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ON THE NON-EMPTINESS OF THE CORE OF A COALITIONAL PRODUCTION ECONOMY WITHOUT ORDERED PREFERENCES

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SUR LA NON-VACUITE DU COEUR D'UNE ECONOMIE DE PRODUCTION PAR COALITION, LORSQUE LES PREFERENCES NE SONT NI TRANSITIVES NI COMPLETES

RESUME

La non-vacuité du coeur d'une économie de production par coalition est démontrée sous des hypothèses faibles, notamment en ce qui concerne la continuité des préférences (non transitives et non complètes) des agents. Le résultat est énoncé pour une économie dont l'espace des biens n'est pas de dimension finie. Il est appliqué à la non-vacuité du coeur de l'économie coalitionnelle associée à une économie de propriété privée.

Mots clés : Economie de production par coalition - Préférences non transitives - Economie balancée - Coeur - Théorèmes de point fixe.

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ABSTRACT

Sufficient conditions are given for a coalitional production economy to have a non-empty core. Preferences may not be transitive or complete and satisfy a weak continuity property. The result is stated for an economy with an infinite dimensional commodity space. It is applied to the coalitional production economy associated with a private ownership economy.

Key words : Coalitional production economy - Non ordered preferences-Balanced economy - Core - Fixed-point theorems.

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1. INTRODUCTION

When the preference orderings of consumers are representable by real valued utility functions, the non-emptiness of the core of a coalitional production economy under assumptions which guarantee that it is representable as a balanced game, is a result of Boehm [4] which relies on a theorem of Scarf [13]. Boehm's result has been extended by Border [6] to the case where preferences are not assumed to be transitive or complete. Borrowing from Shafer and Sonnenschein [16] the construction of a "pseudo-utility", Border defines an analogue of the characteristic function of a game without side payments and then adapts to the non transitive case Shapley's [15] and Ichiishi's [12] arguments.

The approach followed in this paper is different and related to the approach of Gale and Mas-Colell [10] in their proof of existence of equilibria for an economy without ordered preferences. The non-emptiness of the core is directly proved from a fixed-point theorem applied to a product of correspondences suitably defined. By making clear the role played by each of the assumptions, this direct and short approach has the merit of allowing a significant weakening of that is needed for a core existence theorem. However, it should be pointed out that, as in Border, the existence theorems assume the convexity of the total production set, an assumption which is not needed in the transitive case.

The results are stated for economies with an infinite dimensional commodity space. They have as corollary existence of individually rational Pareto optimal allocations under the same mild assumptions which allow non-emptiness of the core.

2. THE MODEL

In a Hausdorff linear topological space L as commodity space, the economic framework is essentially that of Boehm [4]. Let M denote the set of m positive integers $\{1,\ldots, m\}$ and $\mathcal{M}$ the family of all nonempty subsets of M.

Given a finite set M, a <u>coalitional production economy</u> in L is a list of specified data :

$$\mathcal{E}^C = ((X^i, \omega^i, P^i)_{i \in M}\ , (Y^B)_{B \in \mathcal{M}}, Y).$$

M is identified to the set of agents and a member of $\mathcal{M}$ is interpreted as a coalition of agents. To each agent $i \in M$ is associated a consumption set $X^i \subset L$, an initial endowment $\omega^i \in L$ and a preference correspondence $P^i : \prod_{k=1}^{m} X^k \to X^i$. If $x = (x^k) \in \prod_{k=1}^{m} X^k$, $P^i(x)$ is interpreted as the set of the elements of X^i which are (strictly) preferred by agent i to x^i when the consumption of each agent $k \neq i$ is equal to x^k. To each coalition $B \in \mathcal{M}$ is associated a production possibility set $Y^B \subset L$. $Y \subset L$ is the total production possibility set of the economy.

Let $X = \prod_{i=1}^{m} X^i$. An allocation $x \in X$ is attainable for the economy if $\sum_{i \in M} x^i - \sum_{i \in M} \omega^i \in Y$. We will denote by $\hat{X}$ the set of all attainable allocations of the economy.

