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A NOTE ON THE BELLMAN EQUATION OF THE "OVERTAKING CRITERION"

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UNE NOTE SUR L'EQUATION DE BELLMAN DU CRITERE "OVERTAKING"

Résumé. Dans ce papier nous étudions les propriétés des solutions de l'équation de Bellman du critère "overtaking" et montrons que toute trajectoire optimale peut être représentée par un système dynamique continu.

Mots clé. Equation de Bellman, Fonction valeur, Critère "overtaking", Semi-continu supérieurement, système dynamique.

A NOTE ON THE BELLMAN EQUATION OF THE"OVERTAKING CRITERION"

Abstract. In this paper we discuss properties of solutions to the Bellman equation of the "overtaking criterion" and also prove that the optimal trajectory can be represented by a continuous dynamical system.

Key words. Bellman's equation, Value Function, "Overtaking criterion", Upper semi-continuous, Dynamical system.

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1. Introduction

Since the early work of Ramsey (ref 1), optimal growth models have been extensively studied in the economic literature in particular in the case of stationary utilities.

On the one hand when the criterion is to maximize the discounted sum of utilities, existence of optimal programs can be trivially proved and the hard question is to discuss the long term properties of optimal trajectories (refs 2, 3, 4, 5). One of the main tool of this approach is to use Bellman's equation to show that the optimal solutions can be characterized as dynamical systems parameterised by the discount factor and related to the value function of the problem.

On the other hand in the case of a discount factor equal to one, two criterions based on the "overtaking criterion" of Von Weizsacker (ref 6) (optimality is assigned to a path whose partials sums "eventually dominate" when compared with an alternate path from the same initial stock) were developed by Gale (ref 7) and Brock (ref 8). Their proofs (not any more trivial) of existence of "optimal" and "weakly maximal" paths used the fact that the asymptotic properties of a large class of trajectories were known. Once existence proved, the optimal trajectory was known to converge (or converge in the average). Bellman's equation was therefore not discussed.

The purpose of this note is to characterize the solutions of Bellman's equation and to show that optimal solutions can still be viewed as trajectories of a dynamical system in the case of Gale's criterion. We explain why the proofs break down if one uses a weaker criterion.

The note is organised as follows. In section 2, we set the model and recall results already proved in the literature. In section 3, we prove the main theorems of the note : under technical conditions, the value function is the unique solution to Bellman's equation. Note that we do not use a contraction mapping technique as in the discounted case, but we use as in the proofs of existence of Gale (ref 7) and Brock (ref 8) the fact that asymptotic properties of trajectories are known. We then prove that the optimal path depends continuously on the initial stock. Section 4 includes an appendix.

2 - THE MODEL AND ITS PROPERTIES

We consider a triple (X, T, u) where

A1 : X is a compact convex subset of R^n_+, $n \geq 1$, with non empty interior that contains 0.

A2 : $x \to T(x)$ is a continuous set valued correspondence from X into X with non empty compact convex images. Its graph is convex with a non empty interior.

A3 : If $y \in T(x)$ and $x' \geq x$ and $y' \leq y$ then $y' \in T(x')$ (free disposal)

A4 : There is (x,y), $y \in T(x)$, $y >> x$ (existence of an expensible stock)

A5 : The utility function $u : X \times X \to R$ is a concave continuous function. $u(x,y)$ is strictly increasing in x, strictly decreasing in y.

Let $\bar{u} = \max_{x \in T(x)} \{u(x,x)\}$. It follows from A5 that $\bar{u} = \max_{\substack{y \in T(x) \\ y \geq x}} \{u(x,y)\}$.

Let us first recall the following property which is proved in McKenzie (Ref.9).

<u>Proposition 2.1</u> - There is a $p \geq 0$ such that $u(x,y) + py - px \leq \bar{u}$ for all $y \in T(x)$.

Let the value loss $\delta(x,y) = \bar{u} - u(x,y) + px - py$ for any (x,y), $y \in T(x)$. By proposition 2.1, $\delta(x,y) \geq 0$.

A programme is a sequence $\underset{\sim}{x} \in X^\infty$ such that $x_{t+1} \in T(x_t)$. We shall denote by Γ the set of programmes and endowed it with the product topology. $\Gamma(x_0)$ will denote the set of programmes with initial stock x_0.

