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A FUNCTIONAL LIMIT THEOREM FOR FRACTIONAL PROCESSES

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**   CEPREMAP.

We hope that this paper will give a correct answer to a question by P.C.B. Phillips.

Abstract

A FUNCTIONAL LIMIT THEOREM FOR FRACTIONAL PROCESSES

We establish a functional limit theorem for non stationary fractional processes defined by :

$$(1 - L)^d \Phi(L) X_t = \Theta(L) \varepsilon_t ,$$

where Φ, Θ are polynomials in the lag operator L, ε is a white noise with variance σ^2 and d is a real number satisfying $d > \frac{1}{2}$. We prove that the process :

$\left[X_T^*(r) = \frac{1}{T^{d-1/2}} X_{(Tr)} , r \in (0, 1)\right]$ weakly converges to the process

$\left[\sigma \frac{\Theta(1)}{\Phi(1)} \frac{1}{\Gamma(d)} \int_o^r (r - s)^{d-1} dB(s), r \in [0, 1]\right]$, where B is a brownian motion on [0, 1].

These results are used to examine the behaviour of the periodo gram for low frequencies, which is a first step towards estimation of the differencing order d.

Keywords : fractional process, functional limit theorem, non stationarity, identification.

Résumé

UN THEOREME LIMITE FONCTIONNEL POUR DES PROCESSUS FRACTIONNAIRES

Nous établissons un théorème limite fonctionnel pour un processus fractionnaire non stationnaire défini par :

$$(1 - L)^d \Phi(L) X_t = \Theta(L) \varepsilon_t ,$$

où Φ, Θ sont des polynômes dans l'opérateur retard L, ε est un bruit blanc de variance σ^2 et d est un nombre réel satisfaisant $d > \frac{1}{2}$. Nous montrons que le processus :

$\left[X_T^*(r) = \frac{1}{T^{d-1/2}} X_{(Tr)} , r \in (0, 1)\right]$ converge faiblement vers le processus :

$\left[\sigma \frac{\Theta(1)}{\Phi(1)} \frac{1}{\Gamma(d)} \int_o^r (r - s)^{d-1} dB(s), r \in [0, 1]\right]$, où B est un mouvement brownien sur [0, 1].

Ces résultats sont utilisés pour examiner le comportement du periodo gramme aux basses fréquences, ce qui est une premier pas vers l'estimation du degré de non stationnarité d.

Mots clefs : processus fractionnaire, théorème limite fonctionnel, non stationnarité, identification.

Nomenclature JEL : 211.

I - INTRODUCTION.

The dynamic aspects of macromodeling : expectations, description of the adjustment processes, causality between variables, separation between short and long term...have been the topic of a number of theoretical and/or practical papers. These dynamic aspects are generally studied by means of ARMA or ARIMA models applied to the endogenous and exogenous processes. This kind of formulation is adapted for describing the stationary parts of the series, since every stationary process may be well approximated by an ARMA process.

However, concerning non stationary parts, the usual practice consisting in choosing integer orders of differencing introduces discontinuities, which often have drawbacks for the interpretation (this explains for instance the under or overdifferencing phenomenas) or for the properties of the statistical estimation methods (this explains the difficulty of the so-called identification step).

The idea of suppressing the discontinuity by introducing a fractional exponent is due to Mandelbrot-Van Ness (1968) and Mandelbrot (1971). In the case of discrete time process, it consists in considering the process satisfying a difference equation of the form :

$$(1 - L)^d \phi(L) X_t = \theta(L) \varepsilon_t \quad ,$$

where ε is a white noise with variance σ^2, L is the lag-operator, $\phi(L)$ and $\theta(L)$ are two lag polynomials satisfying the usual condition :

$\phi(o) = \theta(o) = 1$, the roots of ϕ and θ are outside the unit circle. The exponent d is a real number, not necessarily equal to an integer. The probabilistic properties of these fractional processes (also called long memory models when $d > o$) have been given by Granger (1978), Granger-Joyeux (1980), Hosking (1981), Gonçalves (1987) among others].

This class of processes seems now interesting for macromodelling. Firstly it gives a better description of non stationary aspects and in particular seems adapted to the study of cointegration properties of a serie [Granger (1986), Engel - Granger (1987)].
Secondly this type of processes is naturally introduced, when we consider the aggregation of heterogenous time series [Granger (1980)] and therefore is adapted for studying the heterogeneity bias or for proposing heterogeneïty tests in time series [Gonçalves-Gourieroux (1987)].

Of course the development of estimation or test procedures involving fractional processes is only possible if the ergodic properties of such processes are known. The case of stationary fractional processes, corresponding to $d < \frac{1}{2}$, has been studied in particular by [Rosenblatt (1976), Fox-Taqqu (1986), Geweke-Porter-Hudak (1983), Sowell (1987)].

On the other hand the case of non stationary fractional processes does not seem to be so well known. In such a case it is necessary to establish a functional limit theorem. Such a theorem will give the asymptotic behaviour of the process after some change of time unit and some reduction.

More precisely if we consider the transformed process :

$$\left[X_T^*(r) = \frac{1}{T^{d-1/2}} X_{[Tr]} \quad , \quad r \in [o,1]\right]$$

, where $[Tr]$ denotes the integer part of its argument, we want to prove that this process weakly converges to the process :

$$\left[X_\infty^*(r) = \frac{\sigma\ \theta(1)}{\phi(1)\ \Gamma(d)} \int_o^r (r-s)^{d-1}\ dB(s) \quad , \quad r \in [o,1]\right]$$

, where B is a brownian motion on the interval $[o,1]$.

A first idea to prove this kind of result is to follow the approach proposed by Phillips [(1986),(1987)] for the cases where d is an integer. This approach is based on a functional limit theorem given in Herrndorf (1984) and on the continuous mapping theorem given in Billingsley (1968). However this last theorem may only be directly applied for $d \geqslant 1$. Therefore it is necessary to develop a new proof for the remaining cases $\frac{1}{2} < d < 1$.

