t t^i = 1 \quad \forall\, i \in M \quad$ and $\mu_t \geq 0 \quad \forall t \in T_r$.

Let ε^i, $i \in M$ and $\alpha_t \geq 0$, $t \in T_r$ be a system of multipliers for the first order conditions :

$$(4) \qquad \bar{p}_t = -\alpha_t + \sum_{i\in M} \varepsilon^i t^i \quad ; \quad \alpha_t \lambda_t = 0 \quad \forall\, t \in T_r.$$

For each $i \in M$, setting $t = e^i$, the i^{th} vector of the natural basis of $\mathbb{R}^M$, we get $\varepsilon^i \geq 0$. Then let t_o be such that $\bar{p}_{t_o} > 0$, which implies

$\sum_{i\in M} \varepsilon^i t^i_o > 0$, and let i_o be such that $\varepsilon^{i_o} t^{i_o} > 0$. From (1), we deduce : $\bar{x}^{i_o} = \sum_{t\in T_r} \bar{\lambda}_t t^{i_o} \bar{z}^{i_o t}$ and from (4) and (2) :

$\bar{\lambda}_t t^{i_o} > 0 \Rightarrow \bar{p}_t = \sum_{i\in M} \varepsilon^i t^i > 0 \Rightarrow t \in I(\bar{x}) \Rightarrow \bar{z}^t \in Q^t(\bar{x})$.

Then $\bar{x}^{i_o} \in \text{co}\, P^{i_o}(\bar{x})$, which contradicts assumption A.1.

□

<u>Proposition 2</u> Assume A_1, A_2, A_3. Then if r is any positive integer, $C'^r(\mathcal{E}) \neq \Phi$.

<u>Proof</u>

Let $\mathcal{F}$ be the collection of all finite dimensional subspaces of L containing ω^i, $i \in M$. For each $F \in \mathcal{F}$, we set : $X^i_F = X^i \cap F$; $X_F = \prod_{i\in M} X^i_F$; if $x \in X_F$, $P^i_F(x) = P^i(x) \cap X^i$; $Y^j_F = Y^j \cap F$; $Y'_F = Y \cap F$ and we consider the economy

$$\mathcal{E}_F = ((X^i_F, P^i_F, \omega^i)_{i\in M}, (Y^j_F)_{j\in N}, (\theta^{ij})_{i\in M, j\in N}).$$

Note that $\hat{X}'_F = \hat{X} \cap F^M$. If F is endowed with the topology induced by the topology of L, it is easily checked that $\mathcal{E}_F$ satisfies assumptions A_1, A_2, A'_3. As F is finite dimensional, it follows from proposition 1 that there exists $\bar{x}_F \in \hat{X} \cap F^M$ such that

$\hat{X}_F^t \cap Q_F^t\ (\bar{x}_F) = \Phi \quad \forall\ t \in T_r.$

Now the collection $\mathcal{F}$, ordered by inclusion, is directed. Since $\hat{X}$ is σ^m-compact, by passing to subnets if necessary, we can assume $\bar{x}_F \overset{\sigma}{\to} \bar{x} \in \hat{X}$. If $t \in T_r$ and $x^t \in \ \bar{X}^t \cap Q^t(\bar{x})$, there exists F_o such that $F \supset F_o \Rightarrow x^t \in \hat{X}_F^t \cap Q_F^t\ (\bar{x}_F)$ which yields a contradiction.

□

<u>Proposition 3</u> Assume $A_1 - A_3$. Then $C'^{e}\ (\mathcal{E}) \neq \Phi$

<u>Proof</u>

From the definition, it is easily seen that for every positive integer r, $C'^{r}(\mathcal{E})$ is a closed subset of $\hat{X}$. On the other hand, if $r' > r$, $t' = \dfrac{r}{r'}\ t \ \in T_r$, with $Y^{t'} = \dfrac{r}{r'} Y^t$, $\hat{X}^{t'} = \hat{X}^t$, so that $C'^{r'}(\mathcal{E}) \subset C'^{r}(\mathcal{E})$. Then the non-emptiness of $\bigcap_{r \geqslant 0} C'^{r}(\mathcal{E})$ follows from the compactness of $\hat{X}$.

□

<u>Proposition 4</u> Assume $A_1 - A_4$. Then $C'^{f}(\mathcal{E}) \neq \Phi$

<u>Proof</u>

For each $j \in N$, set $Y'^{j} = \text{co}\ Y^j$. Let $\mathcal{E}'$ be the private ownership economy $\mathcal{E}' = ((X^i, P^i, \omega^i)_{i \in M}\ ,\ (Y'^{j})_{j \in N},\ (\theta^{ij})_{i \in M, j \in N}\ )$.

Since Y is convex, $\mathcal{E}$ and $\mathcal{E}'$ have the same attainable allocations ; hence $\mathcal{E}'$ satisfies $A_1 - A_3$ and it follows from proposition 3 that $C'^{e}\ (\mathcal{E}') \neq \Phi$. We show now that $C'^{e}(\mathcal{E}') \subset C'^{f}(\mathcal{E})$. Indeed let $\bar{x} \in C'^{e}\ (\mathcal{E}')$. If $\bar{x} \notin C'^{f}\ (\mathcal{E})$, there exists $t \in T$ and

$x^t \in \prod_{t^i > 0} X^i$ such that :

$$\sum_{t^i > 0} t^i\ x^{it} \in \sum_{i \in M} t^i\ \omega^i + \sum_{i \in M} t^i \sum_{j \in N} \theta^{ij}\ Y^j$$

$x^{it} \in \text{co}\ P^i(\bar{x})\ \forall i\ :\ t^i > 0.$

By A-4, for every i, co P^i $(\bar{x})$ is τ-open in X^i.

