SEARCH MARKET EQUILIBRIUM,

BILATERAL HETEROGENEITY

AND REPEAT PURCHASES

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A B S T R A C T

This paper develops a general model of a search market with heterogeneity among both buyers (through search costs) and sellers (through production costs). It provides a unifying framework for previous models with unilateral heterogeneity, and is thereby able to simultaneously account for : a) price dispersion which does not rest on the indeterminacy of individual prices ; b) active search in equilibrium ; c) entry of firms ; d) the matching of sellers with high costs, hence high prices, with buyers whose search costs are also high. These results are then extended from single to repeated purchase markets — possibly with buyer renewal — by embedding the previous model into a dynamic game with incomplete information between sellers and buyers. Stationary equilibria (Bayesian perfect) of this game are shown to be isomorphic, in a certain sense, to single-purchase equilibria, while offering an even more realistic description of a monopolistically competitive search market : an endogenous number of firms charge constant but fully optimal prices, reflecting the interplay of preferences, production costs and search costs ; in every period a flow of consumers enter and invest in a thorough search for a suitable seller, with which they then form a stable long-run relationship until leaving the market.

EQUILIBRE DE PROSPECTION, HETEROGENEITE BILATERALE

ET ACHATS REPETES

R E S U M E

Ce papier développe un modèle général d'équilibre de prospection qui prend en compte à la fois l'hétérogénéité des acheteurs (à travers leurs coûts de prospection) et celle des vendeurs (à travers leurs coûts de production). Il généralise les modèles préexistant qui confinaient la diversité à un seul côté du marché et permet ainsi de rendre compte à la fois de : a) une dispersion de prix qui ne repose pas sur l'indétermination des prix individuels ; b) une prospection active à l'équilibre ; c) l'entrée endogène de nouveaux vendeurs ; d) la composition de la clientèle de chaque type de vendeur. On étend ensuite ces résultats au cas d'achats répétés, en incorporant le modèle précédent dans un jeu dynamique à information incomplète entre acheteurs et vendeurs. On montre que les équilibres (Bayésiens parfaits) stationnaires de ce jeu sont isomorphes, en un certain sens, aux équilibres du modèle à achat unique, tout en fournissant une description encore plus réaliste du fonctionnement d'un marché de concurrence monopolistique : un nombre endogène d'entreprises tarifient des prix constants mais reflétant optimalement l'interaction des préférences, des coûts de production et des coûts de prospection ; à chaque période, de nouveaux acheteurs arrivent sur le marché et investissent dans une recherche approfondie, jusqu'à trouver un vendeur qui leur convienne, avec lequel ils forment alors une relation de long-terme stable jusqu'à leur sortie du marché.

INTRODUCTION

Monopolistically competitive markets are characterized by substantial diversity among both sellers and buyers, and often by stable long-run relationships between the two. Sellers with higher costs charge higher prices, and a larger fraction of their customers is composed of buyers who have a higher valuation of time, or more limited alternatives. In the case of repeat purchases, buyers do not search each time for a suitable seller, but rather invest in an initial search to find a long-term supplier.

This paper develops a general model of search market equilibrium with bilateral heterogeneity, for both cases of single and repeat purchases. Its first objective is to account for the above stylized facts ; the second one is to unify previous models of search market equilibrium within a more general theoretical framework ; the third one is to provide a model able to simultaneously explain : a) Price dispersion, reflecting the variety of production and search technologies and their interplay with preferences ; b) active search by buyers, constraining the optimal pricing behaviour of firms ; c) determinacy of individual price strategies, leading to the stability of the equilibrium with respect to small perturbations ; d) endogenous entry of firms ; e) robustness of the equilibrium configuration across single and repeated purchase markets, including those in which the population of buyers is renewed over time.

A market where identical buyers search sequentially among identical sellers who can freely set prices is subject to the well known monopoly price result of Diamond [1971] : as long as the search cost is not exactly zero, the unique equilibrium is for all firms to charge the monopoly price. Thus there is

neither price dispersion nor search, no matter how small the search cost or how

many firms there are. The literature on sequential search has remedied this

problem by allowing heterogeneity in consumers' search costs (Axell [1977], von

zur Muehlen [1980], Rob [1985], Stiglitz [1987]), in their tastes (Diamond

[1987]) or in firms' production costs (Reinganum [1979]).[1] In Bénabou [1986b]

it is shown that in fact price dispersion can exist, and search matter, even

with identical consumers and firms, provided the frictions in the functioning

of the market are not one-sided (consumer search cost) but also affect firms

(costly price adjustment in an inflationary environment), even in arbitrarily

small amounts.

Departing from this minimalism, one will introduce here differences

among <u>both</u> firms and consumers. Diversity is indeed an important feature of

actual markets, and it is therefore not surprising that (as with market

frictions) models which confine it to only one set of actors are in a certain

way "unbalanced" :

- Models with consumer heterogeneity generate price dispersion and search, but

firms are indifferent between the various prices charged in the market. As the

equilibrium rests on indeterminacy, it can only arise by chance (if just enough

sellers choose to charge each price)[2] and its stability is problematic. An

exception is the repeated purchase model of Mc Millan and Morgan [1984], but

there, on the contrary, firms are forever stuck by consumer loyalty at their

initial prices, which are non-optimal and therefore can again only have arisen

by chance.

- The model of firm heterogeneity of Reinganum [1979] does have sellers solving

well-behaved optimisation problems leading to different prices, but still does

not fully reconcile search with price dispersion : all firms with cost below a critical level charge their monopoly price, as in Diamond [1971], while those with higher cost bunch at consumers' common reservation price, so that no search takes place in equilibrium. Thus, there is no effect of search on the firms among which price dispersion exists, and conversely, there is no price dispersion among those which are affected by search.

Clearly, the strenghs and weaknesses of these two types of models are complementary. This paper combines both approaches into a general model of two-sided heterogeneity [3] which generates equilibria possessing all the desired properties. The methodology and results are presented in terms of consumer search in a good's market, but they are easily recast in terms of job search by heterogeneous workers in a labour market where firms differ by their productivities.

Section I presents the general framework and provides a full characterisation of equilibria as solutions to a functional fixed-point equation. Section II is devoted to the case of uniformly distributed search costs, for which existence, stability and uniqueness results are established ; moreover, a very simple closed-form solution is derived — as in Mc Minn [1980] — when individual demand in the absence of search is inelastic (II.1), while the Reinganum [1979] model is precisely generalized when it is isoelastic (II.2). Section III provides a general method for easily constructing closed form equilibria with complex cost (of search and production) and price distributions, but simple pricing rules. Section IV examines repeat purchases and customer renewal, extending the previous results in the form of Bayesian perfect equilibria of a dynamic game with both incomplete and imperfect

information. Most proofs are given in appendix at the end of the paper.

I - THE GENERAL MODEL

I.1. Firms and consumers : A continuum of firms can produce and sell a homogeneous product with constant returns to scale. Their unit costs are distributed on some $[c,\bar{c}]$, where $0 < c < \bar{c} < + \infty$, with cumulative distribution function (c.d.f.) $G(c)$. There is a continuum of consumers, each of whom has indirect utility function $V(p)$ ($V' < 0$, $V'' > 0$) for buying the firms' product at a price p . In an equilibrium, consumers know the distribution of prices in the market, but not the prices charged by individual sellers ; the cumulative distribution function of prices will be denoted by $F(p)$, with support in $[p,\bar{p}]$, $p > 0$. Except for the first one, which is free, price quotations are obtained only through visits to sellers, each of which entails a cost to the consumer. This search cost represents the resources expended on each visit (time, transportation), to which could be added the cost of waiting for another try when consumers are impatient. Consumers are identical except for their unit search costs, which are distributed on some $[\sigma,\bar{\sigma}]$, where $\sigma < \bar{\sigma} < + \infty$, with a c.d.f. $Q(\sigma)$ admitting a finite density $q(\sigma)$. Atoms in the distribution of search costs are thus excluded, but this entails no significant loss of generality, since they can be obtained as limits of finite densities (cf. Section II.2). Similarly, there is no cost in assuming that q is right-continuous and that $x(p) = - V'(p)$ is positive for all p within the relevant range ; $x(p)$ is the consumer's demand (derived from preferences) in the absence of search, or conditional on buying at the price p , and will be referred to as his "conditional demand" from here on.

I.2. Search : There is a single period, during which prices remain fixed and all search takes place. The standard following assumptions are made :

a) consumers' preferences are additively separable into the (indirect) utility derived from the good $V(p)$, and search costs ; b) There is no limit to the number of searches which can be conducted. Thus wealth effects from search costs are treated as negligible. Under these two assumptions, the optimal search strategy for a consumer with search cost σ is independent of whether or not previous offers can be recalled (recall is never used), and characterized by the reservation price r given by : $^{(4)}$

(1) $$V(r) = -\sigma + \int_0^r V(p) \, dF(p) + \int_r^\infty V(r) \, dF(p)$$

when a solution to this equation exists. It expresses indifference between accepting an offer of r and pursuing search with the same decision rule, which yields an expected utility of $V(r)$. Equivalently :

$$\sigma = \int_0^r (V(p) - V(r)) \, dF(p) = \int_0^r - V'(p) \, F(p) \, dp \quad \text{or :}$$

(2) $$\Gamma_F(r) \equiv \int_0^r x(p) \, F(p) \, dp = \sigma$$

by Roy's identity. The function Γ_F is continuous and strictly increasing on $[\underline{p}, +\infty)$, hence it has an inverse, denoted R_F , on $[0, \Gamma_F(+\infty))$. For σ in this interval, (2) has a unique solution $r = R_F(\sigma)$. For $\sigma > \Gamma_F(+\infty)$, (2) has no solution as an equality : the search cost is so high that any offer is preferable to search. For these consumers, define $r = R_F(\sigma) = +\infty$. Thus we have for all $\sigma \geqslant 0$:

(3) $$R_F(\sigma) = \sup\langle r \in R_+ \mid \Gamma_F(r) = \int_0^r x(p) \, F(p) \, dp \leqslant \sigma\rangle$$

The subscript F will be dropped from Γ_F and R_F when no confusion is possible.

Note from (3) that an upward shift in conditional demand x(p) lowers r for any consumer : a more desirable good involves a larger purchase at any given price and therefore justifies more search.

I.3. Demand : Denote by $1/\theta$ the endogenous density of operating firms (per consumer) in the market. The demand curve facing firms will now be derived by aggregating the individual search rules of all consumers, as in Bénabou [1986a], [1987]. Consider first consumers with search cost $\sigma < \Gamma(+\infty)$. By (2), the density v(r) of reservation prices in this population is :

(4) $v(r) = (q \circ \Gamma)(r) \ \Gamma'(r) = (q \circ \Gamma)(r) \ F(r) \ x(r)$

Consumers with reservation price in [r,r+dr] each have a probability F(r) of being successful in any single search. By the law of large numbers, each firm is therefore visited by $\theta v(r)$ dr of them on their first search, $\theta v(r)(1-F(r))$ on their second search, $\theta v(r)(1-F(r))^{k-1}$ on their k-th search, etc., hence a total of $\theta v(r)/F(r)$ dr = $\theta(q \circ \Gamma)(r) \ x(r)$ dr individuals with $R_F(\sigma) \in$ [r,r+dr] . Summing all those who accept an offer of p , plus the non-searching individuals (r = $+\infty$, or $\sigma > \Gamma(+\infty)$) yields the number $s_F(p)$ of consumers who buy from any firm charging a price p :

$$s_F(p) = \theta \langle \int_p^\infty (q \circ \Gamma)(r) \ x(r) \ dr + \int_{\Gamma(+\infty)}^\infty q(\sigma) \ d\sigma \rangle$$

Since each buyer purchases x(p) units, the demand curve facing each firm is:

(5) $D_F(p) = \theta x(p) \langle \int_p^\infty (q \circ \Gamma)(r) \ x(r) \ dr + \int_{\Gamma(+\infty)}^\infty q(\sigma) \ d\sigma \rangle$

Contrary to the formula used by Axell [1977], D(p) is not simply the product of x(p) and the demand curve which obtains when conditional demand is inelastic ; this is because preferences also enter into the determination of

the optimal reservation price (cf. equation (3)). The properties of this

demand function will now be examined. First, define $\bar{r} = R(\bar{\sigma})$ and $\underline{r} = R(\underline{\sigma})$,

respectively the highest and lowest reservation prices (possibly infinite) in

the population. Demand is zero for $p > \bar{r}$, positive and decreasing below. (5)

For $p < \underline{r}$, the bracketed term in (5) is independent of p , so that $D(p)$ is

simply proportional to a monopolist's demand curve $x(p)$, as in Reinganum

[1979]. Finally, D_F is almost everywhere differentiable, and its kinks

correspond to the left discontinuities of q (Stiglitz [1987] focused in

particular on the kink at $\underline{r}$ resulting from $q(0) > 0$). Its elasticity is :

$$(6) \qquad e_F(p) = - \frac{px'(p)}{x(p)} + \frac{(q \circ r)(p)x(p)p}{\int_p^\infty (q \circ r)(r)x(r)dr + \int_{r(+\infty)}^\infty q(\sigma)d\sigma}$$

i.e. the sum of the elasticity resulting from preferences, and that resulting

from search, so that firms' pricing problem may be well behaved even with very

inelastic conditional demand (cf. Section II.1).

I.4. Pricing rules : Since no consumer buys above $\bar{r}$, only those firms with
(6)
$c < \bar{r}$ can operate profitably. Hence :

$$(7) \qquad 1/\theta = G(\bar{r}) = (G \circ R_F)(\bar{\sigma})$$

These firms maximize the continuous function $\pi(p,c) \equiv (p-c)\, D_F(p)$ over $[c,\bar{r}]$,

where $D_F(p)$ is given by (5). For $c < \bar{r}$, any solution $p*$ must clearly be

interior ; thus, if π is twice differentiable at $p*$, the usual first and

second-order conditions must be satisfied :

$$(8). \qquad C_F(p*) = c , \qquad C'_F(p*) \geq 0$$

where $C_F(p) \equiv p(1-1/C_F(p))$ is marginal revenue (with respect to output) as a

function of price. By (6), the right-continuous function C_F is defined for

$p \in (0,\bar{r})$ by :

(9)
$$C_F(p) = p - \left[\frac{(q \circ \Gamma)(p)x(p)}{\int_p^\infty (q \circ \Gamma)(r)x(r)dr + \int_{\Gamma(+\infty)}^\infty q(\sigma)d\sigma} - \frac{x'(p)}{x(p)} \right]^{-1} .$$

When π is not twice differentiable at $p*$, similar conditions apply, involving left and right limits of C_F (cf. proof of Theorem I).