In section 2, we describe the theorem and the main steps of the proof.

In section 3, we propose a precise study of the moving average coefficients of a fractional process.

In section 4, we study weak convergence properties of Rieman sums towards stochastic integrals.

The proof of the theorem is then given in section 5 .

Finally we consider in section 6 some possible applications of the functional limit theorem. We discuss essentially some results concerning the periodogramm associated with X . Some other possible applications are given in [Gourieroux-Maurel-Monfort (1987].

II - THE THEOREM.

In order to treat the non stationary case, it is useful to only consider the fractional model, from a given initial date $t = o$. Fpr this reason, we consider in the paper a process $X = (X_t \, , \; t > o)$, defined by :

$$(1) \qquad X_t = (1-L)^{-d} \, \phi(L)^{-1} \, \theta(L) \, \tilde{\tilde{\varepsilon}}_t \, ,$$

where $\tilde{\tilde{\varepsilon}}_t = \begin{cases} \varepsilon_t \, , & \text{if } t > o \, , \\ o \, , & \text{otherwise} \, , \end{cases}$

$\varepsilon = (\varepsilon_t)$ is a white noise with variance σ^2, and where

$$(1-L)^{-d} = \sum_{k=o}^{\infty} \frac{\Gamma(k+d)}{\Gamma(k+1)\,\Gamma(d)} L^k \, .$$

To impose that $\tilde{\tilde{\varepsilon}}_t = o$ for non positive values of t gives a meaning to the operator $(1-L)^{-d} \, \phi(L)^{-1} \, \theta(L)$ applied to process $\tilde{\tilde{\varepsilon}}$, even if some roots of the autoregressive operator are equal to one.

In this case it is possible to consider the series expansion of :

$$(1-z)^{-d} \, \phi(z)^{-1} \, \theta(z) = \sum_{k=o}^{\infty} \pi_k^{(d)} z^k$$

, and to deduce the (infinite) moving average of process X :

$$X_t = \sum_{k=o}^{\infty} \pi_k^{(d)} \tilde{\tilde{\varepsilon}}_{t-k} = \sum_{k=o}^{t-1} \pi_k^{(d)} \varepsilon_{t-k} \, .$$

$$= \sum_{k=1}^{t} \pi_{t-k}^{(d)} \varepsilon_k \, .$$

The study of the asymptotic behaviour of process X may be sketched as follows.

i) Let us first introduce the cumulated errors :

$$S_k = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k \quad , \quad k > o \quad , \quad S_o = o \quad .$$

The process X may also be written as :

$$X_t = \sum_{k=1}^{t} \Pi_{t-k}^{(d)} (S_k - S_{k-1}) \quad .$$

ii) After a change of time unit and an homothety, we have :

$$X_T^*(r) = \frac{1}{T^{d-1/2}} X_{[Tr]}$$

$$= \frac{1}{T^{d-1/2}} \sum_{k=1}^{[Tr]} \Pi_{[Tr]-k}^{(d)} (S_k - S_{k-1}) \quad .$$

iii) Now we can examine the moving average coefficients $\Pi_{t-k}^{(d)}$ and we can prove that these coefficients are asymptotically equivalent to :

$$\Pi_{t-k}^{(d)} \approx \frac{\theta(1)}{\phi(1)\,\Gamma(d)} (t-k)^{d-1} \quad .$$

iv) Moreover it is a consequence of results on cumulated errors [see e.g. Komlos - Major - Tusnady (1976)] that the sums $\frac{S_k}{\sigma}$ can be approximated by a brownian motion $B(k)$: $\frac{S_k}{\sigma} \approx B(k)$.

v) Considering these approximations, we have :

$$X_T^*(r) = \frac{1}{T^{d-1/2}} \sum_{k=1}^{[Tr]} \frac{\sigma\,\theta(1)}{\phi(1)\,\Gamma(d)} ([Tr] - k)^{d-1} [B(k) - B(k-1)]$$

$$= \frac{\sigma\,\theta(1)}{\phi(1)\,\Gamma(d)} \sum_{k=1}^{[Tr]} \frac{([Tr] - k)^{d-1}}{T^{d-1}} \left[B(\frac{k}{T}) - B(\frac{k-1}{T})\right] \text{(equality in distribution)}$$

$$\approx \frac{\sigma\,\theta(1)}{\phi(1)\,\Gamma(d)} \sum_{k=1}^{[Tr]} \left(r - \frac{k}{T}\right)^{d-1} \left[B(\frac{k}{T}) - B(\frac{k-1}{T})\right] \quad .$$

vi) Finally the Rieman-Stieljes sum : $\sum_{k=1}^{[Tr]} \left(r - \frac{k}{T}\right)^{d-1} \left[B(\frac{k}{T}) - B(\frac{k-1}{T})\right]$ is an approximation of the stochastic integral $\int_o^r (r-s)^{d-1} \, dB(s)$ and we have :

$$X_T^*(r) \approx \frac{\sigma \, \theta(1)}{\phi(1) \, \Gamma(d)} \int_o^r (r-s)^{d-1} \, dB(s) \quad .$$

Clearly the different steps have to be justified and this is the aim of the following sections. The difficult parts of the proof are of course to justify the various equivalences $\approx$ and to obtain results which are uniform in r between o and 1 .

The theorem which is proved is the following :

THEOREM 1 :

Let us consider the process defined by

$$X_t = (1 - L)^{-d} \, \phi(L)^{-1} \, \theta(L) \, \tilde{\tilde{\varepsilon}}_t \quad , \qquad d > \frac{1}{2} \quad ,$$

where $\tilde{\tilde{\varepsilon}}_t = \begin{cases} \varepsilon_t & , \quad \text{if} \quad t > o \quad , \\ o & , \quad \text{otherwise} \quad , \end{cases}$

(ε_t) is an independent white noise, with moment of order p strictly greater than $\text{Max}\left(\frac{1}{d - \frac{1}{2}} , 2\right)$.