Then let $\varepsilon > 0$ be such that

$1 - \varepsilon < \lambda < 1 \Rightarrow \lambda x^{it} + (1-\lambda)\, \omega^i \in \text{co } P^i(\bar{x}) \quad \forall i : t^i > 0.$

Let $s \in T_Q = T \cap Q^M$ be such that $t^i = 0 \Rightarrow s^i = 0$ and

$t^i > 0 \Rightarrow 1 - \varepsilon < \dfrac{t^i}{s^i} < 1.$

Set, for each $i : t^i > 0$, $x^{is} = \dfrac{t^i}{s^i} x^{it} + \left(1 - \dfrac{t^i}{s^i}\right) \omega^i$

and $x^s = (x^{is}) \in \prod_{s^i > 0} X^i$.

$$\sum_{s^i>0} s^i x^{is} \in \sum_{i\in M} s^i \omega^i + \sum_{i\in M} s^i \sum_{j\in N} \theta^{ij} \frac{t^i}{s^i} Y^j \subset \sum_{i\in M} s^i \omega^i + \sum_{i\in M} s^i \sum_{j\in N} \theta^{ij} Y'^j$$

and $x^{is} \in \text{co } P^i(\bar{x}) \quad \forall\, i : s^i > 0$, which contradits $\bar{x} \in C'^e(\mathcal{E}')$.

□

Proposition 3 extends the theorem 4.7 of Aliprantis et al (1987) at several instances ; in particular, the preferences are not assumed to be transitive or complete. The definition given in section II for the fuzzy core of E extends to the non-ordered case the similar definition given by Aubin (1979) for the fuzzy core of an appropriated economy and proposition 4 extends at several instances the non-emptiness results which can be deduced from the non-emptiness theorems of the fuzzy core of a balanced game.

IV - EQUIVALENCE THEOREMS

Let us denote now by τ the vector space topology considered on L. Let L' be the conjugate space of (L, τ). For each p of L', consider the functions :

$\forall j \in N \quad , \quad \pi^j(p) = \sup p.\, Y^j$

and the correspondences :

$$\forall\, i \in M \quad , \quad \gamma^i(p) = \{x^i \in X^i /\ p.x^i \leq p.\omega^i + \sum_j \theta^{ij}\ \pi^j(p)\}$$

$$\delta^i(p) = \{x^i \in X^i /\ p.x^i < p.\omega^i + \sum_j \theta^{ij}\ \pi^j(p)\}$$

A quasi-equilibrium of $\mathcal{E}$ is a point $(\bar{x}, \bar{y}, \bar{p}) \in \prod_{i \in M} X^i \times \prod_{j \in N} Y^j \times L' \backslash \{0\}$ such that

(1) $\forall i \in M \quad , \quad \bar{x}^i \in \gamma^i\ (\bar{p})$ and $P^i(\bar{x}) \cap \delta^i\ (\bar{p}) = \Phi$

(2) $\forall j \in N \quad , \quad \bar{p}\ .\ \bar{y}^j = \pi^j(\bar{p})$

(3) $\sum_{i \in M} \bar{x}^i = \sum_{j \in N} \bar{y}^j + \sum_{i \in M} \omega^i$

An equilibrium of $\mathcal{E}$ is a quasi-equilibrium $(\bar{x}\ , \bar{y}\ , \bar{p})$ such that

$\forall\, i \in M \quad , \quad P^i(\bar{x}) \cap \gamma^i(\bar{p}) = \Phi$. In this case, $\bar{x}$ is said to be a Walrasian allocation of $\mathcal{E}$.

It is easily seen that every Walrasian allocation of $\mathcal{E}$ belongs to $C^f(\mathcal{E})$. The purpose of this section is to prove some converse statements under the following assumptions :

B_1 - $\forall\, i \in M$, X^i is convex

$\forall\, x \in X$, $P^i(x)$ is τ-open in X^i, convex and $x^i \notin P^i(x)$

B_2 - $\forall j \in N$, Y^j is convex and $0 \in Y^j$

B_3 - If $x \in \hat{X}$ then $x^i \in \overline{P^i(x)}$ (the τ-closure of $P^i(x)$) for every i

and an additional assumption, to be specified later, in the infinite dimensional case.

Let us first remark that under B_1, B_2, B_3, $C^f(\mathcal{E})$ and $C^e(\mathcal{E})$ the fuzzy core of $\mathcal{E}$ and the set of Edgeworth equilibria, coincide (see the proof of proposition 4).

If L is $\mathbb{R}^l$, the l-dimensional Euclidian space, we have the following result, the proof of which does not differ from the proof given in the ordered case.

Proposition 5 Assume B_1, B_2, B_3 and that $L = \mathbb{R}^l$. Then if $\bar{x} \in C^f(\mathcal{E})$, there exists $\bar{y} \in \prod_{j\in N} Y^j$ and $\bar{p} \in \mathbb{R}^l$ such that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$. Moreover $(\bar{x}, \bar{y}, \bar{p})$ is an equilibrium provided that $\bar{p}.\omega^i + \sum_j \theta^{ij}\ \bar{p}.\bar{y}^j > \inf p\ .\ X^i \quad \forall\ i \in M$.