I.5. Equilibrium : For F to be an equilibrium distribution, it must coincide with the distribution which arises when each operating firm with cost c charges a solution $p(c)$ to (9). By (9), $C_F(0^+) < 0$ and $C_F(\bar{r}^-) = \bar{r}$. (7) Suppose for the moment that C_F is increasing and continuous (the usual assumption of a marginal revenue decreasing continuously with the quantity sold). Then (8) defines for all $c < \bar{r}$ a unique optimal price $p_F(c) = C_F^{-1}(c)$. For all p , let $e_x(p) = -px'(p)/x(p)$; by (9), $p(1-1/e_x(p)) < C_F(p) < p$, with equality if and only if, respectively, $p < \underline{r}$ and $p = \bar{r}$. Hence (8) :

(10)
$$c < p_F(c) < p^m(c)$$

(where $p^m(c) < +\infty$ is the firm's monopoly price), with equality if and only if, respectively, $c = \bar{r}$ and $c < C_F(\underline{r})$. Finally, since C_F is increasing, the resulting price distribution is given by ϕ_F below :

Definition : For any $F : R_+ \to R_+$, define Γ_F as in (2) , C_F as in (9) , (9)
$\bar{r}_F = \sup\langle r > 0 \mid \Gamma_F(r) < \bar{\sigma}\rangle$, and $\phi_F : R_+ \to [0,1]$ by :

(11)
$$\begin{cases} \phi_F(p) = \dfrac{G(C_F(p))}{G(\bar{r}_F)} & \text{for } p < \bar{r} \\[2mm] \phi_F(p) = 1 & \text{for } p > \bar{r}_F . \end{cases}$$

More generally, by restricting attention to "well-behaved" equilibria, in which

firms' profit functions do not possess multiple local maxima (i.e. are strictly quasi-concave) one can ensure that C_F is non-decreasing (possibly discontinuous, where q is) over $\langle p|C_F(p) \in \text{support}(G)\rangle$; the above line of reasoning then remains valid and leads to following characterization result.

Theorem I : Let the distribution G of firm costs be continuous. A function $F : R_+ \rightarrow R_+$ is an equilibrium price distribution where almost all firms' profit functions are strictly quasi-concave if and only if :

a) F is a fixed point of the functional mapping $\blacklozenge : F \rightarrow \blacklozenge_F$.

b) F is non-decreasing.

The equilibrium is then generated by each firm with cost below $\bar{r}_F$ charging the (almost always) unique solution $p_F(c)$ to $C_F(p^-) < c < C_F(p^+)$, while the others (if any) do not operate. Consumers search with reservation prices $R_F(\sigma)$.

Proof : cf. appendix.

The standard monopoly pricing problem is characterized by the conjunction of the familiar first-order condition, $p = 1/(1-1/e_x(p))$, which can be viewed as a one-dimensional fixed-point problem, and of a monotonicity condition which ensures that the profit function is locally concave. Theorem 1 in fact generalizes these fixed point and monotonicity conditions to the infinite dimensional space of functions from R_+ into itself. Interestingly, the global monotonicity condition (b) on F is equivalent (given (a)) to the local monotonicity of marginal revenue $C_F(p)$ at (almost) every firm's solution to its first order condition. If G were discontinuous, the possibility of a positive mass of firms' being indifferent between several prices would lead to a similar characterization of equilibrium distributions as fixed points of a

functional correspondence.

While Theorem I fully characterizes (well behaved) equilibria, and provides a basis to search for them numerically, it does not address the question of existence ; as can be seen from the combination of (2), (9) and (11) it is in general a nearly insoluble one. In particular, with an endogenous demand function D_F , marginal revenue C_F need not be increasing (in price) for all F (cf. (9)), so that ϕ does not generally map increasing functions into themselves.$^{(10)}$ The conditions which guarantee existence and/or uniqueness in the standard monopoly problem can therefore not be assumed here, but have to be derived (as equilibrium properties) from assumptions about the two cost distributions and consumer preferences ; this will be done in Sections II and III.

<u>I.6. Discussion</u> : Figure 1 describes the essential features of a typical equilibrium. Firms with cost above $\bar{r}_F$ cannot retain any consumer at a profitable price and therefore stay out of the market. Firms with cost below $c_o \equiv C_F(\bar{r}_{-f})$ can charge their monopoly price $(p_F(c) = p^m(c))$ without inducing any consumer to leave, while all others are constrained by search to price below their monopoly level $(c < p_F(c) < p^m(c))$. Conversely, it is the latters' prices (for $c > c_1 = r_{-f} > c_o$) which trigger active consumer search.

A discontinuity in q at some point $\underline{\sigma}$ (in the case of Figure 1, at $\underline{\sigma}$) causes a similar discontinuity in C_F at $R_F(\sigma)$ (here at r_{-f}) , leading to a bunching of prices by a whole segment $[c_o,c_1] = [C_F(\bar{r}_F) , C_F(\bar{r}_F)]$ of firms; $^{(11)}$ equivalently, each of these firms' price is insensitive to cost variations in the $[c_o,c_1]$ range (cf. Stiglitz [1987] for a discussion of this "rigidity"). The combination of this bunching with the monopolistic behaviour of firms with
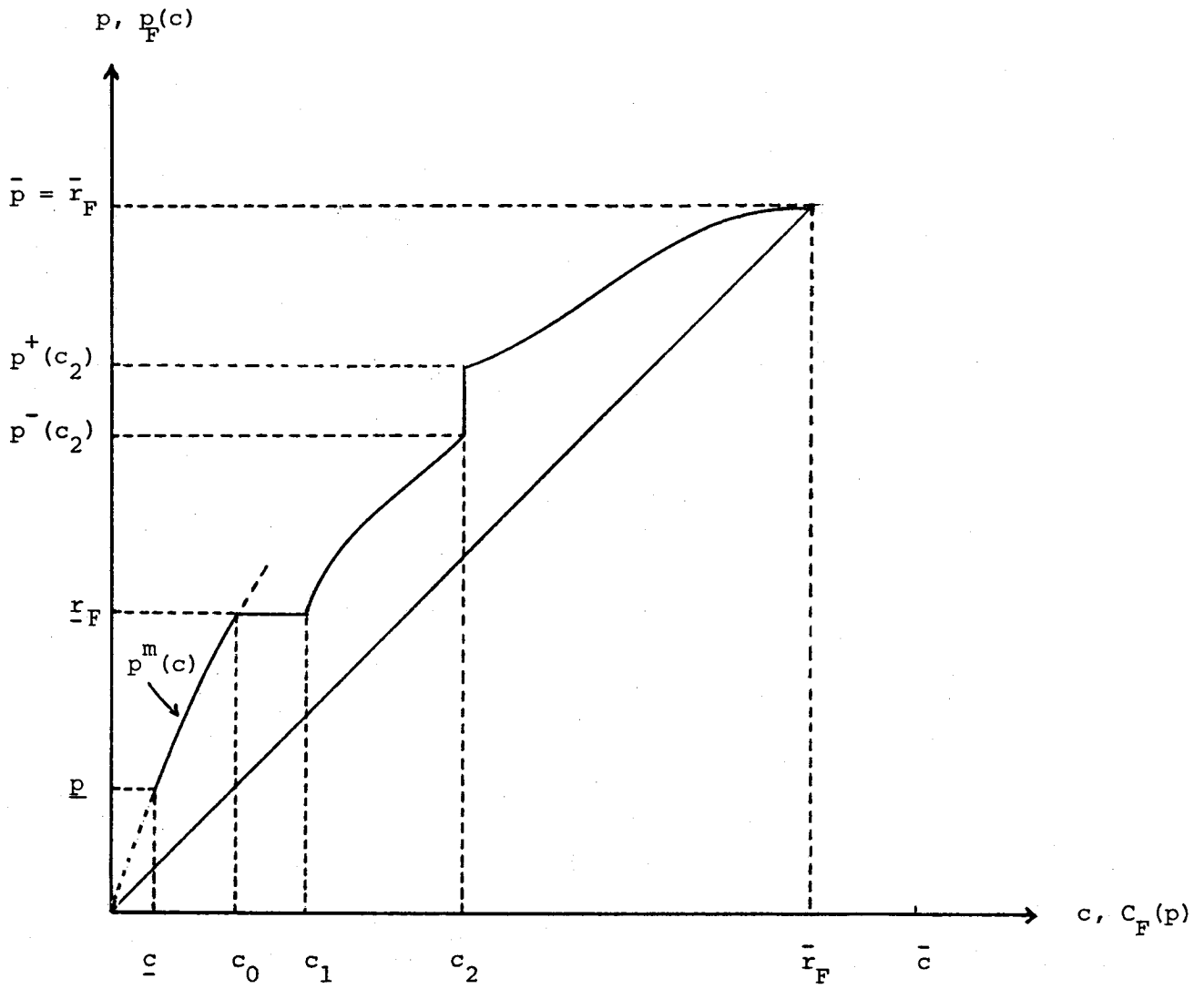
Figure 1: The marginal revenue function $C_F(p)$ and the optimal pricing rule $p_F(c)$.

cost $c < c_0 = C_F(r_F)$ is the central feature of the Reinganum [1979] model.

Conversely, a flat section in $C_F(p)$ corresponds to a multivalued optimal price $[p^-(c_2), p^+(c_2)]$ at a point like c_2 on Figure 1. If the cost distribution happens to have an atom at a point like c_2, then a positive mass of firms are indifferent between all prices in $[p_2^-(c), p_2^+(c)]$; this indeterminacy is the central feature of models with identical firms such as those of Axell [1977], von zur Muehlen [1980], Rob [1985] or Stiglitz [1987]. For instance, to obtain the differential equation which the search cost density q must satisfy in order for an equilibrium with continuous dispersion to exist with identical firms ($\underline{c} = \overline{c}$) , as in the last two of these references, it suffices to equate $(p-c) D_F(p)$ — as defined by (5) — to a constant, and require that F be non-decreasing from zero to one.

Discontinuities or flat sections in C_F are non-generic, limiting cases ; when the distributions q and G are slightly perturbed, the equilibrium reverts to a single-valued, continuous and increasing function associating to each cost type c the corresponding optimal price $p_F(c)$. In the sequel, the subscript F will be dropped from r_F , $\overline{r}_F$, p_F, etc.. when no confusion is possible.

I.7. Job search : The above methodology, and all the paper's results, are directly applicable to a job search problem. Let workers derive utility $V(w)$ ($V'(w) \equiv x(w) > 0$, $V''(w) < 0$) from a wage w, firms have constant marginal productivities of labor distributed according to $G(\lambda)$ and offer wages distributed according to $F(w)$. It then essentially suffices to replace in the preceding formulas $V(p)$ by $V(w)$ and, since the reservation wage is now a minimum acceptable offer, ($G(c)$, $F(p)$, $\phi_F(w)$) by ($1-G(\lambda)$, $1-F(w)$, $1-\phi_F(w)$),

$\Gamma_F(+\infty)$ by $\Gamma_F(0)$, and to transpose the bounds of integration $\int_r^\infty$ into $\int_0^r$ (and

vice versa) in any integral over $dF(p)$. Thus for instance, reservation prices

are determined by :

$$(3') \qquad R_F(\sigma) = \inf\{r \in R_+ \mid \Gamma_F(r) = \int_r^\infty x(w)(1-F(w))\ dw \leqslant \sigma\}$$

(Γ_F is now decreasing), and firms face the labor supply curve :

$$(5') \qquad S_F(w) = \theta x(w)\{\int_0^w (q\circ\Gamma)(r)\ x(r)\ dr + \int_{\Gamma(0)}^\infty q(\sigma)\ d\sigma\}$$

with $1/\theta = (1-G)(R_F(\bar\sigma))$, and so on.

II - UNIFORMLY DISTRIBUTED SEARCH COSTS

In this section, we assume :

$$(12) \qquad\qquad (\forall\ \sigma \in [\underline\sigma,\bar\sigma))\quad (q(\sigma) = 1/(\bar\sigma-\underline\sigma))$$

II.1. Inelastic conditional demand :

To show how the elasticity from search alone can generate an equilibrium

with price dispersion, assume that $x(p) = 1$ for all p $^{(5)}$ and $\underline\sigma = 0$ $^{(12)}$. This

case, which is treated here as a natural application of Theorem I, has in fact

been examined previously by Mc Minn [1980]. While the characterization of the

equilibrium derived here (independently) is similar to his, the two sets of

results turn out to be complementary : existence, stability and (conditions

for) uniqueness of equilibrium are formally established here, while he focuses

on the comparative statics of price dispersion with respect to search costs,

assuming existence (in the free entry case). Under the above assumptions,

$\Gamma(+\infty) = \int_0^\infty F(p)\ dp = +\infty$, since $\lim\limits_{p\to\infty} F(p)=1$, and the demand curve (5) becomes

(for $p \leqslant r$) :

(13)
$$D(p) = \frac{\theta}{\sigma} \int_{max(p,\underline{r})}^{\bar{r}} dr = \frac{\theta}{\sigma} (\bar{r} - max(p,\underline{r}))$$

For <u>any</u> distribution of prices in the market, firms therefore face a

<u>piecewise linear</u> demand curve, leading to an optimal price, for $c < \bar{r}$, of :

(14)
$$p(c) = max \langle \frac{1}{2}(c+\bar{r}), \underline{r} \rangle$$

As was shown in section I.6, the bunching which may occur at $\underline{r}$ is an artefact

due to the discontinuity of the uniform distribution at its lower bound $\underline{\sigma} = 0$;

it would be smoothed out if q were continuous. To avoid such a degeneracy,

attention will from now on be focused on equilibria (if any) which do not

involve bunching $^{(13)}$, i.e. such that the firms with the lowest costs find it

optimal to charge a price greater than $\underline{r}$: $c_1 \equiv C_F(\underline{r}) < c$, cf. Figure 1. Then :

(14')
$$p(c) = \frac{1}{2}(c+\bar{r}) \text{ , for all } c < \bar{r} \text{ ,}$$

which does not involve any bunching, and indeed coincides with the optimal rule

(14) for all firms with $c \in [\underline{c},\bar{r}]$, because $\underline{\sigma} = 0$ implies that $\underline{r} = \underline{p}$, hence

$C(\underline{r}) = 2\underline{p}-\bar{r} = 2p(\underline{c}) - \bar{r} = \underline{c}$. Denote by $E_F(.)$ and $E_G(.)$ expectations with

respect to the distributions F and G respectively. Since $\bar{p} < \bar{r}$, (2) yields :

$$\bar{\sigma} = \int_0^{\bar{r}} F(p) \, dp = \int_0^{\bar{r}}(\bar{r}-p) \, dF(p) = \bar{r} - E_F(p) \text{ .}$$

Each firm therefore charges :

(14")
$$p(c) = \frac{1}{2} (c + E_F(p) + \bar{\sigma}) = \frac{1}{2}(c + E_G(c \mid c < \bar{r})) + \bar{\sigma}$$

which clearly expresses the <u>monopolistically competitive</u> nature of the

equilibrium. Finally, the functional fixed point requirement (11) on F becomes:

(15) $F(p) = \min \left\{ \dfrac{G(2p-\bar{r})}{G(\bar{r})}, 1 \right\}$, with $\bar{r} = \Gamma^{-1}(\bar{\sigma})$

which reduces to an implicit equation in $\bar{r}$:

$$\bar{\sigma} = \int_0^{\bar{r}} \frac{G(2p-\bar{r})}{G(\bar{r})}\, dp = \frac{1}{2} \int_0^{\bar{r}} \frac{G(c)}{G(\bar{r})}\, dc \qquad (14')$$

since G has support in R_+ ; equivalently, defining $H(r) = \int_0^r G(c)\, dc$:

(16) $H(\bar{r})/G(\bar{r}) \equiv \int_0^{\bar{r}} \dfrac{G(c)}{G(\bar{r})}\, dc = 2\bar{\sigma}$

II.1.1. Existence and stability : An equilibrium (with no bunching) will now be

shown to be stable, under a reasonable adjustment process, if and only if the

curve H/G cuts the horizontal $2\bar{\sigma}$ from below at that point. Assume that this

condition is satisfied, and suppose that $\bar{r}_0$ is slightly above the equilibrium

value $\bar{r}$. Then $(H/G)(\bar{r}_0) > 2\bar{\sigma}$, or by (14")-(15), $\Gamma_F 0(\bar{r}_0) > \bar{\sigma}$, where F^0 is the

distribution associated to $\bar{r}_0$ by (15). Consumers with search cost (close to) $\bar{\sigma}$

then adjust their reservation price to $\bar{r}_1 < \bar{r}_0$. Faced with a reduction in

demand, expensive firms cut their prices (or even close down) resulting in a

lower price distribution F^1 . Subsequently all reservation prices fall, and $\bar{r}_1$

decreases to $\bar{r}_2$, forcing more firms to cut prices, and so on until the

decreasing sequence $(\bar{r}_n)$ has converged to $\bar{r}$. If (H/G) is decreasing in a right-

neighborhood of $\bar{r}$, on the contrary, then $\Gamma_F 0(\bar{r}_0) < \bar{\sigma}$ and the adjustment

process involves less search, higher prices, more firms, and a sequence $(\bar{r}_n)$

which increases away from $\bar{r}$. A similar reasoning holds for $\bar{r}_0 < \bar{r}$; therefore

$\bar{r}$ is stable if and only if (H/G) is increasing in its neighborhood.

