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ASYMPTOTIC PROPERTIES OF A LEONTIEF ECONOMY

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PROPRIETES ASYMTOTIQUES D'UNE ECONOMIE DE LEONTIEF

Résumé. Dans ce papier nous étudions les propriétés asymtotiques d'un équilibre d'une économie de Leontief où les agents ont des fonctions d'utilité séparables. Nous donnons une estimation des taux de croissance des quantités et des prix d'équilibre qui tendent vers zéro ou l'infini ou restent bornés inférieurement et supérieurement par des vecteurs strictement positifs. Dans le cas des préférences homogènes nous montrons que tout équilibre d'une économie de Leontief converge vers un équilibre qui croît à taux constant.

Mots Clé. Economie de Leontief, Equilibre général intertemporel, Prix de production, Préférences homogènes.

ASYMTOTIC PROPERTIES OF A LEONTIEF ECONOMY

Abstract. In this paper we discuss the asymtotic properties of equilibrium quantities of a Leontief economy when agents have additively separable utilities. We estimate the rate of growth of equilibrium quantities and prices that either tend to zero or infinity or remain bounded below and above by strictly positive vectors. In the case of homogeneous preferences we prove that the quantities of any equilibrium converge to those of an equilibrium that grows at a constant rate.

Key words. Leontief Economy, Intertemporal General Equilibrium, Production Prices, Homogeneous Preferences.

I. INTRODUCTION

This paper attempts to link the theory of intertemporal general equilibrium and the neo-Ricardian theory of prices of production.

Prices of production have been introduced originally in the works of the classics, and later formalized by neo-Ricardian economists. Prices of production associate with a given wage rate a profit rate which is the same for every producer. In section five of its "Elements", entitled "The Theory of Capitalization and Credit", Walras defined equilibrium by the two following properties : i) all markets for commodities and factors clear and ii) a uniform rate of return is established for each entreprise. Actually, within a static framework of analysis, these two properties are not compatible. Prices of production in general differ from general equilibrium prices. For this reason, the definition of Walrasian equilibrium after Walras was restricted to the consideration of the first property. Prices of production were later introduced in the Von Neumann literature as prices sustaining paths of maximal balanced growth. Turnpike results were then discussed when the objective was to maximize terminal stocks (see Morishima [1961], McKenzie [1963]). This literature does not involve consumption.

The question of whether in an intertemporal setting equilibrium prices could lead to a uniform rate of profit was only adressed recently.

Hahn (1982) first attempted to draw a relationship between the two types of prices in the short run. He argued that prices of production are equilibrium prices, for a two period general equilibrium model with a Leontief technology, for well chosen values of the initial endowments. He concluded that prices of production correspond only to a particular case of general equilibrium prices and, therefore, are not very interesting from the point of view of economic theory. In a rejoinder to Hahn, Dumenil and Levy (1985) considered an infinite horizon model with a unique consumer, a fixed coefficient technology and an exogenous given real wage and claimed that prices supporting an optimal plan of consumption could be chosen so as to converge to the prices of production.

Dana et al. [1987] carried their attempt further. They considered an intertemporal general equilibrium model with finitely many, infinitely lived consumers, and finitely many, infinitely lived firms. On the production side, their assumptions are those of a Leontief economy, in an intertemporal framework. There are as many firms as goods; at each period, each firm produces one good with constant returns to scale. For each good, a technology which is stationary over time relates substitutable inputs of the previous period to the production of one unit of output of the present period. On the consumption side, heterogeneous consumers have endowments of goods in the first period; their preferences for present and future consumption are represented by utility functions.

They obtained the following results : Under standard assumptions (but without assuming additively separable utilities), this intertemporal model has an equilibrium such that at each period the rate of maximum profit is uniform. The second result is in the spirit of Samuelson Solow's [1953] theorem. Let us consider prices of production as stationary prices which equalize the rate of maximum profit ; then the sequence of the undiscounted prices of the equilibrium defined above converges to vector of prices of production.

In this paper we discuss the asymptotic properties of equilibrium quantities when agents have additively separable utilities. Following Peleg and Ryder [1972], we characterize competitive equilibrium by intertemporally decentralised conditions. We then estimate the rate of growth of equilibrium quantities and prices that either tend to zero or to infinity or remain bounded below and above by strictly positive rectors.

In the case of homogeneous preferences we show the existence of equilibria sustained by prices of production which grow at a constant rate. We then prove that the quantities of any equilibrium converge to those of an equilibrium that grows at a constant rate. This result can be compared to those obtained by Bewley [1982], Yano [1984], [1985], Coles [1985, 1986] but is based on a duality argument that uses the asymptotic properties of equilibrium prices. The paper is organised as follows.

In section two we define the intertemporal model.

In section three we recall results obtained on existence of equilibria in our previous paper and characterize equilibria by means of decentralised conditions.

In section four, we study the asymptotic properties of the model.

