ON THE THEORY OF INCOMPLETE MARKETS*

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SUR LA THEORIE DES MARCHES INCOMPLETS

R E S U M E

Les échecs de la coordination par le marché peuvent être formulés en les termes des théories des marchés incomplets, des équilibres avec rationnement et des modèles à générations imbriquées. Ce n'est pas un hasard. Il est montré que les deux premières approches constituent la forme réduite d'une forme extensive (paradigmatique) pour laquelle il y a non pas un agent de chaque type, mais un grand nombre d'agents de chaque type et non pas un sous-marché par marché mais un grand nombre de sous-marchés par marché. Il est alors requis que deux agents ne peuvent se rencontrer sur plus d'un sous-marché. Il appert alors que ce qui est important dans les trois domaines, c'est la structure de rencontres des agents et que l'intégration de ces trois théories peut être rendue complète. Nous ne devons pas être piégés par les différences dans la formulation des concepts d'équilibre prévalant dans ces trois domaines.

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Mots clefs : Echecs de la coordination par le marché.
Marchés incomplets
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S U M M A R Y

Market coordination failures can be formulated in terms of the three theories of incomplete markets, of equilibria with possible rationing and of models with overlapping generations. It is shown that the two first approaches can be conceved as the reduced form of a (paradigmatic) extensive form for which there are many agents (and not one) for each type, many submarkets (and not one) for each market. It is then required that two agents cannot meet on more than one submarket. One can then conclude that what is important in these three fields is the pattern of meetings of agents and that the integration of these three theories can be made complete. We must not be trapped by the differences of formulation of equilibrium concepts which prevail in these three fields.

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Key words : Market coordination failures
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It is, by now, very well known that, when the market structure is incomplete (R. Radner (1968 and 1972)), it may happen that economic agents are stuck to allocations which are low from the Pareto point of view, but which can be called equilibria. That implies that, sometimes, competitive equilibria for the same economy can be ranked according to the Pareto criterion (See O. Hart 1975)). However, it is obvious that the theory of incomplete markets has something general to say about the way exchanges are made and cannot be reduced to the study of the model with spot markets and securities, and if one wants to be able to discriminate between different sources of market coordination failures, one needs a generalization in two directions, (a generalization which establishes a correspondence among three fields : incomplete market structures, overlapping generations models and equilibria with possible rationing) :

(i) A definition of the notion of the market structure which is more general than the one linked with spot markets and securities.For instance, consider a model in which there is one good (let us call it money) which is a necessary medium of exchange (a necessary counterpart in every transaction). We would like to say that the market structure is incomplete even if the reference to spot markets and securities is not obvious.

(ii) A definition of the notion of collective rationality ("collective rationality" is preferred to "optimality" because of the very possibility of Pareto-ranking collectively rational allocations for the same economy) which is independant of assumptions of competition and convexity. This point of view is necessary if one wants to appraise the relative weights of monopolistic competition, increasing returns and incompleteness of the market structure in models which exhibit coordination failures (See e.g. O. Hart (1982), W. Heller (1985) and M. Weitzman (1982)).

In this perspective a useful concept of collective rationality has to meet at least three requirements.

(i) A Pareto optimum has to be, always, collectively rational given a market structure.
(ii) Given a market structure, a competitive equilibrium has to be collectively rational.
(iii) Given a market structure, if convexity is assumed, a collectively rational allocation has to be a (compensated) competitive equilibrium.
Various concepts of "collective rationality"(optimality) have already been proposed for the case of incomplete market structures (S. Grossman (1977) and D. Gale (1982), for instance). However, even for exchange economies, these definitions are too weak because a no trade allocation which is collectively rational in this sense is not necessarily a competitive equilibrium. I propose a notion of collective rationality which allows to get the two theorems of welfare economics in an incomplete market setting. This improvement is not merely technical because :
(i) by giving a richer definition of agents behaviour when the market structure is incomplete, it provides some insight in the reasons why agents can be stuck to a low level allocation, <u>independantly of any assumption of competition and convexity</u>.
(ii) <u>by injecting a large degree of multilateralism "across markets"</u>, it allows us to take into account production in a meaningful way.

I interpret the way an incomplete market structure constrains the possibilities of coordination among agents in terms of a paradigmatic pattern of meetings which is, in some sense, a generalization of the pattern of overlapping generations models for which agents live for two periods, there is one type of agents for each generation and which makes multilateralism a difficult job to implement. The idea is that the model with m agents i and n markets α is the reduced form of a model with many agents γ for the same type i and many submarkets β for the same market α, with identical treatment for agents of the same type. It is then required that if two agents meet on some submarket, they cannot meet on another one. Stability with respect to Pareto improvements which can be implemented only through changes of trades on any finite number of submarkets (Y. Balasko and K. Shell (1982)) constitutes already the first step to get a stronger notion than that of efficiency which considers only changes of trades on one submarket. Due to the nature of the pattern of meetings, one refers alternatively to stability w.r.t. "inward induction" because considering that trades can be modified only on a finite number of submarkets amounts to analyze the spreading of autonomous offers (of type I) : an autonomous offer on market α implies that trades are kept fixed on all other markets $\alpha' \neq \alpha$. Taking into account the symmetric nature of the framework, it is possible to define stability w.r.t. "inward induction" in terms of a stationary point of a concrete mechanism (L. Hurwicz (1959)) for which a message is a set of trade offers : Each agent, when making offers on market α, has the possibility of matching offers made by other agents on markets $\alpha' \neq \alpha$. Alternatively, this set of equilibrium sets of trades offers can be conceived as a set of coherent forecasts of induced (by autonomous ones) sets of trade offers which cannot prime the pump.

Arguing in terms of the paradigmatic pattern of meetings, one introduces the possibility of "outward induction" which, roughly speaking, allows for Pareto improvements implied by attempts made by any agent who tries to perturb the system of forecasts by making offers on many markets simultaneously.Collective rationality is then defined as stability with respect to inward and outward inductions.

From this point of view note that the fact that agent i's offers on market α depend on offers he expects rationally other agents j make on other markets α' is very much linked to the Marshallian notion of pecuniary externalities. Conversely, the case of purely "technological" externalities can be often cast in an incomplete market setting by expanding the set of goods à la mode Arrow (See W. Heller and D.A. Starett (1976)).

It appears, then, that the field of equilibria with possible rationing [1] has very close formal relationships with the classical theory of incomplete markets as developed by R. Radner (1968 and 1972), F. Hahn (1973), O. Hart (1975), S. Grossman (1977), D. Gale (1982) and D. Cass (1987) among other economists. However these links are also substantial. It is often assumed in models with not totally flexible prices that all kinds of exchange are not possible. This is the incomplete market hypothesis and the consequence is that in the two fields, one can find the same kind of coordination failures i.e. one can exhibit examples of equilibria for the same economy which are Pareto ranked, in a competitive setting with convexity assumptions.

Before announcing the plan of this paper, let us emphasize that the analysis of the deep reasons for which the market structure is incomplete are beyond the scope of this paper.

In the first part, one presents the framework and define a general notion of market structure and concepts of competitive equilibrium for the cases of perfectly flexible prices and not totally flexible prices. In the second and third parts, we define the notion of collective rationality and show that it satisfies the three requirements. The last part is devoted to the study of (game-theoretic) foundations to the notion of competitive equilibrium when the market structure is incomplete and of relationships with the theory of equilibria with rationing. Existence problems are briefly approached, but relationships between market structure, technological externalities and increasing returns are not even adumbrated.

I

The objective of this part is to describe the model and the framework which allows to unify the usual theory of incomplete market structures and that of equilibria with possible rationing (more than a beginning of integration with the overlapping generations approach is provided in Example 6).

Consider an exchange economy [2] with r goods h in the sense of Arrow-Debreu and m consumers i. Each consumer i is endowed with a utility function $u_i(x)$ defined on $X_i \subset \mathbb{R}^r$, and an initial endowment w_i.

For each i, one assumes

A1 : $w_i > 0$ [3] and $w_i \in X_i$

A2 : X_i is closed and bounded from below. $\sum_i w_i \in \text{int} \sum_i X_i$. u_i is not decreasing and continous.

Consider n (non empty) subsets of $\mathbb{R}^r$, $[Z^1,..,Z^\alpha,..,Z^n]$, such that for each α : Z^α is a vector subspace of $\mathbb{R}^r$. (See D. Gale (1982)).

Define $M = [Z^1,..,Z^\alpha,..,Z^n]$ and call it the market structure. Also $z_i \equiv x_i - w_i$. The market structure M constrains each consumer i's net rades in the sense that his total net trade has to be the sum of his net trades on all markets α. Denote z_i^α as the net trade made by agent i on market α. Obviously we require that $z_i^\alpha \in Z^\alpha$, $\forall \alpha$.

Let be $\tilde{\tilde{z}} \equiv \left[z_i^1,\ldots,z_i^\alpha,\ldots,z_i^n\right]$ and $\tilde{\tilde{z}} = \left[\tilde{\tilde{z}}_1,\ldots,\tilde{\tilde{z}}_m\right]$. $\tilde{\tilde{z}}_i$ is possible for i, given M, and we write $\tilde{\tilde{z}}_i \in B_i(M)$ iff :

(i) $\forall\alpha : z_i^\alpha \in Z^\alpha$

(ii) $w_i + \sum_\alpha z_i^\alpha \in X_i$

$\tilde{\tilde{z}}$ is balanced iff $\forall\alpha : \sum_i z_i^\alpha = 0$

$\tilde{\tilde{z}}$ is feasible, given M, iff

(i) $\tilde{\tilde{z}}$ is balanced

(ii) $\forall i : \tilde{\tilde{z}}_i \in B_i(M)$

When there is no ambiguity, one shall write B_i instead of $B_i(M)$.

Let us define :

$$C = \left\{z \in \mathbb{R}^r \,\middle|\, z = \sum_\alpha z^\alpha \text{ with } z^\alpha \in Z^\alpha,\ \forall\,\alpha\right\}.$$

The problem comes from the fact that C does <u>not</u> characterize M.