Proof

Let $G = \mathrm{co}(\bigcup_{i\in M} (P^i(\bar{x}) - \sum_j \theta^{ij} Y^j - \omega^i))$. G is well-defined since $\bar{x} \in \hat{X}$ and assumption B_3 imply that $P^i(\bar{x}) \neq \Phi$ $\forall\ i \in M$. We first prove that $0 \notin G$. Indeed if not, there exists $\lambda = (\lambda_i)_{i\in M}$ such that $\lambda_i \geqslant 0 \quad \forall i \in M$, $\sum_{i\in M} \lambda_i = 1$ and $x \in \prod_{\lambda_i > 0} X^i$ such that :

$$\sum_{i\in M} \lambda_i\ x^i \in \sum_{i\in M} \lambda_i \omega^i + \sum_{i\in M} \lambda_i \sum_{j\in N} \theta^{ij}\ Y^j$$
$$x^i \in P^i\ (\bar{x}) \quad \forall\ i : \lambda_i > 0.$$

Thus the fuzzy coalition λ blocks $\bar{x}$, which contradicts $\bar{x} \in C^f(\mathcal{E})$. Then let $\bar{p} \in \mathbb{R}^l \setminus \{0\}$ be such that $\bar{p}.g \geq 0 \quad \forall\ g \in G$. For each $i\in M$, for every $j\in N$,

$x^i \in P^i(\bar{x})$ and $y^j \in Y^j \Rightarrow \bar{p}.x^i \geqslant \bar{p}.\omega^i + \sum_j \theta^{ij}\ \bar{p}.y^j$.

Since $\bar{x} \in \hat{X}$, let $\bar{y} \in \prod_{j\in N} Y^j$ be such that $\sum_{i\in M} \bar{x}^i = \sum_{j\in N} \bar{y}^j + \sum_{i\in M} \omega^i$.

From B_3, one deduces $\bar{p}.\bar{x}^i = \sum_j \theta^{ij}\ \bar{p}.\bar{y}^j + \bar{p}.\omega^i \geq \bar{p}.\omega^i + \sum_j \theta^{ij}\ \bar{p}\ .y^j$ $\forall y^j \in Y^j$ and $\forall\ j\in N$. Then $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.

If $\bar{p}.\omega^i + \sum_j \theta^{ij}\ \bar{p}.\bar{y}^j > \inf \bar{p}\ .\ X^i \quad \forall\ i \in M$, it follows from the openess of $P^i(\bar{x})$ in X^i for every $i \in M$ that $(\bar{x}, \bar{y}, \bar{p})$ is an equilibrium of $\mathcal{E}$.

□

If (L,τ) is any Hausdorff linear topological space, we need an interiority assumption in order to apply a separation argument as in the proof of proposition 5.

Let us first define the correspondences $P : X \to X$ and $R : X \to X$ by $P(x) = \{x \in X / x'^i \in P^i(x) \quad \forall i\in M\}$

$R(x) = \{x \in X/P^i(x') \subset P^i(x) \quad \forall i \in M\}$.

Note that the definition of R does not imply by itself any transitivity property on the preference correspondences. In the transitive case, i.e. if for each i P^i can be identified to the asymetric part of a complete preorder R^i on X^i, then

$R(x) = \{x \in X \ / \ x'^i \in R^i \ (x^i) \quad \forall \ i \in M\}$.

We posit the following assumption :

C - There exists a convex cone Z(with vertex 0), non equal to L, with a non-empty τ-interior i(Z), such that :

either 1 - $x \in \prod_{i \in M} X^i$ and $\sum_{i \in M} x^i \in \omega + Y + Z \Rightarrow R(x) \cap \hat{X} \neq \phi$

or 2 - $x \in \prod_{i \in M} X^i$ and $\sum_{i \in M} x^i \in \omega + Y + Z \Rightarrow (P(x) \cup \{x\}) \cap \hat{X} \neq \phi$.

Now let us consider the economy $\mathcal{E}_Z$ deduced from $\mathcal{E}$ by the addition of a fictitious producer which has Z as production set ; we assume also that $\theta^{iz} = \dfrac{1}{\text{card } M} \quad \forall \ i \in M$.

$\mathcal{E}_Z = ((X^i, P^i, \omega^i)_{i \in M}, (Y^j)_{j \in N}, Z, (\theta^{ij})_{i \in M, j \in N}, (\theta^{iz})_{i \in M})$.

Obviously if $\bar{x} \in C(\mathcal{E}_Z)$ (resp. $C^f(\mathcal{E}_Z)$), then, under assumption C_1, $\bar{\bar{x}} \in R(\bar{x}) \cap \hat{X}$ belongs to $C(\mathcal{E})$ (resp. $C^f(\mathcal{E})$) ; under assumption C_2, $C(\mathcal{E}_Z) \subset C(\mathcal{E})$ and $C^f(\mathcal{E}_Z) \subset C^f(\mathcal{E})$.

If $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ is an equilibrium of $\mathcal{E}_Z$, then under C_1 there exists $\bar{\bar{x}} \in R(\bar{x}) \cap \hat{X}$ and $\bar{\bar{y}} \in \prod_{j \in N} Y^j$ such that $(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{p}})$ is an equilibrium of $\mathcal{E}$; under C_2, $(\bar{x}, \bar{y}, \bar{p})$ is an equilibrium of $\mathcal{E}$.

The next proposition gives an infinite dimensional analogue of proposition 5.

<u>Proposition 6</u> Assume B_1, B_2, B_3 and let $\bar{x} \in C^f(\mathcal{E}_Z)$.

Then, under C_1, there exist $\bar{\bar{x}} \in R(\bar{x}) \cap \hat{X}$, $\bar{y} \in \prod_{j \in N} Y^j$ and

$\bar{p} \in L' \setminus \{0\}$ such that $(\bar{\bar{x}}, \bar{\bar{y}}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.

Under C_2, there exist $\bar{y} \in \prod_{j \in N} Y^j$ and $\bar{p} \in L' \setminus \{0\}$ such that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.

In the both cases, the quasi-equilibrium of $\mathcal{E}$ is an equilibrium of $\mathcal{E}$ provided that $\bar{p}.\omega^i + \sum_j \theta^{ij} \pi^j(\bar{p}) > \inf \bar{p}.X^i \quad \forall i \in M$.