2. Definition and properties of the intertemporal general equilibrium model

Consider m consumers, n producers, n goods producible by means of goods. Assume that the time horizon of the economy is infinite. Let $x_t \in R_+^n$ and $y_t \in R_+^n$ denote commodity bundles at time t and let p_t be a vector of prices at period t. Elements of the commodity space and price space of the economy will be denoted by $\underset{\sim}{x} = (x_t)$, $\underset{\sim}{y} = (y_t)$ and $\underset{\sim}{p} = (p_t)$ and belong to $(R^n)_*^\infty$.

$\underset{\sim}{p} \cdot \underset{\sim}{x} = \sum_{t=0}^{\infty} p_t \cdot x_t$ will denote the __present value__ of the commodity bundle path $\underset{\sim}{x}$ for the price sequence $\underset{\sim}{p}$.

We will use in R^n and $(R^n)^\infty$ the following conventional notations :

in R^n, $z' \geq z \Leftrightarrow \forall k = 1,\ldots,n, z'_k \geq z_k$; $z' > z \Leftrightarrow z' \geq z$ and $z' \neq z$

$z' \gg z \Leftrightarrow \forall k = 1,\ldots,n, z'_k > z_k$.

In $(R^n)^\infty$, if $\underset{\sim}{z'} = (z'_t)$ and $\underset{\sim}{z} = (z_t)$

$\underset{\sim}{z'} \geq \underset{\sim}{z} \Leftrightarrow \forall t \geq 0, z'_t \geq z_t$; $\underset{\sim}{z'} > \underset{\sim}{z} \Leftrightarrow \underset{\sim}{z'} \geq \underset{\sim}{z}$ and $\underset{\sim}{z'} \neq \underset{\sim}{z}$

Economy $\mathcal{E}$ is described by the list :

$$\mathcal{E} = (((R_+^n)^\infty, U^i, S^i), A^j , i=1,\ldots,m ; j=1,\ldots,n)$$

each term of which is defined below.

Production

There are n producers and n producible goods. At each period, producer j produces, with constant returns to scale, an output $y^j \in R_+$ of good j from inputs of labour and goods available at the previous period. For all $j = 1,\ldots,n$, let us denote by A^j the set of inputs (a^j), to produce one unit of good j.

Define :

$$A = \prod_{j=1}^{n} A^j$$

Note that $a = (a^j) \in A$ belongs to R^{n^2} and can be identified to a $(n \times n)$-matrix.

We make on A the following assumptions :

ı P_1 - $\forall j = 1,\ldots,n$, A^j is a non-empty, convex and closed subset of R^n_+

P_2 - All matrices in A have the same strictly positive row

P_3 - For every proper and non empty subset J of $\left\{1,\ldots,n\right\}$, there exist $k \in J$ and $j \notin J$ such that $\inf_{a^j \in A^j} a^j_k > 0$.

P_2 assumes that one good is necessary for the production of all goods.

Assumption P_3 can be interpreted as a "non-splitability" condition for the production system. It means that for every non -trivial subgroup of producers, some good produced by a producer of this subgroup is a strongly necessary input (an input which cannot tend to 0) for the production of some producer of the complementary subgroup.

Let $\lambda(a)$ denote the dominant eigenvalue of matrix a. The following result is proved in Dana et al. [1987].

Proposition 2.1 - Under assumptions P_1 - P_3

1) $\lambda* = \inf \left\{\lambda(a), a \in A\right\}$ is strictly positive and there exists a* such that $\lambda* = \lambda(a*)$

2) There exists a unique $p* \gg 0$ such that $\lambda* \ p* = p* \ a*$ and $(p*a*)_j = \inf \left\{p* \ a^j, a^j \in A^j\right\}$ for every j.

We assume moreover

P_4 There exists a unique a* in A such that $\lambda* \ p* = p* \ a*$

Remark : P_4 is fulfilled if we assume that A^j is strictly convex for every j.

Consumption

There are m consumers who own firms. Consumer i has at time 0 an initial endowment $S^i \in R^n_+$. A _consumption path_ for consumer i is a sequence $\underset{\sim}{x}^i = (x^i_t) \in (R^n_+)^\infty$. Intertemporal preferences of consumers are assumed to be represented by a utility function $U^i : (R^n_+)^\infty \to \bar{R}_+$ and we make the following assumptions :

C_1 - $\forall i = 1,\ldots,m$, every point of $(R^n_+)^\infty$ is a feasible consumption path.

C_2 - $\forall i = 1,\ldots,m$, $S^i \geqslant 0$ and $S = \sum_{i=1}^{m} S^i \gg 0$.

C_3 - $\forall i = 1,\ldots,m$, $U^i(\underset{\sim}{x}^i) = \sum_{t=0}^{\infty} (\delta^i)^t u^i(x^i_t)$ where $u^i : R^n_+ \to R_+$ is

strictly increasing, strictly concave, continuous and satisfies $u^i(0) = 0$.