Let us write also

$U_i(\tilde{\tilde{z}}_i) \equiv u_i(w_i + \sum_\alpha z_i^\alpha)$

For each α, it is useful to look at Z^α from another point of view. Define, for each α :

$$\bar{T}^\alpha = \left\{\bar{t} \in \mathbb{R}^r \,\middle|\, \|\bar{t}\| = 1 \text{ and } \exists\, z \in Z^\alpha \text{ and } a \in \mathbb{R}_+ \text{ such that } z = a\,\bar{t}\right\}$$

For each $\bar{T}^\alpha$, let us identify the direction $\bar{t}$ and the direction $-\bar{t}$, choosing one as the representative of its equivalence class (we denote it by t) and define T^α accordingly. Thus, $\forall\alpha$, given T^α:

$$Z^\alpha = \left\{z \in \mathbb{R}^r \mid \exists\, t \in T^\alpha \text{ and } a \in \mathbb{R} \text{ such that } z = a\,t\right\}$$

In this sense, a market Z^α is a set of directions of exchange or activities of exchanges in $\mathbb{R}^r$. Define $T = \bigcup_\alpha T^\alpha$. Thus T defines the technology of exchange in the same way as activity analysis treats the technology of production (See T.C. Koopmans ed. (1951)). For each $t \in T, a_i(t)$ is thus the intensity of use of activity of exchange t by agent i .

$a_i \equiv (a_i(t))_{t\in T}$ is then a (signed) measure on T.

By convention one shall say that i buys along t when $a_i(t) > 0$ and that he sells when $a_i(t) < 0$.

I emphasize that T is more fundamental than M because M constitudes only a way of grouping elements of T . Moreover, as n is assumed to be finite, one is constrained by this approach, to argue in a class of T's which is not the class of all T's. However, generalizing would require the use of the notion of signed measure.
One shall discriminate between two polar cases for the market structures M one considers (see footnote 5 and Example 2) :

A4(i) : $M = [Z^1,.,Z^\alpha,.,Z^n]$ is such that each Z^α has at least dimension two and $\nexists\, p \in \mathbb{R}^r_{++}$ such that $p.z = 0 \quad \forall z \in Z^\alpha$.

A4(ii) : $M = [Z^1,.,Z^\alpha,.,Z^n]$ is such that each Z^α has dimension one.
Clearly, the associated T of a M satisfying A4(ii) is finite (choose a basis composed of one column vector for each α).
Given $M = [Z^1,.,Z^\alpha,.,Z^n]$, one shall say that good h appears on market α if there is $z^\alpha \in Z^\alpha$ with $z^\alpha_h \neq 0$. <u>$\forall\alpha$, at lesat two goods appears on α</u>.

<u>Definition 1</u> : A market structure M satisfying A4(i) is complete if its associated T contains all directions of $\mathbb{R}^r$.
Before defining notions of (competitive) equilibria, let us introduce convexity and no satiation.
A3 : For each i, X_i is convex and u_i is semi strictly quasi-concave[(4)] in the sense that if $u_i(x^1) > u_i(x^2)$ then for $\beta \in]0,1]$, $u_i\{(1-\beta)\, x^1 + \beta\, x^2\} > u_i(x^2)$ and if $u_i(x^1) \geqslant u_i(x^2)$ then for $\beta \in [0,1]$, $u_i\{(1-\beta)\, x^1 + \beta\, x^2\} \geqslant u_i(x^2)$. Moreover, given M for which A4(i) holds, $\forall\alpha$, $\exists i \equiv i(\alpha)$: $\forall\ \tilde{\tilde{z}} \in B_i(M)$, there is ${}_1\tilde{\tilde{z}} \in B_i(M)$ such that $U_i\ ({}_1\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}})$ and ${}_1z^{-\alpha} = z^{-\alpha}$. (Where $z^{-\alpha} = [z^1,..,z^{\alpha-1},z^{\alpha+1},..,z^n]$).

A3 is similar to assumptions (4d) and (4e) in K. Arrow and F. Hahn (1971). Assume A4(i). For each α, choose a normalized basis of Z^α :

$\left\{\lambda^\alpha_1,..,\lambda^\alpha_k,..,\lambda^\alpha_{q(\alpha)}\right\}$, with $\left\|\lambda^\alpha_k\right\| = 1$, the matrix of which is Λ^α.
Λ^α has rank $q(\alpha)$ $(2 \leqslant q(\alpha) \leqslant r)$. Each column vector λ^α_k of Λ^α is a bundle of goods. Let be π^α_k the "price" of bundles λ^α_k on market α . Let be

$\pi^{\alpha} = \left[\pi_1^{\alpha},..,\pi_k^{\alpha},..,\pi_{q(\alpha)}^{\alpha}\right]$ and a^{α} the net trade in bundles $\lambda_1^{\alpha},.,\lambda_{q(\alpha)}^{\alpha}$ on market α. As $q(\alpha) \geqslant 2$, one can say that $q(\alpha)$ bundles are exchanged one against other ones on each market α at "prices" π^{α}.

Define also $q = \sum_{\alpha} q(\alpha)$ and $\Lambda = [\Lambda^1,..,\Lambda^{\alpha},..,\Lambda^n]$. One can thus define for each i .

$$V_i(a_i) \equiv u_i(w_i + \Lambda a) \text{ with } a_i \in P_i$$

where $P_i = \left\{a_i \in \mathbb{R}^q \mid w_i + \Lambda a_i \in X_i\right\}$.

Consumer i's problem is then

$$\begin{aligned} &\text{Max } V_i(a_i) \text{ subject to} \\ &\pi^{\alpha} a_i^{\alpha} \leqslant 0 \qquad \alpha = 1,..,n \\ &a_i \in P_i \end{aligned}$$

Notions of equilibria can then be defined along these lines. The advantage of this approach is that we have then a perfect homology with the simplest model of incomplete markets : that with separate "spot" markets only. Its drawback is that it is dependant on the particular basis Λ^{α} chosen for each α. One prefers thus to state definitions in a more general way.

<u>Definition 2</u> : Consider $M = [Z^1,..,Z^n]$ satisfying A4(i). $\left[\approx{z}^*, (\pi^{*\alpha})_{\alpha}\right]$ is a Compensated Equilibrium (C.E.) with respect to M if :

(i) z^* is feasible.

(ii) $\forall\alpha$: $\pi^{*\alpha}$ is linear form on Z^{α} and for at least one α, there is $z^{\alpha} \in Z^{\alpha}$ with $\pi^{*\alpha} z^{\alpha} \neq 0$

(iii) $\forall i$: there is no $\approx{z}_i \in B_i$ with $U_i(\approx{z}_i) > U_i(\approx{z}_i^*)$ and $\pi^{*\alpha} z_i^{\alpha} \leqslant 0 \quad \forall\alpha$ with at least one strict inequality.

Also :

<u>Definition 3</u> : Consider $M = [Z^1,..,Z^n]$ satisfying A4(i) $\left[\approx{z}^*, (\pi^{*\alpha})_{\alpha}\right]$ is a Pseudo-Equilibrium (P.E.) if

(i) $\approx{z}^*$ is feasible

(ii) $\forall\alpha$: $\pi^{*\alpha}$ is a linear form on Z^{α} and for at least one α, there is $z^{\alpha} \in Z^{\alpha}$ with $\pi^{*\alpha} z^{\alpha} \neq 0$

(iii) $\forall i$: $\approx{z}_i^*$ maximizes $U_i(\approx{z}_i)$ subject to $\approx{z}_i \in B_i$ and $\pi^{*\alpha} a^{\alpha} \leqslant 0 \quad \forall\alpha$.

These definitions are a little unsatisfactory because the no-arbitrage condition does not appear. If a basis Λ^α has been chosen for each α, the no arbitrage condition takes the following form : There is $p \in \mathbb{R}^r_+ \setminus \{0\}$ – a non trivial linear form on $\mathbb{R}^r$ – such that

$$p' \Lambda^\alpha = \pi^\alpha \qquad \forall\alpha$$

This condition can be added either to definition 2 or to definition 3. For simplicity sake, one adds it only to definition 3.

Definition 4 : Given $M = [Z^1,..,Z^n]$ satisfying A4(i), a Quasi-Equilibirum (Q.E.) with respect to M is a couple $(\tilde{\tilde{z}}^*, p^*)$ with $p^* \in \mathbb{R}^r_+ \setminus \{0\}$ such that

(i) $\tilde{\tilde{z}}^*$ is feasible

(ii) For at least one α, there is $z^\alpha \in Z^\alpha$ with $p^*.z^\alpha \neq 0$

(iii) $\forall i$ $\tilde{\tilde{z}}^*_i$ maximizes $U_i(\tilde{\tilde{z}}_i)$ subject to $\tilde{\tilde{z}}_i \in B_i$ and $p^*.z^\alpha \leqslant 0 \quad \forall\alpha$.

Notice that, in this definition, one requires that each good has one price, whatever is the market α on which it appears.

Remark : For defining a C.E., a P.E. or a Q.E. with transfers, one proceeds as usually : $\forall\alpha$, let be $d^\alpha = \left[d^\alpha_1,..,d^\alpha_m\right] \in \mathbb{R}^m$ with $\sum_i d^\alpha_i = 0$ .The budgetary constraints are then (for a Q.E.) for each i :

$$p^* z^\alpha \leqslant d^\alpha_i \qquad \forall\alpha$$

The relationships between these three notions are classical.

A5 : A C.E. $(\tilde{\tilde{z}}^*, (\pi^{*\alpha})_\alpha)$ w.r.t. M satisfies A5 if for each i, there is $\tilde{\tilde{z}} \in B_i$ with $\pi^{*\alpha} z^\alpha \leqslant 0 \quad \forall\alpha$ and strict inequality for some α.

A6 : Given an allocation $\tilde{\tilde{z}}$ feasible w.r.t. M satisfying A4(i) :
For each good h, there is i(h) whose utility function is strictly increasing in good h, at $\tilde{\tilde{z}}_{i(h)}$.

Proposition 1 : Assume A1-A3 and that M satisfies A4(i). A Q.E. is a P.E. and a P.E. is a C.E.. Under A5, a C.E. is a p.E. and under A6 a P.E. is a Q.E.

Proof : See Annex.

D. Gale (1982) has extended the notion of "resource related" to an incomplete market setting.

A7 : All pairs (i_1,i_2) of agents are resource related at $\tilde{\tilde{z}}^*$, if for every market α, there is $\tilde{\tilde{z}} = [z^\alpha,0]$ such that
$U_{i_1}(\tilde{\tilde{z}}^*_{i_1} + \tilde{\tilde{z}}) > U_{i_1}(\tilde{\tilde{z}}^*_{i1})$ and $(\tilde{\tilde{z}}^*_{i2} - \tilde{\tilde{z}}) \in B_{i_2}$.
In view of the Paradigm, this definition must be generalized.