Then the process $\left(X_T^*(r) = \frac{1}{T^{d-1/2}} \, X_{[Tr]} \; , \; r \in [o,1] \right)$

weakly converges to the process :

$$\left(\frac{\sigma\, \theta(1)}{\phi(1)\, \Gamma(d)} \int_o^r (r-s)^{d-1}\, dB(s) \quad , \quad r \in [o,1] \right) \quad ,$$

where B is a brownian motion on $[o,1]$

The processes X_T^* and B are indexed by $r \in [o,1]$.

The associated sample paths are elements of $D[o,1]$: the space of real functions on $D[o,1]$ which are right continuous and have finite left limits. the space $C[o,1]$ is a metric space when we introduce the Skorohod topology [see Billingsley (1968) Chapter 3]. The weak convergence appearing in theorem 1 is associated with this particular topology. Let us recall that the restriction of this topology to the set $C[o,1]$ of continuous functions on $[o,1]$ is nothing else than the uniform topology.

III - ASYMPTOTIC BEHAVIOUR OF THE MOVING AVERAGE COEFFICIENTS

In this section, we prove that the coefficient $\pi_k^{(d)}$ is asymptotically equivalent to : $\dfrac{\sigma\, \theta(1)}{\phi(1)\, \Gamma(d)}\, k^{d-1}$, and we give the associated convergence rate.

Property 2 :

There exists a constant c , $c > o$, such that :

$$\forall \quad k \in N^* \quad : \quad \left| \pi_k^{(d)} - \frac{\theta(1)}{\phi(1)\, \Gamma(d)}\, k^{d-1} \right| \leqslant c\, k^{d-2} \quad .$$

Proof :

i) Since the roots of the autoregressive polynomial ϕ are outside the unit circle, the coefficients α_j of $A(z) = \sum_{j=o}^{\infty} \alpha_j z^j = \frac{\theta(z)}{\phi(z)}$, decrease at an exponential rate :

$$\exists \ \varrho \ , \ o < \varrho < 1 \ , \ \exists \ c_o > o \ : \ \forall j \ \ |\alpha_j| < c_o \varrho^j \ .$$

Moreover it is possible to express the coefficients $\pi_k^{(d)}$ in terms of the coefficients α_j . We have :

$$\sum_{k=o}^{\infty} \pi_k^{(d)} z^k = (1 - z)^{-d} A(z)$$

$$= \left[\sum_{k=o}^{\infty} \frac{\Gamma(k+d)}{\Gamma(k+1)\ \Gamma(d)} z^k \right] \left[\sum_{j=o}^{\infty} \alpha_j z^j \right] \ .$$

We deduce that :

$$\pi_k^{(d)} = \sum_{j=o}^{k} \alpha_{k-j} \frac{\Gamma(j+d)}{\Gamma(j+1)\ \Gamma(d)} \ .$$

ii) The asymptotic behavior of $\pi_k^{(d)}$ depends on the asymptotic behavior of $\frac{\Gamma(j+d)}{\Gamma(j+1)\ \Gamma(d)}$. We have the following lemma.

Lemma 3 : There exists c_1 such that :

$$\left| \frac{\Gamma(j+d)}{\Gamma(j+1)\ \Gamma(d)} - \frac{j^{d-1}}{\Gamma(d)} \right| \leqslant c_1 j^{d-2} \ , \quad \forall \ j \geqslant 1 \ .$$

Proof of the lemma : From [Abramowitz-Stegun (1970) formula 6.1.47], we know that for z tending to infinity :

$$z^{b-a}\ \frac{\Gamma(z+a)}{\Gamma(z+b)} = 1 + \frac{(a-b)\ (a+b-1)}{2\ z} + o\left(\frac{1}{z}\right) \quad .$$

Applying this result, we see that :

$\dfrac{\Gamma(j+d)}{\Gamma(j+1)} = j^{d-1}\left(1 + \dfrac{(d-1)d}{2j} + o\left(\dfrac{1}{j}\right)\right)$, and we deduce the lemma.

iii) Now, we have :

$$\left|\ \pi_k^{(d)} - \frac{\theta(1)}{\phi(1)}\ \frac{k^{d-1}}{\Gamma(d)}\ \right|$$

$$= \left|\ \sum_{j=o}^{k} \alpha_{k-j}\ \frac{\Gamma(j+d)}{\Gamma(j+1)\ \Gamma(d)} - A(1)\ \frac{k^{d-1}}{\Gamma(d)}\ \right|$$

$$= \left|\ \alpha_k + \sum_{j=1}^{k} \alpha_{k-j}\left[\frac{\Gamma(j+d)}{\Gamma(j+1)\ \Gamma(d)} - \frac{j^{d-1}}{\Gamma(d)}\right] + \sum_{j=1}^{k}\alpha_{k-j}\left[\frac{j^{d-1}}{\Gamma(d)} - \frac{k^{d-1}}{\Gamma(d)}\right]\right.$$

$$\left. - \sum_{j=k}^{\infty} \alpha_j\ \frac{k^{d-1}}{\Gamma(d)}\ \right|$$

$$\leqslant A_k + B_k + C_k + D_k \quad ,$$

where : $A_k = |\alpha_k|$,

$$B_k = \sum_{j=1}^{k} |\alpha_{k-j}| \left|\frac{\Gamma(j+d)}{\Gamma(j+1)\ \Gamma(d)} - \frac{j^{d-1}}{\Gamma(d)}\right| \quad ,$$

$$C_k = \sum_{j=1}^{k} |\alpha_{k-j}|\ \frac{\left|j^{d-1} - k^{d-1}\right|}{\Gamma(d)} \quad ,$$

$$D_k = \left|\ \sum_{j=k}^{\infty} \alpha_j\ \right|\ \frac{k^{d-1}}{\Gamma(d)} \quad .$$

iv) Since the coefficients $|\alpha_j|$ are smaller than $c_o\ \rho^j$, we immediately see that there exist $\tilde{\tilde{\rho}}$, $\rho < \tilde{\tilde{\rho}} < 1$ and K_1 , K_2 such that : $A_k < K_1 \tilde{\tilde{\rho}}^k$, $D_k < K_2 \tilde{\tilde{\rho}}^k$, $\forall k$.

v) Let us now examine the second term B_k . From lemma 3 and from the exponential decrease of α_j , we deduce :

$$B_k \leqslant \sum_{j=1}^{k} c_o \rho^{k-j} c_1 j^{d-2} = c_o c_1 \sum_{j=o}^{k-1} \rho^j (k-j)^{d-2} .$$

Two cases have to be distinguished.