C_4 - $\forall i = 1,\ldots,m$, $\exists \mu_i \in]0,1[$, $\exists M_i > 0$, $\exists m_i > 0$

$$m_i \|x\|^{\mu_i} \leqslant u^i(x) \leqslant M_i \|x\|^{\mu_i}, \quad \forall x \in R^n_+$$

C_5 - $\max_i (\delta^i)^{\frac{1}{\mu_i}} < \min (\lambda*, 1)$

Attainable allocation paths

An _attainable allocation path_ is a triple

$$(\underset{\sim}{x}, \underset{\sim}{y},\underset{\sim}{a}) \in \prod_{i=1}^{m} ((R^n_+)^\infty) \times (R^n_+)^\infty \times A^\infty$$

which satisfies the following relations :

$$(1) \quad \sum_{i=1}^{m} x_0^i + a_1 \, y_1 \leq S$$

$$(2) \quad \sum_{i=1}^{m} x_t^i + a_{t+1} \, y_{t+1} \leq y_t \, , \quad \forall t \geq 1$$

We will say that each of $\underset{\sim}{y}, \underset{\sim}{a}, \underset{\sim}{x}$ is __attainable__ if it is a part of an attainable allocation path.

Let $\hat{X}(s)$ be the set of all x attainable with a total initial endowment equal to s, $\hat{X}^i(s)$ the set of all $\underset{\sim}{x}^i$ attainable with s and

$$\hat{C}(s) = \left\{ c \in (R_+^n)^\infty \; / c = \sum_{i=1}^{m} x_i, \, x \in \hat{X}(s) \right\}.$$

__Proposition 2.2__ - Under assumptions $P_1 - P_3$, $C_1 - C_2$

1) $\quad \forall s \in R_+^n, \hat{C}(s) = \left\{ \underset{\sim}{c} \in (R_+^n)^\infty | c_0 + \sum_{t=1}^{\infty} (\prod_{\tau=1}^{t} a_\tau) \, c_t \leq s, \, \underset{\sim}{a} = (a_t) \in A^\infty \right\}$

2) For every s, there exists M(s) such that for every $\underset{\sim}{c} \in \hat{C}(s)$,
$$||c_t|| < \frac{M(s)}{\lambda^{*t}}.$$

3) For all s in R_+^n, $\hat{C}(s)$ is convex and compact in the product topology of $(R^n)^\infty$.

__Proof__ : The proof of 1) may be found in Dana et al. [1987] lemma 4.4. To prove 2, let $\underset{\sim}{c} \in \hat{C}(s)$, then by 1); for some $\underset{\sim}{a}, c_0 + \sum_{1}^{\infty} (\prod_{\tau=1}^{t} a_\tau) \, c_t \leq s$. Let p* be as in proposition 2.1.

Then $p* \, c_0 + \sum_{1}^{\infty} \lambda^t \, p* \, c_t < p* \, c_0 + \sum_{1}^{\infty} p* \, (\prod_{\tau=1}^{t} a_\tau) \, c_t < p* \cdot s$

Therefore $\qquad p* \, c_t \leqslant \dfrac{p* \cdot s}{\lambda^{*t}}.$

As $\widehat{C}(s)$ is trivially closed for the product topology, the proof of 3) follows easily from 1) and 2).

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

For further use let us introduce the following cone

$$F_+ = \left\{ \underset{\sim}{x} \in (R_+^n)^\infty \,\middle|\, x_0 + \sum_{t=1}^{\infty} \left(\prod_{\tau=1}^{t} a_\tau \right) x_t < \infty \,,\, \underset{\sim}{a} = (a_t) \in A^\infty \right\}$$

It follows from proposition 2.2 that $F_+ = \underset{s \in R_+^n}{U} \widehat{C}(s)$. We also have:

<u>Proposition 2.3</u> - Under assumptions $P_1 - P_3$, $C_1 - C_5$

For every i, U^i is well defined on F_+ and is continuous on $\widehat{C}(s)$ endowed with the product topology.

<u>Proof</u> - From proposition 2.2. there exists $M(s) > 0$ such that

$\| C_t \| \leqslant \dfrac{M(s)}{(\lambda*)^t}$. From C_4 and C_5, one has :

$$\sum_{t=0}^{\infty} (\delta^i)^t \, u^i(c_t) \leqslant M(s)^{\mu_i} \, M_i \, \sum_{t=0}^{\infty} \left(\frac{\delta^i}{\lambda*^{\mu_i}} \right)^t < +\infty$$

Hence $\displaystyle\sum_{t=T}^{\infty} (\delta^i)^t \, u^i(c)$ converges uniformly to $U^i(c)$; U^i is therefore continuous.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. Existence of competitive equilibria

Competitive equilibrium paths

We define an __equilibrium path__ $(\tilde{x}, \tilde{y}, \tilde{a}, \tilde{p})$ by the following conditions:

(3) (utility maximization) - $\forall i$, $\tilde{x}^i$ maximizes $U^i(\tilde{x}^i)$ subject to $\tilde{x}^i \in (R_+^n)^\infty$ and $\tilde{p} \cdot \tilde{x}^i \leq \bar{p}_o \cdot s^i$.