<u>Proof</u>

Let $\bar{x} \in C^f(\mathcal{E}_Z)$ and let $G = \mathrm{co}(\bigcup_{i \in M} (P^i(\bar{x}) - \sum_j \theta^{ij} Y^j - Z - \omega^i))$

Using the fact that Z is a convex cone, one sees as in the proof of proposition 5 that $0 \notin G$. Since G has a non-empty τ-interior, there exists $\bar{p} \in L' \setminus \{0\}$ such that $\bar{p}.g \geq 0 \quad \forall g \in G$.

Under C_1, let $\bar{\bar{x}} \in R(\bar{x})$ and let $\bar{\bar{y}} \in \prod_{j \in N} Y^j$ be such that $\sum_{i \in M} \bar{\bar{x}}^i = \sum_{i \in M} \omega^i + \sum_{j \in N} \bar{\bar{y}}^j$. Then for each $i \in M$ and for every $j \in N$,

$x^i \in P^i(\bar{\bar{x}}),\ y^j \in Y^j,\ z \in Z \Rightarrow x^i \in P^i(\bar{x})$ and $\bar{p}.x^i \geq \bar{p}.\omega^i + \sum_{i \in N} \theta^{ij} \bar{p}.y^j + \bar{p}.z$.

Since $\sum_{i \in M} \bar{\bar{x}}^i = \sum_{i \in M} \omega^i + \sum_{j \in N} \bar{\bar{y}}^j$ and $0 \in Z$, one deduces from B_3 that $(\bar{\bar{x}}, \bar{\bar{y}}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.

Under C_2, let $\bar{\bar{x}} \in (P(\bar{x}) \cap \{\bar{x}\}) \cap \hat{X}$. Since $\bar{x} \in C^f(\mathcal{E}_Z)$, $\bar{\bar{x}} = \bar{x}$ and $\bar{x} \in \hat{X}$. Then let $\bar{y} \in \prod_{j \in N} Y^j$ be such that $\sum_{i \in M} \bar{x}^i = \sum_{j \in N} \bar{y}^j + \sum_{i \in M} \omega^i$. As previously, one sees that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.

In the both cases, it should be noticed that $\bar{p}.z \leq 0 \quad \forall z \in Z$.

If $\bar{p}.\bar{\omega}^i + \sum_j \theta^{ij} \pi^j(\bar{p}) > \inf \bar{p}.X^i \quad \forall i \in M$, it follows from the τ-openess of $P^i(\bar{x})$ in X^i for every $i \in M$ that the quasi-equilibrium is an equilibrium.

□

To end this section, let us remark that proposition 6 is not *stricto sensu* an equivalence theorem between the fuzzy core and the set of walrasian allocations of the economy $\mathcal{E}$.

Actually, in the infinite dimensional case, proposition 6 proves that <u>some</u> allocations in $C^f(\mathcal{E})$, but not necessarily all of them, can be decentralized by a price system as competitive equilibria of $\mathcal{E}$. However, it will be seen later that, in some applications, $C^f(\mathcal{E}_Z)$ co incides with $C^f(\mathcal{E})$.

V - APPLICATIONS

Proposition 6 can be applied as well to economies which satisfy some interiority assumption *à la Duffie* (1986) as to economies which satisfy some uniform properness assumption *à la Mas-Colell* (1986).

More precisely, let AY and, for every j, AY^j denote the asymptotic cones of Y and Y^j ; if, for each i, P^i can be identified to the asymetric part of a *complete* preorder R^i on X^i , let $\mathcal{D}$ be the *preference generated set* defined as in Debreu (1962) :

$$\mathcal{D} = \{\sum_{i\in M} x^i - \sum_{i\in M} \omega^i \;/\; x^i \in P^i(\hat{X}) \quad \forall\, i \in M\}$$

and D the cone (with vertex 0) generated by $\mathcal{D}$.

<u>Proposition 7</u> If AY has a non-empty τ-interior, assumption C_1 of proposition 6 can be satisfied with Z=AY (or any convex cone with a non-empty τ-interior contained in AY).

In the transitive case, if (AY-D) has a non-empty τ-interior, assumption C_1 of proposition 6 can be satisfied with Z=AY-D (or any convex cone with a non-empty τ-interior contained in AY-D).

Moreover in this last case, if $\sum_{j\in N} AY^j - D$ has a non-empty τ-interior, then $C^f(E_Z)$ coïncides with $C^f(\mathcal{E})$.

<u>Proof</u>

The easy proof of the two first statements of proposition 7, which is given in Florenzano (1987), is omitted.

If $Z = \sum_{j\in N} AY^j - D$ has a non-empty τ-interior, we show that

$\mathcal{C}^f(\mathcal{E}) \subset \mathcal{C}^f(\mathcal{E}_Z)$. Let $\bar{x} \in \mathcal{C}^f(\mathcal{E})$.

If there exists $t \in T$ and $x^t \in \prod_{t^i>0} X^i$ such that

$$\sum_{i\in M} t^i x^i \in \sum_{i\in M} t^i \omega^i + \sum_{i\in M} t^i \sum_{j\in N} \theta^{ij} Y^j + \sum_{j\in N} AY^j - D$$

$$x^i \in P^i(\bar{x}) \ \forall i : t^i > 0$$

then let $\lambda \geqslant 0$ and, for each i, $x'^i \in P^i(\hat{X})$ be such that

$$\sum_{i\in M} (t^i x^i + \lambda x'^i) \in \sum_{i\in M} (t^i+\lambda)\, \omega^i + \sum_{i\in M} t^i \sum_{j\in N} \theta^{ij} Y^j + \sum_{j\in N} AY^j.$$

Using the assumption B_1 and B_2, an easy calculation shows that if $t'^i = \dfrac{t^i + \lambda}{\sum_{i\in M}(t^i+\lambda)}$, the fuzzy coalition $t' = (t'^i)_{i\in M}$ blocks $\bar{x}$ in economy $\mathcal{E}$, which contradicts $\bar{x} \in \mathcal{C}^f(\mathcal{E})$.