<u>Definition 1.1</u> - A programme $\underset{\sim}{x}^* \in \Gamma(x_0)$ is optimal if for any $\underset{\sim}{x} \in \Gamma(x_0)$

$$\overline{\lim} \sum^{T} (u(x_t, x_{t+1}) - u(x_t^*, x_{t+1}^*)) \leq 0$$

<u>Definition 1.2</u> · A programme $\underset{\sim}{x}$ is good if for any other programme $\underset{\sim}{x}'$

$$\underline{\lim} \sum_{0}^{T} (u(x_t, x_{t+1}) - u(x'_t, x'_{t+1})) > -\infty$$

<u>Proposition 2.2</u> · If $\underset{\sim}{y}$ is a good programme, then $\underset{\sim}{x}$ is good iff

$$\underline{\lim} \sum_{0}^{T} (u(x_t, x_{t+1}) - u(y_t, y_{t+1})) > -\infty$$

<u>Proof</u> . To prove the sufficiency condition let us remark that

$$\underline{\lim} \sum_{0}^{T} [u(x_t, x_{t+1}) - u(x'_t, x'_{t+1})] \geq \underline{\lim} \sum_{0}^{T} [u(x_t, x_{t+1}) - u(y_t, y_{t+1})] +$$

$$\underline{\lim} \sum_{0}^{T} [(y_t, y_{t+1}) - u(x'_t, x'_{t+1}]) > -\infty$$

□

Let us denote by Γ_G (resp. $\Gamma_G(x_0)$) the set of good programs (resp.starting from x_0).The following proposition follows from proposition 2.2.

<u>Proposition 2.3</u> . If $\Gamma_G(x_0) \neq \phi$, then an optimal programme from x_0 is good.

Let $\bar{x} \in \text{Argmax} \left\{ u(x,x),\ x \in T(x) \right\}$.

<u>Proposition 2.4</u> . The optimal stationary progam $\underset{\sim}{\bar{x}} = (\bar{x}, \bar{x}, \ldots)$ is a good programme

<u>Proof</u> . See Gale (Ref.7, Theorem 4 page 10).

□

Proposition 2.5. If the programme $\underset{\sim}{x}$ is not good then :

$$\sum_0^{\infty} [u(x_t, x_{t+1}) - u(\bar{x}_t, \bar{x}_{t+1})] = -\infty$$

Proof. see Gale (Ref.7, Theorem 5 page 11)

□

Proposition 2.6. Γ_G is convex compact. If $\Gamma_G(x_0) \neq \phi$ then it is convex compact.

Proof. It suffices to remark that $\Gamma \setminus \Gamma_G$ the set of programmes which are not good is relatively open.

□

Let us now on assume the following assumption :

A6 : $\bar{x}$ is unique and for any $x \in X$

$$u(\lambda\bar{x} + (1-\lambda)x, \lambda\bar{x} + (1-\lambda)\ x) > \lambda u(\bar{x},\bar{x}) + (1-\lambda)\ u(x,x)$$

The following proposition may be deduced from A1-A6.

Proposition 2.7. If $\underset{\sim}{x}$ is a good program then $x_t \to \bar{x}$.

Proof. See Gale (Ref.7)

□

Remark 2.1. As it was announced in the introduction, it is this very strong property which implies existence of an optimal program and the properties that are developed in the paper,

Proposition 2.8. Let $\gamma : \Gamma \to [-\infty, +\infty[$ be defined by

$$\gamma(\underset{\sim}{x}) = \lim_{T\to\infty} \sum_0^T [\, u(x_t,x_{t+1}) - u(\bar{x},\bar{x})]$$

Then γ ix a concave, upper semi-continuous function.

Proof. Let us first show that γ is well defined. If $\underset{\sim}{x}$ is not good then $\gamma(\underset{\sim}{x}) = -\infty$. If $\underset{\sim}{x}$ is good then

$$\sum_0^T [u(x_t, x_{t+1}) - u(\bar{x}, \bar{x})] = \sum_0^T - \delta(x_t, x_{t+1}) + px_0 - px_{T+1}.$$

As $\delta(x_t, x_{t+1}) \geq 0$, $\lim_{T\to\infty} [\sum_0^T - \delta(x_t, x_{t+1})]$ exists. As $\underset{\sim}{x}$ is bounded, if