(4) (profit maximization) - $\forall j$, $\forall \tilde{a}^j \in (A^j)^\infty$, $\forall y^j \in (R_+)^\infty$,

$$\sum_{t=1}^{\infty} (\bar{p}_t^j - \bar{p}_{t-1} \cdot a_t^j) \, y_t^j \leq \sum_{t=1}^{\infty} (\bar{p}_t^j - \bar{p}_{t-1} \cdot \bar{a}_t^j) \, \bar{y}_t^j = 0$$

(5) (market clearing) - $(\tilde{x}, \tilde{y}, \tilde{a})$ is an attainable allocation path with $\leq$ replaced by $=$ in relations (1) and (2).

Equilibrium price paths $\tilde{p} = (\bar{p}_t)$ must belong to the cone

$$\left\{ p \in (R_+^n)^\infty / p_t^j \leq \inf_{a^j \in A^j} p_{t-1} \cdot a^j \, , \, \forall j = 1, \ldots, n \right\}.$$ In the following, we will

concentrate on equilibrium price paths which allow every producer to get at each period a maximum profit (equal to 0) whatever may be the level y_t^j of production. Such prices satisfy the induction relation :

(6) $- \forall t \geq 1, \forall j = 1, \ldots, n, \quad \bar{p}_t^j = \min_{a^j \in A^j} \bar{p}_{t-1} \cdot a^j.$

Let us denote by $f : R_+^n \to R_+^n$ the function defined by : $f_j(p) = \inf_{a^j \in A^j} p \cdot a^j$.

If $p \gg 0$, $f_j(p)$ equals the minimum cost to produce one unit of commodity j.

Proposition 3.1

Under assumptions $P_1 - P_3$ and $C_1 - C_5$ let $(\tilde{x}, \tilde{y}, \tilde{a}, \tilde{p})$ be a competitive equilibrium path of $\mathcal{E}$. If we assume $\bar{p}_0 \gg 0$ and if we define

$\bar{\bar{p}} = (\bar{\bar{p}}_t)$ by $\bar{\bar{p}}_0 = \bar{p}_0$ and $\bar{\bar{p}}_t = f^t(\bar{p}_0)$ $\forall t \geq 1$, then $(\underset{\sim}{\bar{x}}, \underset{\sim}{\bar{y}}, \underset{\sim}{\bar{a}}, \underset{\sim}{\bar{\bar{p}}})$ is a competitive equilibrium path of $\mathcal{E}$.

Therefore every competitive equilibrium can be replaced by another competitive equilibrium with same quantities but which prices satisfy the relation $p_t = f^t(\bar{p}_0)$.

proposition 3.2

Under assumptions P_1-P_3 and C_1- C_5, $\mathcal{E}$ admits an equilibrium $(\underset{\sim}{\bar{x}}, \underset{\sim}{\bar{y}}, \underset{\sim}{\bar{a}}, \underset{\sim}{\bar{p}})$ with $\bar{p}_t = f^t(\bar{p}_0)$, $p_0^- \gg 0$ and

$$\bar{y}_t = \bar{c}_t + \sum_{k=t+1}^{\infty} (\prod_{\tau=t+1}^{k} \bar{a}_\tau) \bar{c}_k .$$

The proofs of proposition 3.1 and 3.2 may be found in Dana et al. [1987]. Following Peleg and Ryder [1972], in order to study the asymptotic properties of equilibria we shall characterize them by conditions decentralized in time.

Characterization of competitive equilibria

Proposition 3.3 - Assume P_1- P_3 , C_1- C_5

$(\underset{\sim}{\bar{x}}, \underset{\sim}{\bar{y}}, \underset{\sim}{\bar{a}}, \underset{\sim}{\bar{p}})$ with $\bar{p}_0 \gg 0$ and $\bar{p}_t = \bar{p}_{t-1} \bar{a}_t$ is a competitive equilibrium iff :

(i) For every i such that $S^i > 0$, there exists $d^i > 0$ such that $\forall t$, $\forall c \in R_+^n$:

(7) $(\delta^i)^t u^i(\bar{x}_t^i) - d^i \bar{p}_t \bar{x}_t^i \geqslant (\delta^i)^t u^i(c) - d^i \bar{p}_t c$

(ii) $\bar{c}_0 + \sum_{1}^{\infty} (\prod_{\tau=1}^{t} \bar{a}_\tau) \bar{c}_t = s$; $\bar{y}_t = \bar{c}_t + \sum_{k=t+1}^{\infty} (\prod_{\tau=t+1}^{k} \bar{a}_\tau) \bar{c}_k$

(iii) $\underset{\sim}{\bar{p}} . \underset{\sim}{\bar{x}^i} = \bar{p}_0 . S^i$, $\forall i$

<u>Proof</u> - Let $(\tilde{x}, \tilde{y}, \tilde{a}, \tilde{p})$ be a competitive equilibrium path, then it is optimal and (ii) is satisfied. Moreover $\tilde{\bar{x}}^i$ maximizes $U^i(\tilde{x}^i)$ under the constraint $\tilde{x}^i \in F_+$, $\bar{p} \, \tilde{x}^i \leqslant \bar{p}_0 \, S^i$. The functions $U^i(\tilde{x}^i)$ and $\bar{p} \, \tilde{x}^i$ are well defined on F_+. Since $S^i > 0$ and $p_0 \gg 0$ the Slater condition is satisfied and therefore we can use theorem 5.3.1.2 of Hurwicz [1958]. Thus for every i there exists $d^i \in R_{++}$ such that :

$$(8) \qquad U^i(\tilde{\bar{x}}^i) - d^i \, \bar{p} \, \tilde{\bar{x}}^i \geqslant U^i(\tilde{c}) - d^i \, \tilde{\bar{p}} \cdot \tilde{c}, \; \forall \tilde{c} \in F_+$$

Let us consider the sequence $c_\tau = \bar{x}_\tau$ if $\tau \neq t$ and $c_t = c$.