<u>Proposition 2</u> : Assume that $\left[\tilde{\tilde{z}}^*, (\pi^{*\alpha})_\alpha\right]$ is a C.E. w.r.t. M at which all pairs of agents are resource related on each market α. Then $\tilde{\tilde{z}}^*$ is a P.E. if A1-A3 hold, M satisfies A4(ii) and $0 \in \operatorname{int} \sum_i B_i$.

Proof : Classical.

It is essential to relate, in the case of the canonical model with spot markets and securities, these definitions to the usual one.

<u>Example 1</u> : (Spot markets and Hart-Securities) : There are two periods, one state of nature in period 0 and σ states s in period 1 . There are r′ physical goods at each state period. There are thus $r = r'.(\sigma+1)$ goods à la Arrow-Debreu. Assume that there are $\bar{q}$ securities j with $\bar{q} \leqslant \sigma$. A security j has to be paid in period 0 (its price is $\bar{\pi}^j$) and it gives $\lambda^j_h(s) \geqslant 0$ units of good h in state s of period 1 . There are moreover $\sigma + 1$ spot markets $s = 0, 1, \dots, \sigma$. Define $\lambda^j(s) = \left[\lambda^j_1(s), \dots, \lambda^j_{r'}(s)\right]$.

Let be z_{io} agent i′s net trade in goods of period 0 and z_{is} agent i′s net trade in goods of state s of periode 1 $(s = 1,\dots,\sigma)$.

Thus the spot market $\alpha = s$ $(0 \leqslant s \leqslant \sigma)$ is defined as :

$$Z^\alpha = \left\{z = [z_0, z_1, \dots, z_\sigma] \in \mathbb{R}^r \mid z_{s'} = 0 \text{ for } s' \neq \alpha\right\}.$$

With each security j, one associates a market Z^α with $\alpha = \sigma + j$.

$$Z^\alpha = \left\{z = [z_0, z_1, \dots, z_\sigma] \in \mathbb{R}^r \;\middle|\; \exists\, \mu^j \in \mathbb{R} \text{ such that } z_s = \mu^j \lambda^j(s) \text{ for } s \geqslant 1\right\}$$

Notice that the market structure $M = \left[Z^0, .., Z^s, .., Z^\sigma ; Z^{\sigma+1}, .., Z^{\sigma+\bar{q}}\right]$ satisfies A4(i).

Choose for each Z^{α} the following basis. For $\alpha = s \leqslant \sigma$, choose for Z^{α} the canonical basis Λ^{s} for which each vector λ_k^s contains 0 in all entries except 1 for the entry corresponding to good (s,h) with k = h . For $\alpha = \sigma + j (j = 1,..,\bar{q})$, choose the basis, the r x (r'+1) matrix of which is :

$$\Lambda^{\sigma+j} = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & & 0 \\ 0 & & & \\ & & 1 & \\ & 0 & & \lambda^{j} \end{bmatrix}$$

Notice that no vector of Λ contains a strictly negative element.

<u>Definition 5</u> : A Walrasian Equilibrium with $\bar{q}$ securities j is a 4-uple $[v^*, \bar{\pi}^*, x^*, \mu^*]$ where $v^* = [v^{*0}, v^{*1}, \ldots, v^{*\sigma}]$ is the set of spot price vectors, $\bar{\pi}^* = \left[\bar{\pi}^{*1}, \ldots, \bar{\pi}^{*\bar{q}}\right]$ is the vector of prices of securities, $(v^{*s}, \bar{\pi}^*) > 0$, $x^* = \left[x_1^*, \ldots, x_m^*\right]$ is the m-uple of consumption of goods and $\mu^* = \left[\mu^*_1, \ldots, \mu^*_m\right]$ is the m-uple of net trades of securities, such that :

(i) (x^*, μ^*) is feasible i.e. possible for each i and balanced :

$$\sum_i x^*_{is} = \sum_i w_{is} \quad s = 0,1,..,\sigma \quad \text{and} \quad \sum_i \mu_i^{*j} = 0 \quad j = 1,..,\bar{q} \,.$$

(ii) $\forall i$: (x_i^*, μ_i^*) maximizes $u_i(x_i)$ subject to

$$v^{*0} x_{i0} + \sum_j \bar{\pi}^{*j} \mu_i{}^j \leqslant v^{*0} w_{i0} \,, \quad v^{*s} x_{is} \leqslant v^{*s}(w_{is} + \sum_j \lambda^j(s) \, \mu_i{}^j) \quad \forall s \neq 0 \text{ and}$$

$$\left[x^*_{i0}, x^*_{i1}, \ldots, x^*_{i\sigma}\right] = x_i^* \in X_i .$$

For verifying that for this particular M, a Q.E. $\{p^*, \bar{z}^*\}$ is a W.E. with securities, choose for each s, v^{*s} colinear with p^{*s} (the projection of p^* on Z^{α} for $\alpha = s \leqslant \sigma$) and define for j = 1,.., q :

$$\bar{\pi}^{*j} = \sum_{s \geqslant 1} p^{*s} \lambda^j(s).$$

Define also

$$x_i^* = w_i + \sum_\alpha z_i^{*\alpha}$$

Finally, for $\alpha = \sigma + j (j = 1,..,\bar{q})$ represent $z_i^{*\sigma+j}$ with the help of $\Lambda^{\sigma+j}$, i.e. :

$$z_i^{*\sigma+j} = \Lambda^{\sigma+j} a^{*\sigma+j}$$

Define then $\mu_i^{*j} = a^{*\sigma+j}_{q(\sigma+j)}$ for $j = 1,..,\bar{q}$.

Conversely, consider a W.E. with securities $[v^*, \bar{\pi}^* ; x^*, \mu^*]$. It is helpful to use Λ^α in order to represent z^α, $\forall\alpha$. Choose $\pi^{*\alpha} = v^{*s}$ for $\alpha = s = 0, 1,.., \sigma$. For $\alpha = \sigma + j(j = 1,..,\bar{q})$, define $\pi_k^{*0} = v_k^{*0}$ for $k = h(h = 1,..,r')$ and $\pi^{*\sigma}_{r'+1} = \bar{\pi}^{*j}$. One gets a P.E. $\left[\tilde{z}^*, (\pi^{*\alpha})_\alpha\right]$.
Under A1-A3 and if A6 is assumed (Λ contains no negative elements) it is straighforward to show that this P.E. $\left[\tilde{z}, (\pi^{*\alpha})_\alpha\right]$ is actually a Q.E. (See Proof of Proposition 1 in the Annex). ▭

Assume now A4(ii) choose for each α, <u>the</u> normalised basis $t^\alpha (\|t^\alpha\| = 1)$. Following <u>formally</u> the above argument, let be $\pi^\alpha \in \mathbb{R}$ the "price" of the unique bundle t^α on market α .

Assume that $a_i = \left(a_i^1 ,..., a_i^\alpha ,..., a_i^n\right)$ with $a_i^\alpha = 0$ $\forall\alpha$ is a solution of the following problem :

Max $V_i(a_i)$ subject to
$\pi^\alpha a^\alpha \leqslant 0 \quad \forall\alpha$
and $a_i \in P_i$

This leads to the definition of a no trade Dreze Generalized Equilibrium (D.G.E.)[5] which is a fundamental notion for this paper.

<u>Definition 6</u> : Given $M = [Z^1,...,Z^n]$ which satisfies A4(ii), $a^* = 0$ is a no trade D.G.E. iff there is $\pi^* = \left[\pi^{*1} ,..., \pi^{*\alpha} ,..., \pi^{*n}\right]$ with $\pi^{*\alpha} \in \mathbb{R}$ $\forall\alpha$, such that for each i :

a_i^* maximizes $V_i(a_i)$ subject to
$\pi^{*\alpha} a_i^\alpha \leqslant 0 \quad \forall\alpha$
and $a_i \in P_i$

Clearly if $\pi^{*\alpha} > 0$, $\pi^{*\alpha} a^\alpha \leq 0 \iff a^\alpha \in \mathbb{R}_-$

if $\pi^{*\alpha} = 0$, $\pi^{*\alpha} a^\alpha \leq 0 \iff a^\alpha \in \mathbb{R}$

if $\pi^{*\alpha} < 0$, $\pi^{*\alpha} a^\alpha \leq 0 \iff a^\alpha \in \mathbb{R}_+$

Thus $\pi^{*\alpha} < 0$ corresponds to the case where demand is rationed on market α, $\pi^{*\alpha} > 0$ to the case where supply is rationed on market α and $\pi^{*\alpha} = 0$ to the case where neither supply nor demand are rationed on market α .

<u>Remark</u> : It is tempting to refer to the case of flexible prices when A4(i) is satisfied and to the case of not totally flexible prices when A4(ii) holds. In the latter case, one can imagine that each Z^α is, originally, a two-dimensional subspace. Thus two bundles are exchanged on α . If we fix the exchange ratio on each market α, one gets, $\forall\alpha$, a one dimensional subspace.

<u>Exemple 2</u> : Consider an economy with three goods. Thus $r = 3$. Assume ${}_1M = [Z]$ where Z is defined through the following basis.

$$\begin{matrix} 1 & 1 \\ -\tfrac{1}{2} & -2 \\ 0 & 1 \end{matrix}$$

Thus one could say that this is a case of flexible prices because these two bundles can be exchanged at any exchange ratio. However, notice that Z can be defined also through the relation

$$\bar{p}_1 z_1 + \bar{p}_2 z_2 + \bar{p}_3 z_3 = 0$$

with $\bar{p}_1 = 1$, $\bar{p}_2 = 2$ and $\bar{p}_3 = 3$. Thus, one can say that actually, Z formulates the fact that prices are fixed but that there are, othervise, no restriction on the way exchanges are made. Technically, this argument has some substance because if one searches for existence of a Q.E. for this M, one encounters the following difficulty. When $p \neq \bar{p}$, subspace $\{z \in Z \mid p z = 0\}$ has dimension 1 and when $p = \bar{p}$, it has dimension 2 . One has thus <u>increases</u> of rank, and not decrease of rank, at some critical points. (Increases of rank cannot happen if bundles on the same market α do not overlap and do not contain negative elements).