$\lim_{T\to\infty} \sum_0^T - \delta(x_t, x_{t+1}) = -\infty$ then $\lim_{T\to\infty} \sum_0^T [u(x_t, x_{t+1}) - u(\bar{x}, \bar{x})] = -\infty$

contradicting $\underset{\sim}{x}$ good. Therefore $\lim_{T\to\infty} \sum_0^T - \delta(x_t, x_{t+1}) > -\infty$ and from proposition 2.7

$$\gamma(\underset{\sim}{x}) = \lim_{T\to\infty} [\sum_0^T - \delta(x_t, x_{t+1})] + px_0 - p\bar{x}$$

As $\gamma(\underset{\sim}{x})$ is the decreasing limit of continuous functions $\sum_0^T - \delta(x_t, x_{t+1}) + px_0 - p\bar{x}$, γ is upper semi-continuous and concave.

□

Remark 2.2.

If x is such that $T(x)$ contains a strictly positive vector one can show that $\Gamma_G(x) \neq \phi$ if one assumes that the subgradient $\partial u(x,y)$ for $y \in T(x)$ is non empty and uniformly bounded. (This assumption is trivially fulfilled if one assumes that u is continuously differentiable).

Corollary 2.1. If $\Gamma_G(x) \neq \phi$, $\underset{\sim}{x}^*$ is optimal from x iff $\underset{\sim}{x}^*$ solves the following problem :

$$\max \lim_{T\to\infty} \sum_0^T [u(x_t, x_{t+1}) - u(\bar{x}, \bar{x})], \text{ subject to } \underset{\sim}{x} \in \Gamma(x)$$

3 - THE MAIN RESULTS

We are now ready for the main theorems of the paper.

Theorem 3.1 . Let $V(x) = \sup \left\{ \gamma(\underset{\sim}{x}) \, , \, \underset{\sim}{x} \in \Gamma(x) \right\}$. Then

a) $V(\bar{x}) = 0$. V is a concave non decreasing upper semi-continuous function which satisfies Bellman's equation (β) :

$$V(x) = \sup \left\{ u(x,y) + V(y), \; y \in T(x) \right\} - u(\bar{x},\bar{x})$$

b) If W is any other solution of (β) that is upper semi-continuous and satifies $W(\bar{x}) = 0$ then $W(x) \leq V(x)$.

c) If $(\bar{x},\bar{x}) \in$ int graph T then V is the unique solution of (β).

Proof . a) The upper semi-continuity of V follows from Berge (Ref.10,p.122) and the fact that the correspondence $x \to \Gamma(x)$ has a closed graph and is therefore upper semi-continuous.

It follows from proposition 2.8 that $V(x) > -\infty$ iff $\Gamma_G(x) \neq \phi$. If $V(x) > -\infty$, as $\gamma(\underset{\sim}{x})$ is upper semi-continuous and $\Gamma_G(x)$ is compact there exists an optimal program starting from x. By the usual argument $V(x) = \max \left\{ u(x,y) + V(y), \; y \in T(x) \right\} - u(\bar{x},\bar{x})$. If $V(x) = -\infty$ then $V(y) = -\infty$ for every $y \in T(x)$ (if not there would exist a good program from x) and indeed Bellman's equation holds.

b) Let W be any other upper semi-continuous solution such that $W(\bar{x}) = 0$. Let x be such that $W(x) > -\infty$. Then for some programme $x_1, x_2, \ldots x_T \ldots$ with $x_t \in T(x_{t-1})$, $W(x) = \sum_{0}^{T} [u(x_i, x_{i+1}) - u(\bar{x},\bar{x})] + W(x_{T+1})$.

As W is upper semi-continuous, W is bounded above ; thus for some M, $W(x) - M \leqslant \sum_0^T [u(x_i,x_{i+1}) - u(\bar{x},\bar{x})]$ for every T. Therefore $\underset{\sim}{x}$ is good and $x_t \to \bar{x}$ (proposition 2.7). By definition of V,

$\lim \sum_0^T [u(x_i,x_{i+1}) - u(\bar{x}, \bar{x})] \leqslant V(x)$. Therefore $W(x) \leqslant V(x) + \overline{\lim}\, W(x_{T+1})$.

As W is upper semi-continuous and $x_{T+1} \to \bar{x}$, $\overline{\lim}\, W(x_{T+1}) \leqslant W(\bar{x}) = 0$ and $W(x) \leq V(x)$.

c) Let us assume that $(\bar{x}, \bar{x}) \in$ int graph T.