(*) If $d \geqslant 2$, we have : $B_k \leqslant c_o c_1 \sum_{j=o}^{k-1} \rho^j k^{d-2} \leqslant K_3 k^{d-2}$.

(**) If $d \leqslant 2$, we have

$$B_k \leqslant c_o c_1 \sum_{j=o}^{[k/2]} \rho^j (k-j)^{d-2} + c_o c_1 \sum_{j=[k/2]+1}^{k-1} \rho^j (k-j)^{d-2} ,$$

$$B_k \leqslant c_o c_1 \left[\sum_{j=o}^{[k/2]} \rho^j\right] (k - [k/2])^{d-2} + c_o c_1 \sum_{j=[k/2]+1}^{k-1} \rho^j ,$$

$$B_k \leqslant c_o c_1 \frac{1}{1-\rho} (k - [k/2])^{d-2} + c_o c_1 \frac{\rho^{[k/2]}}{1-\rho} \leqslant K_4 k^{d-2} .$$

vi) Finally let us study the third term. From mean value theorem, we have :

$$j^{d-1} - k^{d-1} = (j-k)(d-1)\,\theta^{d-2} \quad \text{with} \quad \theta \in [j,k] .$$

(*) If $d \geqslant 2$, we have :

$$C_k \leqslant \sum_{j=1}^{k} |\alpha_{k-j}| (k-j)(d-1) k^{d-2} \leqslant K_5 k^{d-2} ,$$

since the serie with general term $j \alpha_j$ is absolutely convergent.

(**) If $d \leq 2$, we have :

$$C_k \leq \sum_{j=1}^{k} |\alpha_{k-j}| \,(k-j)(d-1)\, j^{d-2} \leq K_6 \sum_{j=1}^{k} \tilde{\varrho}^{\,k-j}\, j^{d-2} \leq K_6 \sum_{j=o}^{k-1} \tilde{\varrho}^{\,j}\, (k-j)^{d-2} ,$$

where : $\varrho < \tilde{\varrho} < 1$. Then the same approach as in v) (**) gives the result :

$C_k \leq K_6\, k^{d-2}$.

□

IV - WEAK CONVERGENCE OF THE RIEMAN - STIELJES SUM TO THE STOCHASTIC INTEGRAL

In this subsection we are interested in the last step discussed when sketching the proof, i.e. in the weak convergence of :

$\sum_{k=1}^{[Tr]} \left(r - \frac{k}{T}\right) \left[B\left(\frac{k}{T}\right) - B\left(\frac{k-1}{T}\right)\right]$ to the stochastic integral $\int_o^r (r-s)^{d-1} dB(s)$,

for $d > \frac{1}{2}$. (For the first process to be defined, we adopt the convention $x^{d-1} = 1$, if $o < x < 1$).

It is first interesting to note that a partial result might be obtained by both integrating by part the sum and the integral. The integral is for instance equal to :

$$\int_o^r (r-s)^{d-1}\, dB(s) = (d-1) \int_o^r (r-s)^{d-2}\, B(s)\, ds .$$

Then we could use the fact that the mapping defined on $C\,[o,1]$ by :

$$(b(r) \;,\; r \in [o,1]) \longrightarrow \left(\int_o^r (r-s)^{d-2}\, b(s)\, ds \;,\; r \in [o,1] \right)$$

is uniformly continuous, if $\int_0^r (r-s)^{d-2} ds < \infty$, i.e. $d > 1$, and then the continuous mapping theorem [see Billingsley (1968) Theorem 5.1] could be applied. Unfortunately such an approach would only give the result for $d > 1$ and not for $d > \frac{1}{2}$. Therefore it is necessary to develop another approach and to explicitly use the fact that the two processes are gaussian ones.

Property 4 :

The processes $\left(Z_T(r) = \sum_{k=1}^{[Tr]} \left(r - \frac{k}{T}\right)^{d-1} \left[B\left(\frac{k}{T}\right) - B\left(\frac{k-1}{T}\right)\right] , r \in [o,1]\right)$

weakly converge to the process :

$\left(Z(r) = \int_0^r (r-s)^{d-1} d B(s) , r \in [o,1]\right)$, when T goes to infinity and $d > \frac{1}{2}$.

Proof : i) The proof will be based on Prohorov's theorem [Billingsley (1968), p. 35-40). Weak convergence is satisfied if :

(*) $\forall r_1,\dots,r_n$, $[Z_T(r_1),\dots,Z_T(r_n)]$ weakly converges to $[Z(r_1),\dots,Z(r_n)]$,

(**) the sequence $[Z_T(r)]$ is tight.

ii) The convergence of the finite dimensional distributions is a direct consequence of the definition of the stochastic integral $Z(r)$ as the limit of $Z_t(r)$ in quadratic mean. This implies that $[Z_T(r_1),\dots,Z_T(r_n)]$ converges in quadratic mean to $[Z(r_1),\dots,Z(r_n)]$ and therefore the convergence is also valid in distribution.

iii) For the condition of tightness it is necessary to use a criterion easy to apply. One possible one is the following. [See the restrictive version of theorem 15.6 given at the bottom of page 128 in Billingsley and note that Z is continuous].