<u>Theorem II.1</u> : Let consumers have perfectly inelastic conditional demand,

and search costs uniformly distributed on $[0,\bar{\sigma}]$. For any distribution of firm

costs $G(c)$, there exists a stable no-bunching equilibrium. Any such equilibrium

corresponds uniquely to a maximum consumer reservation price $\bar{r}$ solution to :

$(H/G)(\bar{r}) = 2\bar{\sigma}$ (with the intersection occuring from below), and results from

each firm with cost $c \leqslant \bar{r}$ charging :

$$(17) \qquad p(c) = \frac{1}{2} (c + E_G(c|c \leqslant \bar{r})) + \bar{\sigma}$$

while no firm with cost $c > \bar{r}$ operates. The resulting price distribution is :

$$(18) \qquad F(p) = \min \left\langle \frac{G(2p-r)}{G(\bar{r})} , 1 \right\rangle$$

and consumers search with reservation prices $R(\sigma)$ given by : $\sigma = \int_0^{R(\sigma)} F(p)dp$.

<u>Proof</u> : cf. appendix.

No matter how complex the distribution of production costs (including

atoms, "holes" etc .) and the resulting distribution of prices in the market,

the equilibrium pricing rule is extremely simple : each firm charges the

<u>half-sum of its own cost and the average cost</u> of all firms in the market,

plus a constant measuring <u>search costs</u>. When $\bar{\sigma}$ decreases, both the number

of profitable firms and the prices they charge decrease, with only the most

efficient firms remaining and behaving competitively $(c = \underline{c} = p(\underline{c}))$ in the

limit, as $\bar{\sigma}$ goes to zero. Mc Minn [1980] shows that price dispersion is a non-

decreasing function of $\bar{\sigma}$, provided the cost distribution G has a decreasing

density ; this condition is also one of the cases covered by Theorem II.1.2

below, which guarantees that the equilibrium is unique.

II.1.2. Uniqueness : The solution to (16) is unique, for all values of

$\bar{\sigma} > 0$, if and only if H/G is increasing, or G/H decreasing on $[\underline{c},+\infty)$; note

that this requires the c.d.f. G to be continuous on this interval.

Theorem II.2 : The no-bunching equilibrium is unique for all $\bar{\sigma}$ if and only if

the integral of G : $H(r) = \int_0^r G(c)$ dc has a decreasing hazard rate H'/H = G/H ,

on I $\equiv [\underline{c},+\infty)$. A sufficient condition is that the distribution G itself have a

continuous density G' and a decreasing hazard rate G'/G .

Proof : cf. appendix.

Any distribution with non-increasing density, and many usual

distributions such as (on any support) the uniform, normal, exponential,

Pareto, logistic, or extreme value, have a decreasing hazard rate, and

therefore generate a unique, stable, equilibrium without bunching.

II.2. Isoelastic conditional demand

Search alone can thus generate well-behaved demand curves and price dispersion,

even when conditional demand is very inelastic. The dual case of (constant)

$e_x > 1$ highlights the interplay of search and preferences in the determination

of the equilibrium, and provides an exact generalization of Reinganum's [1979]

model to the case where consumers are heterogeneous. In addition, the number of

firms in the market is endogenized. In particular, it confirms her conjecture

that this would lead to increased price dispersion accompanied by active

consumer search.

Let therefore $V(p) = p^{1-\alpha}/(\alpha-1)$ with $\alpha > 1$, and σ be distributed unifor-

ly on $[\underline{\sigma},\bar{\sigma}]$, $0 < \underline{\sigma} < \bar{\sigma}$. The equilibrium demand curve (5) becomes, for $p < \bar{r}$:

$$D(p) = \frac{\theta p^{-\alpha}}{\bar\sigma - \underline\sigma}\left(\int_{\max(p,\underline r)}^{\bar r} r^{-\alpha}\,dr + \max(\bar\sigma - \Gamma(+\infty),\ 0)\right),\ \text{or}:$$

$$(19)\qquad \begin{cases} D(p) = Ap^{-\alpha}(p^{1-\alpha} - \bar r^{1-\alpha} + \beta) & \text{for } p \in [\underline r, \bar r] \\[2ex] D(p) = Ap^{-\alpha}(\underline r^{1-\alpha} - \bar r^{1-\alpha} + \beta) & \text{for } p < \underline r \end{cases}$$

where $1/A = (\alpha-1)(\bar\sigma - \underline\sigma)/\theta$ and $\beta = (\alpha-1)\max(\bar\sigma - \Gamma(+\infty),\ 0)$. Note that $\beta > 0$ requires $\bar r = +\infty$ while, by (3), $\beta = 0$ implies $\Gamma(\bar r) = \bar\sigma$; thus one can also write : $\beta = (\alpha-1)\max(\bar\sigma - \Gamma(\bar r),\ 0)$. Equation (19) shows how search both increases the elasticity of the original demand curve $x(p) = p^{-\alpha}$, and shifts it vertically — down if $\bar\sigma$ is not so large as to result in an infinite reservation price $\bar r$, up otherwise. The general formula (9) yields marginal revenue :

$$(20)\qquad C(p) = \begin{cases} p(1 - \dfrac{1}{\alpha}) & \text{for } p < \underline r \\[3ex] C^*(p|\bar r,\beta) \equiv p(1 - \dfrac{1}{\alpha})\left(1 + \dfrac{1}{2\alpha-1-\alpha(\bar r^{1-\alpha}-\beta)p^{\alpha-1}}\right) & \text{for } p \in [\underline r, \bar r] \end{cases}$$

Lemma 1 : For all $(\alpha,\beta,\bar r,c)$ with, $\alpha > 1$, $\beta \geqslant 0$, $0 < \bar r < +\infty$ and $\beta/\bar r = 0$, the function $p \to C^*(p|\bar r,\beta)$ is continuous and increasing from $[0,\bar r]$ into itself; its inverse will be denoted as $c \to p^*(c|\bar r,\beta)$. The function $p \to C(p)$ shares the same properties, except for a discontinuity at $\underline r$.

Proof : cf. Appendix.

From Theorem I, each firm with cost $c < \bar r$ must charge the unique solution to the first-order-condition $C(p^-) < c < C(p^+)$ (the second order condition in (8) is always satisfied because C is increasing). The optimal pricing rule is therefore :

(21a) - Firms with cost $c < \underline{r}(1-1/\alpha)$ price as monopolists : $p(c) = c/(1-1/\alpha)$;

(21b) - Firms with cost $c \in [\underline{r}(1-1/\alpha)$, $C^*(\underline{r}|\bar{r},\beta)]$ bunch at $p(c) = \underline{r}$;

(21c) - Firms with cost $c \in (C^*(\underline{r}|\bar{r},\beta)$, $\bar{r}]$ price as search-constrained

          monopolistic competitors : $p(c) = p*(c|\bar{r},\beta)$;

(21d) - Firms with cost $c > \bar{r}$ (if any) do not operate.

The resulting price distribution is :

$$(22) \quad \begin{cases} F(p) = \dfrac{G(p(1-1/\alpha))}{G(\bar{r})} & \text{for } p < \underline{r} \\[3mm] F(p) = \dfrac{G(C^*(p|\bar{r},\beta))}{G(\bar{r})} & \text{for } p \in [\underline{r},\bar{r}] \\[3mm] F(p) = 1 & \text{for } p > \bar{r} , \end{cases}$$

from which consumers search with reservation prices $R(\sigma)$ given by :

$$(23) \qquad R(\sigma) = \sup\langle r \in R_+ \mid \int_0^r p^{-\alpha} F(p) \, dp \leqslant \sigma \rangle \ .$$

Finally, the functional fixed point requirement closing the model takes the

form $R(\sigma) = \underline{r}$, $R(\bar{\sigma}) = \bar{\sigma}$, where $R(.)$ is given by (23). Replacing F by its

expression (22) and remembering that β is itself a function of F , hence of $\underline{r}$

and $\bar{r}$, leads to the characterisation of an equilibrium as a solution $(\underline{r},\bar{r},\beta)$

to the following fixed-point system :

$$(24a) \quad \underline{r} = \sup\langle r > 0 \mid \int_0^r p^{-\alpha} \cdot \frac{G((1-1/\alpha)p)}{G(\bar{r})} \, dp \leqslant \underline{\sigma} \rangle$$

$$(24b) \quad \bar{r} = \sup\langle r > 0 \mid \int_{\underline{r}}^r p^{-\alpha} \cdot \frac{G(C^*(p|\bar{r},\beta))}{G(\bar{r})} \, dp \leqslant \bar{\sigma} - \underline{\sigma} \rangle$$

$$(24c) \quad \beta = (\alpha-1) \max\left\{ 0 \ , \ \bar{\sigma} - \int_0^{\underline{r}} p^{-\alpha} \cdot \frac{G((1-1/\alpha)p)}{G(\bar{r})} \, dp - \int_{\underline{r}}^{\bar{r}} p^{-\alpha} \cdot \frac{G(C^*(p|r,\beta))}{G(\bar{r})} \, dp \right\}$$

Theorem II.3 : Let consumers have isoelastic conditional demand $x(p) = p^{-\alpha}$,

$\alpha > 1$, and search costs uniformly distributed on $[\sigma, \bar{\sigma}]$, with $0 < \underline{\sigma} < \bar{\sigma}$.

For any distribution of firm costs $G(c)$ with $\underline{c} > 0$, (15) there exists a stable

equilibrium. Any such equilibrium corresponds uniquely to a triplet $(\underline{r}, \bar{r}, \beta)$

solution to the system (24a)-(24c), where $\underline{r}$ and $\bar{r}$ represent respectively the

lowest and highest reservation prices in the population, and $\beta/(\alpha-1)$ the mass

of consumers with infinite reservation price ($\beta = 0$ if $\bar{r} < +\infty$). The

corresponding equilibrium distribution of prices is then (22) and results from

the optimal pricing rule (21a)-(21d) and the optimal search rule (23).

Proof : cf. appendix.

Such an equilibrium is illustrated in Figure 2a. The more efficient

firms ($\underline{c} < c < \underline{r}(1-1/\alpha)$) can charge their monopoly price, because it is too

low for any consumer to reject. Less efficient firms ($\underline{r}(1-1/\alpha) < c < \bar{r}$) are

increasingly constrained by search, which forces their optimal price further

and further down below their monopoly price ; at $c = \bar{r}$, the markup has totally

disappeared, and less efficient firms cannot operate profitably. At the point

$p = \underline{r}$ where search becomes binding, i.e. where consumers start to leave, the

demand curve has a (concave) kink, leading to the same price $\underline{r}$ for a range of

costs $[\underline{r}(1-1/\alpha), C^*(\underline{r})]$ and generating an atom in the distribution of

prices. (16) As was shown in Section I.6, this results from the discontinuity

of the uniform density q at $\underline{\sigma}$, and would be smoothed out with a continuous

one. The pricing behaviour of firms in $[\underline{c}, C^*(\underline{r}|\bar{r}, \beta)]$ corresponds exactly to