In the same way, if $B \in \mathcal{M}$ and if $X^B = \prod_{i \in B} X^i$, we will denote by $\hat{X}^B$ the attainable set of coalition B :

$$\hat{X}^B = \{x^B = (x^{iB})_{i \in B} \in X^B \ / \ \sum_{i \in B} x^{iB} - \sum_{i \in B} \omega^i \in Y^B\} .$$

For each B, a preference correspondence $P^B : X \to X^B$ can be defined by

$$P^B(x) = \{z^B = (z^{iB}) \in X^B \ / \ z^{iB} \in P^i(x) \ \forall\ i \in B\}.$$

$P^B(x)$ is interpreted as the set of the elements of X^B which are unanimously preferred to x by the members of coalition B.

A coalition B is said to block an attainable allocation $x \in \hat{X}$ if there exists $z^B \in \hat{X}^B \cap P^B(x)$.

The core of economy $\mathcal{E}^C$ is defined as the set of attainable allocations which are blocked by no coalition.

If we assume $Y = Y^M$, the concept of core of an economy can be linked to two other concepts which do not need for their definition a complete specification of the production possibilities of the coalitions.

An attainable allocation x is said <u>Pareto-optimal</u> if x cannot be blocked by the grand coalition M.

An attainable allocation x is said <u>individually rational</u> if there is no $i \in M$ such that $\omega^i \in P^i(x)$.

Obviously the core of an economy E^C is included in the set of all Pareto-optimal allocations. If for each $i \in M$, $\omega^i \in X^i$ and $0 \in Y^{\{i\}}$, the same is true for the set of all individually rational Pareto optimal allocations.

3. EXISTENCE THEOREMS.

Let $\mathcal{B} = \{B \in M / \hat{X}_B \neq \phi\}$. Consider the following subset of $\mathbb{R}^{\mathcal{B}}$:

$$\Delta^{\mathcal{B}} = \{\lambda = (\lambda_B) \in \mathbb{R}^{\mathcal{B}} \;/\; \lambda_B \geq 0 \;\forall\, B \in \mathcal{B} \text{ and } \sum_{\substack{B \supset \{i\} \\ B \in \mathcal{B}}} \lambda_B = 1 \;\forall\, i = 1, \ldots, m\} .$$

To each $\lambda \in \Delta^{\mathcal{B}}$, one can associate

$\mathcal{B}(\lambda) = \{B \in \mathcal{B} \;/\; \lambda_B > 0\}$.

A collection C of members of $\mathcal{B}$ is said <u>balanced</u> if there exists $\lambda \in \Delta^{\mathcal{B}}$ such that $C = \mathcal{B}(\lambda)$.

An economy E is said <u>balanced</u> if, for each $\lambda \in \Delta^{\mathcal{B}}$, $\sum_{B \in \mathcal{B}} \lambda_B Y^B \subset Y$.

It should be noted here that an exchange economy ($Y^B = \{0\}$ $\forall$ $B \in M$ and $Y = \{0\}$) is obviously balanced. It will be see later that a balanced coalitional production economy can be associated to a private ownership production economy.

Assume that X and each X^B, $B \neq M$, are endowed with the topologies induced by the product topology on L^M and L^B. We make on E the following assumptions :

A-1 $\forall$ $i \in M$, X^i is convex and $\hat{X}^{\{i\}}$ is non-empty

$\forall$ $x \in X$, $x^i \notin$ co $P^i(x)$ (the convex hull of $P^i(x)$)

P^i has open lower sections (i.e. for each $z^i \in X^i$,

the set $(P^i)^{-1}(z^i) = \{x \in X / z^i \in P^i(x)\}$ is open in X)

A-2 Y is convex and $\hat{X}$ is compact

A-3 E is balanced

We will first prove the non-emptiness of the core in the case where the commodity space is the 1-dimensional Euclidian space (proposition 1). Then following Bewley ideas, by considering traces of economy $\mathcal{E}$ on finite dimensional spaces, we will deduce from this first result a result of existence in the infinite dimensional case (proposition 2)