Let x be such that $W(x) > -\infty$. Consider the good programme defined in b) then $x_T \to \bar{x}$. By Lemma 4.1 in the appendix, $\bar{x} \in T(x_t)$ for t large enough. As W fulfills

$W(x_t) \geqslant u(x_t,\bar{x}) - u(\bar{x},\bar{x}) + W(\bar{x})$ for t large enough, thus

$\underline{\lim}\, W(x_t) \geqslant 0$. As $\overline{\lim}\, W(x_t) \leqslant 0$ we have that $\lim W(x_t) = 0$ and therefore $W(x) = V(x)$.

Let us now assume that $W(x) = -\infty$. Then for any $y \in T(x)$, $W(x) = u(x,y) - u(\bar{x},\bar{x}) + W(y)$ and $W(y) = -\infty$.

For any programme starting from x,

$$W(x) = \sum_0^T \{u(x_i,x_{i+1}) - u(\bar{x},\bar{x})\} + W(x_{T+1}) \text{ and } W(x_{T+1}) = -\infty \quad \forall T.$$

Assume that there exists a good programme from x; then $x_T \to \bar{x}$. Therefore by the proof above $\underline{\lim}\, W(x_T) \geqslant 0$ which contradicts $W(x_T) = -\infty$ $\forall T$. Thus there exists no good programme from x and therefore $V(x) = -\infty$ and $W(x) = V(x)$.

□

Let us furthermore assume the following assumptions :

A8 : For $x > 0$, $T(x)$ is strictly convex in R_+^n.

A9 : $u(x,.)$ is strictly concave.

We can now prove

Theorem 3.2. For every $x \in \mathrm{dom}\, V$, define $\tau(x) = \mathrm{Argmax}\left\{u(x,y)+V(y), y \in T(x)\right\}$. Then $\tau : \mathrm{dom}\, V \to X$ is continuous and the optimal path from x is $\bar{x}_t = \tau^t(x)$

Proof. Let us first remark that if $x \in \mathrm{dom}\, V$, as V is upper semi-continuous, $\tau(x)$ is well defined. To prove that τ is continuous let $x_n \underset{n \to \infty}{\to} x$, $x \in \mathrm{dom}\, V$ and let $\tau(x_n) \underset{n \to \infty}{\to} \bar{y}$.

Let $y \in \mathrm{int}\, T(x)$; then by lemma 4.1 in appendix, $y \in T(x^n)$, n large enough. Therefore

$$u(x_n, \tau(x_n)) + V(\tau(x_n)) \geqslant u(x_n, y) + V(y);\ \text{thus}$$

$$u(x,\bar{y}) + V(\bar{y}) \geqslant u(x,\bar{y}) + \overline{\lim}\, V(\tau(x_n)) \geqslant u(x,y) + V(y)$$

Assume that for some $y \in T(x)$

$$u(x,y) + V(y) > u(x,\bar{y}) + V(\bar{y}).$$

Define $y_\lambda = \lambda y + (1-\lambda)\,\bar{y}$, with $\lambda \in \,]0, 1[$.

As $u(x,.)$ is strictly concave, one has

$$u(x,y_\lambda) + V(y_\lambda) > u(x,\bar{y}) + V(\bar{y}).$$

Define $\varepsilon = u(x,y_\lambda) + V(y_\lambda) - u(x,\bar{y}) - V(\bar{y})$

Since $T(x)$ is strictly convex in R^n_+, one can find y' in $\mathrm{int}\, T(x)$ such that

$$y' \gg y_\lambda$$

and $u(x,y_\lambda) \leq u(x,y') + \frac{\varepsilon}{2}$.

As V is non decreasing,

$$\frac{\varepsilon}{2} + u(x,y') + V(y') \geq u(x,y_\lambda) + V(y_\lambda) = u(x,\bar{y}) + V(\bar{y}) + \varepsilon$$

Hence $u(x,y') + V(y') \geq u(x,\bar{y}) + V(\bar{y}) + \frac{\varepsilon}{2} > u(x,\bar{y}) + V(\bar{y})$ yielding a contradiction since $y' \in T(x)$. Therefore $u(x,\bar{y})+V(\bar{y}) \geq u(x,y)+V(y)$, for every $y \in T(x)$.