Clearly $\tilde{c} \in F_+$ and we get (7).

Conversely assume (i), (ii) and (iii). Then summing up inequalities we have :

$$\sum_{t=0}^{T} (\delta^i)^t \, u^i \, (\bar{x}_t) - d^i \sum_{t=0}^{T} \bar{p}_t \cdot \bar{x}_t$$

$$\geqslant \sum_{t=0}^{T} (\delta^i)^t \, u^i \, (c_t) - d^i \sum_{t=0}^{T} \bar{p}_t \, c_t \, , \; \forall c_t \in R_+^n$$

As $\bar{\tilde{c}}$ fulfills (ii), $\bar{\tilde{c}} \in F_+$, and therefore all $\tilde{\bar{x}}^i \in F_+$. Let $\tilde{c}^i \in (R_+^n)^\infty$ be such that $\bar{p} \cdot \tilde{c}^i \leqslant \bar{p}_0 \, S^i$, then $\tilde{c}^i \in F_+$. Let $T \to \infty$, then all the infinite sums are well defined. We thus have (8). Using (iii) we obtain $U^i(\bar{\tilde{c}}^i) \geqslant U^i(\tilde{c}^i)$. Therefore $(\bar{\tilde{x}}, \bar{\tilde{y}}, \bar{\tilde{a}}, \bar{\tilde{p}})$ is a competitive equilibrium.

$$\square$$

4. Asymptotic properties of equilibrium prices

4.1. Convergence properties of equilibrium paths

Let $f : R_+^n \to R_+^n$ be defined by $f_j(p) = \inf_{a^j \in A^j} p \, a^j$. For further use let us introduce the following functions.

Let $p, q \in R_+^n$ be strictly positive. We denote :

$$M(p,q) = \inf \left\{ \lambda / p \leq \lambda \, q \right\} = \max_k \frac{p_k}{q_k}$$

$$m(p,q) = \sup \left\{ \lambda / p \geq \lambda \, q \right\} = \min_k \frac{p_k}{q_k}$$

Then, if $\lambda > 0$ we have for all p and $q \gg 0$:

$$M(\lambda p, q) = \lambda M(p,q) \; ; \quad m(\lambda p, q) = \lambda m(p,q) \; ; \quad m(p,q) q \leq p \leq M(p,q) \, q.$$

Let us recall that f is an increasing, homogeneous of degree 1 continuous mapping and

$$m(p,q) < m(f(p), f(q)) < M(f(p), f(q)) < M(p,q)$$

and that under $P_1 - P_3$ f has a unique strictly positive eigenvector $p*$ such that $f(p*) = \lambda* \, p* = p* \, a*$ ($\lambda*$ and $p*$ are those defined in proposition 2.1). Let us prove the following result.

<u>Proposition 4.1</u> - Assume $P_1 - P_3$ then for every $p_0 \gg 0$, the sequence $\dfrac{f^t(p_0)}{||f^t(p_0)||}$ converges to $\dfrac{p*}{||p*||}$ and $\dfrac{f^t(p_0)}{\lambda*^t}$ converges to $\mu \, p*$ for some $\mu > 0$.

<u>Proof</u> - The proof of the first statement may be found in Dana et al. [1987]. We prove now the second statement.

It follows from $f(p_0) < p_0$ a* that :

$$f^t(p_0) < p_0 (a*)^t < M(p_0, p*) \; p*(a*)^t = (\lambda*)^t \; M(p_0, p*) \; p*$$

$$f^t(p_0) \geq f^t(p*) \; m(p_0, p*) = (\lambda*)^t \; m(p_0, p*) \; p*.$$

Therefore $\dfrac{f^t (p_0)}{\lambda^{*t}}$ is bounded above and below. Let p_k be a cluster point

of $\dfrac{f^t (p_0)}{\lambda^{*t}}$. As $\dfrac{f^t (p_0)}{||f^t(p_0)||} \rightarrow \dfrac{p*}{||p*||}$, $p_k = \alpha_k \; p*$ for some $\alpha_k > 0$.