□

Assume now that the commodity space (L,τ) is a locally convex-solid topological vector lattice (we use here the terminology of Aliprantis and Burkinshaw (1978)). We write $\leqslant$ the order relation on L, $<$ the associated strict relation, $\wedge$ and $\vee$ the classical lattices notations for infimum and supremum. As usually, for an element x of L, x^+, x^- and $|x|$ denote respectively the <u>positive part</u>, the <u>negative part</u>, and the <u>absolute value</u> of x ; L^+ is the <u>positive cone</u> of L. Let $\mathcal{V}_\tau(0)$ be a basis of convex and solid o-neighborhoods. We give here two slightly different formulations for uniform properness of preferences in the transitive case and in the general case.

In the transitive case, we say that preferences are <u>uniformly proper</u> if the following assumption is satisfied :

D_1 - $\forall i \in M$, $X^i = L^+$, $\omega^i \in X^i$ and there exists $v^i > 0$ and $V^i \in \mathcal{V}_\tau(0)$ such that for all $x^i \in X^i$ and $\lambda > 0$

$$X^i \cap (\{x^i\} + \lambda(\{v^i\} + V^i)) \subset R^i(x^i)$$

In the general case, we follow Zame (1987) and say that preferences are <u>uniformly proper</u> if the following assumption is satisfied :

D_2 - $\forall i \in M$, $X^i = L^+$, $\omega^i \in X^i$ and there exists $v^i > 0$ and $V^i \in \mathcal{V}_\tau(0)$ such that for all $x \in \prod_{i \in M} X^i$ and $\lambda > 0$

$$X^i \cap (\{x^i\} + \lambda(\{v^i\} + V^i)) \subset P^i(x)$$

v^i is then interpreted as a direction of <u>strict desirability</u> for i.

For uniform properness of production, we follow Richard (1986) and say that each production set is <u>uniformly proper</u> if the following assumption is satisfied :

D_3 - $\forall\ j \in N$, there exist $v^j > 0$ and $V^j \in \mathcal{V}_\tau(0)$ such that for all y^j in Y^j, $\lambda \geqslant 0$ and $u \in V^j$

$(y^j - \lambda\ v^j + \lambda\ u)^+ \leqslant y^{j+} \Rightarrow y^j - \lambda\ v^j + \lambda\ u \in Y^j$

(or, equivalently, $\lambda\ u^+ \leqslant y^{j-} + \lambda v^j \Rightarrow y^j - \lambda\ v^j + \lambda\ u \in Y^j$)

But uniform properness of production can also be stated for the total production set Y in the assumption :

D_4 - There exists $v^Y > 0$ and $V^Y \in \mathcal{V}_\tau(0)$ such that for all $y \in Y$, $\lambda \geqslant 0$ and $u \in V^Y$

$(y - \lambda v + \lambda u)^+ \leqslant y^+ \Rightarrow y - \lambda v^Y + \lambda u \in Y$.

(or, equivalently, $\lambda u^+ \leqslant y^- + \lambda\ v^Y \Rightarrow y - \lambda\ v^Y + \lambda\ u \in Y$).

Assumption D_3 (resp D_4) means that each Y^j (resp. Y) has an almost asymptotic cone with a non-empty τ-interior : every point of Y^j (resp. Y) is the vertex of a τ-open cone, the points of which can be be produced as far as they correspond to a least output than the initial point.

Obviously D_3 implies D_4 with $v^Y = \sum_{j \in N} v^j$ and $V^Y = \bigcap_{j \in N} V^j$

Indeed let $y \in Y$, $\lambda \geqslant 0$ and $u \in \bigcap_{j\in N} V^j$; if $\lambda u^+ \leqslant y^- + \lambda v^Y \leqslant \sum_{j\in N} (y^{j-} + \lambda v^j)$, it follows from the decomposition property of vector lattices that $\lambda u^+ = \sum_{j\in N} w^j$ with $0 \leqslant w^j \leqslant y^{j-} + \lambda v^j \quad \forall\ j \in N$.

Then, for $j=1$, $|w^1 - \lambda u^-| \leqslant w^1 \vee \lambda u^- \leqslant \lambda |u|$ and $w^1 - \lambda u^- \in \lambda V^1$.

As $(w^1 - \lambda u^-)^+ \leqslant w^1 \leqslant y^{1-} + \lambda v^1$, it follows D_3 that

$y^1 - \lambda v^1 + w^1 - \lambda u^- \in Y^1$.

For $j \neq 1$, it follows from D_3 that $y^j - \lambda v^j + w^j \in Y^j$.

By sommation, $y - \lambda v^Y + \lambda u \in Y$.

Proposition 8 in the analogue of proposition 7. Its proof uses extensively the decomposition property of vector lattices.

Proposition 8 Assume D_1 and D_4 in the transitive case, D_2 and D_4 in the general case. Then the assumptions C_1 in the transitive case, C_2 in the general case of proposition 6 can be satisfied with

$$Z=\{\lambda(-v+u)/\lambda \geqslant 0,\ u \in V\}, v = v^Y + \sum_{i\in M} v^i,\ V \subset V^Y \cap \left(\bigcap_{i\in M} V^i\right),$$

$V \in \mathcal{V}_T(0)$ such that $v \notin V$.