Let us keep the assumption that "prices" are fixed at $\bar{p}$, and let us add the idea that commodity 1 plays the role of money. One gets then the market structure $_2M = [Z^1, Z^2]$ where $_2M$ satisfies A4(ii) :

$$Z^1 = \left\{ z \in \mathbb{R}^3 \;\middle|\; \bar{p}\, z = 0 \text{ and } z_3 = 0 \right\}$$

$$Z^2 = \left\{ z \in \mathbb{R}^3 \;\middle|\; \bar{p}\, z = 0 \text{ and } z_2 = 0 \right\}$$

Note that $_1T$ (associated with $_1M$) is an infinite set and that $_2T$ (associated with $_2M$) contains only two elements . (The philosophy of this paper is that T is the fundamental characteristics of a market structure).

II

In this part, one will define the notion of stability w.r.t. inward induction and try to justify it intuitively. I emphasize that this notion is an intermediate one in the sense that only the stronger notion of collective rationality allows us to get the two theorems of Welfare Economics in an incomplete market setting. However, it is maybe wise, from an expositional point of view, to distinguish this intermediate step.

First recall L. Hurwicz's (1959) approach. Consider an economy with two consumers and two goods. z^* is Pareto optimal, in an Edgeworth box.

On figure 1, Y_1 is the set of net trades at least as good for consumer 1 as z_1^* and Y_2 is the set of trades at least as good as z_2^*, for consumer 2.

If $\bar{B}_i$ is the set of possible trades for i, it is equivalent to write down

(i) $Y_1 = \left\{ z \in \bar{B}_1 \mid U_1(z) \geq U_1(\bar{z}) \;,\; \forall\, \bar{z} \in -Y_2 \right\}$

$Y_2 = \left\{ z \in \bar{B}_2 \;\middle|\; U_2(z) \geq U_2(\bar{z}) \;,\; \forall\, \bar{z} \in -Y_1 \right\}$

(ii) $z_1^* \in Y_1$ and $z_2^* \in Y_2$

Thus a Pareto optimum is defined in terms of sets of trade offers which are themselves defined in a recursive way. Notice that (i) and (ii) imply that :

$$z_1^* \text{ maximizes } U_1(z) \text{ in } \left\{ z \in \bar{B}_1 \mid z \in -Y_2 \right\}$$

$$z_2^* \text{ maximizes } U_2(z) \text{ in } \left\{ z \in \bar{B}_2 \mid z \in -Y_1 \right\}$$

FIGURE 1

Define now, in a classical way, a Pareto optimum[6].

Definition 7 : Given $M = [Z^1,..,Z^n]$, a feasible $\tilde{\tilde{z}}^* = \left[\tilde{\tilde{z}}_1^*,..,\tilde{\tilde{z}}_m^*\right]$ is a (weak) Pareto optimum iff there are m sets $[Y_1,..,Y_m]$ such that

(i) $\forall i : Y_i = \left\{ \tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}_i^*) \right\}$

(ii) There is no balanced $\tilde{\tilde{z}}$ such that $\tilde{\tilde{z}}_i \in Y_i \quad \forall i$.

In this definition, the market structure constrains trades only through feasibility. On the contrary, one would like to write down that the market structure constrains also the way economic agents coordinate their mutually profitable additional trades. In this perspective, coordination failures arise because economic agents cannot coordinate across markets their profitable additional trades and may fail to achieve Pareto optimal allocations, given M . In order to justify this idea, one has to describe (loosely) in what sense the market structure constrains the way agents meet. Let us offer tentatively the following paradigm.

Paradigm :

One assumes that there are many submarket β for each market α and many agents γ for each type i.

The pattern of meetings satisfies the following requirements

(i) Each agent (i,γ) participates to one submarket (α,β) for each α .

(ii) If two agents meet on some submarket, they cannot meet on another one.

(iii) More strongly : identify an agent (i,γ) with a node and draw a link between two nodes if the agents, whom these nodes are identified with, meet on some submarket. Then there is no cycle of agents who do not participate, all, to the same submarket.

Obviously, one assumes that agents of the same type are treated identically for any feasible allocation.

For concreteness, the set of indices β is a subset $\bar{N}$ of N^n, N being the set of (positive and negative) integers. Thus one can write
$\beta = [\beta_1, . , \beta_\alpha, . , \beta_n]$ and $\gamma = [\gamma_1, . , \gamma_\alpha, . , \gamma_n]$ and

$\forall\gamma$: agent $(1,\gamma)$ attends submarkets (α,β) for $\gamma = \beta$; $\forall\alpha$.
$\forall\gamma$: agent (i,γ) attends submarkets (α,β) for

$\beta_\alpha = \gamma_\alpha + i-1$ and $\beta_{\alpha'} = \gamma_{\alpha'}$ $\forall\, \alpha' \neq \alpha$; $\forall\alpha$.

Then :

$$\bar{N} = \left\{\beta \in N^n \;\middle|\; \sum_\alpha \beta_\alpha = 0\right\} \sim N^{n-1}$$

For instance, when m = 2 and n = 3, one gets by considering all (α,β) for $1 \leq \beta_1 \leq 2$, $1 \leq \beta_2 \leq 2$ and $\beta_3 = -\sum_{\alpha\neq 3} \beta_\alpha$ and ranking the β's according to the increasing lexicographic order.

Market 1

Subm.	Meetings
(1,11-2)	(1,11-2)$\leftrightarrow$(2,01-2)
(1,12-3)	(1,12-3)$\leftrightarrow$(2,02-3)
(1,21-3)	(1,21-3)$\leftrightarrow$(2,11-3)
(1,22-4)	(1,22-4)$\leftrightarrow$(2,12-4)

Market 2

Subm.	Meetings
(2,11-2)	(1,11-2)$\leftrightarrow$(2,10-2)
(2,12-3)	(1,12-3)$\leftrightarrow$(2,11-3)
(2,21-3)	(1,21-3)$\leftrightarrow$(2,20-3)
(2,22-4)	(1,22-4)$\leftrightarrow$(2,21-4)

Market 3

Subm.	Meetings
(3,11-2)	(1,11-2)$\leftrightarrow$(2,11-3)
(3,12-3)	(1,12-3)$\leftrightarrow$(2,12-4)
(3,21-3)	(1,21-3)$\leftrightarrow$(2,21-4)
(3,22-4)	(1,22-4)$\leftrightarrow$(2,22-5)

Given a feasible $\approx{z} = \left[\approx{z}_1, . . , \approx{z}_m\right]$, define $\forall i$:

$\approx{z}_{i,\gamma} = \approx{z}_i$ for each γ. Then, it is straightforward to check balance :

$$\sum_{(i,\gamma)\,\in\, M(\alpha,\beta)} z^{\alpha,\beta}_{i,\gamma} = 0 \qquad \forall(\alpha,\beta)$$

where $M(\alpha,\beta)$ is the set of agents (i,γ) who attend market (α,β).

In terms of the Paradigm, for efficiency it is assumed that trades are kept constant, except on all submarkets of one market. In terms of the original model, when on market α, each agent i considers as given his own actions on other markets $\alpha' \neq \alpha$. As $z_i^{\alpha'} = - \sum_{j \neq i} z_j^{\alpha'}$, that means that he considers as given other agents' actions on other markets α'.

The following definition is similar to those ones given in P.R. Nayak (1980), D. Gale (1982) and E. Maskin and J. Tirole (1984).

<u>Definition 8</u> : Given M, a feasible $_1\tilde{\tilde{z}} = \left[{}_1\tilde{\tilde{z}}_1, \ldots, {}_1\tilde{\tilde{z}}_m\right]$ is efficient with respect to M iff there are m.n. sets $\left\{{}^{\alpha}W_i\right\}_{(i,\alpha)}$ such that

(i) $\forall (i,\alpha)$:

$${}^{\alpha}W_i = \mathrm{Cl}\left[\left\{\tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) > U_i({}_1\tilde{\tilde{z}}_i) \text{ and } z^{\alpha'} = -\sum_{j \neq i} {}_1z_j^{\alpha'} \;\; \forall \alpha' \neq \alpha\right\} \cup \; {}_1\tilde{\tilde{z}}_i\right]$$

(ii) $\forall \alpha$:

There is no balanced $\tilde{\tilde{z}}$ with $\tilde{\tilde{z}}_i \in {}^{\alpha}W_i \;\; \forall i$ and $U_i(\tilde{\tilde{z}}_i) > U_i({}_1\tilde{\tilde{z}}_i) \;\; \forall i$.

Here Cl [A] stands for closure of A and one writes ${}_1\tilde{\tilde{z}}_i$ instead of $\left\{{}_1\tilde{\tilde{z}}_i\right\}$ in order to make lighter the notations.

Note that when convexity (A3) is assumed, each ${}^{\alpha}W_i$ is convex because each constraint $z^{\alpha'} = \sum_{j \neq i} {}_1z^{\alpha'}_j$ defines a convex set. Moreover, one has to discriminate between an offer (made by i) $\tilde{\tilde{z}} \in {}^{\alpha}W_i$ related to market α and its projection $z^{\alpha} \in {}^{\alpha}W^{\alpha}$ (${}^{\alpha}W^{\alpha}$ is the projection of ${}^{\alpha}W$ on Z^{α}, noted also W^{α}) on market α. As for $\alpha' \neq \alpha$, $z^{\alpha'}$ matches only other agents' actions, $z^{\alpha} \in W_i^{\alpha}$ can be called the "active" part of $\tilde{\tilde{z}} \in {}^{\alpha}W_i$. For each (i,α), an element z^{α} of W_i^{α}, different from ${}_1z^{\alpha}$ is called an autonomous offer (of type I).

The following proposition is well known.

<u>Proposition 3</u> : Assume A1-A3 and $\forall i$: u_i is strictly increasing and differentiable and $w_i \in \mathrm{int}\, X_i$. Then if w is efficient with respect to M satisfying A4(i) there is p^* such that $(0,p^*)$ is a Q.E. with respect to M.

Proof : See annex.