In order to prepare the fixed-point argument, let us introduce some definitions borrowed with slight modifications from Borglin and Keiding [7]. Let for each k=1,..., n, T^k be a non-empty convex and compact subset of some finite dimensional Euclidian space, $T = \sum_{k=1}^{n} T^k$ and, for some k, $\psi : T \to T^k$ a correspondence. ψ is said KF(Ky Fan) if ψ is convex-valued, has open lower sections and satisfy $t^k \notin \psi(t)$. $\psi_t : T \to T^k$ is a <u>KF-majorant of ψ at t</u> if ψ_t is KF and there is some open neighborhood U_t of t such that for $t' \in U_t$, $\psi(t') \subset \psi_t(t')$. ψ is <u>KF-majorized</u> if for each $t \in T$ such that $\psi(t) \neq \phi$, there is a KF majorant of ψ at t. As in Borglin and Keiding (see in [7] the end of the proof of corollary 3), one can show that if $\psi : T \to T^k$ is KF-majorized, there is a KF correspondence $\psi' : T \to T^k$ with $\psi(t) \subset \psi'(t)$ $\forall t \in T$.

<u>Proposition 1</u> Assume A-1 _ A-3 and that $L = \mathbb{R}^{\ell}$. Then the core of economy $\mathcal{E}$ is non-empty.

<u>Proof</u>. For each $(x, z, \lambda) \in \hat{X} \times \prod_{B \in \mathcal{B}} \overline{co}\, \hat{X}^B \times \Delta^{\mathcal{B}}$, let us define :

- $\Theta(z, \lambda) = (x'^i)_{i \in M}$ with for each $i \in M$, $x'^i = \sum_{\substack{B \in \mathcal{B} \\ B \supset \{i\}}} \lambda_B z^{iB}$

- $\varphi(x) = (\varphi^B(x))_{B \in \mathcal{B}}$ with for each $B \in \mathcal{B}$, $\varphi^B(x) = \overline{co}\, \hat{X}^B \cap co\, P^B(x)$

- $\psi(x, \lambda) = \bigcap_{I(x)} \{\mu \in \Delta^{\mathcal{B}} / \mu_B > \lambda_B\}$ with $I(x) = \{B \in \mathcal{B} / \hat{X}^B \cap P^B(x) \neq \phi\}$

It is easily seen that $\Delta^{\mathcal{B}}$ is a non-empty, convex and compact subset of $\mathbb{R}^{\mathcal{B}}$, that $\hat{X}$ is non-empty and that each $\hat{X}^B$ is relatively compact. Hence for each $B \in \mathcal{B}$, the closed convex hull $\overline{co}\, \hat{X}^B$ is compact and $\hat{X} \times \prod_{B \in \mathcal{B}} \overline{co}\, \hat{X}^B \times \Delta^{\mathcal{B}}$ is a non-empty, convex and compact subset of some finite dimensional Euclidian space. It follows from the convexity of X^i for each $i \in M$, the convexity of Y and the balancedness assumption that $\theta(\prod_{B\in\mathcal{B}} \text{co}\ \hat{X}^B, \Delta^{\mathcal{B}}) \subset \hat{X}$. Since $\hat{X}$ is compact, θ is a continuous function from $\prod_{B\in\mathcal{B}} \overline{\text{co}}\ \hat{X}^B \times \Delta^{\mathcal{B}}$ into $\hat{X}$. Each of correspondences φ_B has open lower sections and convex values. It is trivial to check that ψ is KF-majorized (if $\psi(x, \lambda) \neq \phi$, take $\psi_{(x,\lambda)}(x', \lambda') = \bigcap_{I(x)} \{\mu \in \Delta / \mu_B > \lambda_B\}$).

Then let $\psi' : \hat{X} \times \Delta^{\mathcal{B}} \to \Delta^{\mathcal{B}}$ be a KF correspondence such that

$$\psi(x, \lambda) \subset \psi'(x, \lambda) \qquad \forall\ (x, \lambda) \in \hat{X} \times \Delta^{\mathcal{B}}.$$

We define $\chi : \hat{X} \times \prod_{B\in\mathcal{B}} \overline{\text{co}}\ \hat{X}^B \times \Delta^{\mathcal{B}} \to \hat{X} \times \prod_{B\in\mathcal{B}} \overline{\text{co}}\ \hat{X}^B \times \Delta^{\mathcal{B}}$ by

$$\chi(x, z, \lambda) = (\{\theta\ (z, \lambda)\}\ , \varphi(x), \psi'(x, \lambda)).$$