Tightness criterion

There exist $c > o$, $\gamma > o$, $\alpha > 1$ such that :

$$\forall\, r_1 < r < r_2 : E\left[|Z_T(r) - Z_T(r_1)|^{\gamma} |Z_T(r_2) - Z_T(r)|^{\gamma}\right] \leqslant c(r_2 - r_1)^{\alpha} .$$

iv) Let us first show that another tightness criterion is :

There exist $c > o$, $\beta > o$ such that :

$$\forall\, r_1 < r < r_2 \quad : \quad E\left[Z_T(r) - Z_T(r_1)\right]^2 E\left[Z_T(r_2) - Z_T(r)\right]^2 \leqslant c(r_2 - r_1)^{\beta} .$$

Since we are only interested in gaussian processes, we have :

$$E\left[Z_T(r) - Z_T(r_1)\right]^{2\gamma} = K_{\gamma} \left[E\left[Z_T(r_1) - Z_T(r)\right]^2\right]^{\gamma} ,$$

where K_{γ} is the absolute moment of order 2γ of a standard normal. Let us now choose a value of γ such that $\gamma > \frac{2}{\beta}$. If the previous criterion is satisfied, we have : $\left[E\left[Z_T(r) - Z_T(r_1)\right]^{2\gamma}\right]\left[E\left[Z_T(r_2) - Z_T(r)\right]^{2\gamma}\right] \leqslant c\, K_{\gamma}^2 (r_2 - r_1)^{\beta\gamma}$.

From the Cauchy-Schwarz inequality we deduce that :

$$E\left[|Z_T(r) - Z_T(r_1)|^{\gamma}\, |Z_T(r_2) - Z_T(r)|^{\gamma}\right] \leqslant \sqrt{c}\; K_{\gamma}\, (r_2 - r_1)^{\beta\gamma/2} .$$

Since $\beta\gamma/2 > 1$, we see that the Billingsley's tightness criterion is satisfied.

v) Therefore we have to compute the second order moment of $Z_T(r) - Z_T(r_1)$, $r > r_1$.

We have :

$$E\left[Z_T(r) - Z_T(r_1)\right]^2$$

$$= E\left[\sum_{k=1}^{[Tr]}\left(r - \frac{k}{T}\right)^{d-1}\left[B\left(\frac{k}{T}\right) - B\left(\frac{k-1}{T}\right)\right] - \sum_{k=1}^{[Tr_1]}\left(r_1 - \frac{k}{T}\right)^{d-1}\left[B\left(\frac{k}{T}\right) - B\left(\frac{k-1}{T}\right)\right]\right]^2$$

$$= \frac{1}{T}\sum_{k=1}^{[Tr_1]}\left[\left(r - \frac{k}{T}\right)^{d-1} - \left(r_1 - \frac{k}{T}\right)^{d-1}\right]^2 + \frac{1}{T}\sum_{k=[Tr_1]+1}^{[Tr]}\left(r - \frac{k}{T}\right)^{2d-2} \mathbb{1}_{[Tr]>[Tr_1]} .$$

$= D_1 + D_2$ (Say), where $\mathbb{1}_A$ denotes the indicator function of A .

vi) The first term is always smaller than a positive power of $r - r_1$.

(*) If $d \geqslant 2$, we have for $x \in [o, r_1]$:

$$\left|(r - x)^{d-1} - (r_1 - x)^{d-1}\right|$$

$$= (d-1)(r - r_1)\,\xi^{d-2} \qquad \text{(from mean value theorem)}$$

$$\leqslant (d-1)(r - r_1) \qquad (\text{since } \xi \leqslant r \leqslant 1)$$

Therefore : $D_1 \leqslant (d-1)^2 (r - r_1)^2$.

(**) If $1 < d < 2$, we can note that the function x^{d-1} is a concave function. Therefore the maximum of $(r - x)^{d-1} - (r_1 - x)^{d-1}$ on $[o, r_1]$ is obtained for $x = r_1$. Then we have : $D_1 \leqslant (r - r_1)^{2d-2}$.

(***) Finally if $d \leqslant 1$, we have :

$$D_1 \leqslant \int_o^{r_1} \left[(r-x)^{d-1} - (r_1-x)^{d-1}\right]^2 dx ,$$

$$D_1 \leqslant \int_o^{r_1} \left[(r-r_1 + r_1-x)^{d-1} - (r_1-x)^{d-1}\right]^2 dx ,$$

$$D_1 \leqslant (r-r_1)^{2d-2} \int_o^{r_1} \left[\left(1 + \frac{r_1-x}{r-r_1}\right)^{d-1} - \left(\frac{r_1-x}{r-r_1}\right)^{d-1}\right]^2 dx ,$$

$$D_1 \leqslant (r-r_1)^{2d-1} \int_o^{r_1/(r-r_1)} \left[(1+v)^{d-1} - v^{d-1}\right]^2 dv ,$$

$$D_1 \leqslant (r-r_1)^{2d-1} \int_o^{\infty} \left[(1+v)^{d-1} - v^{d-1}\right]^2 dv .$$

We can note that the integral of the RHS exists, as soon as $d < 3/2$.

vii) Let us now examine the second term $D_2 = o$.

(*) If $[Tr] = [Tr_1]$, we have $D_2 = o$.

(**) In the case $[Tr] \neq [Tr_1]$, we have :

$$D_2 = \frac{1}{T} \sum_{k=[Tr_1]+1}^{[Tr]} \left(r - \frac{k}{T}\right)^{2d-2}$$

$$= \frac{1}{T} \sum_{k=[Tr_1]+1}^{[Tr]-1} \left(r - \frac{k}{T}\right)^{2d-2} + \frac{1}{T}\left([Tr] - \frac{k}{T}\right)^{2d-2}$$

$$\leqslant \int_{r_1-(1/T)}^{r} (r-x)^{2d-2} dx + \frac{1}{T^{2d-3}}$$

$$= \frac{(r-r_1+\frac{1}{T})^{2d-1}}{2d-1} + \frac{1}{T^{2d-3}} .$$

We get in particular $D_2 \leqslant K$, where K is a well chosen constant

viii) Finally let us consider simultaneously the majorations obtained in vi) and vii) . We have :

$$E[Z_T(r) - Z_T(r_1)]^2 = D_1 + D_2 ,$$

$$E[Z_T(r) - Z_T(r_2)]^2 = D'_1 + D'_2 .$$

(*) If $r_2 - r_1 < \frac{1}{T}$, we have either $[Tr] = [Tr_1]$ or $[Tr] = [Tr_2]$ and then either $D_2 = o$ or $D'_2 = o$.