In terms of the Paradigm, it is straightforward to get a stronger concept by arguing as in Capital theory. See E. Malinvaud (1953). Given a feasible $\left\{\tilde{\tilde{z}}^{*}_{i,\gamma}\right\}_{(i,\gamma)}$, a feasible $\left\{\tilde{\tilde{z}}_{i,\gamma}\right\}_{(i,\gamma)}$ dominates finitely $\left\{\tilde{\tilde{z}}^{*}_{i,\gamma}\right\}_{(i,\gamma)}$ if there is a sequence of finite sets of submarkets (connected by agents) $\{L^1, L^2, \ldots, L^\nu, \ldots,\}$ with $L^{\nu'} \supset L^\nu$ if $\nu' > \nu$ and $\bigcup_\nu L^\nu = L$ (the set of all submarkets) such that : for each L^ν, $\left\{\tilde{\tilde{z}}_{i,\gamma}\right\}_{(i,\gamma)|L^\nu}$ (the "truncation" gotten by keeping trades fixed outside L^ν) is feasible and $U_i(\tilde{\tilde{z}}_{i,\gamma}) > U_i(\tilde{\tilde{z}}^{*}_{i,\gamma})$ for each $(i,\gamma) \in M(L^\nu)$ (where $M(L^\nu)$ is the set of agents (i,γ) who attend markets of L^ν). One has only to consider a set of typical arrangements. Consider the example with $m = 2$ and $n = 3$.

(i) The first type of arrangements is the one for which trades can be modified on only one submarket. For $n = 3$, we have three arrangements of this type. Alternatively, for these arrangements, one takes into account only autonomous offers (of type I) W_i^α and for each α, one considers $\sum_i W_i^\alpha$. Define $D_i^\alpha(1) = W_i^\alpha$, $\forall(i,\alpha)$. Let us use the generic symbol $L^\alpha(1)$ for any submarket of market α. When $m = 2$, a submarket $L^\alpha(1)$ can be represented by a graph with two nodes (See Figure 2a). One denotes, on the graph, the type of agents by Roman letters.

FIGURE 2a

When $m \geq 3$, a submarket $L^\alpha(1)$ can be represented by a polyhedron. Notice that when $m \geq 3$, there are cycles on the same submarket because there is multilateralism on each submarket. On figure (2b), one pictures $L^\alpha(1)$ for $m = 3$.

FIGURE 2b

(ii) The second type of arrangements is defined in terms of a set of submarkets (on which trades can be modified) in such a way that, for each type i, there is one agent (i,γ) who participates to n submarkets and all other agents participate to one submarket of this set. For the case $m = 2$ and $n = 3$, one can consider the n $(= 3)$ arrangements : $L^1(2)$, $L^2(2)$ and $L^3(2)$.

Notice that for $L^1(2)$, $L^2(2)$ $L^3(2)$ respectively, there are four agents whose trades can be modified on only one submarket and for whom only autonomous offers (of type I) can be taken into account. One can then say that they are at the periphery of $L^\alpha(2)$ and refer to "inward induction" of autonomous offers (of type I). (See Figure (2c)).

FIGURE 2c

The question of the existence of Pareto improvements by modifying trades on only the submarkets of the second type of arrangements can be cast in the following framework :

Define $\forall(i,\alpha)$:

$$\mathcal{D}_i(2) = \mathrm{Cl}\left[\left\{\tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}_i^*) \text{ and } z^{\alpha'} \in -\sum_{j\neq i} D_j^{\alpha'}(1)\ \forall\alpha' \neq \alpha\right\} \cup \tilde{\tilde{z}}_i^*\right]$$

where $D_i^\alpha \equiv \mathcal{D}_i^\alpha$ is the projection of $\mathcal{D}_i$ on Z^α.

Notice that $D_i^\alpha(2) \supset D_i^\alpha(1) \equiv W_i^\alpha$ $\forall(i,\alpha)$.

It is then straighforward to verify that for iteration 2 :

There is a Pareto improvement iff for some α : there is $\tilde{\tilde{z}} = \left[\tilde{\tilde{z}}_1,\ldots,\tilde{\tilde{z}}_m\right]$ with $\tilde{\tilde{z}}_i \in \mathcal{D}_i(2)$ $\forall i$, $\sum_i z_i^\alpha = 0$ and $U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}_i^*)$ $\forall i$.

Alternatively, there is a Pareto improvement iff there is a feasible $\tilde{\tilde{z}} = \left[\tilde{\tilde{z}}_1,\ldots,\tilde{\tilde{z}}_m\right]$ with $z_i^\alpha \in D_i^\alpha(2)$ $\forall(i,\alpha)$ and $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}_i^*)$ $\forall i$.

(Notice that $z_i^{*\alpha} \in D_i^\alpha(\tau)$ $\forall\tau$ and $\forall(i,\alpha)$).

(iii) The third type of arrangements corresponds obviously to iteration 3.

$\forall(i,\alpha)$ define :

$$\mathcal{D}_i(3) = \mathrm{Cl}\left[\left\{\tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}_i^*) \text{ and } z^{\alpha'} \in -\sum_{j\neq i} D_j^{\alpha'}(2)\ \forall\alpha' \neq \alpha\right\} \cup \tilde{\tilde{z}}_i^*\right]$$

An example of $L^1(3)$ is pictured as Figure 2d.

FIGURE 2d

Sticking still to the assumption of identical treatment of agents of the same type, one gets thus a concrete process and one defines stability w.r.t. inward induction as stability w.r.t. inward induction for every finite set of submarkets. Going to the limit by considering a fixed point of this concrete process does not mean that one considers Pareto optimality, as it will be clear in the sequel.

<u>Definition 9</u> : Given $M = [Z^1,..,Z^n]$, a feasible $\tilde{\tilde{z}}^* = \left[\tilde{\tilde{z}}^*_1,..,\tilde{\tilde{z}}^*_m\right]$ is stable w.r.t. inward induction if there are m.n. sets $D^\alpha_i \equiv {}^\alpha\mathcal{D}^\alpha_i$ such that

(i) $\forall(i,\alpha)$:

$D^\alpha_i = \mathrm{Cl}\left[\left\{z^\alpha \in Z^\alpha \;\middle|\; \exists\, \tilde{\tilde{z}} = [z^1,..,z^\alpha,..,z^n] \in B_i \text{ with } U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}^*_i)\right.\right.$
and $\left.\left. z^{\alpha'} \in -\sum_{j\neq i} D^{\alpha'}_j \;\; \forall\alpha' \neq \alpha \right\} \cup z^{*\alpha}_i\right]$.

Under convexity, it is assumed that $\forall(i,\alpha)$: $\sum_{j\neq i} D^\alpha_j$ is convex.

(ii) There is no feasible $\tilde{\tilde{z}} = \left[\tilde{\tilde{z}}_1,..,\tilde{\tilde{z}}_m\right]$ such that $\forall i$:

$z^\alpha_i \in D^\alpha_i\ \forall\alpha$ and $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}^*_i)$.

Because of our construction, there is nothing illogical in requiring that, under convexity, $\sum_{j\neq i} D^\alpha_j$ is convex, $\forall(i,\alpha)$.

<u>Remark</u> : One can substitute for (ii) of Definition 9 : (ii') $\forall\alpha$: There is no balanced $\tilde{\tilde{z}} = \left[\tilde{\tilde{z}}_1,..,\tilde{\tilde{z}}_m\right]$ such that $\forall i$ $\tilde{\tilde{z}}_i \in {}^\alpha\mathcal{D}_i$ and $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}^*_i)$.

The two following statements are related to Definition 9 :

(i) Stability w.r.t. inward induction is strictly stronger than efficiency. Example 3 illustrates this fact.

(ii) Definition 9 can be written in terms of the Paradigm. Consider a feasible $(\tilde{\tilde{z}}^*_{i,\gamma})_{(i,\gamma)}$ with $\tilde{\tilde{z}}^*_{i,\gamma} = (z^{*\alpha,\beta}_{i,\gamma})_{(\alpha,\beta)\in N(i,\gamma)}$ where $N(i,\gamma)$ is the set of submarkets that (i,γ) visits.

<u>Definition 9'</u> : Given $M = [z^1,..,z^n]$, a feasible $(\tilde{\tilde{z}}^*_{i,\gamma})_{(i,\gamma)}$ is stable w.r.t. inward induction if there is a system of sets of offers $\left\{D^{\alpha,\beta}_{i,\gamma}\right\}$ such that :

(i) $\forall(i,\gamma)$ and $\forall(\alpha,\beta)$:

$D^{\alpha,\beta}_{i,\gamma} = \mathrm{Cl}\left[\left\{z^\alpha \in Z^\alpha \;\middle|\; \exists\, \tilde{\tilde{z}} = [z^1,..,z^\alpha,..,z^n] \in B_i \text{ with } U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}^*_i) \text{ and}\right.\right.$

$\left.\left. z^{\alpha',\beta'} \in -\sum_{\substack{(i',\gamma')\in M(\alpha',\beta') \\ \neq(i,\gamma)}} D^{\alpha',\beta'}_{i',\gamma'} \;\; \forall(\alpha',\beta') \in N(i,\gamma) \setminus \{\alpha,\beta\} \right\} \cup z^{*\alpha,\beta}_{i,\gamma}\right]$

(ii) There is no feasible $(\tilde{\tilde{z}}_{i,\gamma})_{(i,\gamma)}$ such that $\forall(i,\gamma)$ $U_i(\tilde{\tilde{z}}_{i,\gamma}) > U_i(\tilde{\tilde{z}}^*_{i,\gamma})$ and $\forall(\alpha,\beta) \in N(i,\gamma)$: $\tilde{\tilde{z}}^{\alpha,\beta}_{i,\gamma} \in D^{\alpha,\beta}_{i,\gamma}$.

Definition 9′ shows clearly that the system of sets $\{D^{\alpha,\beta}_{i,\gamma}\}$ can be considered as a coherent (rational) system of forecasts of sets of offers : When on submarket (α,β), each agent $(i,\gamma) \in M(\alpha,\beta)$ offers $D^{\alpha,\beta}_{i,\gamma}$ knowing that he can match elements of (rational) sets of offers which are proposed by other agents on markets $(\alpha',\beta') \in N(i,\gamma) \setminus \{(\alpha,\beta)\}$.