the case treated by Reinganum [1979], while the others are effectively

competing through active consumer search. Indeed, the Reinganum model and

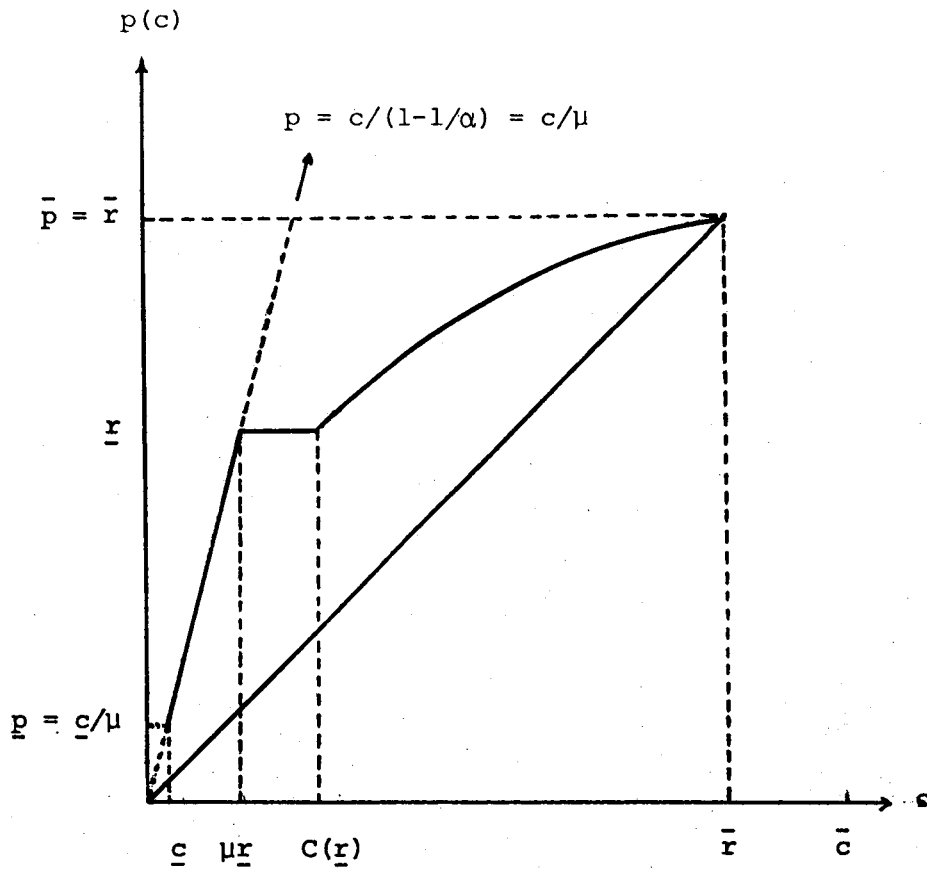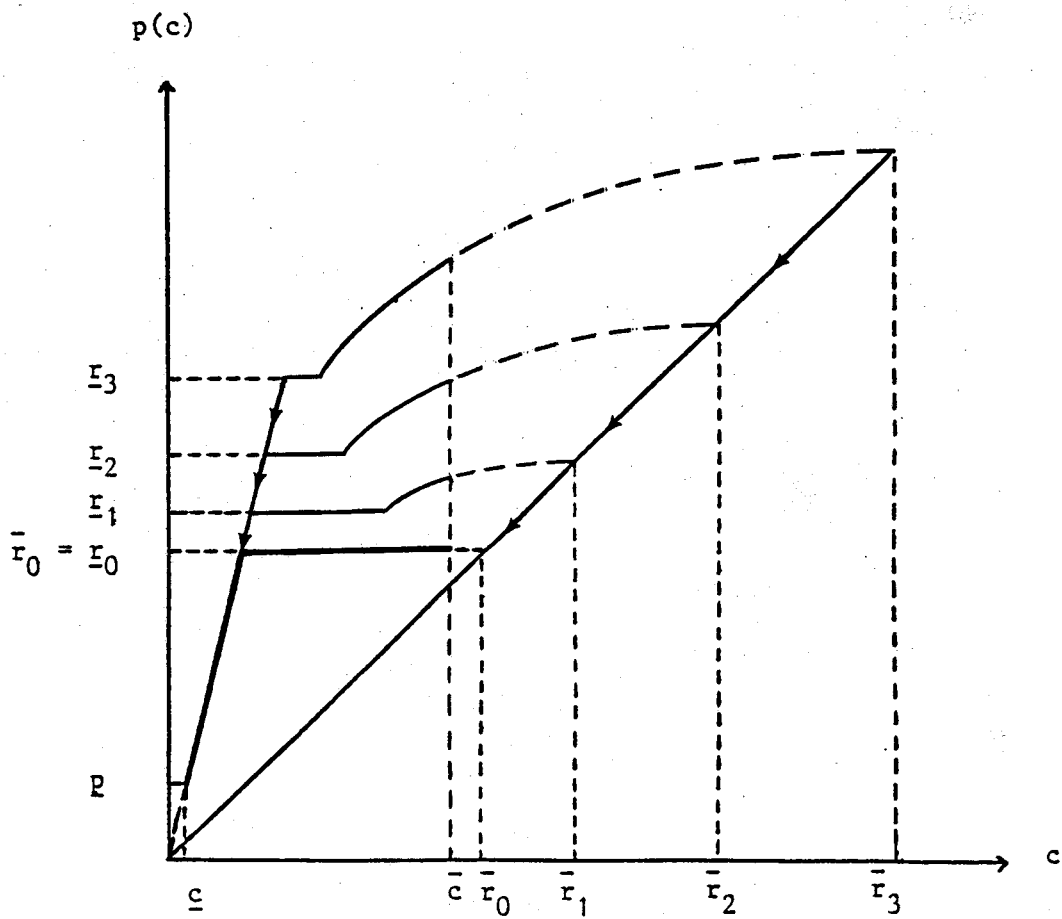results — fixed number of firms, identical consumers, no search in equilibrium

Figure 2a

Figure 2b

- are easily obtained as a limiting case, when $\bar{\sigma} - \underline{\sigma}$ (hence also $\bar{r} - \underline{r}$)

decreases to zero in equations (24a)-(24c) and (21a)-(21d) (with the additional

constraint $G(\bar{r}) = 1$: all firms operate). Figure 2b illustrates this convergence

for the case where $\bar{\sigma}$ decreases to a fixed $\underline{\sigma}$.

Each firm in the market creates an externality for others (and consu-

mers) by altering the distribution of prices, so that multiple equilibria may

arise. Unlike that of Section II.1, the equilibrium fixed point problem is here

too complicated to derive general conditions guaranteeing uniqueness. One will

focus instead on whether there exists an equilibrium in which all firms operate

$(G(\bar{r}) = 1)$ as in Reinganum [1979]. As intuition suggests, such will be the case

if search costs are "high enough" with respect to production costs.

<u>Theorem II.4</u> : There exists an equilibrium with all firms in $[\underline{c},\bar{c}]$ operating,

if :   (i) $\displaystyle\int_0^{\frac{\bar{c}}{\ }} p^{-\alpha} G((1-1/\alpha)p)\ dp \leqslant \underline{\sigma}$, or if : (ii) $\displaystyle\int_{\underline{c}}^{\bar{c}} p^{-\alpha} G(p)\ dp \leqslant \bar{\sigma}$

<u>Proof</u> : cf. appendix.

Condition (i) in fact ensures that an equilibrium exists with $\bar{r} \geqslant \underline{r} \geqslant \bar{c}$.

It is implied by Reinganum's [1979] assumption that $\bar{c}(1-1/\alpha) < \underline{c}$; in her case

$\bar{r} = \underline{r} > \bar{c}$. If (i) does not hold but (ii) does, there exists an equilibrium with

$+\infty \geqslant \bar{r} \geqslant \bar{c} > \underline{r} > \underline{c}$.

III - CONSTRUCTING EQUILIBRIA WITH SIMPLE PRICING RULES

Under no distribution of search costs other than the uniform is the

fixed-point problem (11) solvable for an arbitrary distribution G. But, given

some G (and x), it is possible to find search cost distributions leading to an

equilibrium with a simple pricing rule, such as :

(25) $\qquad p(c) = \dfrac{c+b}{\mu}$; $\mu > 0$

If firms face an equilibrium demand curve of the type :

(26) $\qquad D(p) = k(\max(p,\underline{r}) + b\alpha)^{-\alpha}$

with $\alpha \equiv 1/(1-\mu) > 1$ and $b \geqslant 0$, then (25) indeed generates an equilibrium

where all firms participate ($\theta=1$), provided $\sigma = 0$, so that $p(c) \geqslant p(\underline{c}) = \underline{p} = \underline{r}$

for all $c \geqslant \underline{c}$. Assume inelastic conditional demand for simplicity. Then

$\Gamma(+\infty) = +\infty$ and by (5), demand is of the required form (26) if :

(27) $\qquad (q\circ\Gamma)(r) = k'(r + b\alpha)^{-(1+\alpha)}$ for all $r \geqslant \underline{r}$

From (2) and (26) :

$$\Gamma(r) = \int_0^r F(p)\ dp = \int_0^r (G\circ C)(p)\ dp = \int_0^r G(\mu p - b)\ dp$$

$$= \mu^{-1}\int_{-b}^{\mu r-b} G(c)\ dc = \mu^{-1}\int_0^{\mu r-b} G(c)\ dc = \mu^{-1} H(\mu r-b)$$

where, as previously, H is the integral of G , which is continuous and

increasing from $[\underline{c},+\infty)$ into $[0,+\infty)$. Therefore :

(28) $\qquad \Gamma^{-1}(\sigma) = \mu^{-1}(b + H^{-1}(\mu\sigma))$

maps $[0,+\infty)$ into $[(\underline{c}+b)/\mu,+\infty) = [\underline{r},+\infty)$, and the solution to (27) is :

(29) $\qquad q(\sigma) = K(\alpha b + H^{-1}(\mu\sigma))^{-(1+\alpha)}$, for all $\sigma \geqslant 0$.

Such a function is continuous, constant on $[0,\underline{c}/\mu]$ and then decreasing. Since

$H^{-1}(\mu\sigma) \geqslant \underline{c} + \mu\sigma$, it is summable if and only if $\alpha b+\underline{c} > 0$. Hence :

Theorem III : Let consumers have inelastic conditional demand. For any distribution of firm costs $G(c)$, if the density of consumer search costs on $[0,+\infty)$ is:

$$(30) \qquad q(\sigma) = [\alpha b + H^{-1}(\mu\sigma)]^{-(1+\alpha)} / \int_0^\infty [\alpha b + H^{-1}(\mu y)]^{-(1+\alpha)} dy$$

where $\alpha > 1$, $\mu = 1-1/\alpha$, $\min(b,c) > 0$ and H is the integral of G, then there exists an equilibrium where all firms operate and charge $p(c) = (c+b)/\mu$. The resulting price distribution is $F(p) = G(\mu p - b)$ on $[(c+b)/\mu , +\infty)$, and consumers search with reservation prices $R(\sigma)$ given by : $\sigma = \int_0^{R(\sigma)} F(p) \, dp$.

The usefulness of this result in constructing equilibria with search and continuous price dispersion is best illustrated by a couple of examples.

Example 1 : Let $G(c) = 1-1/c^2$ on $[1,+\infty)$ and, for all $\sigma > 0$:

$$q(\sigma) = K\left[\sigma + 2\mu + (\sigma^2 + 4\mu\sigma)^{1/2}\right]^{-(1+\alpha)}$$

with $\mu = 1-1/\alpha$ in $(0,1)$ and K normalized so that q sums to 1 . In equilibrium, $D(p) = k(\max(p,1/\mu))^{-\alpha}$, $p(c) = c/\mu$, $F(p) = 1-1/\mu^2 p^2$ on $[1/\mu, +\infty)$.

Example 2 : Let $G(c) = th(c/2) = (e^c-1)/(e^c+1)$ on $[0,+\infty)$ and, for all $\sigma > 0$:

$$q(\sigma) = K\left\{\alpha b + Ln\left[2e^{\mu\sigma} - 1 + 2e^{\mu\sigma/2}(e^{\mu\sigma}-1)^{1/2}\right]\right\}^{-(1+\alpha)}$$

with the same notation as above, and $b > 0$. In equilibrium, $D(p) = k(p + \alpha b)^{-\alpha}$, $p(c) = (c+b)/\mu$, and $F(p) = th((\mu p-b)/2) = (e^{\mu p-b}-1)/(e^{\mu p-b}+1)$ on $[b/\mu , +\infty)$.

IV - REPEAT PURCHASE EQUILIBRIUM

IV.1. Permanent population of consumers

Many goods, especially non-durables, are consumed and purchased repeatedly. Buyers do not search each time for a suitable seller, but rather invest in an initial search to find a permanent or long-term supplier. Similar

behaviour prevails in the labor market, where job search is followed by a long

term implicit or explicit contractual relationship ; recall, at this point,

that the methodology and results presented here in terms of consumer search for

a commodity could equivalently be cast in terms of job search by workers (cf.

Section I.7). Implicit in the above behaviour is some inference of future

prices on the basis of past ones : if yesterday's price was acceptable

(respectively, too high), it can be expected that today's and tomorrow's will

also be acceptable (respectively, too high), and consumers can (resp. can not),

economize on search costs by coming back to the same seller in the following

period. The type of search costs considered here correspond to resources

required to find out about a seller (where he is located, whether the good he

sells is indeed the desired one and of appropriate quality, and what price he

charges), i.e. "inspection costs" rather than expenses incurred every time a

visit or a purchase is made (trips or ordering costs).

The aim of this section is to formalize the inference and decision

processes of agents in a repeat purchase market in order to account for such

long-term relationships and examine their consequences on equilibrium prices.

Mc Millan and Morgan [1984] examine such a market, where identical firms

charge constant, different prices, and consumers purchase repeatedly and

loyally. However, these results arise in their model through pricing behaviour

which is only partly optimal and realistic : in some all-important first

period, identical firms somehow distribute themselves appropriately among

prices which yield different levels of profits, hence are suboptimal ;

consumers then allocate themselves among sellers, and repeat purchases in

effect lock firms forever into their initial heterogeneity of clientele and

price. This lock-in occurs because : a) a firm cannot profitably lower its

price after the first period, as competitors' clients would never learn about it ; b) the allocation of consumers resulting from the initial distribution of prices is such that a price increase would cause a loss of consumers outweighing the gain realized on those who stayed. Thus the first period, with its arbitrary prices, looms excessively large, especially since there generally exists an infinity of such equilibria. Moreover, none of those involving price dispersion is likely to be stable to small perturbations (firms have no reason to return to the equilibrium prices), nor robust to the entry of new consumers in the market, which gives firms a chance to correct their initial prices. The class of models presented here, on the other hand, can be extended to repeat purchases in a game theoretic framework, generating more realistic and fully optimal equilibria.

IV.1.1. Description of the game :

Strategies and Payoffs. Time is discrete and firms set prices in every period (no committment is possible). They will be indexed by $f \in [0,1]$, with associated cost c^f and price p_t^f in period t . Buyers live forever, derive in each period a utility $V(p)$ from purchasing the good at a price p , and share with firms a discount factor of $\delta < 1$. Since Q is assumed to be atomless, they will be indexed by their search cost $\sigma \in \Sigma \equiv [\underline{\sigma}, \overline{\sigma}]$. Buyers initially search randomly (search is instantaneous, i.e. there is no limit on the number of searches which can be carried out within one period). In every following period, they decide whether to return to any previously encountered seller, or to search again randomly. Given any price quotation generated by this decision, they decide whether to buy or to continue searching, among previous suppliers

or randomly. At the beginning of the first period they receive one price

quotation for free ; all others, which must be obtained through search, entail

a cost of σ . In every subsequent period, consumers may costlessly return to

any previously visited store, or search again, with the same unit cost σ as

before.

The cost or absence of cost of returning to a previously visited seller

turns out to be important and to have complex consequences. If there are

positive return costs, buyers' decision is different when they are faced with a

given price in their search and when they consider whether or not to go back to

a seller believed to charge that same price. For instance, a consumer who is

indifferent between accepting some reservation price R(σ) and searching, will

not return to a firm believed to be charging R(σ) if this entails a positive

cost. As shown by Bagwell [1985] in a monopolistic context, firms' incentives

to fool customers into coming back and then exploit them may then result in

complex, non-stationary price strategies. Incorporating such behavior into the

general model presented here would be too difficult. It is assumed instead, as

mentioned above, that returning to any previously visited seller is costless

(as in Mc Millan and Morgan [1984]). This, however, raises a more subtle issue

which has to do with how many firms a consumer returns to, off the equilibrium

path ; it will be discussed at the end of section IV.1.2. [17]

Information structure. When setting its price, a firm has not yet observed how

many previous clients have decided to come back and check its price again, nor

does it know the prices charged by its competitors. Similarly —these decisions

are simultaneous— consumers must decide whether or not to return to a

previously visited seller before having observed its current price. They also

do not know its cost type, and must form beliefs about it as well. Thus they have both incomplete (about firms' types) and imperfect (about unobserved past and/or present prices) information.

IV.1.2. Equilibrium : The equilibrium concept used is that of Bayesian perfect equilibrium (BPE ; cf. Harsanyi [1968], Fudenberg and Tirole [1987], [1988]) : at any information set a player's strategy must be optimal given his beliefs, and these beliefs must be obtained by updating priors, using Bayes' rule, the history of observed moves, and the strategies of all other players' types. Refinements such as sequential equilibrium (Kreps and Wilson [1982]) are not available for games with continua of types and actions, but it will still be possible to check that off-the-equilibrium-path beliefs are "reasonable".