It follows from Gale and Mas-Colell (see the theorem of section 2 of [10] as corrected by [11]) that there exists $(\bar{x}, \bar{z}, \bar{\lambda}) \in \hat{X} \times \prod_{B\in\mathcal{B}} \overline{\text{co}}\ \hat{X}^B \times \Delta^{\mathcal{B}}$ such that

$$\bar{x} = \theta(\bar{z}, \bar{\lambda}) \tag{1}$$

$$\forall\ B \in \mathcal{B},\ \bar{z}^B \in \overline{\text{co}}\ \hat{X}^B \cap \text{co}\ P^B(\bar{x}) \text{ or } \overline{\text{co}}\ \hat{X}^B \cap \text{co}\ P^B(\bar{x}) = \phi \tag{2}$$

$$\psi'(\bar{x}, \bar{\lambda}) = \phi \text{ and a fortiori } \psi\ (\bar{x}, \bar{\lambda}) = \phi. \tag{3}$$

To complete the proof, we show that $I(\bar{x}) = \phi$. Indeed if not, by a classical separation argument, there exists $\bar{p} = (\bar{p}_B) \in \mathbb{R}_+^{\mathcal{B}} \setminus \{0\}$, $\bar{p}_B = 0\ \forall\ B \notin I(\bar{x})$, such that $\bar{\lambda}$ is a solution of the linear programming problem :

$$\max \sum_{B\in\mathcal{B}} \bar{p}_B\ \mu_B$$

$$\sum_{\substack{B\supset\{i\} \\ B\in\mathcal{B}}} \mu_B = 1 \qquad \forall\ i \in M \text{ and } \mu_B \geq 0 \qquad \forall\ B \in \mathcal{B}.$$

Let ε_i, $i \in M$ and $\alpha_B \geq 0$, $B \in \mathcal{B}$ be a system of multipliers for the first order conditions : $\alpha_B = -\bar{p}_B + \sum_{i\in B} \varepsilon^i$; $\alpha_B\ \bar{\lambda}_B = 0\ \ \forall\ B\in\mathcal{B}$.

For each $i \in M$, $\varepsilon^i \geq \overline{p}_{\{i\}} \geq 0$. Then let B_0 be such that $\overline{p}_{B_0} > 0$ and $i \in B_0$ such that $\varepsilon^i > 0$. For each $B \notin I(\overline{x})$, $B \in \mathcal{B}$ and $B \supset \{i\}$, we have $\alpha_B > 0$ and hence $\overline{\lambda}_B = 0$. From (1), we deduce $\overline{x}^i = \sum_{\substack{B \in I(\overline{x}) \\ B \supset \{i\}}} \overline{\lambda}_B \overline{z}^{iB}$ and from (2) $\overline{z}^{iB} \in \text{co } P^i(\overline{x})$ $\forall B \in I(\overline{x})$, $B \supset \{i\}$, which contradicts $\overline{x}^i \notin \text{co } P^i(\overline{x})$.

□

<u>Proposition 2</u> Assume A-1– A-3. Then the core of economy $\mathcal{E}$ is non-empty.

<u>Proof</u>. Let for each $i=1,\dots,m$ $\underline{x}^i \in \widehat{X}^{\{i\}}$. Let $\mathcal{F}$ be the collection of all finite dimensional subspaces of L containing $\underline{x}^i$ and ω^i, $i \in M$. For each $F \in \mathcal{F}$, we set : $X_F^i = X^i \cap F$; $X_F = \prod_{i \in M} X_F^i$; if $x \in X_F$, $P_F^i(x) = P^i(x) \cap X_F^i$; $Y_F^B = Y^B \cap F$; $Y_F = Y \cap F$ and we consider the coalitional production economy

$$\mathcal{E}_F = ((X_F^i, \omega^i, P_F^i)_{i \in M}, (Y_F^B)_{B \in \mathcal{M}}, Y_F).$$