It follows from Bellman's equation that the optimal path from x is $x_t = \tau^t(x)$.

□

Remark 3.1 . A weaker criterion has been studied by Brock (Ref.8). More precisely a programme $\underset{\sim}{x} \in \Gamma(x_0)$ is "weakly maximal" if

$$\forall \underset{\sim}{x}' \in \Gamma(x_0), \underline{\lim} \sum_0^T (u(x_t,x_{t+1}) - u(\bar{x}_t,\bar{x}_{t+1})) \leq 0.$$

Brock (Ref.8) shows existence of a weakly maximal programme from x_0 if $\Gamma_G(x_0) \neq \phi$ and under the following weaker assumption than A6

A6' : $\bar{x}$ is unique.

Indeed under A6', the turnpike property of good programs (proposition 2.7) doesn't hold anymore. However an average turnpike property of good programs holds : $\sum_0^T \frac{x_i}{T} \to \bar{x}$. For a sequence $\{a_n\}$, a generalised Cesaro criterion leads to

$$\underline{\lim}\, a_n \leq \lim \sum_0^T \frac{a_i}{T} \leq \overline{\lim}\, a_n.$$

Therefore if $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are two good programmes, $\underline{\lim}\,(px_t - px_t') \leq 0$ and

$$\underline{\lim} \sum_0^T \left[u(x_t, x_{t+1}) - u(x_t', x_{t+1}') \right] = \lim \sum_0^T \left[\delta(x_t', x_{t+1}') - \delta(x_t, x_{t+1}) \right]$$

$$+ \underline{\lim}\,(px_t' - px_t)$$

$$\leq \lim \sum_0^T \left[\delta(x_t', x_{t+1}') - \delta(x_t, x_{t+1}) \right]$$

Any programme from x_0 that maximizes the upper semi-continuous function $\lim_T \left[\sum_0^T - \delta(x_t, x_{t+1}) \right]$ is therefore weakly maximal.

Define $\gamma(\underset{\sim}{x}) = \underline{\lim} \sum_0^T \left[u(x_t, x_{t+1}) - u(\bar{x}, \bar{x}) \right] = \lim \left[\sum_0^T - \delta(x_t, x_{t+1}) \right] + px_0 - \overline{\lim}\, px_T$ and $V(x) = \sup \left\{ \gamma(\underset{\sim}{x}),\ \underset{\sim}{x} \in \Gamma(x) \right\}$.

As: $-\overline{\lim}\, px_t \leq p\bar{x}$ for every good programme, $V(\bar{x}) = 0$.

However it is not straighforward to give a condition that will insure that the map from $\Gamma_G(x_0) \to R$, $\underset{\sim}{x} \to -\overline{\lim}\, px_t$ is upper semi-continuous. This is why theorem 3.1 cannot be generalised under A6'.

4 - APPENDIX

<u>Lemma 4.1</u> · Let G be a continuous, convex compact valued correspondence from R_+^n into R_+^n. Assume $x^0 \in \text{int } G(k^0)$, then there exist a neighbourhood $V(k^0)$ of k^0 such that for every k in $V(k^0)$, x^0 belongs to G(k).

<u>Proof</u>· Let $x^0 \in \text{int } G(k^0)$. Then there exists a ball $B(x^0, \varrho)$ centered at x^0 and with radius ϱ included in $G(k^0)$. Since G is lower semi-continuous, there exists a neighbourhood of k^0 such that for every k in $V_1(k^0)$, $G(k) \cap B(x^0, \varrho) \neq \phi$.

Assume that the conclusion is false. Then there exists a sequence k^n, $k^n \to k^0$ such that $G(k^n) \cap B(x^0, \varrho) \neq \phi$ but $x^0 \notin G(k^n)$.

Let $\bar{x}^n$ denote the projection of x^0 on $G(k^n)$ and let y^n be diametrically opposed to $\bar{x}^n$ in $B(x^0,\varrho)$, so that $\bar{x}^n$ is also the projection of y^n on $G(k^n)$. Let y be a cluster point of the sequence y^n. We have $d(y, G(k^0))=\lim_n d(y_n, G(k^n)) \geq \varrho$ so that $y \notin G(k^0)$. On the contrary by construction $y \in S(x^0, \varrho) \subset G(k^0)$, a contradiction.

$\square$

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