Let us consider the case $D'_2 = o$, we have :

$$(D_1 + D_2)(D'_1 + D'_2) \leqslant [K_1(r-r_1)^\gamma + K][K_1(r_2-r)]^\gamma$$

$$\leqslant C(r_2-r)^\gamma \leqslant C(r_2-r_1)^\gamma .$$

(**) If $r_2 - r_1 > \frac{1}{T}$, there exists some positive constants K_2 and γ_2 such that $D_2 < K_2(r_2 - r_1)^{\gamma_2}$.

Then $(D_1 + D_2)(D'_1 + D'_2) \leqslant [K_1(r - r_1)^\gamma + K_2(r_2 - r_1)^{\gamma_2}] K_3$

$$\leqslant K_4(r_2 - r_1)^{\gamma_4} .$$

V - PROOF OF THE THEOREM

i) The proof is based on the existence of a sequence of independent standard normal variables $(e_t \; , \; t \geq 1)$ such that $B(t) = e_1 + \ldots + e_t$ is close to $\frac{1}{\sigma} S$. More precisely the following property has been obtained by [Komlos-Major-Tusnady (1976), Theorem 4].

Property 5 :

Let us consider a sequence $\varepsilon_1, \ldots, \varepsilon_t, \ldots$ of i.i.d random variables with zero mean and variance equal to σ^2 .

If H is a function such that :

i) $H(x) > 0$, if $x > 0$;

ii) H is an increasing continuous function ;

iii) $\exists \; \delta > 0 \; : \; \frac{H(x)}{x^{3+\delta}}$ is increasing for $x > x_0$;

iv) $\frac{\text{Log } H(x)}{x}$ is decreasing for $x > x_0$,

v) $E \, H(|\varepsilon_t|) < +\infty$.

Then for each T there exist positive constants C_1, C_2, a (only depending on the distribution of ε) and a finite sequence $e_1, \ldots, e_T$ of i.i.d. r.v. with standard normal distributions such that :

$\forall x$, $H^{-1}(T) < x < C_1 \sqrt{T \text{ Log } T}$:

$$P\left[\sup_{k \leq T} \left| \frac{S_k}{\sigma} - B(k) \right| > x \right] \leq C_2 \frac{T}{H(ax)} ,$$

where : $S_k = \varepsilon_1 + \ldots + \varepsilon_k$ and $B(k) = e_1 + \ldots + \varepsilon_k$.

This property may for instance be applied with $H(x) = x^p$, with $p > 3$ and $x = T^{\frac{1}{p}+\gamma}$, $o < \gamma < \frac{1}{2} - \frac{1}{p}$. The inequality becomes :

$$(6) \qquad P\left[\sup_{k\leqslant T}\left|\frac{S_k}{\sigma} - B(k)\right| > T^{\frac{1}{p}+\gamma}\right] \leqslant C\,\frac{1}{T^{\rho\gamma}} .$$

This property may be extended using theorem 2 of Major (1976) to the case :

$E|\varepsilon_t|^p < \infty$, with : $2 < p \leqslant 3$.

ii) We may now decompose the transformed process X_T^* into several components:

$$X_T^*(r) = \frac{1}{T^{d-1/2}} X_{[Tr]}$$

$$= \frac{1}{T^{d-1/2}} \sum_{k=1}^{[Tr]} \pi_{[Tr]-k}^{(d)} (S_k - S_{k-1})$$

$$= \xi_{1T}(r) + \xi_{2T}(r) + \xi_{3T}(r) ,$$

where : $$\xi_{1T}(r) = \frac{\sigma\, A(1)}{\Gamma(d)} \sum_{k=1}^{[Tr]} \left(r - \frac{k}{T}\right)^{d-1} \frac{B(k) - B(k-1)}{\sqrt{T}} ,$$

$$\xi_{2T}(r) = \frac{\sigma}{T^{d-1/2}} \sum_{k=1}^{[Tr]} \pi_{[Tr]-k}^{(d)} \left[\frac{S_k - S_{k-1}}{\sigma} - [B(k) - B(k-1)]\right] ,$$

$$\xi_{3T}(r) = \frac{\sigma}{T^{d-1/2}} \sum_{k=1}^{[Tr]} \left[\pi_{[Tr]-k}^{(d)} - \frac{A(1)}{\Gamma(d)} (Tr-k)^{d-1}\right] [B(k) - B(k-1)] .$$

The process ξ_{1T} has the same distribution as $\frac{\sigma A(1)}{\Gamma(d)} Z_T$, since $\left[\frac{B(k) - B(k-1)}{\sqrt{T}}\right]$ has the same distribution as $\left[B\left(\frac{k}{T}\right) - B\left(\frac{k-1}{T}\right)\right]$.

Therefore we deduce from property 4 that ξ_{1T} weakly converges to :

$$\left[\frac{\sigma A(1)}{\Gamma(d)} \int_o^r (r-s)^{d-1} dB(s) \quad , \quad r \in [o,1]\right] .$$

Then theorem 1 is proved if we verify that ξ_{2T} and ξ_{3T} converge in probability to zero.

iii) Convergence of ξ_{2T} to zero.