Now, Definition 9 and 9′ characterize a feasible trade as being stable w.r.t. inward induction by saying that there exists a coherent system of sets of offers which supports it, not by saying that it can be supported by any coherent set of offers. An obvious candidate is the limit of the sequence $\{D^{\alpha}_i(\tau)\}$, which is "minimal" (in the sense of inclusion of sets) in the set of coherent systems which satisfy (i) of Definition 9, knowing that in any case $Y^{\alpha}_i \supset D^{\alpha}_i \supset W^{\alpha}_i$ $\forall(i,\alpha)$. Notice also that Definition 9 is Definition 9′ for which one has assumed that $\forall(i,\alpha)$: $\tilde{\tilde{z}}^{*\alpha,\beta}_{i,\gamma} = \tilde{\tilde{z}}^{*\alpha}_i$ and $D^{\alpha,\beta}_{i,\gamma} = D^{\alpha}_i$ $\forall(\gamma,\beta)$. Finally, let us sketch the relationships between stability w.r.t. inward induction and "finite" optimality in the sense of Capital theory.

(i). Consider a feasible $(\tilde{\tilde{z}}^*_{i,\gamma})_{i,\gamma}$ which satisfies Definition 9′ with $\forall(i,\alpha)$: $\tilde{\tilde{z}}^{*\alpha,\beta}_{i,\gamma} = \tilde{\tilde{z}}^{*\alpha}_i$ and $D^{\alpha,\beta}_{i,\gamma} = D^{\alpha}_i$ $\forall(\gamma,\beta)$. It is straighforward to check that there is no feasible $(\tilde{\tilde{z}}_{i,\gamma})$ which dominates it, finitely.

(ii). The converse statement requires some compacity argument in order to be made easily. Consider $\rho > 0$ and for each (i,α), a ball $B(\rho,z^{*\alpha}_i)$ containing $z^{*\alpha}_i$. One assumes that ρ is large enough in order that $B(\rho,z^{*\alpha}_i)$ contains the projection of the set of feasible $\tilde{\tilde{z}}$'s with $U_i(\tilde{\tilde{z}}_i) \geq U_i(\tilde{\tilde{z}}^*_i)$, if this set is compact. Notice that if A3 obtains, there is no problem because if a feasible $\tilde{\tilde{z}}$ Pareto dominates $\tilde{\tilde{z}}^*$, then $\mu\tilde{\tilde{z}} + (1-\mu)\tilde{\tilde{z}}^*$ Pareto dominates $\tilde{\tilde{z}}^*$ for every $\mu \in [0,1[$. Then, in relevant definitions consider $\bar{W}^{\alpha}_i = W^{\alpha}_i \cap B(\rho,z^{*\alpha}_i)$ and

$$\bar{D}^{\alpha}_i = Cl\left[\left\{ z^{\alpha} \in B(\rho,z^{*\alpha}_i) \;\middle|\; \exists\, \tilde{\tilde{z}} = (z^{\alpha},z^{-\alpha}) \in B_i \text{ with } U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}^*_i) \right.\right.$$

$$\left.\left. \text{and } z^{\alpha'} \in -\sum_{j\neq i} \bar{D}^{\alpha'}_j \;\; \forall\alpha' \neq \alpha \right\} \cup z^{*\alpha}_i \right].$$ Now, consider a feasible $(\tilde{\tilde{z}}^*_{i,\gamma})_{i,\gamma}$ with $\forall(i,\alpha)$:

$$\tilde{\tilde{z}}^{*\alpha,\beta}_{i,\gamma} = \tilde{\tilde{z}}^{*\alpha}_i \quad \forall(\gamma,\beta)$$

which is not finitely dominated.

Define $\bar{D}_i^\alpha(1) = \bar{W}_i^\alpha \; \forall (i,\alpha)$ and $\forall \tau > 1$:

$$\bar{D}_i^\alpha(\tau) = Cl\left[\left\{ z^\alpha \in G(\rho, z_i^{*\alpha}) \;\middle|\; \exists\, \tilde{\tilde{z}} = [z^\alpha, z^{-\alpha}] \in B_i \text{ with } U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}_i^*) \text{ and } z^{\alpha'} \in -\sum_{j \neq i} \bar{D}_j^{\alpha'}(\tau - 1) \;\forall \alpha' \neq \alpha \right\} \cup z_i^{*\alpha}\right].$$

One has for each (i,α) :

$$\bar{D}_i^\alpha(\tau) \supset \bar{D}_i^\alpha(\tau - 1).$$

Consider a limit point of the sequence, $\bar{D}_i^\alpha$, for each (i,α). One can check that there is not a feasible $\tilde{\tilde{z}}$ with $z_i^\alpha \in \bar{D}_i^\alpha(i,\alpha)$ which Pareto dominates $\tilde{\tilde{z}}^*$.

Example 3 : (E. Maskin's lecture notes).

This example was originally built by E. Maskin in order to show that a Social Nash Optimum (S. Grossman (1977)) is not necessarily an equilibrium. In our language, this is an example of an efficient allocation which is not a C.E. and not stable w.r.t. inward induction.

There are four goods ($r = 4$) and two consumers ($m = 2$).

$u_1(.) = 4 \operatorname{Min}\{x_1, x_3\} + x_2 + x_4$ and $w_1 = [0,1,0,1]$

$u_2(.) = x_1 + x_2 + x_3 + x_4$ and $w_2 = [1,0,1,0]$

The market structure is such that good 1 can be exchanged only against good 2 and good 3 only for good 4.

In this case $M = [Z^1, Z^2]$ with

$$Z^1 = \left\{ z \in \mathbb{R}^4 \mid z_3 = z_4 = 0 \right\}$$

$$Z^2 = \left\{ z \in \mathbb{R}^4 \mid z_1 = z_2 = 0 \right\}$$

Let us show that the initial allocation ($\tilde{\tilde{z}}^* = 0$) is efficient. Choose :

$${}^1W_1 = \left\{ \mathbb{R}_+^2 \times \{0,0\} \right\} \text{ and } {}^2W_1 = \left\{ \{0,0\} \times \mathbb{R}_+^2 \right\}$$

$${}^1W_2 = \left\{ \tilde{\tilde{z}} = [z^1, 0] \;\middle|\; \text{either } z^1 \geq 0 \text{ or } z_2^1 \geq -z_1^1 \geq 0 \right\}$$

$${}^2W_2 = \left\{ \tilde{\tilde{z}} = [0, z^2] \;\middle|\; \text{either } z^2 \geq 0 \text{ or } z_4^2 \geq -z_3^2 \geq 0 \right\}$$

These are sets of autonomous offers.

FIGURE 3

On figure 3, sets ${}^{1}W_2^1$ and ${}^{2}W_2^2$ are pictured. Thus $\approx\!\!\!z^* = 0$ is efficient. It is straightforward to show that its is not stable w.r.t. inward induction. Ad absurdum assume that there are four sets $\{D_i^\alpha\}_{(i,\alpha)}$ which satisfy conditions (i) and (ii) of definition 9. It would follow that $D_2^\alpha \supset W_2^\alpha$ because W_2^α contains only autonomous offers.

$[-\varepsilon,\ 3/2\ \varepsilon\ ;\ 0,0] \in W_2^1 \subset D_2^1$ and also
$[0,0\ ;\ -\varepsilon\ ,\ 10/9\ \varepsilon] \in W_2^2 \subset D_2^2$

As $[-\varepsilon,\ 3/2\varepsilon\ ;\ 0,\ 0] \in W_2^1$, then : $[0,\ 0\ ;\ \varepsilon\ ,\ -\ 10/9\ \varepsilon] \in D_1^2(2) \subset D_1^2$.
As $[0,\ 0\ ;\ -\varepsilon\ ,\ 10/9\ \varepsilon] \in W_2^2$, then : $[\varepsilon,\ -3/2\varepsilon\ ;\ 0,0] \in D_1^1(2) \subset D_1^1$.
Choose then $\tilde{\tilde{z}}_1 = [\varepsilon,\ -3/2\varepsilon\ ;\ \varepsilon\ ,\ -\ 10/9\ \varepsilon]$ and
$\tilde{\tilde{z}}_2 = [-\varepsilon,\ 3/2\varepsilon\ ;\ -\varepsilon\ ,\ 10/9\ \varepsilon]$
$\tilde{\tilde{z}} = \left[\tilde{\tilde{z}}_1, \tilde{\tilde{z}}_2\right]$ is feasible and $U_i(\tilde{\tilde{z}}_i) > U_i(0)$ $i = 1,\ 2.$

In terms of the Paradigm, it is obvious : on Figure 2 f, agent (1,1-1) trades with (2,0-1) on submarket (1,1-1) and with (2,1-2) on submarket (2,1-1). There is a Pareto improvement for the set of these agents. Extend now sequentially both sides of Figure 2 f, as it is done in capital theory. We can generate then a Pareto improvement for which all agents of the same type are treated identically. Each agent $(1,\gamma)$ gets : $\tilde{\tilde{z}}_1 = [\varepsilon,\ -3/2\varepsilon\ ;\ \varepsilon,\ -10/9\ \varepsilon]$ and each agent $(2,\gamma)$ gets
$\tilde{\tilde{z}}_2 = [-\varepsilon,\ 3/2\varepsilon\ ;\ -\varepsilon,\ 10/9\ \varepsilon]$. □

III

The objective of this part is to show (in terms of the Paradigm) how to go farther than stability w.r.t. inward induction without going as far as full optimality which allows for unrestricted changes of trades on an infinite set of submarkets.

The best way of introducing the idea of "outward induction" is to argue on an example.

<u>Example 4</u> :

$m = n = 2$ and $r = 3$

$M = [Z^1, Z^2]$ with

$Z^1 = \left\{ z \in \mathbb{R}^3 \mid z_2 = 0 \text{ and } z_1 = -z_3 \right\}$

$Z^2 = \left\{ z \in \mathbb{R}^3 \mid z_1 = 0 \text{ and } z_2 = -z_3 \right\}$

$u_1 = 4 \text{ Min } \{x_1, x_2\} + x_3 \qquad w_1 = [0,0,1]$

$u_2 = 5/4 \text{ Min } \{x_1, x_2\} + x_3 \qquad w_1 = [1,1,0]$

The initial allocation is not Pareto optimal. Consumer 1 would like to buy goods 1 and 2 simultaneouly but not either good 1 or good 2. Consumer 2 would like to sell goods 1 and 2 simultaneously but not either good 1 or good 2. Thus ${}^1W_1 = {}^1W_2 = {}^2W_1 = {}^2W_2 = \{[0,0 \; ; \; 0,0]\}$. Also ${}^\alpha D_i = {}^\alpha W_i \quad \forall (i,\alpha)$.