As $M(f^t(p_0), \lambda* \; p*) = M(\dfrac{f^t(p_0)}{\lambda*^t} , p*) < M(f^{t-1}(p_0), \lambda*^{t-1} p*) = M(\dfrac{f^{t-1}(p_0)}{\lambda*^{t-1}}, p*)$

the sequence $M(\dfrac{f^t(p_0)}{\lambda*^t} , p*)$ is therefore non increasing. Let

$\mu = \underset{t}{\inf} \; M(\dfrac{f^t(p_0)}{\lambda*^t} , p*)$. As M is continuous we have : $\mu = M (\alpha_k \; p*, p*) = \alpha_k$.

Therefore the sequence $\dfrac{f^t (p_0)}{\lambda*^t}$ has a unique cluster point μ p* and

is therefore convergent.

□

<u>Proposition 4.2</u> - Assume P_1 - P_4. Let $(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{y}},\underset{\sim}{\bar{a}},\underset{\sim}{\bar{p}})$ be a competitive equilibrium with $\bar{p}_0 \gg 0$ and $\bar{p}_t = \bar{p}_{t-1} \bar{a}_t$ for every t. then $\bar{a}_t \to a*$.

<u>Proof</u> Let $\bar{p}_t = \bar{p}_{t-1} \bar{a}_t$. Then $\lambda* \dfrac{\bar{p}_t}{(\lambda*)^t} = \dfrac{\bar{p}_{t-1}}{(\lambda*)^{t-1}} \bar{a}_t$. There exists $\mu_1 > 0$ and

$\mu_2 > 0$ such that for t large enough

$$0 \ll \mu_1 \ p* < \dfrac{\bar{p}_t}{(\lambda*)^t} < \mu_2 \ p*. \text{ Therefore } \lambda* \ \mu_2 \ p* > \mu_1 \ p* \ \bar{a}_t.$$

The sequence $\underset{\sim}{\bar{a}}$ is therefore uniformmy bounded.

Let $\bar{\bar{a}}$ be a cluster point of $\underset{\sim}{\bar{a}}$. Since $\lambda* \dfrac{\bar{p}_t}{(\lambda*)^t} = \dfrac{\bar{p}_{t-1}}{(\lambda*)^{t-1}} \bar{a}_t$

and $\dfrac{\bar{p}_t}{(\lambda*)^t} \underset{t \to \infty}{\to} \mu \ p*$, $\bar{\bar{a}}$ satisfies $\lambda* \ p* = p* \ \bar{\bar{a}}$. Under P_4, $\bar{\bar{a}} = a*$.

$\square$

4.2. Asymptotic properties of the equilibrium consumption and production path

We have the following proposition.

<u>Proposition 4.3</u>

Assume P_1 - P_4 , C_1 - C_5.

Define $g_i = (\dfrac{\delta^i}{\lambda*})^{\frac{1}{1-\mu_i}}$ for $i=1,\ldots,m$

and $g = \max_i g_i$

Let $(\tilde{x},\tilde{y},\tilde{a},\tilde{p})$ be an equilibrium path with $\bar{p}_t = f^t(\bar{p}_0)$, $\forall t$.

Then there exist $A_i > 0$, $B_i > 0$, $A > 0$, $B > 0$ such that :

1) $\forall i$, $B_i < \inf_t \dfrac{||\bar{x}_t^i||}{g_i^t} < \sup_t \dfrac{||\bar{x}_t^i||}{g_i^t} < A_i$

2) $B < \inf_t \dfrac{||\bar{y}_t||}{g^t} < \sup_t \dfrac{||\bar{y}_t||}{g^t} < A$

Proof

1 - By proposition 3.3, there exists $d^i > 0$ such that

$$(\tfrac{\delta^i}{\lambda*})^t u^i(\bar{x}_t^i) - d^i \dfrac{f^t(\bar{p}_0)}{(\lambda*)^t} \bar{x}_t^i > (\tfrac{\delta^i}{\lambda*})^t u^i(c) - d^i \dfrac{f^t(\bar{p}_0)}{(\lambda*)^t} \cdot c, \; \forall c \in R_+^n$$

Define $\bar{\bar{x}}_t^i = \bar{x}_t^i (g^i)^{-t}$. By proposition 4.1, for t sufficiently large

$$\mu_1 p* > \dfrac{f^t(\bar{p}_0)}{(\lambda*)^t} > \mu_2 \; p* \gg 0, \text{ for some } \mu_1 > 0, \mu_2 > 0.$$

Under C_4, one has, for t large enough :

$$M_i ||\bar{\bar{x}}_t^i||^{\mu_i} - d^i \mu_2 \; p* \; \bar{\bar{x}}_t^i > m_i ||c'||^{\mu_i} - d^i \mu_1 \; p* \; c' \; , \; \forall c' \in R_+^n$$

In other words

$$M_i ||\bar{\bar{x}}_t^i||^{\mu_i} - d'^i ||\bar{\bar{x}}_t^i|| > \max_r \left\{ M_i r^{\mu_i} - d''^i r \right\} > 0$$

where $d'^i = d^i \mu_2 \; \min_j p_j^*$ and $d''^i = d^i \mu_1 \; \max_j p_j^*$.