We have :

$$\xi_{2T}(r) = \frac{\sigma}{T^{d-1/2}} \left(\frac{\varepsilon_{[Tr]}}{\sigma} - e_{[Tr]}\right) + \frac{\sigma}{T^{d-1/2}} \sum_{k=1}^{[Tr]-1} \pi_{[Tr]-k}^{(d)} \left[\frac{S_k - S_{k-1}}{\sigma} - [B(k)-B(k-1)]\right]$$

$$\xi_{2T}(r) = \frac{\sigma}{T^{d-1/2}} \left(\frac{\varepsilon_{[Tr]}}{\sigma} - e_{[Tr]}\right) + \frac{\sigma}{T^{d-1/2}} \sum_{k=1}^{[Tr]-1} \pi_{[Tr]-k}^{(d-1)} \left[\frac{S_k}{\sigma} - B(k)\right] .$$

The process $\left(\frac{\sigma}{T^{d-1/2}} \left(\frac{\varepsilon_{[Tr]}}{\sigma} - e_{[Tr]}\right) \quad , \quad r \in [o,1]\right)$ tends in probability to zero, and we only have to examine the second term :

$$\xi_{2T}^{*}(r) = \frac{\sigma}{T^{d-1/2}} \sum_{k=1}^{[Tr]-1} \pi_{[Tr]-k}^{(d-1)} \left[\frac{S_k}{\sigma} - B(k)\right] .$$

We have :

$$\sup_{r\in[o,1]} \left|\xi^*_{2T}(r)\right| \leqslant \sup_{r\in[o,1]} \frac{\sigma}{T^{d-1/2}} \sum_{k=1}^{[Tr]-1} \left|\pi^{(d-1)}_{[Tr]-k}\right| \sup_{k\leqslant T} \left|\frac{S_k}{\sigma} - B(k)\right| ,$$

$$\sup_{r\in[o,1]} \left|\xi^*_{2T}(r)\right| \leqslant \frac{\sigma}{T^{d-1/2}} C_o \sum_{k=1}^{T-1} (T-k)^{d-2} \sup_{k\leqslant T} \left|\frac{S_k}{\sigma} - B(k)\right| \text{ (from property 2)}$$

$$\sup_{r\in[o,1]} \left|\xi^*_{2T}(r)\right| \leqslant C_1 \frac{1}{T^{\mathrm{Min}(d-1/2,1/2)}} \sup_{k\leqslant T} \left|\frac{S_k}{\sigma} - B(k)\right|$$

(Since $1^{d-2} + \ldots + (T-1)^{d-2}$ is convergent for $d < 1$, and is of order T^{d-1} for $d > 1$) .

Comparing to Komlos-Major-Tusnady' and Major's results on $\sup_{k\leqslant T} \left|\frac{S_k}{\sigma} - B(k)\right|$, we see that convergence in probability holds as soon as :
$\frac{1}{p} < \mathrm{Min}\left(d - \frac{1}{2} , \frac{1}{2}\right) \Longleftrightarrow p > \mathrm{Max}\left(\frac{1}{d-1/2} , 2\right)$, if ε_t has a moment of order strictly greater than 2 .

iv) <u>Convergence of</u> ξ_{3T} <u>to zero</u>

(*) $$\xi_{3T}(r) = \frac{\sigma}{T^{d-1/2}} \left[1 - \frac{A(1)}{\Gamma(d)} (Tr - [Tr])^{d-1}\right] e_{[Tr]} + \xi^*_{3T}(r) .$$

It is directly seen that the first term tends in probability to zero and thus we only have to examine the second one.

(**) We have :

$$\sup_{r\in[o,1]} \left|\xi^*_{3T}(r)\right| \leqslant \frac{\sigma}{T^{d-1/2}} \sup_{r\in[o,1]} \sum_{k=1}^{[Tr]-1} \left|\pi^{(d)}_{[Tr]-k} - \frac{A(1)}{\Gamma(d)} (Tr-k)^{d-1}\right| \max_{k\leqslant T} |e_k| ,$$

$$\sup_{r\in[o,1]} \left|\xi^*_{3T}(r)\right| \leqslant \frac{\sigma}{T^{d-1/2}} \left\{ \sup_{r\in[o,1]} \sum_{k=1}^{[Tr]-1} \left|\pi^{(d)}_{[Tr]-k} - \frac{A(1)}{\Gamma(d)} ([Tr]-k)^{d-1}\right| \right.$$

$$+ \sup_{r\in[o,1]} \sum_{k=1}^{[Tr]-1} \frac{|A(1)|}{\Gamma(d)} \left| (Tr-k)^{d-1} - ([Tr]-k)^{d-1} \right| \Bigg\} \operatorname{Max}_{k\leq T} |e_k| \quad .$$

(***) From property 2, we know that :

$$\left| \pi^{(d)}_{[Tr]-k} - \frac{A(1)}{\Gamma(d)} ([Tr]-k)^{d-1} \right| \leq C_o \, ([Tr]-k)^{d-2} \quad .$$

On the other hand using the mean value theorem, we have :

$$\left| (Tr-k)^{d-1} - ([Tr]-k)^{d-1} \right| = (Tr - [Tr]) \, |d-1| \, \xi^{d-2} \quad ,$$

where $[Tr]-k \leq \xi \leq Tr-k \leq [Tr]-k+1$.

Therefore $\left| (Tr-k)^{d-1} - ([Tr]-k)^{d-1} \right| \leq |d-1| \operatorname{Max}\left[([Tr]-k)^{d-2}, ([Tr]-k+1)^{d-2} \right]$.