The m.n sets D_i^α constitute a system of coherent sets of forecasts of offers which do not lead to any Pareto improvement. Let us argue in terms of the Paradigm (Figure 2f) and consider, e.g., consumer (1,1-1) who meets consumer (2,0-1) on submarket (1,1-1) and consumer (2,1-2) on submarket (2,1-1). Without matching any offer, he proposes (autonomous offers of type II) simultaneously $[\varepsilon,0,-\varepsilon]$ on submarket (1,1-1) and $[0,\varepsilon,-\varepsilon]$ on submarket (2,1-1). Notice that these offers are proposed to two agents who do no meet and that there are no cycles of agents. In order that there is a chance that this offer induces a wave of trades which are Pareto improving, (any Pareto improvement implies, in this example, that trades are modified on all submarkets), it is necessary that both agents (2,0-1) (on the left side) and (2,1-2) (on the right side) modify consequently their offers in order to create a wave of offers on both sides. One of them (e.g. (2,0-1)) will not do that if he has already taken into account the offer made by (1,1-1) to him (e.g. $[\varepsilon,0,-\varepsilon]$) in order to make offers on the other submarket he attends. The same argument can be made for agent (2,0-1) who can increase his utility only by trading with two distinct agents of type 1.

Before showing that in example 4, the initial allocation is also stable w.r.t. outward induction, i.e. collectively rational, let us formulate two definitions. ▭

We give first a (rough) Definition in terms of the Paradigm :

<u>Definition 10'</u> : A feasible $\left\{\tilde{\tilde{z}}^*_{i,\gamma}\right\}$ is collectively rational iff there are sets $\left\{\tilde{\tilde{Z}}^{\alpha,\beta}_{i,\gamma}\right\}$ and $\left\{\bar{Z}^{\alpha,\beta}_{i,\gamma}\right\}$ such that :

(i)

$$\tilde{\tilde{Z}}^{\alpha,\beta}_{i,\gamma} = \left\{ z^{\alpha,\beta} \in Z^{\alpha} \;\middle|\; \exists\, \tilde{\tilde{z}} = [z^{\alpha,\beta}, z^{-(\alpha,\beta)}] \in B_i,\ U_i(\tilde{\tilde{z}}) \geq U_i(\tilde{\tilde{z}}^*_{i,\gamma}) \right.$$

$$\left. \text{and } z^{\alpha',\beta'} \in -\left[\sum_{\substack{(i',\gamma')\neq(i,\gamma) \\ (i',\gamma')\in M(\alpha',\beta')}} \tilde{\tilde{Z}}^{\alpha',\beta'}_{i',\gamma'} \cup \bar{Z}^{\alpha',\beta'}_{i,\gamma} \right] \forall(\alpha',\beta') \in N(i,\gamma)\setminus(\alpha,\beta) \right\}.$$

(ii) There is no balanced $\left\{\tilde{\tilde{z}}_{i,\gamma}\right\}$ with $\tilde{\tilde{z}}^{\alpha,\beta}_{i,\gamma} \in \tilde{\tilde{Z}}^{\alpha,\beta}_{i,\gamma}$ $\forall(i,\gamma)$ and $(\alpha,\beta) \in N(i,\gamma)$ with $U_i(\tilde{\tilde{z}}_{i,\gamma}) \geq U_i(\tilde{\tilde{z}}^*_{i,\gamma})$ $\forall(i,\gamma)$ with strict inequality for at least one (i,γ).

(iii) $\forall(i,\gamma)$, $\forall(\alpha,\beta)$, $\bar{Z}^{\alpha,\beta}_{i,\gamma}$ satisfies :

$\forall(i,\gamma)$ and $\forall N_2 \subset N(i,\gamma)$ ($N_2 \neq \emptyset$ and $\bar{N} \in N(i,\gamma)\setminus N_2$), define :

$$^{N_2}\bar{\bar{Z}}_{i,\gamma} = \left\{ \tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) \geq U_i(\tilde{\tilde{z}}^*_{i,\gamma})\ ;\ \forall(\alpha',\beta') \in \bar{N} : \right.$$

$$\left. z^{\alpha',\beta'} \in -\left[\sum_{\substack{(i',\gamma')\neq(i,\gamma) \\ (i',\gamma')\in M(\alpha',\beta')}} \tilde{\tilde{Z}}^{\alpha',\beta'}_{i',\gamma'} \cup \bar{Z}^{\alpha',\beta'}_{i,\gamma} \right] \right\}.$$

For each (i,γ) and $N_2 \subset N(i,\gamma)$: For each $\tilde{\tilde{z}} \in {}^{N_2}\bar{\bar{Z}}_{i,\gamma}$, one can find $(\alpha,\beta) \in N_2(i,\gamma)$ such that for each $(i',\gamma') \in M(\alpha,\beta)$:

$$z^{\alpha,\beta} \in \left[\sum_{\substack{(j',\gamma)\neq(i',\gamma') \\ (j',\gamma)\in M(\alpha,\beta)}} \tilde{\tilde{Z}}^{\alpha,\beta}_{j',\gamma} \cup \bar{Z}^{\alpha,\beta}_{i',\gamma'} \right].$$

Definition (10') can be read as follows. When agent (i,γ) makes offers on submarket $(\alpha,\beta) \in N(i,\gamma)$, he takes into account not only offers made by other agents on submarkets (α',β'), different from (α,β), which he attends but also "trembles" or "forecasts" of offers made by others. Condition (iii) of Definition (10') holds for these forecasts. It indicates that if an agent (i',γ') makes a profitable offer ^{N_2}z <u>simultaneously</u> on markets $\alpha \in N_2(i',\gamma') \subset N(i',\gamma')$, for at least one $(\alpha',\beta') \in N_2(i',\gamma')$, all agents

(i,γ) different from (i',γ') and attending submarket (α',β') take into account this offfer in $\bar{Z}^{\alpha',\beta'}_{i,\gamma}$. From (ii) of Definition 10' this attempt made by (i',γ') does not lead to a Pareto improvement, given the pattern of meetings. (See the end of the continuation of Example 4).

<u>Definition 10</u> : Given $M = [Z^1,.,Z^m]$ a feasible $\tilde{z}^*$ is collectively rational if there are m.n sets $\tilde{Z}^\alpha_i$ and m.n sets $\bar{Z}^\alpha_i$ such that :

(i) $\forall(i,\alpha)$: $\tilde{Z}^\alpha_i = \mathrm{Cl}\left[\left\{ z^\alpha \in Z^\alpha \;\middle|\; \exists\, \tilde{z} = [z^1,.,z^\alpha,.,z^n] \in B_i \text{ with } U_i(\tilde{z}_i) > U_i(\tilde{z}^*_i) \text{ and } z^{\alpha'} \in \left[\sum_{j\neq i} \tilde{Z}^{\alpha'}_j \cup \bar{Z}^{\alpha'}_i\right] \forall \alpha' \neq \alpha \right\} \cup z^{*\alpha}_i \right]$. If convexity is assumed, then $\forall(i,\alpha)$, $\sum_{j\neq i} \tilde{Z}^\alpha_j \cup \bar{Z}^\alpha_i$ is convex[7].

(ii) $\forall(i,\alpha)$, $\bar{Z}^\alpha_i$ satisfies the following requirements

(a) $\bar{Z}^\alpha_i \cup \sum_{j\neq i} \tilde{Z}^\alpha_j \supset \mathrm{Cl}\left[\sum_{j\neq i} \bar{\bar{Z}}^\alpha_j\right]$ where for each j, $(\bar{\bar{Z}}^\alpha_j)_\alpha$ satisfies the following conditions : If $\tilde{z} \in B_j$ with $U_j(\tilde{z}) > U_j(\tilde{z}^*_j)$ and $z^{\alpha'} \in -\left[\bar{Z}^{\alpha'}_j \cup \sum_{i'\neq j} \tilde{Z}^{\alpha'}_{i'}\right]$ for $\alpha' \in \bar{N} \subset \{1,..,n\}$ then for at least one $\alpha \notin \bar{N}$, $z^\alpha \in \bar{\bar{Z}}^\alpha_j$.

(b) sets $\bar{Z}^\alpha_i$ are compatible in the sense that if $\exists$ i with $z^{*\alpha}_i + a_i t \in \bar{Z}^\alpha_i \cup \sum_{j\neq i} \tilde{Z}^\alpha_j$, $a_i \in \mathbb{R}_{++}$ and $\|t\| \neq 0$, then $\forall i' \neq i$, $\exists\, a_{i'} \in \mathbb{R}_{++}$ with $z^{*\alpha}_{i'} + a_{i'} t \in \bar{Z}^\alpha_{i'} \cup \sum_{j\neq i'} \tilde{Z}^\alpha_j$.

(iii) There is no feasible $\tilde{z}$ such that $\forall i$: $z^\alpha_i \in \tilde{Z}^\alpha_i \;\forall\alpha$ and $U_i(\tilde{z}_i) > U_i(\tilde{z}^*_i)$.

Remark : Requirement (iib) is a mild regularity condition which can, probably, be relaxed. Roughly, sets $\tilde{Z}^\alpha_i$ contain autonomous offers (of type I) and induced offers while sets $\bar{Z}^\alpha_i$ contain autonomous offers of type II. However, this terminology is loose because Definition 10 does not require that $\forall(i,\alpha)$: $\sum_{j\neq i} \tilde{Z}^\alpha_j \cap \bar{Z}^\alpha_i = \emptyset$. However, as $\forall(i,\alpha)$, $\tilde{Z}^\alpha_i$ depends only on

$\sum_{j\neq i} \bar{\bar{Z}}^{\alpha}_j \cup \bar{Z}^{\alpha}_i$, it is always possible to redefine $\bar{Z}^{\alpha}_i$ in such a way that $\sum_{j\neq i} \bar{\bar{Z}}^{\alpha}_j \cap \bar{Z}^{\alpha}_i = \emptyset$.

<u>Example 4</u> (continued) :

The initial allocation is collectively rational.