Therefore $\inf_t \|\bar{x}^i_t\| = B_i > 0$ and $\sup_t \|\bar{x}^i_t\| < A_i$ with $A_i = (\dfrac{M_i}{d'_i})^{\frac{1}{1-\mu_i}}$

2. By proposition 3.2, one has :

$$\bar{y}_t = \bar{c}_t + \sum_{k=t+1}^{\infty} (\prod_{\tau=t+1}^{k} \bar{a}_\tau) \bar{c}_k$$

where $\bar{c}_k = \sum_{i=1}^{m} \bar{x}^i_k$ for $k > t$

Let $I = \left\{ i \mid g^i = g \right\}$

For t sufficiently large

$$\frac{\bar{y}_t}{g^t} > \frac{\sum_i \bar{x}^i_t}{g^t} > \frac{\sum_{i \in I} \bar{x}^i_t}{g^t} + \varepsilon > \sum_{i \in I} B_i + \varepsilon$$

On the other hand, from C_5, one has $g \lambda* < 1$

Let ε' be such that $g \lambda*(1+\varepsilon') < 1$. For t large enough, since $\bar{a}_t$ converges to $a*$:

$$\frac{\bar{y}_t}{g^t} < \frac{\bar{c}_t}{g^t} + \sum_{k=t+1}^{\infty} (g(1+\varepsilon')a*)^{k-t} \frac{\bar{c}_k}{g^k}$$

Let $e = (1,1,\ldots,1)$

Then $\dfrac{\bar{y}_t}{g^t} < \sup_t \dfrac{\|\bar{c}_t\|}{g^t} (I - g(1+\varepsilon') a*)^{-1} e$

$\square$

Corollary 4.1

The equilibrium consumption and production paths tend to infinity if $\delta^i > \lambda*$, to zero if $\delta^i < \lambda*$ and remain bounded below and above by strictly positive vectors if $\delta^i = \lambda*$.

4.3. A particular case : the case of homogeneous utilities

In this part, we assume :

C_4' : $\forall i$, there exists $\mu^i \in]0,1[$ such that $\forall x \in R_+^n$, $\forall \lambda > 0$, $u^i(\lambda x) = \lambda^{\mu_i} u(x)$

Remark

Since $u^i(x) = u^i(||x|| \frac{x}{||x||})$, one has $m_i ||x||^{\mu_i} < u^i(x) < ||x||^{\mu_i} M_i$,

with $m_i = \min\limits_{||x||=1} u^i(x)$ and $M_i = \max\limits_{||x||=1} u^i(x)$.

Hence C_4' implies C_4.

We next prove that under C_4' every competitive equilibrium converges to a competitive equilibrium with constant growth rate that we define next.

Definition

$(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{y}},\hat{\underset{\sim}{a}},\hat{\underset{\sim}{p}})$ is competitive equilibrium with constant growth rate g if it is a competitive equilibrium associated with some initial endowments and if there exists $g > 0$ such that $\hat{x}_t^i = g^t x_0^i$ for every i and $\hat{y}_t = g^t y_0$.

Proposition 4.4

Assume $C_1 - C_4'$, C_5, $P_1 - P_4$.

Let g be defined as in proposition 4.3.

and $(\bar{x},\bar{y},\bar{a},\bar{p})$ be a competitive equilibrium with $\bar{p}_t = f^t(\bar{p}_0)$, $\forall t$. There exists a competitive equilibrium $(\hat{x},\hat{y},\hat{a},\hat{p})$ with constant growth rate g such that one has :

$$\hat{a}_t = a*, \forall t, \text{ and } \hat{x}_t^i = g^t \hat{x}_0^i \quad , \forall t, \text{ such that } \forall i = 1,\ldots,m \quad \frac{\bar{x}_t}{g^t} \to \hat{x}_0^i.$$

$$\forall i = 1,\ldots,n \quad \frac{\bar{y}_{t,i}}{\hat{y}_{t,i}} \to 1 \quad , \quad \frac{\bar{a}_t}{\hat{a}_t} \to a* \text{ and } \quad \frac{\bar{p}_t^j}{\hat{p}_t^j} \to 1 \quad , \forall j = 1,\ldots n.$$

Proof.

Let x_0^i be a cluster point of $\left\{ \dfrac{\bar{x}_t^i}{g^t} \right\}$.

By proposition 3.3, there exists $d^i > 0$ such that

$$(\frac{\delta^i}{\lambda*}) u^i(\bar{x}_t^i) - d^i \frac{f^t(\bar{p}_0)}{(\lambda*)^t} \bar{x}_t^i > (\frac{\delta^i}{\lambda*})^t u^i(c) - d^i \frac{f^t(\bar{p}_0)}{(\lambda*)^t} c \quad , \forall c \in R_+^m.$$

Since u^i is homogeneous and $\dfrac{f^t(\bar{p}_0)}{(\lambda*)^t} \to \mu p*$, one has :

$$u^i(x_0^i) - d^i \mu p* \cdot x_0^i > u^i(c) - d^i \mu p* \, c \quad , \forall c \in R_+^n$$

Since u^i is strictly concave, such a x_0^i is unique. Hence $\dfrac{\bar{x}_t}{g_t^i}$ converges to x_0^i.