To simplify the problem, note first that when looking for equilibrium paths of Bayesian perfect equilibria, attention can be restricted to the specification of strategies and beliefs for histories of the game which do not involve any simultaneous deviations by a positive mass of players (cf. Gul, Sonnenschein and Wilson [1986]).[18] Secondly, in their search, return and buying decisions, consumers now compare not only current prices but expected present values of price paths. To keep the analysis tractable, attention will be restricted to equilibria with stationary price paths (as in Mc Millan and Morgan [1984]) ; this condition also avoids a Folk-theorem type multiplicity of equilibria. Note that only the equilibrium outcome, and not allowable price strategies, is being restricted here. The (stationary) distribution of equilibrium prices will again be denoted by F(p) . Consumers' initial decision problem thus reduces to a once and for all search between the present values of constant price paths, for which the appropriate reservation

price is (by (3)) :

(31) $\qquad R_F^\delta(\sigma) \equiv R_F(\sigma(1-\delta)) = \sup\langle r \geqslant 0 \mid \int_0^r F(p) \, x(p) \, dp \leqslant \sigma(1-\delta)\rangle$.

By the optimality principle, they will keep buying (as initially planned) from

any firm thus selected as long as its price remains constant, i.e. forever on

the equilibrium path. Denote by Q^δ : $\sigma \in \mathfrak{L} \rightarrow Q(\sigma/(1-\delta))$ the distribution of

scaled-down search costs, and q^δ the corresponding density. Any constant price

strategy then generates a demand per period $D_F^\delta(p)$ given by (5), with q replaced

by q^δ . If firm f's optimal price strategy is a constant p^f, then in particular

p^f must maximize $\pi_F^\delta(p) = (p-c) \, D_F^\delta(p)/(1-\delta)$. Therefore :

Theorem IV.1 : Along the equilibrium path of any repeat purchase BPE with

stationary prices : (i) Each consumer σ initially searches with reservation

price $R_F^\delta(\sigma)$ given by (31), then keeps buying from the same firm in all subse-

quent periods ; (ii) Each firm f charges $p_F^\delta(c^f)$, its optimal price in an

equilibrium of the one-shot purchase market for production and search costs

distributions G(c) and $Q_\delta(\sigma) \equiv Q(\sigma/(1-\delta))$.

The converse result is established in Theorem IV.2 below : every one-

shot purchase equilibrium (such as those described in Sections I to III) can be

transformed into the outcome of a BPE of the repeat purchase game, in which

prices on the equilibrium path (but not outside) are stationary. This result is

more difficult to prove, because off the equilibrium path, firms and consumers

may have quite complicated strategies ; in particular, the latter may not just

buy loyally or search at random, but also return to previously visited firms.

Intuitively, the equilibrium works as follows : i) consumers interpret any

upward deviation of a firm's price as a signal that its cost is higher than

previously thought ; firms' equilibrium strategies then leads these consumers to view this price increase as permanent. Those with low reservation prices then leave and attach themselves to some other seller, thereby effectively depriving the deviator from any incentive to lower its price again. ii) On the contrary, any downward deviation of the price leaves consumers' beliefs unaffected and is therefore viewed (rightly so) as temporary. Thus price cuts do not ensure repeat purchases, while price hikes result in irreversible customer loss, so that both types of deviations (from $p_F^\delta(c^f)$) are unprofitable. In order to formally establish this result, some definitions are required. Let H_t^f and $H_{t,n}^\sigma$ denote respectively the information sets of f in period t, and of σ just after his n-th search (or return) step of period t . H_t^f consists of the firm's past prices $\langle p_s^f | s < t \rangle$ and experienced demands $\langle D_s^f | s < t \rangle$. $H_{t,n}^\sigma$ consists of the "addresses" $I_{t,n}^\sigma$ of all previously visited firms and for each of them, the history of prices observed there — in particular the highest price $P_{t,n}^\sigma(f)$ ever observed at each of them. The subsets $J_{t,n}^\sigma = \langle f \in I_{t,n}^\sigma | P_{t,n}^\sigma(f)$ $\langle R_F^\delta(\sigma) \rangle$ and $K_{t,n}^\sigma = \text{argmin} \langle P_{t,n}^\sigma(f) | f \in J_{t,n}^\sigma \rangle$ are thus also part of the information available at $H_{t,n}^\sigma$.

Theorem IV.2 : Let $G(c)$ and $Q(\sigma)$ define production and search cost distributions such that $G(c)$ and $Q^\delta(\sigma) \equiv Q(\sigma/(1-\delta))$ generate an equilibrium of the one-shot search market, with increasing pricing rule $p_F^\delta(c)$ and associated price distribution $F(p)$. The following strategies and beliefs constitute a Bayesian perfect equilibrium with stationary path :

(i) Each firm f with cost $c^f < \bar{r}_F$ initially charges $p_0^f(c^f) = p_F^\delta(c^f)$; at any information set H_t^f (t > 1) , it expects all consumers who bought there in the previous period to visit it again first in the current one, and charges:

(32) $\qquad \tilde{p}(H_t^f) = \max \langle p_F^\delta(c^f) , \max \langle p_s^f | 0 \leqslant s \leqslant t - 1 \rangle \rangle$

(ii) At any information set $H_{t,n}^\sigma$, consumer σ has beliefs distributed according

to $G(c)$ and $F(p)$ over the cost type and price of firms $f' \notin I_{t,n}^\sigma$.

He believes (with probability one) each firm $f \in I_{t,n}^\sigma$ to be of

cost type $C_F(P_{t,n}^\sigma(f))$ and to have charged $P_{t,n}^\sigma(f)$ in all periods

$s \leqslant t$ in which he did not visit it. His strategy is given by the

following program:

(a) If $J_{t,n}^\sigma = \emptyset$, he samples a firm at random, then updates $H_{t,n}^\sigma$ into $H_{t,n+1}^\sigma$

(b) If $J_{t,n}^\sigma \neq \emptyset$, he returns to any firm $f \in K_{t,n}^\sigma \neq \emptyset$ and buys from it if:

(33) $\qquad p_t^f \leqslant \min \langle P_{t,n}^\sigma(f') | f' \in J_{t,n}^\sigma \backslash \langle f \rangle \rangle$

where the r.h.s. is taken to mean $R_F^\delta(\sigma)$ when $J_{t,n}^\sigma \backslash \langle f \rangle = \emptyset$. If (33) does not

hold, he pursues the program after updating $H_{t,n}^\sigma$ into $H_{t,n+1}^\sigma$, and in particular:

$P_{t,n+1}^\sigma(f) = \max \langle p_t^f , P_{t,n}^\sigma(f) \rangle$;

$J_{t,n+1}^\sigma = J_{t,n}^\sigma$ or $J_{t,n}^\sigma \backslash \langle f \rangle$, according to $p_t^f \lessgtr R_F^\delta(\sigma)$;

$K_{t,n+1}^\sigma = K_{t,n}^\sigma$ or $K_{t,n+1}^\sigma \backslash \langle f \rangle$, according to $p_t^f \lessgtr \min \langle P_{t,n}^\sigma(f') | f' \in J_{t,n}^\sigma \backslash \langle f \rangle \rangle$;

<u>Proof</u> : cf. appendix.

A consumer thus goes back to the firm (if any) which he expects to have

the lowest price $P_{t,n}^\sigma(f) \leqslant R_F^\delta(\sigma)$, and buys there is its actual price p_t^f is

still the lowest he knows of (and below his reservation price). On the

equilibrium path, firms charge constant prices and buyers purchase repeatedly

from the same store. The stationarity of firms' and consumers' dynamic

problems plays, together with the latter's beliefs, an essential role in this

type of equilibrium. Contrary to Bagwell's [1985] model of introductory

pricing, for instance, firms have no incentive to misrepresent their cost by

initially charging low prices, because the consumers who were fooled into

coming back and discovered a high price would leave without buying, and search

for another permanent supplier, expecting (rightly) the current one to continue

charging high prices in the future. Indeed, let a firm raise its price from p

to p+ε; customers with reservation price r ∈ [p,p+ε) now believe it to have

cost C_F(p+ε) and expect it to charge (forever in the future) the corresponding

optimal price, equal to p+ε by (32) ; since $\dfrac{V(p+\varepsilon)}{1-\delta} < V(p+\varepsilon) + \delta\,\dfrac{V(r)}{1-\delta} < \dfrac{V(r)}{1-\delta}$,

they prefer to leave rather than remain patrons in the future, or even buy from

the firm before resuming search for future purchases. Two important sources of

the difference with Bagwell [1985] are the possibility of costless return and

consumers' infinite horizon, which together make their decision problem

stationary, eliminating in particular any last period in which they could be

exploited[19] .

While all players' strategies are optimal — on or off the equilibrium

path — given their beliefs, close scrutiny of this BPE reveals that buyers use

in fact a weakly dominated strategy (w.d.s.). Consumer σ , after returning to

the firm $f \in J^{\sigma}_{t,n}$ which he expects to have the lowest price, buys from f if

its actual price p^f_t is lower than the prices $P^{\sigma}_{t,n}$(f') he expects at the

other firms he knows of (and lower than $R^{\delta}_F(\sigma)$). This decision to purchase is

expected-utility maximizing, but weakly dominated by the following one :

visiting all known firms to find out (at zero cost) their actual price before

deciding from which one to purchase. The same criticism applies to Mc Millan

and Morgan [1984]. This point, however, is relatively minor, for three

reasons. First, the players using a w.d.s. are not those who do the signalling, i.e. not firms but consumers ; the usual restriction of out-of-equilibrium beliefs by elimination of strategies which are weakly dominated for certain signalling types is thus not applicable here. Secondly, the beliefs of consumers off the equilibrium path (after witnessing a price change) which sustain their w.d.s. are very reasonable ; in particular, they satisfy Tirole and Fudenberg's [1988] more restrictive definition of BPE : a) a deviation by any player (firm) affects beliefs about his type only ; b) beliefs obey Baye's rule not just on the equilibrium path, starting from priors, but also starting from the new beliefs triggered by any deviation, and until the next zero-probability event occurs. Finally, if consumers did keep informed of the prices of all previously visited firms, the latter's strategies would become much more complicated, since lost customers could now be "recalled" by lowering the price. If an equilibrium existed with non stationary price paths, it would involve consumers repeatedly switching between firms, coming back, etc, and would be intractable. On the other hand, if the modified consumer and firm strategies still led to an equilibrium with stationary price paths, it would have the same outcome (cf. Theorem IV.1) as the above BPE, so that the latter's use of a w.d.s. would then be of very minor significance. (20)

IV.1.3. Repeat purchase and price dispersion : The relevant search cost when comparing constant price paths is $\sigma(1-\delta)$. Consumers thus have lower reservation prices, and initially search more than for a single purchase (one can think of local residents versus tourists). They are in fact investing in a thorough search, so as to benefit from a low price and save search costs in all future

periods. As a result, prices are lower, and the amount of price dispersion

smaller, the more frequently the good is purchased (the closer δ is to one).

IV.2. Repeat purchases with consumer renewal

The type of equilibrium presented here differs from that of Mc. Millan

and Morgan [1984] in two important and related aspects : a) it involves

optimal prices in every period ; b) as a result, it can accomodate a renewal of

the consumer population, which would destroy any equilibrium in which firms are

stuck at suboptimal prices. As before, let firms charge constant prices, but

assume now that consumers leave the market, or "die" with a constant

probability $\mu \in (0,1]$ per period ; a constant "number" (mass) $\lambda > 0$ of new

ones enter the market or "are born" in every period. Let the initial (stock)

density of consumers at time zero equal its long-run value λ / μ ; equivalently,

one could consider the market as having been in operation since $t = -\infty$. All

cohorts are identical in terms of tastes and distribution of search costs,

hence also (since they face the same distribution F) of reservation prices.

Along the equilibrium path, a firm therefore has the same optimal price with

respect to each cohort, i.e. there is no incentive for intergenerational price

discrimination [19] ; neither is there any incentive to fool consumers into

coming back, because in that case they will leave without buying (given the

appropriate beliefs). These arguments suggest that the same type of BPE path as

in Theorems IV.1-IV.2 will still prevail, with consumers searching only upon

entering the market. The formal proof is substantially more complicated,

because off the equilibrium path there generally are incentives to discriminate

between generations : whereas a firm who had deviated and charged $p > p_F^\delta(c^f)$

would previously not have benefited from lowering back its price (equation

(32)), it now has conflicting incentives to maintain it because its existing customer base is composed of high-reservation price individuals, and to lower it in order to restore the optimal mix of consumers corresponding to its cost type, by attracting and retaining new clients. The following lemma establishes that there exists an optimal price strategy for any previous sequence of prices, and gives bounds for this strategy.

Lemma 2 : Assume that almost all firms f charge $p_F^{\delta\mu}(c^f)$ in every period, and that consumers (once in the market) have the same strategies as in Theorem IV.2, except that δ is replaced by $\delta\mu$. For a firm f at any information set H_t^f , the present value of profits to be expected from any sequence of prices $\langle p_s | s \geqslant t \rangle$ is then : (21)

$$(34)\ \pi(\langle p_s | s \geqslant t \rangle | H_t^f) = \theta\lambda \sum_{s=t}^{+\infty} \delta^{s-t} (p_s - c^f) \sum_{\tau=0}^{+\infty} (1-\mu)^\tau D^{\delta\mu}(\max\langle p_k | s -\tau \leqslant k \leqslant s \rangle)$$

where $p_k \equiv p_k^f$ for k < t - 1 . Moreover, there exists a sequence $\langle p*(H_t^f) | s \geqslant t \rangle$ maximizing this present value, with :

$$(35) \qquad (\forall\ s \geqslant t) : p_F^{\delta\mu}(c^f) \leqslant p*(H_t^f) \leqslant \max\langle p_F^{\delta\mu}(c^f) , \max\langle p_k^f | k \leqslant t - 1 \rangle\rangle .$$

Proof : cf. appendix.

Note that if $p_F^{\delta\mu}(c^f)$ was charged in the past, it remains optimal in the future. Moreover, the lemma shows that f will never cut its price below $p_F^{\delta\mu}(c^f)$ in an attempt to attract new customers : those with $R_F^{\delta\mu}(\sigma) < p_F^{\delta\mu}(c^f)$ can never be exploited since they leave permanently as soon as p is raised above $R_F^{\delta\mu}(\sigma)$. Lemma 2 also serves as the foundation for the following result.

Theorem IV.3 : The results of Theorems IV.1 and IV.2, showing the equivalence

(one to one mapping) between stationary paths of repeat purchase Bayesian

perfect equilibria and equilibria of the one-shot market, still hold in the

presence of buyer renewal, provided that δ is replaced by $\delta\mu$ and firms' price

strategy (32) by $p^* : H_t^f \rightarrow p^*(H_t^f)$ defined in Lemma 2.

Proof : cf. appendix.

The type of equilibrium described in this theorem provides a comprehensive and

realistic description of a heterogeneous, monopolistically competitive search

market : in every period, consumers come in, invest in an initial search for a

suitable seller — of which they then become loyal customers until leaving the

market — while an endogenous number of firms charge constant, fully optimal

prices reflecting the interplay of preferences, production costs, and search

costs. (22)

CONCLUSION

This paper developed a general model of search market equilibrium with

diversity among both sellers and buyers, which is able to account for many

empirical facts (price or wage dispersion, active search, allocation of types

of buyers across types of sellers, stable clienteles in a repeat purchase

context). By unifying and "smoothing out" previous models of unilateral

heterogeneity, it shed light on the sources of their unsatisfactory features

(indeterminacy, absence of search, bunching), and showed how these could be

eliminated as nongeneric, limiting cases. Finally, it extended the single

purchase, or static results, to the case of repeated purchases, by

incorporating the model into a game-theoretic structure with both imperfect and

incomplete information.