(****) We deduce that there exists some constant C such that :

$$\sup_{r\in[o,1]} \left| \xi^{*}_{3T}(r) \right| \leq \frac{\sigma}{T^{d-1/2}} \, C_o \, T^{d-1} \operatorname{Max}_{k\leq T} |e_k| = \frac{\sigma \, C}{\sqrt{T}} \operatorname{Max}_{k\leq T} |e_k| \quad .$$

Since the random variables e_k are i.i.d. standard normal variables, we have :

$$P\left[\sup_{r\in[o,1]} \left| \xi^{*}_{3T}(r) \right| \geq b \right] \leq P\left[\operatorname{Max}_{k\leq T} |e_k| \geq \frac{b\sqrt{T}}{\sigma \, C} \right] \quad .$$

If Φ and φ are respectively the cumulative function and the density function of the standard normal, we have :

$$P\left[\sup_{r\in[o,1]} \left| \xi^{*}_{3T}(r) \right| \geq b \right] \leq \left[1 - \left[2\Phi\left(\frac{b\sqrt{T}}{\sigma \, C} \right) - 1 \right]^{T} \right] \quad .$$

Asymptotically : $1 - \Phi(x) \approx x \, \varphi(x)$ and the upper bound is :

$$1 - \left[\Phi \left(\frac{b\sqrt{T}}{\sigma C}\right) \right]^T \approx 1 - \left[1 - 2 \frac{b\sqrt{T}}{\sigma C} \varphi \left(\frac{b\sqrt{T}}{\sigma C}\right)\right]^T$$

$$\approx 1 - \exp \left\{ T \operatorname{Log} \left[1 - 2 \frac{b\sqrt{T}}{\sigma C} \varphi \left(\frac{b\sqrt{T}}{\sigma C}\right)\right] \right\}$$

$$\approx 1 - \exp \left\{ - 2 \frac{bT^{3/2}}{\sigma C} \varphi \left(\frac{b\sqrt{T}}{\sigma C}\right) \right\}$$

$$\approx 1 - 1 = o \quad .$$

Therefore, we have :

$$\lim_{T\to\infty} P \left[\operatorname{Sup}_{r\in[o,1]} \left|\xi^*_{3T}(r)\right| \geq b \right] = o \quad .$$

□

VI - STUDY OF THE PERIODOGRAM FOR LOW FREQUENCIES [See Janacek (1982)]

The second order properties of a stationary process can be analysed in frequency domain, i.e. using the spectral density function and its empirical counterpart the periodogram. In the non stationary case of fractional processes, it is possible to introduce by analogy the pseudo-spectral density function :

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{|\theta(\exp i\omega)|^2}{|1 - \exp i\omega|^{2d} |\Phi(\exp i\omega)|^2} \quad .$$

In a neighbourhood of $\omega = o$, this function is equivalent to :

$$f_X(\omega) \approx \frac{\sigma^2}{2\pi} \left[\frac{\theta(1)}{\phi(1)}\right]^2 \frac{1}{\omega^{2d}} \quad .$$

This implies that a study at low frequencies may give some information concerning the order of differencing.

In practice the pseudo-spectrum is unknown and in this section we replace it by the periodogram :

$$I_T(\omega) = \sum_{t=1}^{T} X_T \exp it\omega \quad .$$

Since we have simultaneously T tends to infinity and ω to zero, we are going to relate the two rates of convergence.

Property 7 :

If X is a fractional process satisfying the assumption of theorem 1, the process $\left(\frac{1}{T^{d+1/2}} \; I_T\left(\frac{\omega}{T}\right) \; , \; \omega \in [o,\pi] \right)$ weakly converges to

$$\left(\frac{\sigma \, \theta(1)}{\phi(1) \, \Gamma(d)} \int_o^1 \left[\int_s^1 \exp i\omega \, r \, (r - s)^{d-1} \, dr \right] dB(s) \quad , \quad \omega \in [o,\pi] \right) \quad .$$

Proof : We have :

$$\frac{1}{T^{d+1/2}} \; I_T\left(\frac{\omega}{T}\right) = \frac{1}{T^{d+1/2}} \sum_{t=1}^{T} \exp\left(\frac{it\omega}{T}\right) X_t$$

$$= \frac{1}{T} \sum_{t=1}^{T} \exp\left(\frac{it\omega}{T}\right) \frac{X_t}{T^{d-1/2}}$$

$$= \frac{1}{T} \sum_{t=1}^{T} \exp\left(\frac{it\omega}{T}\right) X_T^*\left(\frac{t}{T}\right)$$

$$= \int_o^1 \left[\sum_{t=1}^{T} \exp\left(\frac{it\omega}{T}\right) X_T^*\left(\frac{t}{T}\right) \mathbb{1}_{t-1/T \leq s < t/T} \right] ds \quad .$$

Since $(X_T^*(r) \ , \ r \in [o,1])$ weakly converges to $(X_\infty^*(r) \ , r \in [o,1])$, a direct application of the continuous mapping theorem gives the weak convergence of $\left(\frac{1}{T^{d+1/2}} I_T\left(\frac{\omega}{T}\right) \ , \ \omega \in [o,\pi] \right)$ to

$$\left(\int_o^1 \exp i\omega s \, X_\infty^*(s) \, ds \ , \ \omega \in [o,\pi] \right)$$

$$= \left(\frac{\sigma \, \theta(1)}{\phi(1) \, \Gamma(d)} \int_o^1 \exp i\omega r \left[\int_o^r (r-s)^{d-1} dB(s) \right] dr \ , \ \omega \in [o,\pi] \right) .$$

The property is finally obtained by interverting the integrals

□

Property 7 may be used to derive the asymptotic correlations between two values of the shortened periodogram. For instance we have :

$$\mathrm{Cov}_{as}\left[\frac{1}{T^{d+1/2}} I_T\left(\frac{\omega_1}{T}\right) \ , \ \frac{1}{T^{d+1/2}} I_T\left(\frac{\omega_2}{T}\right) \right]$$

$$= \int_o^1 \left(\int_s^1 \exp(i\omega_1 r_1) \, (r_1 - s)^{d-1} \, dr_1 \right) \left(\int_s^1 \exp(-i\omega_2 r_2) \, (r_2 - s)^{d-1} \, dr_2 \right) ds$$

$$= \int_o^1 \int_o^1 \int_o^1 \exp i(\omega_1 r_1 - \omega_2 r_2)(r_1 - s)^{d-1} (r_2 - s)^{d-1} \mathbb{1}_{s \leq \mathrm{Min}(r_1, r_2)} \, dr_1 dr_2 ds \quad .$$

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