Let $I = \left\{ i \mid g^i = g \right\}$ Define $\hat{x}_0^i = 0$ for $i \notin I$ and $\hat{x}_0^i = x_0^i$ for $i \in I$.

Obviously $\dfrac{\bar{x}_t^i}{g^t} \to \hat{x}_0^i$

Consider now the feasibility relation :

$$\frac{\Sigma \; \bar{x}_t^i}{g^t} + g \; \bar{a}_t \; \frac{\bar{y}_{t+1}}{g^{t+1}} = \frac{\bar{y}_t}{g^t}.$$

Let γ_0 be a cluster point of $\left\{ \dfrac{\bar{y}_t}{g^t} \right\}$, γ_1 be one of $\left\{ \dfrac{\bar{y}_{t+1}}{g^{t+1}} \right\},\ldots,$

γ_k be one of $\left\{ \dfrac{\bar{y}_{t+k}}{g^{t+k}} \right\}.$

Let $\hat{c}_0 = \Sigma_i \; \hat{x}_0^i.$ Since $\bar{a}_t \rightarrow a\ast$ (proposition 4.2) then

$$\hat{c}_0 + g \; a\ast \; \gamma_1 = \gamma_0$$

. . . .

$$\hat{c}_0 + g \; a\ast \; \gamma^K = \gamma^{k-1}$$

Hence $\gamma^t - \gamma^{t+1} = g \; a\ast \; (\gamma^{t+1} - \gamma^{t+2})$ for $0 < t < k-2$ and
$\gamma^0 - \gamma^1 = (g \; a\ast)^{k-1}(\gamma^{k-1} - \gamma^k)$

Since $\gamma^t < A'$, $\forall t$, (proposition 4.3)

$\|\gamma^0 - \gamma^1\| < 2 \; \|(g \; a\ast)^{k-1} \; A' \| \rightarrow 0.$ Therefore

$\gamma^0 = \gamma^1 = (I - g \; a\ast)^{-1} \; \hat{c}$ and .The sequence $\left\{ \dfrac{\bar{y}_t}{t_g} \right\}$ converges to $(I - g \; a\ast)^{-1} \; \hat{c}_0$

Define :

$$\hat{s}^i = \Sigma_{t=0}^{\infty} \; (g \; a\ast)^t \; \hat{x}_0^i$$

By applying proposition 3.3, one can see that $(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{y}}, \hat{\underset{\sim}{a}}, \hat{\underset{\sim}{p}})$ defined as follows :

$$\hat{x}_t^i = g^t \, \hat{x}_0^i \, , \, \forall i, \, \forall t$$

$$\hat{y}_t = g^t \, (I - g \, a*)^{-1} \, \hat{c}_0, \, \forall t$$

$$a_t = a*, \, \forall t$$

$$\hat{p}_t = \mu \, p*(\lambda*)^t, \, \forall t$$

is a competitive equilibrium with $\hat{s}^i$ as initial endowments.

$\square$

<u>Remarks</u> :

1. Using proposition 3.3 one can prove that under $P_1 - P_4$, $C_1 - C_3$ and

$$C_5' : \max_i \delta^i < \min (\lambda*, 1)$$

there exists a competitive equilibrium $(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}, \bar{\underset{\sim}{a}}, \bar{\underset{\sim}{p}})$ such that :

$$\bar{a}_t \rightarrow a*, \, \frac{\bar{p}_t}{(\lambda*)^t} \rightarrow \mu \, p*, \, \bar{x}_t \rightarrow 0, \, \bar{y}_t \rightarrow 0.$$

2. One can also prove that, if C_4 and C_5 are replaced by

$$C_4'' : u^i(x) \leqslant M_i \, ||x||, \, \forall x \in R_+^n, \, \forall i$$

$$C_5' : \max_i \delta^i < \min (\lambda*, 1)$$

Then there exists t_0 such that

$$\forall t > t_0 \ , \ \bar{x}_t^i = 0 \quad , \ \forall i, \ \bar{y}_t = 0$$

Where $\bar{x}, \bar{y}$ are equilibrium consumption and production paths.

Indeed, under these assumptions, proposition 3.3 still holds, and in particular, there exists $d^i > 0$ such that relation (7) is verified. Since $\dfrac{\bar{p}_t}{(\lambda*)^t}$ converges to $\mu \, p* \gg 0$ when $t \to \infty$, one has, for t large enough :

$$\forall c \in R_+^n \setminus \{0\} \ , \left(\frac{\delta^i}{\lambda*}\right)^t \frac{u^i(c)}{||c||} - d^i \ \frac{\bar{p}_t}{(\lambda*)^t} \ \frac{c}{||c||} \ < M^i \left(\frac{\delta^i}{\lambda*}\right)^t - d^i \ \frac{\bar{p}_t \cdot c}{(\lambda*)^t ||c||} < 0.$$

Hence, for t large enough, relation (7) is verified iff $\bar{x}_t^i = 0$; from proposition 3.2 one has $\bar{y}_t = 0$.

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