NOTES

(1)   Salop and Stiglitz [1987] also introduce product heterogeneity ; the equilibrium they derive, however, does not feature any price dispersion.

(2)   It is well known that mixed strategy equilibria of a game can be interpreted and justified as limits of pure strategy equilibria of the same game perturbed by an infinitesimal amount of incomplete information over players' types ; this, however, amounts to appealing to their heterogeneity.

(3)   While revising this paper, I became aware of Mc Minn's [1980] paper which already considered a particular model with both firm and consumer heterogeneity, corresponding to the assumptions of in Section II.1. The link between both sets of results is discussed in that section.

(4)   For proofs, see Kohn and Shavell [1974] or Lippman and Mc Call [1976]. The latter also provides a discussion of the complications which arise when (a) or (b) fails ; in particular, the optimal strategy with recall under risk-aversion may not be a reservation price rule any more.

(5)   If consumers have a finite maximum valuation Z for the good, it will be assumed to be large enough not to constrain the equilibrium $(Z > \bar{r})$.

(6)   Firms with $c = \bar{r}$ only matter when $\bar{r}$ coincides with an atom of G . Allowing for this occurrence, the most general form of (7) is :

$G(\bar{r}^-) < 1/\theta < G(\bar{r}^+) = G(\bar{r})$. An equilibrium where double equality does not hold is unstable : a slight change in the distributions will alter $\bar{r}$ and it will not correspond to an atom any more. Also, any group of firms with $c = \bar{r}$ could decide to exit (say), thereby significantly altering the whole equilibrium.

(7)   When $\bar{r} = +\infty$, the bracketed term has a finite limit as $p \to +\infty$, hence $C_F(+\infty) = +\infty$. When $\bar{r} < +\infty$ ($\Gamma(+\infty) > \bar{\sigma}$), it is no smaller than $(q \circ \Gamma)(p) \ x(p) /$

$\int_p^r (q \circ \Gamma)(r) \ x(r) \ dr$; if $q(\bar{\sigma}^-) > 0$ this tends to $+\infty$ as p tends to $\bar{r}$. If

$q(\bar{\sigma}^-) = 0$, the minor additional assumption that q is locally non increasing to

the left of $\bar{\sigma}$ ensures that this term is larger than $1/(\bar{r}-p)$, hence the result.

(8)   Under the standard assumption that $x(p)$ generates a well-behaved

monopoly problem, i.e. that $p(1-1/e_x(p))$ is increasing (e.g. $x(p)$ concave).

(9)   In all generality, $\phi_F(p) = \theta(G \circ C_F)(p)$ for $p < \bar{r}_F$, where $1/\theta$ is any

number satisfying the inequality of footnote (6) ; ϕ_F is discontinuous at $\bar{r}_F$,

and the equilibrium unstable, if equation (7) does not hold.

(10)   It does when $x(p) = 1$ (below some $Z > 0$) and q is increasing. Existence

could probably be established in this case by showing that ϕ maps the space of

c.d.f.'s on $[0,Z]$ into itself, and is continuous for the weak topology. Since

the slightly stronger assumptions of Sections II and III allow a full

resolution of the problem, this exercise does not seem worthwile.

(11)   Only upward discontinuities of q on $[\underline{\sigma},\bar{\sigma})$ are consistent with the

existence of a well behaved equilibrium.

(12)   This is required to prevent all firms from charging the maximum price Z.

(13)   One thus leaves aside many Diamond-type equilibria with bunching, where

all firms charge the same price $p \in [\bar{c}+\bar{\sigma},Z]$, so that $p(c) = \underline{r} = p > (c+p+\bar{\sigma})/2 =$

$(c+\bar{r})/2$ for all $c < \bar{c}$.

(14)   Again, allowing for the coincidence of $\bar{r}$ with a discontinuity of G :

$(H/G)(\bar{r}^-) \geqslant 2\bar{\sigma} \geqslant (H/G)(\bar{r}^-)$. The stability reasoning of paragraph II.1.1 below

can also be used to eliminate any solution without double equality.

(15)   This is required for the integrals defining reservation prices (e.g.

$\int_0^r p^{-\alpha}\, G(p(1-1/\alpha))\, dp)$ to be convergent in the neighborhood of $r = 0$. Otherwise

all consumers will search until they find the lowest price in the market, and

the only equilibrium will be the competitive one, with only the most efficient

firm(s) ($c = \underline{c}$) operating. The same caveat applies to Reinganum [1979].

(16)   Note from (20) that $C(\underline{r}) > \underline{r}(1-1/\alpha) > \underline{p}(1-1/\alpha) = \underline{c}$, so that there is

always bunching, contrary to case I.1.

(17)   An alternative assumption would be that only the last firm where the

good was bought can be revisited costlessly ; however, it is hard to find an

interpretation for such search costs (neither transportation nor inspection

costs will do), when consumers have perfect memory.

(18)   Since other histories can never be reached by a single player's

deviation, the specification of equilibrium play after such a history leaves

the equilibrium path unchanged.

(19)   If consumers had a finite life, their reservation prices (for a constant

price distribution) would increase with age, giving firms an incentive to

conceal their cost type, then exploit loyal (old) consumers by raising prices,

as in Bagwell [1985]. An equilibrium might then involve price cycles.

(20)   As an additional justification, assume that returning to any $f \in I^{\sigma}_{t,n}$,

entails a small cost $\varepsilon > 0$. The strategies and beliefs of Theorem IV.2 then

constitute a (Bayesian perfect) ε-equilibrium, in which not returning to more

than one $f \in K^{\sigma}_{t,n}$ at a time is not a w.d.s. any more, but in which returning to

f instead of searching may be suboptimal by $\max(0,\ \varepsilon - [V(P^{\sigma}_{t,n}(f))- V(R^{\delta}_F(\sigma))]/$

$(1-\delta)) < \varepsilon$. As $\varepsilon \to 0$, the search technology approaches the limiting case

assumed in the Theorem, and the ε-equilibrium becomes a BPE.

(21)   Formula (34) is written for the case where the game has been played

since $-\infty$. When it starts at some initial date 0, (34) still holds provided one

defines $p_k \equiv p_o^f$ for $k < 0$.   .

(22)   The same remarks about consumers' return strategy as those which

followed Theorem IV.2 still apply, however.

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A P P E N D I X

Proof of Theorem I :

The subscript F will be dropped from C_F and r_F ; dG (resp. dF) will denote the measure on the set of firms (resp. prices) defined by G (resp. F) .

A - Sufficiency of the theorems' conditions : Let $F : R_+ \rightarrow R_+$ satisfy

(a)-(b). By construction, $F(p) = \phi_F(p)$ tends to 1 as p tends to $+\infty$, while

$C(0) < 0$, so that $F(0) = 0$; F is thus a c.d.f. on R_+ . For all $c \in (c,\bar{r})$,

C has left and right limits $C(p^-)$ and $C(p^+) = C(p)$ at c ; define therefore :

$$P(c) = \langle p \in [0,\bar{r}) \mid C(p^-) < c < C(p) \rangle$$

Any local maximum of $\pi(.|c)$ must belong to P(c), which generalizes the first

order condition in (8), and is non-empty because $C(0) < 0 < c$ and $C(\bar{r}^-) = \bar{r} > c$.

Lemma A1 : P(c) is a singleton, for dG-almost all c .

Proof : For all $(p_1,p_2) \in [0,r)^2$, since both G and F = GoC are non-decreasing :

(A1) $(p_1 < p_2$ and $C(p_1) > C(p_2)) \Longrightarrow$ G is constant on $[C(p_2) , C(p_1)]$

Since dG has no atom, this means that $[C(p_2) , C(p_1)]$ has measure zero under

dG. Consider now any $c \in (\underline{c},\bar{r})$ and $(p_1,p_2) \in P(c)^2$, with $p_1 < p_2$.

a) If $C(p_1) > C(p_2^-)$, then by (A1) $[C(p_3),C(p_1)]$ with $p_3 = p_2 - \epsilon$ and

ϵ small enough has measure zero for dG . Thus $[C(p_2^-) , C(p_1)]$, which contains

c (since $(p_1,p_2) \in P(c)^2$) , has measure zero for dG .

b) If $C(p_1) < C(p_2^-)$, then $c < C(p_1) < C(p_2^-) < c$ since $(p_1,p_2) \in P(c)^2$;

thus $C(p_1) = C(p_2^-)$. Suppose that C is not identically equal to c on $[p_1,p_2)$,

i.e. that there exists p_3 in this interval such that either :

b1) $C(p_3) > c$; then $[c, C(p_3)]$ has measure zero for dG .

b2) $C(p_3) < c$; then $[C(p_3), c]$ has measure zero for dG .

Summing up cases (a) and (b) establishes that, except for a dG-negligible set of firm costs c , C is constant on all $[p_1, p_2)$ where $(p_1, p_2) \in P(c)^2$. In other words, for dG-almost all c : (i) $P(c)$ is an interval $[p^-(c) , p^+(c)]$ (or $[p^-(c) , p^+(c))$ if C is discontinuous at $p^+(c)$), and if this interval has positive length, then C is constant and equal to c on it ; (ii) for $p < p^-(c)$, $C(p) < c$; (iii) for $p > p^+(c)$, $C(p) > c$ (for dG-almost all c).

Consider now any interval I on which G is increasing, i.e. which does not contain any subinterval of measure zero under dG . Define, for all $c \in I$, the function $p(c) \equiv p^+(c)$. It is non-decreasing, because $c_1 > c_2$ and $p^+(c_1) < p^+(c_2)$ implies $C(p^+(c_1)) < c_2 < c_1 < C(p^+(c_2))$, (by (i) to (iii) above) and it was seen that this requires $[C(p^+(c_1)) , C(p^+(c_2))]$ and its subinterval $[c_2, c_1]$ to have measure zero for dG , a contradiction. As a non-decreasing function, $p(c)$ has at most a countable number of discontinuities ; these discontinuities are easily seen to correspond to points c such that : $\bar{p}(c) < p^+(c)$, so the set of $c \in I$ such that $P(c)$ has more than one element is countable. Since this is true for every interval I which contains a non-negligible mass of firm costs, and since there are no atoms in the cost distribution (G is continuous), the set of $c \in [c, \bar{r}]$ such that $P(c)$ is not a singleton has measure zero under G ; this concludes the proof of Lemma A1.

We now show that F is an equilibrium price distribution.

Let firms with $c > \bar{r}$ not operate. For all $c < \bar{r}$, $\pi(.|c)$ has a global maximum on $[c, \bar{r}]$, which must also be a local maximum (since $\pi(c|c) = \pi(\bar{r}|c)=0$), hence

an element of P(c) . But, for dG almost all firms, P(c) is a singleton

⟨p(c)⟩ (with C(p(c)) = c) , implying so that p(c) is indeed a globally optimal

price for a firm with cost c . Moreover, for any such firm :

* if $p > p(c) = p^+(c)$, then C(p) > c by (i) and (iii) above ;

* if $p < p(c) = p^-(c)$, then C(p) < c by (ii) above ;

so that p(c) < p if and only if C(p) < p , for dG-almost all c . The

number of prices no greater than p charged in the market is therefore G(C(p)) ,

which by hypothesis is $F(p)G(\bar{F})$. Hence : Prob⟨p(c) < p⟩ = F(p) .

B - <u>Necessity of the theorem's conditions</u> : Let F be an equilibrium such that

the set of firms with more than one local maximum in their profit function

π(.|c) has measure zero under dG .

<u>Lemma A2</u> : For all $(p_1, p_2) \in [0, \bar{F}]^2$ such that $p_1 < p_2$ and $C(p_1) > C(p_2)$, the

interval $[C(p_2), C(p_1)]$ has measure zero under dG .

<u>Proof</u> : Since G has no atoms and its support is contained in [0,+∞) , the

proposition is immediate when $C(p_1) = C(p_2)$ or $C(p_1) < 0$. Assume from now on

that $(C(p_2), C(p_1)) \cap R_+^* \neq \emptyset$ and consider any c in this interval.

Denote by $\pi'_d(p|c) = - D'(p)(c - C(p))$ the right derivative of π(.|c) at every

point ; then C(0) < 0 < c < $C(p_1)$ implies that $\pi'_d(0|c) > 0 > \pi'_d(p_1|c)$. The

continuous function π(.|c) restricted to $[0, p_1]$, has a global maximum ; since

$\pi'_d(0|c) > 0$, it cannot occur at zero, so two cases are possible :

    a) π(.|c) possesses an interior, and therefore local, maximum in $(0, p_1)$.

    b) $(\forall p \in [0, p_1])(\pi(p|c) < \pi(p_1|c))$. But since $\pi'_d(p_1|c) < 0$:

        $(\exists \varepsilon > 0) (\forall p \in [p_1, p_1+\varepsilon)) , \pi(p|c) < \pi(p_1|c))$

        hence p_1 is a local maximum of π(.|c) .

Thus in both cases, $\pi(.|c)$ possesses a local maximum in $[0,p_1]$. Similarly,

using the fact that $C(p_2) < c < C(p_1) < p_1 < \bar{r} = C(\bar{r})$, one shows that $\pi(.|c)$

has a local maximum in $(p_2,\bar{r})$. Thus, for all $c \in (C(p_2),C(p_1)) \cap R_+^*$, $\pi(.|c)$

has more than one local maximum, which (by assumption) implies that $(C(p_2)$,

$C(p_1))$ has measure zero under dG . Since G has no atoms, the same holds for

$[C(p_2)$, $C(p_1)]$, which concludes the proof of the lemma.

The number of firms charging a price $p(c) < p$ in equilibrium will now be

computed. For any $c \in [\underline{c},\bar{r})$, consider the following cases, for any given p :

a) If $c < C(p)$ and $p(c) > p$; since any optimal price $p(c)$ must belong

to $P(c)$, $C(p(c)^-) < c$ so : $p_1 \equiv p < p(c) - \epsilon \equiv p_2$ and $C(p_1) > C(p_2)$ for all ϵ

small enough. By Lemma A2, $[C(p_2)$, $C(p_1)]$ has measure zero ; thus $[c$, $C(p)]$

has measure zero under dG .

b) If $c > C(p)$ and $p(c) < p$; since $C(p(c)) > c$ because $p(c) \in P(c)$,

$p_1 \equiv p(c) < p \equiv p_2$ and $C(p_1) > c > C(p_2)$, which requires $[C(p),c]$ to have

measure zero under dG .

Thus, if c_1 and c_2 $(c_1 < c_2)$ satisfy (b) (remember that p is fixed), then

$[c_1,c_2]$, as a subset of $[C(p),c_2]$, has measure zero under dG . Costs

satisfying (b) are therefore separated by intervals of measure zero under dG ;

as dG has no atoms, the set of such costs itself must also have measure zero.

Since the same applies to costs satisfying (a), the following property holds

for dG-almost all costs c :

(A2) $(p(c) < p \iff c < C(p))$

The "number" of firms charging a price no greater than a given p is therefore

$G(C(p))$; normalizing by $1/\theta = G(\bar{F})$ yields $F(p) = \phi_F(p)$ for all p .   Q.E.D.