Moreover assume D_1 and D_3 in the transitive case. If

$Z=\{\lambda(-v+u)/\lambda \geqslant 0, u \in V\}$ with $v = \sum_{i\in M} v^j + \sum_{i\in M} v^i$, $V \subset \left(\bigcap_{j\in N} V^j\right) \cap \left(\bigcap_{i\in M} V^i\right)$

$V \in \mathcal{V}_T(0)$ such that $v \notin V$, then $\mathcal{C}^f(\mathcal{E})$ coïncides with $\mathcal{C}^f(\mathcal{E}_Z)$.

Proof

Let $x \in \prod_{i\in M} X^i$ be such that $\sum_{i\in M} x^i = \omega + y - \lambda v + \lambda u$ with $y \in Y$, $u \in V, \lambda > 0$.

If $\lambda > 0$, $\lambda u^+ \leqslant \sum_{i\in M} (x^i + \lambda v^i) + y^- + \lambda v^y$ and it follows from the decomposition property that $\lambda u^+ = \sum_{i\in M} s^i + s^Y$ with for each $i \in M$

$0 \leqslant s^i \leqslant x^i + \lambda v^i$ and $0 \leqslant s^Y \leqslant y^- + \lambda v^Y$. For each $i \in M$, $x^i + \lambda v^i - s^i \in X^i$ and $\sum_{i\in M} (x^i + \lambda v^i - s^i) = \omega + y - \lambda v^Y + s^Y - \lambda u^-$.

Since $|s^Y - \lambda u^-| \leqslant s^Y \vee \lambda u^- \leqslant \lambda u^+ \vee \lambda u^- \leqslant \lambda |u|$ and since V is solid, $s^Y - \lambda u^- \in V$; on the other hand, $(s^Y - \lambda u^-)^+ \leqslant s^Y \leqslant y^- + \lambda v$.

It follows from assumption D_4 that $y - \lambda v^Y + s^Y - \lambda u^- \in Y$.

In the transitive case, it follows from D_1 that $x^i+\lambda v^i - s^i \in R^i(x^i)$ $\forall i \in M$ and $\hat{X} \cap R(x) \neq \Phi$.

In the general case, it follows from D_2 that $x^i+\lambda v^i - s^i \in P^i(x)$ $\forall i \in M$ and $\hat{X} \cap P(x) \neq \Phi$.

Now assume D_1 and D_3 in the transitive case and that Z is defined as in the last statement of proposition 8. Let $\bar{x} \in C^f(\mathcal{E})$. If $\sum_{i\in M} t^i x^i = \sum_{i\in M} t^i \omega^i + \sum_{i\in M} t^i \sum_j \theta^{ij} y^j + \lambda(-v+u)$ with $\lambda > 0$, $x^i \in X^i$ $\forall i \in M$, $y^j \in Y^j$ $\forall j \in N$, $x^i \in P^i(\bar{x})$ $\forall i : t^i > 0$, then, as previously,

$$\lambda u^+ \leqslant \sum_{i\in M} t^i (x^i + \frac{\lambda}{t^i} v^i) + \sum_{i\in M} t^i \sum_{j\in N} \theta^{ij} (y^{j-} + \frac{\lambda}{mt^i\ \theta^{ij}} v^j)$$

with m = card M.

$$\lambda u^+ = \sum_{t^i>0} s^i + \sum_{\substack{t^i>0\\ j\in N}} s^{ij} \quad \text{with } 0 \leqslant s^i \leqslant t^i (x^i + \frac{\lambda}{t^i} v^i)\ \forall i : t^i > 0$$

and $0 \leqslant s^{ij} \leqslant t^i \theta^{ij} (y^{j-} + \frac{\lambda}{mt^i\ \theta^{ij}} v^j)$ $\forall i : t^i > 0$, $\forall j \in N$.

From D_1, one deduces :

$$x^i + \frac{\lambda}{t^i} v^i - \frac{\lambda}{t^i} s^i \in R^i(x^i) \subset P^i(\bar{x}) \quad \forall i : t^i > 0.$$

On the other hand

$$\sum_{i\in M} t^i (x^i + \frac{\lambda}{t^i} v^i - \frac{\lambda}{t^i} s^i) = \sum_{i\in M} t^i \omega^i +$$

$$\sum_{i\in M} t^i [\theta^{i1} (y^1 - \frac{\lambda}{mt^i\theta^{i1}} v^1 + \frac{s^{i1} - \lambda u^-}{mt^i\theta^{i1}}) + \sum_{j\neq 1} \theta^{ij} (y^j - \frac{\lambda}{mt^i\theta^{ij}} v^j + \frac{s^{ij}}{mt^i\theta^{ij}})]$$

From D_3 it follows that

$$y^1 - \frac{\lambda}{mt^i\ \theta^{i1}} v^1 + \frac{s^{i1} - \lambda u^-}{mt^i\ \theta^{i1}} \in Y^1$$

and $y^j - \frac{\lambda}{mt^i\ \theta^{ij}} v^j + \frac{s^{ij}}{mt^i\ \theta^{ij}} \in Y^j \quad \forall j > 2.$

By addition, one gets

$$\sum_{i\in M} t^i \left(x^i + \frac{\lambda}{t^i} v^i - \frac{\lambda}{t^i} s^i\right) \in \sum_{i\in M} t^i\omega^i + \sum_{i\in M} t^i \sum_{j\in N} \theta^{ij} Y^j$$

and t blocks $\bar{x}$ in $\mathcal{E}$, which contradicts $\bar{x} \in C^f(\mathcal{E})$.

□

VI - EXISTENCE OF EQUILIBRIA REVISITED

Let us now consider simultaneously the two vector space topologies σ and τ on the commodity space L.