Note that $\widehat{X}_F = \widehat{X} \cap F^M$ and $\widehat{X}_F^B = \widehat{X}^B \cap F^B$ for each $B \in \mathcal{M}$. If F is endowed with the topology induced by the topology of L, it is easily checked that $\mathcal{E}_F$ satisfies assumptions A-1–A-3. As F is finite dimensional, it follows from proposition 1 that there exists $\overline{x}_F$ belonging to the core of $\mathcal{E}_F$. Now the collection $\mathcal{F}$, ordered by inclusion, is directed. Since $\widehat{X}$ is compact, by passing to subnets if necessary, we can assume $\overline{x}_F \to \overline{x} \in \widehat{X}$. If $B \in \mathcal{M}$ and $x^B \in P^B(\overline{x}) \cap \widehat{X}^B$, there exists F_0 such that $F \supset F_0 \Rightarrow x^B \in P_F^B(\overline{x}_F) \cap \widehat{X}_F^B$, which contradicts the fact that $\overline{x}_F$ belongs to the core of $\mathcal{E}_F$.

□

Proposition 2 extends Border's result with respect as well to the dimension of the underlying commodity space than to the properties of economy $\mathcal{E}^C$: preference correspondences are not assumed to have an open graph but only open lower sections and the production possibility sets of the coalitions are not assumed to be closed. Compared to Boehm's, these assumptions and also transitivity and completeness of preference relations are traded, as in Border, for the additional assumption that the total production possibility set is convex.

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4. APPLICATION TO THE CASE OF A PRIVATE OWNERSHIP ECONOMY

Boehm's framework and hence the previous result are applicable in a wide variety of (institutional) assumptions on the distribution of production possibilities among the coalitions. But it should be noted that the balancedness condition on economy E^C may be very restrictive. For example, if the same technology is available to all and if inaction is possible ($Y^B = Y \; \forall \; B \in M$ and $0 \in Y$), this assumption leads to a model of production with constant returns to scale (Y is a convex cone). Therefore a core existence theorem based on the balancedness condition is more appropriate for a model where to each coalition is associated a set of parameters which link the coalition production set to the aggregate production set of the economy (see in Boehm's [5] an interpretation in this line of the Scarf's [14] assumption of distributivity).

We will apply here the previous results to the case of a coalitional production economy associated to a private ownership economy in such a way that the balancedness condition is satisfied.

Let L be a Hausdorff linear topological space and
$E = ((X^i, \omega^i, P^i)_{i\in M}, (Y^j)_{j=1,\ldots,n}, (\theta^{ij})_{i\in M, j=1,\ldots,n})$ a private ownership economy in L, standardly defined. As in section II, m consumers are described by their characteristics $(X^i, \omega^i, P^i)_{i\in M}$; n production units are described by the production possibility sets Y^j, $j=1,\ldots,n$.
Shares $\theta^{ij} \geq 0$, which satisfy $\sum_{i\in M} \theta^{ij} = 1$, $\forall j = 1,\ldots, n$, describe the distribution among the consumers of the ownership of each production unit.
To E we associate the coalitional production economy

$$E^C = ((X^i, \omega^i, P^i)_{i\in M}, (Y^B)_{B\in M}, Y)$$

defined by :

$$Y^B = \sum_{i\in B} \sum_{j=1}^{n} \theta^{ij} Y^j, \; B\in M \quad ; \; Y=Y^M = \sum_{j=1}^{n} Y^j$$

and we make on E the following assumptions :

B-1 $\forall \; i \in M$, X^i is convex and $\omega^i \in X^i$
$\forall \; x \in X$, $x^i \notin$ co $P^i(x)$
P^i has open lower sections (in X)

B-2 $\forall j=1,\dots, n$, $0 \in Y^j$

B-3 Y is convex and $\hat{X}$ is compact

Proposition 3 is an immediate consequence of proposition 2

Proposition 3 Under assumptions B-1–B-3 on $\mathcal{E}$, the core of $\mathcal{E}^C$ is non-empty.

Corollary - Under assumptions B-1–B-3, economy $\mathcal{E}$ has individually rational Pareto optimal allocations.

It goes without saying that this last statement could be easily directly proved as in Yannelis [17].

It should also be emphasized that, in Proposition 3, compactness of $\hat{X}$ and openness of the lower sections of the correspondences P^i are required simultaneously with respect to the same Hausdorff vector space topology on the commodity space. Aliprantis et al. give in [2] an example of an economy without core allocations for which one of the both requirements is not satisfied.

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