Proof of Theorem II.1 :

Only existence remains to be established. The function H/G has left and right

limits everywhere on $(\underline{c},+\infty)$ and is right-continuous, because G has these

properties and H is continuous. For $r \geqslant \bar{c}$, $(H/G)(r) = \int_0^c G(c) \, dc + r - \bar{c}$,

hence $(H/G)(+\infty) = +\infty$. Moreover, for all $c \leqslant r$, $G(c) \leqslant G(r)$ so :

$(H/G)(r) \leqslant \int_c^r dc = r - \underline{c}$, hence $(H/G)(\underline{c}^+) = 0$. Therefore for all $\bar{\sigma} > 0$,

there exists $r > 0$ such that : $(H/G)(r^-) \leqslant 2\bar{\sigma} < (H/G)(r^+) = (H/G)(r)$. But any

discontinuity in G is upward, so any discontinuity in H/G is downward. The

preceding double inequality can therefore only be a double equality (with

intersection from below).   Q.E.D.

Proof of Theorem II.2 :

It remains to be shown that if $g/G \equiv G'/G$ is decreasing on $I = [\underline{c},+\infty)$, so is

$H'/H = G/H$. Since H/G is continuous on I, with limits of 0 and $+\infty$ at $\underline{c}^+$ and $+\infty$

respectively (cf. proof of Theorem II.1.1), it is increasing if and only if :

(A3) $(\forall \lambda > 0) \ (\exists! \ c_\lambda > \underline{c}) \ (H(r_\lambda)/G(r_\lambda) = \lambda)$

Define, for all $\lambda > 0$, the function : $J_\lambda(r) = H(r) - \lambda G(r)$ for $c \in [\underline{c},+\infty)$.

Then (A3) is equivalent to :

(A4) $(\forall \lambda > 0) \ (\exists! \ r_\lambda > 0) \ (J_\lambda(r_\lambda) = 0)$.

The function J_λ is continuously differentiable on I , and

(A5) $(\forall r \in I) \ (J'_\lambda(r) = G(r)[1 - \lambda(g/G)(r)])$.

But G is non-decreasing and positive on I , while g/G is decreasing by

assumption ; thus by (A5) there exists a unique $c_\lambda > \underline{c}$, such that J is

decreasing on $[\underline{c}, c_\lambda]$ (which is possibly empty) and increasing (to a limit of

$+\infty$) on $[c_\lambda, +\infty)$. Since $J_\lambda(\underline{c}) = -\lambda G(\underline{c}) < 0$, J_λ has no positive root in

$[\underline{c}, c_\lambda]$. On $[c_\lambda, +\infty)$, on the contrary, J_λ increases from $J_\lambda(c_\lambda) < 0$ to a

limit of $+\infty$. Hence (A4) holds.   Q.E.D.

Proof of Lemma 1 :

Let $k \equiv \beta - r$; $(C^*)'(p)$ has the sign of :

$$(2 + kp^{\alpha-1})(2\alpha - 1 + \alpha kp^{\alpha-1}) - (\alpha-1)kp^{\alpha-1} = \alpha(2 + kp^{\alpha-1})(1 + kp^{\alpha-1}) + 2(\alpha-1)$$

which is clearly positive when $k > 0$. Since $\beta/\bar{r} = 0$ and $\alpha > 1$, the case $k < 0$

requires $\beta = 0$, $k = -r^{1-\alpha}$ so $kp^{\alpha-1} > -1$ hence again $(C^*)'(p) > 0$. Q.E.D.

Proof of Theorem II.3 :

Only existence remains to be shown ; let $\mu = 1 - 1/\alpha$.

Case 1 : $\int_0^\infty p^{-\alpha} G(p(1-1/\alpha)) \, dp \leq \sigma$.

Let all firms charge $p^m(c) = c/\mu$. Then $F(p) = G(\mu p)$ is an equilibrium

distribution, because $r_F(+\infty) \leq \sigma$, or $r_F = +\infty$, so that no consumer searches ;

thus $(r, \bar{r}, \beta) = (+\infty , +\infty , (\alpha-1)[\bar{\sigma} - \int_0^\infty p^{-\alpha} G(\mu p) \, dp])$ solves (24a) to (24c).

Case 2 : $\int_0^\infty p^{-\alpha} G(p(1-1/\alpha)) \, dp > \sigma$.

The function T_0 : $[u \rightarrow \int_0^u p^{-\alpha} G(\mu p) \, dp]$ is continuous and increasing on

$[c/\mu , +\infty)$, from 0 at $\underline{c} > 0$ to a limit $T_0(+\infty) > \underline{c}$. Therefore :

(A6) $(\forall r \geq \underline{c})$ $(\exists ! u(r) \geq \underline{c}/\mu)$ $(\int_0^{u(r)} p^{-\alpha} G(\mu p) \, dp = G(r) \sigma)$

By (A6), $u(.)$ is non-decreasing and right-continuous. Note that, for r large

enough, $G(r)$ is arbitrarily close to 1, hence $u(r)$ close to the finite solution

of $T_0(u) = \underline{\sigma}$, so $r > u(r)$. Define now for all (w,r) with $r \geqslant w \geqslant \underline{c}$:

(A7) $$T(w|r,0) = \underline{\sigma} + \int_{u(r)}^{w} p^{-\alpha} [G(C^*(p|r,0))/G(r)] dp$$

with $C^*(p|\bar{r},\beta)$ defined by (20) and $\beta = 0$ here. This function is continuous in w and right-continuous in r , with left and right limits in r at every point. Let us first concentrate on the case where $w = r$, i.e. on the function $T(r) \equiv T(r|r,0)$. When r is close to $\underline{c}^+$, $u(r) \geqslant \underline{c}/\mu > r$ so :

(A8) $$T(r) = \underline{\sigma} + \int_{u(r)}^{r} p^{-\alpha} [G(C^*(p|r,0))/G(r)] \, dp < \underline{\sigma} < \bar{\sigma}$$

and two cases arise.

<u>Case 2.1</u> : $(\exists \, r \in (\underline{c} , +\infty)) \, (T(\bar{r}^-) < \bar{\sigma} < T(\bar{r}))$.

But $G(\bar{r}^-) < G(\bar{r})$ and $u(\bar{r}^-) < u(\bar{r})$ imply by (A8), $T(\bar{r}^-) > T(\bar{r})$, so the above must be a double equality, with the intersection between $T(r)$ and the horizontal $\bar{\sigma}$ occuring from below (which is easily seen, as in Section II.1.1, to correspond to stability). Assume now that firms behave as claimed in the theorem, with the above defined $\bar{r}$, $\underline{r} \equiv u(\bar{r})$ and $\beta = 0$; the claimed form (22) of $F(p)$ follows immediately (note from (A5) that it is indeed increasing on $[\underline{c}/\mu , +\infty]$), as well as consumers' optimal strategies from the general formula (3). Thus, from the above definition of $\bar{r}$, (22), (A6) and (A7) , $r_F(r)$ is equal to :

(A9)
$$\begin{cases} (\forall \, r < \underline{r}) \, (r_F(r) = T_0(r)/G(\bar{r})) \\ (\forall \, r \in [\underline{r},\bar{r}]) \, (r_F(r) = T(r|\bar{r},0)) \end{cases}$$

(A9') $r_F(\underline{r}) = T_0(u(\bar{r}))/G(\bar{r}) = \underline{\sigma}$; $r_F(\bar{r}) = T(\bar{r}|\bar{r},0) = T(\bar{r}) = \bar{\sigma}$.

Since Γ_F is increasing on $[c/\mu\ ,\ +\infty)$, (A9') implies that $(\underline{r},\bar{r},0)$ indeed solves the equilibrium equations (24a)-(24c).

Case 2.2 : $(\forall\ r \geqslant \underline{c})$ $(T(r) < \bar{\sigma})$.

The fixed-point requirement will bear in this case on β rather than $\bar{r}$.

Define $\bar{r} = +\infty$ and $\underline{r} = u(+\infty)$, i.e. the only solution to :

(A10)
$$\int_0^{\underline{r}} p^{-\alpha}\ G(\mu p)\ dp = G(+\infty).\underline{\sigma} = \underline{\sigma}$$

As r tends to $+\infty$, $u(r)$ tends to r and, since the integrand in (A8) is positive, $T(r)$ tends to a limit $T(+\infty)$, with $T(+\infty) < \bar{\sigma}$ by assumption. For all $\beta \geqslant 0$, define now the following functions on $[\underline{c},+\infty]$:

(A11)
$$\begin{cases} T(w|+\infty,\beta) = T_0(w) = \int_0^w p^{-\alpha}\ G(\mu p)\ dp & \text{for } w \leqslant \underline{r} \\[2mm] T(w|+\infty,\beta) = \underline{\sigma} + \int_{\underline{r}}^w p^{-\alpha}\ G(C^*(p|+\infty,\beta))\ dp & \text{for } w \geqslant \underline{r} \end{cases}$$

(A12) $\qquad L(\beta) \equiv T(+\infty|+\infty,\beta) + \beta/(\alpha-1)$

Note from (A4) that $C^*(p|+\infty,\beta)$ is continuous and decreasing in β ; (A11)-(A12) then imply that L is right-continuous in β , and can only have downward discontinuities. Since $L(+\infty) = +\infty$ and :

$$L(0) = T(+\infty|+\infty,0) = \underline{\sigma} + \int_{\underline{r}}^{\infty} p^{-\alpha}\ G(C^*(p(+\infty,0)) = T(+\infty) < \bar{\sigma}\ ,$$

there exists $\beta* \in [0,+\infty)$ with $L(\beta*) = \bar{\sigma}$ (and intersection from below), i.e. :

(A13) $\qquad \beta* = (\alpha-1)(\bar{\sigma} - T(+\infty|+\infty,\beta*))$

This is nothing else than equation (24c), for $(\underline{r},\bar{r},\beta) = (u(+\infty),+\infty,\beta*)$.

Moreover, from the definition of F in (22), and from (A10) and (A11) , $\Gamma_F(r)$ is equal to $T(r|+\infty,\beta*)$, for all r . In particular, by (A10) and (A13) :

$$\Gamma_F(\underline{r}) = T(\underline{r}|+\infty,\beta\ast) = T_0(\underline{r}) = \underline{\sigma} \ ; \ \Gamma(\bar{r}) = T(+\infty|+\infty,\beta\ast) = \bar{\sigma} - \beta\ast/(\alpha-1) < \sigma$$

so that (24a) and (24b) are satisfied as well by $(u(+\infty),+\infty,\beta\ast)$.   Q.E.D.

Proof of Theorem II.4 :

We examine the same cases as in the proof of Theorem II.2.1, imposing the

additional constraint $G(\bar{r}) = 1$. Case 1, where $\underline{r} = \bar{r} = +\infty$ remains unchanged. In

Case 2, $\underline{r}$ is now simply determined as the unique solution to :

$$\int_0^{\frac{r}{-}} p^{-\alpha} G(\mu p) \ dp = \underline{\sigma} \ ,$$

The rest of the analysis (existence of $\bar{r}$ and β) then goes through as before,

with $G(\bar{r})$ replaced by 1 everywhere ; it remains to be checked, however, that

any solution $\bar{r}$ thus obtained satisfies $\bar{r} > \bar{c}$. Under (i), the above equation

implies $\underline{r} > \bar{c}$, hence $\bar{r} > \underline{r} > \bar{c}$. If (i) does not hold but (ii) does, then

$\underline{r} \in (\underline{c},\bar{c})$ but, for all $r \in [\underline{r},\bar{c})$:

$$\int_{\underline{r}}^{r} p^{-\alpha} G(C^{\ast}(p|\bar{r},0)) \ dp < \int_{\underline{c}}^{r} p^{-\alpha} G(p) \ dp < \int_{\underline{c}}^{\bar{c}} p^{-\alpha} G(p) \ dp < \bar{\sigma} - \underline{\sigma}$$

since $C^{\ast}(p|\bar{r},0) < p$ for all $p < \bar{r}$. Thus $T(r) < \bar{\sigma}$ on $[\underline{r},\bar{c})$ so that $\bar{r}$, which is

either a solution to $T(r) = \bar{\sigma}$ (case 2.1) or $\bar{r} = +\infty$ (case 2.2) if no such

solution exists, must be no smaller than $\bar{c}$.   Q.E.D.

Proof of Theorem IV.2 :

A - Consumer strategies : Firms' strategies (32) prescribe prices $\tilde{p}(H_t^f)$

which — possibly after a single adjustment — remain constant over time,

whatever prices were charged in the past, and which — excluding simultaneous

deviations of positive measure — induce the price distribution $F(p)$. At any

information set, consumer σ thus faces (or at least, believes with probability

one he is facing) a distribution of constant price paths. Using this observation, we now characterize his dynamic programming problem. For any history H of consumer σ , let I(H) denote here the set all previously visited firms, P(f|H) the highest price ever observed at any f $\in$ I(H) , W(H) the value of an optimal program when σ starts from "home" (i.e. is not visiting a seller) with information set H , and W(H;p) the value of an optimal program when σ starts from a firm f $\in$ I(H) charging a price p = P(f|H) which is expected to remain constant forever. In this last situation, σ can either buy from the firm or leave, and in both cases his relevant information remains unchanged, so that we can write :

(A14) $\tilde{W}(H;p) = \max\langle\ V(p) + \delta W(H)\ ,\ W(H)\ \rangle$

When σ is "at home", or between firms, he can either go to some f $\in$ I(H) , which by assumption he expects (with probability one) to charge P(f|H), or sample a firm at random. In the first case he does not expect to gain any new information, while in the second he will discover a new firm f′ charging a price p′ (which he believes to be permanent ; his new information set will then be denoted (for brevity) by H′ = H $\cup$ ⟨p′⟩ . Thus :

(A15) $W(H) = \max \left\{ \max_{f \in I(H)} \tilde{W}(H;P(f|H))\ ,\ -\sigma + \int_{-\infty}^{+\infty} \tilde{W}(H \cup \langle p'\rangle;p')\ dF(p')\ \right\}$

where the maximum over f $\in$ I(H) is taken to be $-\infty$ when this set is empty. The Bellman equation obtained by substituting (A14) into (A15) defines σ′s dynamic programming problem at H . It is easily verified (by substituting them in, and using the definition (31) of $R_F^\delta(\sigma)$) that the functions :

(A16) $\qquad W(H) = \tilde{W}(H,p) = \dfrac{1}{1-\delta} \max \left\{ \max_{f \in I(H)} V(P(f|H)) \ , \ V(R_F^\delta(\sigma)) \right\}$

$(\tilde{W}(H;p)$ is defined for all p such that there exists $f \in I(H)$ with $p = P(f|H))$

are solutions to (A14)-(A15). Moreover, this expected value is clearly reached

by the algorithm described in Theorem IV.2, since, in particular, searching at

random until a price $p < R_F^\delta(\sigma)$ is found and then purchasing repeatedly at this

price yields $V(R_F^\delta(\sigma))/(1-\delta)$. This algorithm therefore defines an optimal

solution, for any consumer history H.

B - Firms' strategies : Simultaneous deviations by consumers are excluded.

Therefore, for any price path $\langle p_0, \ldots, p_{t-1} \rangle$, followed by a firm f (the

superscript f will be omitted for simplicity) only the consumers who

encountered it and have reservation prices above $P_{t-1} \equiv \max\langle p_s | s < t - 1\rangle$

remain at time t , and all of them remain. Since other firms do not deviate, no

new consumers have arrived or can be expected to arrive in the future. As of

period t , the distribution of search reservation prices among the firm's

clientele is therefore the initial one, given by (4), except that it is

truncated below P_{t-1} . Since this also holds for any $s \geq t$, the demand curve

after any firm history H_s^f is $D_s^f(p) = D(\max\langle p, P_{s-1}\rangle)$, where D is given by (5) .