Under C_1, if $C^f(\mathcal{E})_Z$ coïncide with $C^f(\mathcal{E})$, the addition of assumptions A_1, A_2, A_3 (written for L endowed with σ) with B_1, B_2, B_3 (written for L endowed with τ) guarantees the existence of quasi-equilibria for economy $\mathcal{E}$. We saw in the previous section that this condition is satisfied in the transitive case if $Z = \sum_{j\in N} AY^j - D$ has a non-empty τ-interior or, when (L,τ) is a locally convex topological vector lattice, under uniform properness assumptions D_1 on preferences and uniform properness assumptions D_3 on each production set. This last case was addressed in Aliprantis et al (1987) who obtain in theorem 5.10 the same existence theorem as Richard (1986).

In all the other cases, let us formulate the following assumptions on economy $\mathcal{E}$:

A'_1 - $\forall$ i $\in$ M, X^i is convex, σ-closed and $\omega^i \in X^i$

$\forall x \in X$, $x^i \notin \text{co } P^i(x)$

P^i has σ^m -open lower sections and τ-open upper sections

A'_2 - $\forall$ j $\in$ N, Y^j is convex, σ-closed and $0 \in Y^j$

A_3' - $\hat{X}$ is σ^m-compact

A_3'' - $\forall\ j \in N$, $\hat{Y}^j$ is σ-compact

B' - If $x \in \hat{X}$, then $x^i \in \overline{\text{co } P^i(x)}$ (the τ-closure of co $P^i(x)$) for every $i \in M$

C' - There exists a <u>σ-closed</u> convex cone Z (with vertex 0) with a non-empty τ-interior $i(Z)$ such that

either 1) $x \in X$ and $\sum_{i \in M} x^i \in \omega + Y + Z \Rightarrow R(x) \cap \hat{X} \neq \Phi$

or 2) $x \in X$ and $\sum_{i \in M} x^i \in \omega + Y + Z \Rightarrow (P(x) \cup \{x\}) \cap \hat{X} \neq \Phi$

We first prove a non-emptiness theorem which has as corollaries several existence theorems.

<u>Proposition 9</u> Assume A_1' - A_3' and C_1' or C_2' and A_3'' .Then C'^f $(\mathcal{E}_Z)$ is non-empty.

<u>Proof</u>

Let $\mathcal{K}$ be the collection of the convex and σ-compact subsets K of L containing 0, each $\hat{X}^i$ and each $\hat{Y}^j$ in case of assumption A''^3.

For each $K \in \mathcal{K}$, if $x \in \prod_{i \in M} (X^i \cap K)$, we set $P^{ik}(x) = P^i(x) \cap K$ and we consider the economy:

$$\mathcal{E}_Z^k = ((X^i \cap K,\ P^{ik},\ \omega^i)_{i \in M}\ ,\ (Y^j \cap K)_{j \in N}\ ,\ Z,\ (\theta^{ij})_{i \in M, j \in N},\ (\theta^{iZ})_{i \in M}).$$

It is easily seen that $\hat{X}_Z^k$ is σ^m -closed, hence σ^m -compact. Thus it follows from proposition 4 that C'^f $(\mathcal{E}_Z^k) \neq \Phi$. Then let $\bar{x}^k \in C'^f(\mathcal{E}_Z^k)$

It follows from C_1' (resp. C_2' and A_3'') that there exists $\bar{\bar{x}}^k \in R(\bar{x}^k) \cup \hat{X}$ (resp that $\bar{x}^k \in \hat{X}$). Now the collection $\mathcal{K}$ is directed by set-inclusion.

Since $\hat{X}$ is σ^m -compact, we can assume, by passing to subnets if necessary, that $\bar{\bar{x}}^k \to \bar{x} \in \hat{X}$ (resp. that $\bar{x}^k \to \bar{x} \in \hat{X}$). It is easy to see that if $\bar{x} \notin C'^f(\mathcal{E}_Z)$, then there exists K_o such that $K \supset K_o \Rightarrow \bar{x}^k \notin C'^f(\mathcal{E}_Z^k)$, which yieds a contradiction.

□

Corollary 1 Assume $A_1' - A_3'$, B' and C_1' or C_2' and A_3''. Then $\mathcal{E}$ has a quasi-equilibrium.

The proof is an immediate consequence of proposition 6.

Note that under C_1', the assumption that each Y^j is convex and σ-closed can be replaced by : Y is convex and σ-closed.

We have in particular :

Corollary 2 If Y is convex and σ-closed and if AY has a non-empty τ-interior, $\mathcal{E}$ has a quasi-equilibrium under A_1', A_2, A_3' and B'.

□

To go further, we need to make an assumption on the commodity space L, which connects the topologies τ and σ considered on L. We set :

E_1 - (L,τ) is a Hausdorff locally convex topological vector space and τ has a basis $\mathcal{V}_\tau(0)$ of convex, circled and σ-closed o-neighborhoods V whose gauge p_V is a norm.

E_2 - (L,τ) is a Hausdorff locally convex-solid topological vector lattice and τ has has a basis $\mathcal{V}_\tau(0)$ of convex, solid and σ-closed o-neighborhoods V whose gauge p_V is a norm.

Under these two assumptions if $V \in \mathcal{V}_\tau(o)$ and if $v \notin V$, the convex cone generated by $\{v\} + V$ is σ-closed. In view of propositions 7 and 8, proposition 9 has the following corollaries :

Corollary 3 Assume E_1 , A'_1 - A'_3 and B' . Then if (AY-D) has a non-empty τ-interior, $\mathcal{E}$ has a quasi-equilibrium.

If $\omega \in \sum_{i\in M} X^i - i(AY\text{-}D)$, then under an irreducibility assumption on economy $\mathcal{E}$, this quasi-equilibrium is an equilibrium .

Proof

The first statement follows from E_1, proposition 7 and corollary 1 (of proposition 9) since if $v \in i(AY\text{-}D)$, $v \neq 0$. Then if $V \in \mathcal{V}_\tau(0)$ is such that $v \notin V$ and $\{v\} + V \subset AY\text{-}D$, the convex cone generated by $\{v\} + V$ is σ-closed and contained in AY-D, hence satisfies C'_1 .

If $\omega \in \sum_{i\in M} X^i - i(AY\text{-}D)$, v can be choosen in $\sum_{i\in M} X^i - \omega$.

Let $(\bar{x}, \bar{y}, \bar{p})$ be the quasi-equilibrium of $\mathcal{E}$. $\bar{p}.v < 0$ and $\bar{p}.\omega^i + \sum_{i\in M} \theta^{ij}\ \bar{p}.\ \bar{y}^j \geqslant \bar{p}.\omega^i > \inf \bar{p}.X^i$ for some $i \in M$. Then the irreducibility assumption guarantees that $\bar{p}.\omega^i + \sum_{i\in M} \theta^{ij}\ \bar{p}\ .\bar{y}^j >$ $\inf \bar{p}.X^i$ for every $i \in M$.

□

Corollary 4 Assume E_2 , A'_1 - A'_3 and B' . Then under D_1 and D_4 in the transitive case, D_2 , D_4 and A''_3 in the general case, $\mathcal{E}$ has a quasi-equilibrium.

If, in the uniform properness assumptions, each v^i and v^Y can be choosen such that $v^i \leqslant \omega$ $\forall\ i \in M$ and $v^Y \leqslant \omega$, then under an irreducibility assumption on $\mathcal{E}$, this quasi-equilibrium is an equilibrium.

Proof

The first statement is a consequence of E_2, proposition 8 and corollary 1 of proposition 9.

Let $(\bar{x}, \bar{y}, \bar{p})$ be the quasi-equilibrium of $\mathcal{E}$. If $v \leqslant (m+1)\ \omega$, it follows from $\bar{p}.v > 0$ that $\bar{p}.\omega^i + \sum_{i\in M} \theta^{ij}\ \bar{p}.\ \bar{y}^j \geqslant \bar{p}.\omega^i > \inf \bar{p}.X^i$

for some $i \in M$. Then, as previously, $\bar{p}.\omega^i + \sum_{i \in M} \theta^{ij}\ \bar{p}.\ \bar{y}^j > \inf\ \bar{p}.X^i$ for every $i \in M$.

As <u>irreducibility assumption</u> we propose the following :

For every x in $\hat{X}$ and for any proper and non-empty subset J of M, there exist $x' \in X$ and a set of real numbers $\theta^i \geqslant 1$, $i \in M$ satisfying :

- $x'^i \in \overline{P^i(x)} \quad \forall\, i \in J$ with $x'^{i_o} \in P^{i_o}(x)$ for some $i_o \in J$
- $\sum_{i \in M} \theta^i\ (x'^i - \omega^i) \in Y.$

It can easily be checked that some definitions of irreducibility, given in the infinite dimensional setting (see Jones (1987), Zame (1987)) are more restritive.

The remainder of this section is devoted to a short discussion on the admissible commodity spaces in corollary 3 and corollary 4 in relation with the choice of σ.

If $\sigma = \sigma(L,L')$, any Hausdorff locally convex topological vector space L whose the topology τ is generated by a family of <u>norms</u> satisfies assumption E_1. Note that such a space is not necessarily normed. In view of the convexity assumptions, the σ-closedness requirements for each production set and each consumption set can be written for τ ; it is the same for the σ-openess of the lower sections of each P^i if the preferences are convex, transitive and complete. But the $\sigma(L, L')$ - compactness of the attainable sets may be a strong assumption except if L is semi-reflexive ; in this last case, boundedness assumptions guarantee the relative (and thus the) $\sigma(L, L')$ - compactness of the attainable sets. This case covers L_p, $p > 1$ but also $\mathfrak{D}_1$ the space of real functions indefinitely differentiable on [0,1].

If (L, τ) is a normed space, the conjugate space of some other normed space M, and if the norm on L is the dual norm of the norm on M, then (L,τ) satisfies assumption E_1 with $\sigma = \sigma(L, M)$. In this case, norm-boundedness assumptions guarantee the relative compactness of the attainable sets. But if $L \neq M$, the σ-closedness requirements for each consumption and production set and the σ-openess of the lower sections for preference correspondences may be strong assumptions which have natural economic interpretations in commodity spaces of economic interest as L_∞ or ca(K) (see Bewley (1972), Brown-Lewis (1981), Jones (1986).

In the same way if $\sigma = \sigma(L, L')$, any Hausdorff locally convex- solid topological vector lattice L whose the topology τ can be generated by a family of Riesz <u>norms</u> satisfies assumption E_2.If L is a Dedekind complete Lebesgue space,order-boundedness assumptions guarantee the relative $\sigma(L, L')$ - compactness of the attainable sets. This case covers in particular the spaces L_p , $p \geqslant 1$.

If L is a normed Riesz space with the Fatou property (in particular if L is the dual of some normed Riesz space M), let $L_n^\sim$ be the order-continuous dual of L and $L_n' = L' \cap L_n^\sim$. Then if L_n' separates the points of L, L satisfies assumption E_2 for $\sigma = \sigma(L, L_n')$. If, in addition, L is Dedekind-complete, order-boundeness assumptions guarantee the $\sigma(L, L_n')$ - compactness of the attainable sets. And the relation in L : $x^\alpha \uparrow x \Rightarrow x^\alpha \xrightarrow{\sigma} x$ gives rise to natural interpretations of the σ-openess of the lower section of preferences correspondences.

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