The firm after H_t^f thus chooses $\langle p_s | s \geq t\rangle$ to maximize :

(A17) $\qquad \pi(\langle p_s | s \geq t\rangle | H_t^f) = \sum_{s=t}^{\infty} \delta^{s-t} (p_s - c) \ D(\max\langle p_s, P_{s-1}\rangle)$

$\qquad\qquad\qquad\qquad\qquad = \sum_{s=t}^{\infty} \delta^{s-t} (p_s - c) \ D(P_s)$

where $c = c^f$. For any $s \geq t$:

$\qquad\qquad (p_s - c) \ D(P_s) \leq (P_s - c) \ D(P_s) \leq (p(c) - c) \ D(p(c))$

with double equality if and only if $p_s = P_s = p(c)$. So, if $P_{t-1} < p(c)$, the

optimal strategy is to set $p_s = p(c)$, hence $P_s = p(c)$ for all future s . If

$P_{t-1} > p(c)$, on the contrary, then for any sequence $\langle p_s | s \geqslant t \rangle$,

$P_s \geqslant P_{t-1} > p(c)$. Since $(p-c) D(p)$ is decreasing on $(p(c),+\infty)$:

$$(p_s -c) D(P_s) < (P_s -c) D(P_s) < (P_{t-1} -c) D(P_{t-1})$$

for all $s \geqslant t$, with equality if and only if $P_s = P_{t-1}$. The optimal strategy

is therefore to set $p_s = P_s = P_{t-1}$ for all $s \geqslant t$. Thus, for any history H_t^f :

(A18) $\qquad \tilde{p}(H_t^f) \equiv \max\langle p(c) , P_{t-1} \rangle$

C - <u>Updating of beliefs</u>: Along the equilibrium path, each firm f charges $p_F^\delta(c^f)$

in every period ; each consumer σ initially searches according to

$R_F(\sigma(1-\delta))$ and then remains attached to his first supplier. Thus consumers'

beliefs (with probability 1) that a firm's optimal static price $p_F^\delta(c^f)$ is the

highest price observed there and that it is currently charging that same price

do obey Bayes's rule for updating according to firms' strategies (and they are

always true). The same holds for firms' beliefs that all customers from the

previous period have decided to come back. Beliefs off the equilibrium path

are unrestricted by Bayes' rule, but, even there, consumers' beliefs about the

yet unobserved price of a previous supplier are consistent with their belief

about its cost type and their knowledge of each type's strategy (A16).   Q.E.D.

<u>Proof of Lemma 2</u> :

Consumers' strategies and beliefs call for them to leave and never return

whenever they observe a price greater than their reservation price. Consider a

firm f with cost $c^f = c$ at an information set H_s^f . Its clientele is composed of

the sum, for all τ , $0 < \tau < s$, of those consumers who joined at date $s-\tau$, have "survived" since, and have a reservation price above the maximum price charged by the firm since that date : $P^f_{s-\tau,s} \equiv \max\langle p^f_k |s-\tau < k < s\rangle$. In any generation, the demand per firm from consumers with $R^{\delta\mu}_F(\sigma) > P^f_{s-\tau,s}$ is $\theta\lambda\ D^{\delta\mu}_F(P_{s-\tau,s})$, with $D^{\delta\mu}_F(.)$ given by (5). For simplicity the indices f,δ,μ,F will be omitted from c^f , P^f , $p^{\delta\mu}_f$, $D^{\delta\mu}_F$, etc. Summing up over all such generations between 1 and s , and adding consumers from the initial (time zero) stock which has density λ/μ yields :

$$D(p_s|H^f_s) = \theta\ \lambda\ \sum_{\tau=0}^{s-1} (1-\mu)^\tau\ D(P_{s-\tau,s}) + (\theta\ \lambda/\mu)(1-\mu)^s\ D(P_{0,s})$$

$$= \theta\ \lambda\ \sum_{\tau=0}^{s-1} (1-\mu)^\tau\ D(P_{s-\tau,s}) + \theta\ \lambda\ \sum_{\tau=s}^{\infty} (1-\mu)^\tau\ D(P_{s-\tau,s})$$

$$= \theta\ \lambda\ \sum_{\tau=0}^{\infty} (1-\mu)^\tau\ D(P_{s-\tau,s})$$

with the convention that $P_{s-\tau,s} = P_{0,s}$ for all $\tau > s$ (equivalently, one could consider the market in operation since $t = -\infty$) . Summing up the discounted profits corresponding to these demands, over $s > t$ yields (34). A solution to the infinite-horizon program of maximizing this expected present value of profits over $\langle p_s |s > t\rangle$ will now be constructed, as the limit, as T goes to infinity, of the solutions to the finite horizon problems obtained by truncating the original one at $T > t$. Let thus H^f_t be given ; the dependence of π on H^f_t will now be dropped for notational simplicity. For any $T > t$, and any infinite sequence $\langle p_s |s > t\rangle$, denote by $\pi_T(\langle p_s\rangle)$ the firm's discounted profits from t to T only. Since $\pi_T(\langle p_s\rangle)$ depends only on $\langle p_s|t < s < T\rangle$, it trivially defines a function (which will also be denoted as π_T) on R^{T-t+1}_+ . The finite horizon problem consists in maximizing π_T on that space.

Claim 1 : In the finite horizon problem, charging $p < p(c)$ is never optimal.

Indeed, for any sequence $\langle p_s | t \leq s \leq T \rangle$, defining :

$\langle q_s = \max(p_s, p(c)) | t \leq s \leq T \rangle$, one can write for all $s \geq t$ and $\tau \geq 0$:

$$Q_{s-\tau,s} \equiv \max \langle q_k | s-\tau \leq k \leq s \rangle = \max(P_{s-\tau,s}, p(c))$$

and consider the following two cases.

Case 1 : If $P_{s-\tau,s} < p(c)$; since $(p-c)D(p)$ is increasing on $[c,p(c)]$:

$$(p_s - c)D(P_{s-\tau,s}) \leq (P_{s-\tau,s} - c)(D(P_{s-\tau,s})$$

$$< (p(c) - c)D(p(c)) = (q_s - c)D(Q_{s-\tau,s}) .$$

Case 2 : If $P_{s-\tau,s} \geq p(c)$;

$$(p_s - c)D(P_{s-\tau,s}) = (p_s - c)D(Q_{s-\tau,s}) \leq (q_s - c)D(Q_{s-\tau,s})$$

with strict inequality if $p_s < q_s$, i.e. $p_s < p(c)$. Thus in both cases :

$$\pi_T(\langle p_s \rangle) = \theta\lambda \sum_{s=t}^{T} \delta^{s-t} \sum_{\tau=0}^{\infty} (1-\mu)^\tau (p_s - c)D(P_{s-\tau,s})$$

$$\leq \theta\lambda \sum_{s=t}^{T} \delta^{s-t} \sum_{\tau=0}^{\infty} (1-\mu)^\tau (q_s - c)D(Q_{s-\tau,s}) , \text{ or :}$$

(A19) $\pi_T(\langle p_s \rangle) \leq \pi (\langle \max(p(c) , p_s) \rangle)$

with strict inequality if any p_s is smaller than $p(c)$. □

Claim 2 : In the finite horizon problem, $p > \max\langle p(c) , P_{t+1} \rangle$ is never optimal.

Assume first that $P_{t-1} \leq p(c)$; by (A17), $p_s = p(c)$ for all $s \geq t$

maximizes π_T. Assume that on the contrary, $P_{t-1} > p(c)$; redefine the sequences

$\langle q_s \rangle$ and $\langle Q_{s-\tau,s} \rangle$ as : $q_s = \min\langle p_s, P_{t-1} \rangle$, $Q_{s-\tau,s} = \max\langle q_k | s-\tau \leq k \leq s \rangle$

$= \min\langle P_{s-\tau,s}, P_{t-1} \rangle$. For any $s \geq t$ and $\tau \geq 0$, three cases are possible.

Case 1 : If $P_{s-\tau,s} < P_{t-1}$; then $(p_s - c)D(P_{s-\tau,s}) = (q_s - c)D(Q_{s-\tau,s})$.

Case 2 : If $P_{s-\tau,s} > P_{t-1} > p(c)$ and $p_s > P_{t-1}$; then $(p_s - c)D(P_{s-\tau,s}) <$

$(P_{s-\tau,s} - c)D(P_{s-\tau,s}) < (P_{t-1} - c)D(P_{t-1}) = (q_s - c)D(Q_{s-\tau,s})$.

Case 3 : If $\max(p(c),p_s) < P_{t-1} < P_{s-\tau,s}$; then $(p_s - c)D(P_{s-\tau,s}) <$

$(p_s - c)D(P_{t-1}) = (p_s - c)D(Q_{s-\tau,s}) = (q_s - c)D(Q_{s-\tau,s})$.

Thus, in every case, $(p_s - c)D(P_{s-\tau,s}) < (q_s - c)D(Q_{s-\tau,s})$, with strict

inequality if and only if $P_{s-\tau,s} > P_{t-1}$. Hence :

$$\pi_T(\langle p_s \rangle) = \theta\lambda \sum_{s=t}^{T} \delta^{s-t} \sum_{\tau=0}^{\infty}(1-\mu)^{\tau} (p_s - c)D(P_{s-\tau,s})$$

$$< \theta\lambda \sum_{s=t}^{T} \delta^{s-t} \sum_{\tau=0}^{\infty}(1-\mu)^{\tau} (q_s - c)D(Q_{s-\tau,s})$$

with strict inequality if and only if there exists some $s > t$ and $\tau > 0$ such

that $P_{s-\tau,s} > P_{t-1}$, i.e. there exists $k > t$ such that $p_k > P_{t-1}$. Thus :

(A20) $\qquad \pi_T(\langle p_s \rangle) < \pi_T(\langle \min(p_s , P_{t-1}) \rangle)$

with strict inequality if $p_k > P_{t-1}$ for some $k > t$. $\square$

On the compact $\Omega_{t,T} \equiv [p(c), \max(p(c),P_{t-1})]^{T-t+1}$, the function π_T is

continuous, and therefore attains its maximum at some $\langle p_{T,s} | t < s < T \rangle$, with :

(A21) $\qquad p(c) < p_{T,s} < \max(p(c), P_{t-1})$

for all $s > t$. Moreover, for any sequence $\langle p_s | s > t \rangle \in R_+^{T-t+1}$, (A19)-(A20)

imply : $\pi_T(\langle p_s \rangle) < \pi_T(\langle \varrho_s \rangle)$, where $\langle \varrho_s | s > T \rangle \in \Omega_{t,T}$ is defined by

$\varrho_s \equiv \max(p(c), \min(p_s, P_{t-1}))$. As a result :

(A22) $\qquad \pi_T(\langle p_s \rangle) < \pi_T(\langle p_{T,s} \rangle)$

for any arbitrary $\langle p_s | s > t \rangle$. Let now the sequence $\langle p_{T,s} | t < s < T \rangle$ be

extended into an infinite one by setting $p_{T,s} = c$ for all $s > T$. Consider any

fixed $s > t$ and let $T > s$ vary ; the sequence $\langle p_{T,s} | T > s \rangle$ has values in the

compact $[p(c), \max(p(c), P_{t-1})]$, hence there exists a subsequence

$\langle p_{T^k,s} | k \in N \rangle$ converging to a limit p_s^* with :

(A23) $\qquad (\forall\, s > t) : \qquad p(c) < p_s^* < \max\langle p(c) , P_{t-1} \rangle$.

__Claim 3__ : (A24) For any sequence of prices $\langle p_s | s > t \rangle$: $\pi(\langle p_s \rangle) < \pi(\langle p_s^* \rangle)$.

In other words, $\langle p_s^* \rangle$ maximizes π . Fix $T > t$ large enough to have :

(A25) $\qquad \theta\lambda(p(c) - c)D(p(c))\ \delta^{T-t+1}/[(1-\delta)\mu] < \epsilon/2$

For all $s > t$ and $\tau > 0$:

$$(p_s - c)D(P_{s-\tau,s}) < (P_{s-\tau,s} - c)D(P_{s-\tau,s}) < (p(c) - c)D(p(c))$$

hence :

(A26) $\qquad \pi(\langle p_s \rangle) = \pi_T(\langle p_s \rangle) + \theta\lambda \sum_{s=T+1}^{\infty} \delta^{s-t} \sum_{\tau=0}^{\infty}(1-\mu)^{\tau}(p_s - c)\,D(P_{s-\tau,s})$

$$< \pi_T(\langle p_s \rangle) + \epsilon/2$$

But, by definition of $\langle p_{T,s} \rangle$:

$$\pi_T(\langle p_s \rangle) < \pi_T(\langle p_{T,s} \rangle) = \pi_{T'}(\langle p_{T,s} \rangle) \text{ for all } T' > T$$

(because $p_{T,s} = c$ for $s > T$) ; similarly, by definition of $p_{T',s}$:

$$\pi_{T'}(\langle p_{T,s} \rangle) < \pi_{T'}(\langle p_{T',s} \rangle) = \pi(\langle p_{T',s} \rangle) < \pi_T(\langle p_{T',s} \rangle) + \epsilon/2$$

by (A26) applied with $p_s = p_{T',s}$ for all $s > t$. Thus, finally :

(A27) $\qquad \pi(\langle p_s \rangle) < \pi_T(\langle p_{T',s} \rangle) + \epsilon \qquad (\forall\, T' > T)$.

Let now let T' take values T^k , with k going to infinity, so that each

$p_{T',s}$ converges to p^*_s ; since the above sum $\pi_T(\langle p_{T',s} \rangle)$ involves only a fixed,

finite number T-t+1 of such terms, it is licit to take limits to obtain :

(A28) $\pi(\langle p_s \rangle) < \pi_T(\langle p^*_s \rangle) + \epsilon$

which holds for all T large enough to have (A25). Finally, since for all s ,

$p^*_s > p(c) > c$ by (A21), and since demand is always non-negative :

$$\pi(\langle p_s \rangle) < \pi_T(\langle p^*_s \rangle) + \epsilon < \pi(\langle p^*_s \rangle) + \epsilon$$

for all $\epsilon > 0$. Letting now ϵ tend to zero concludes the proof.   Q.E.D.

Proof of Theorem IV.3 :

The optimality of firms' strategies (given their beliefs and consumers'

strategies) obtains by construction of $\langle p^*_s(H^f_t) \rangle$ in Lemma 2. A consumer's

problem is identical to what it was in Theorem IV.1, except that his effective

discount rate is now $\delta\mu$. His beliefs over the cost types of firms and the

prices charged in the past by those he did not visit have also been defined to

be identical to what they were in that same theorem. By (35), firm's

equilibrium strategies then induce the same (point) expectations over current

and future price paths $P^\sigma_{t,n}(f)$. Each consumer's optimal strategy is therefore

unchanged (except for the discount rate). Finally, the Bayesian property of

beliefs also results from Theorem IV.1, since the equilibrium path as well as

beliefs along it are unchanged.   Q.E.D.