Define :

$$\bar{Z}^1 = \left\{ z^1 \in Z^1 \;\middle|\; \exists\, \bar{\bar{z}} = (z^1, z^2) \in B_1 \text{ with } U_1(\bar{\bar{z}}) \geq U_1(0) \right\}.$$

$$\bar{Z}^2 = \left\{ z^2 \in Z^2 \;\middle|\; \exists\, \bar{\bar{z}} = (z^1, z^2) \in B_2 \text{ with } U_2(\bar{\bar{z}}) \geq U_2(0) \right\}.$$

and $\bar{Z}^{\alpha}_i = \bar{Z}^{\alpha}$ $\forall (i,\alpha)$. Also :

$$^{\alpha}\bar{\bar{Z}}_i = \left\{ \bar{\bar{z}} \in B_i \;\middle|\; U_i(\bar{\bar{z}}) \geq U_i(0) \text{ and } z^{\alpha'} \in -\left[\bar{Z}^{\alpha'}_i \cup \bar{\bar{Z}}^{\alpha'}_j \right] \right\}.$$

Notice that the system $\left\{\bar{\bar{Z}}^{\alpha}_i\right\}_{(i,\alpha)}$ is such that agent 1 makes offers of purchasing on market 1 but that $\bar{\bar{Z}}^2_1 = \{0,0,0\}$ and that agent 2 makes offers of selling on market 2 but that $\bar{\bar{Z}}^1_2 = \{0,0,0\}$.

In terms of the Paradigm, agent $(1,\gamma)$ proposes (autonomous offers of type II) Y^1_1 to agent $(2,\gamma')$ with $(2,\gamma') \in M(1,\gamma)$ and Y^2_1 to $(2,\gamma'')$ with $(2,\gamma'') \in M(2,\gamma)$, where Y^{α}_i is the projection on Z^{α} of

$$Y_i = \left\{ \bar{\bar{z}} \in B_i \;\middle|\; U_i(\bar{\bar{z}}) > U_i(0) \right\}.$$

One chooses $\bar{\bar{Z}}^1_1 = \mathrm{Cl}\ Y^1_1$ and $\bar{\bar{Z}}^2_1 = \{0\}$

In the same way, agent $(2,\gamma)$ proposes (an autonomous offer of type II), Y^1_2 to agent $(1,\gamma')$ with $(1,\gamma') \in M(1,\gamma')$ and Y^2_2 to $(1,\gamma'')$ with $(1,\gamma'') \in M(2,\gamma'')$.

We do that for all agents $(1,\gamma)$ for type 1 and all agents $(2,\gamma)$ of type 2. Then condition (iii) of Defintion 10′ is satisfied.

Consider now a Pareto improvement. Given the Paradigm (see Figure 2 f and imagine that it is extended both sides to infinity), a Pareto improvement implies – even if we do not require identity of treatment of agents of the same type, for Pareto improvements - that trades have to be modified on all submarkets. Now, consider an agent $(1,\gamma)$ (e.g. $\gamma = (1\text{-}1)$ on Figure 2 f) who is trying to perturb this system of offers by offering simultaneously to trade with agents $(2,\gamma')$ $(\gamma' = (0\text{-}1))$ and $(2,\gamma'')(\gamma'' = (1\text{-}2))$. For agent $(2,\gamma'')$, this offer is an innovation, and he may consider the possibility of revising his own offers and thus to induce a wave of new offers (going to the right in Figure 2 f). But, by definition, the trade proposed to agent $(2,\gamma')$ is already taken into account and he has no reason to revise his own offers, i.e. given the pattern of meetings, to induce a wave of new offers (going to the left in Figure 2 f). Thus this attempt leads to a blind alley because by assumption a Pareto improvement implies that trades are modified on all submarkets. Note that the same argument can be made for an agent $(2,\gamma)$ of type 2. Thus, as in the field of sequential equilibrium analysis (forward induction), this kind of argument provides a higher degree of stability. □

On the contrary, example 5 pictures a situation for which the initial allocation which is stable w.r.t. inward induction is not stable w.r.t. to outward induction.

Example 5 : (J. Sylvestre (1982)).

This example was built to show that a Dreze Equilibrium is stronger than a K-Equilibrium (J. Benassy (1975)) or a P-Equilibrium (Y. Younès (1975)).

$r = 4$, $m = 2$ and $n = 3$.

$w_1 = [1,0,0,0]$; $w_2 = [1,1,1,1]$

$u_1 = (x_{11}.x_{12}.x_{13})^{1/3} + (x_{11}.x_{12}.x_{14})^{1/3} + (x_{11}.x_{13}.x_{14})^{1/3}$

$u_2 = \text{Min}\ \{x_{21}.x_{22}, x_{21}.x_{23}, x_{21}.x_{24}\}$

Thus $u_1(w_1) = 0$ and $u_2(w_2) = 1$

$T = \{t^1, t^2, t^3\}$ with $t^1 = \{-1, 1, 0, 0\}$, $t^2 = \{-1, 0, 1, 0\}$

$t^3 = \{-1, 0, 0, 1\}$. $M = [Z^1, Z^2, Z^3]$ is constructed accordingly.

Consider the initial allocation. It is straightforward to verify that it is stable w.r.t. inward induction. For that, choose $^{\alpha}\mathcal{D}_i = \{0\}$ for $i = 1, 2$ and $\alpha = 1, 2, 3$. This is enough because consumer 1 would like to buy any pair of the set of goods $\{2, 3, 4\}$ against good 1 and consumer 2 would like to sell any pair of the set $\{2, 3, 4\}$ against good 1 but none would like to trade only one good of the set $\{2, 3, 4\}$ for good 1.

However, the initial allocation is not stable w.r.t. outward induction. According to (ii a) of Definition 10, as for each pair of markets agent 1 would like to buy, the system $\{\bar{Z}_2^\alpha\}_\alpha$ has to be such that agent 2 can match purchase offers

- either on markets 1 and 2
- or on markets 1 and 3
- or on markets 2 and 3

Conversely, as for each pair of markets, agent 2 would like to sell, the system $\{\bar{Z}_1^\alpha\}_\alpha$ has to be such that agent 1 can match sale offers

- either on markets 1 and 2
- or on markets 1 and 3
- or on markets 2 and 3

It is then straightforward to check that one cannot find sets $\{\bar{\bar{Z}}_i^\alpha\}_{(i,\alpha)}$ and $\{\bar{Z}_i^\alpha\}_{(i,\alpha)}$ which support the initial allocation according to definition 10. $\square$

<u>Proposition 4</u> : Given $M = [Z^1, .., Z^n]$, if $\bar{\bar{z}}$ is Pareto optimal, it is collectively rational.

Proof : See annex.

The two following propositions show the link between the notions of D.G.E. and Q.E. on the one hand and of collective rationality on the other hand, in the case where A3 and A4 ((i) or (ii)) are made.

<u>Proposition 5</u> : Assume that $M = [Z^1, .., Z^n]$ satisfies A4(ii). Under A1 and A2 if $\bar{\bar{z}}^*$ is a no trade D.G.E. it is collectively rational. Under A1–A3, if 0 is collectively rational, it is a D.G.E.

Proof : See annex.

<u>Proposition 6</u> : Assume that $M = [Z^1 ,., Z^n]$ satisfies A4(i). Under A1 and A2, if $\overset{\approx}{z}^*$ is a Q.E., it is collectively rational. Under A1-A3, if 0 is collectively rational, it is a C.E. (it is a Q.E. if A5 and A6 hold also). Under A1-A3, if $\overset{\approx}{z}^*$ is collectively rational, it is a C.E. with transfers (it is a Q.E. with transfers if A5 and A6 hold also).

Proof : See annex.

The following example suggests that the homology between models with incomplete markets and overlapping generations models is valid (even when money is present). In this case, "inside money" corresponds to "outside money".

<u>Example 6</u> : Relations with Overlapping Generations Models.

1. There are two consumers and three commodites.

The last commodity is inside money in the sense that
(i) it does not enter in utility functions, initial and final endowments in this commodity are 0 for each i.
(ii) it has to enter in each transaction for goods 1 and 2 separately.
(ii) entails that the market structure $M = [Z^1, Z^2]$is such that

$$Z^1 = \left\{z \in \mathbb{R}^3 \mid z_2 = 0\right\}$$

$$Z^2 = \left\{z \in \mathbb{R}^3 \mid z_1 = 0\right\}$$

Let us assume that $X_i = \mathbb{R}^3_+$ and that $u_i(x_{i1}, x_{i2})$ is strictly increasing, $\forall i$. Assume also that the initial allocation is not Pareto optimal and that there is one W.E. for the economy endowed with a complete market structure such that agent 1 buys good 1.

There are then two Q.E.'s w.r.t. M. The first one corresponds to the initial allocation with $p_3^* = 0$ and the second one corresponds to the W.E.

associated with the complete market structure with $p^*_3 > 0$, with

$$z^{*1}_{i3} = -z^{*2}_{i3} = \frac{-p^*_1}{p^*_3} z^{*1}_{i1} = \frac{p^*_2}{p^*_3} z^{*2}_{i2} \text{ , } \forall i.$$

Obviously, these two equilibria correspond to the two stationary states of the Samuelson Case of the simplest overlapping generations model for which each agent lives for two periods and at each period, there is one good and outside money. More precisely, one has to assume that the two agents are symmetric, i.e. :

$$u_1(x_1, x_2) = u_2(x_2, x_1) \ \forall (x_1, x_2) \in \mathbb{R}^2_+ \text{ and } w_{12} = w_{21}, w_{11} = w_{22} \text{ ,}$$

in order to make complete the homology given the following correspondence between patterns of meetings :

Overlapping generations model		Paradigm of the incomplete markets model	
Periods	Meetings of agents	Subm.	Meetings of agents
⋮	⋮		
-2	-2 ↔ -1	(2,1-1)	(1,1-1) ↔ (2,1-2)
-1	-1 ↔ 0	(1,2-2)	(2,1-2) ↔ (1,2-2)
0	0 ↔ 1	(2,2-2)	(1,2-2) ↔ (2,2-3)
1	1 ↔ 2	(1,3-3)	(2,2-3) ↔ (1,3-3)
⋮	⋮	⋮	⋮

Thus market 2 corresponds to even periods for which an agent born at an odd period is "old", and market 1 corresponds to odd periods for which an agent born at an odd period is "young". Similarly, an agent born at an odd period is of type 1 in the model with incomplete markets, and an agent born at an even period is of type 2.

Finally, notice that if we relax requirement (iii) for the Paradigm, on can assume only a finite number b of submarkets and agents (for each submarket and each type, respectively). Then, one can find, by dropping the requirement of identical treatment for agents of the same type, all periodic solutions of order at most b, of the overlapping general model. For instance, for b = 2, one has the following pattern of meetings :

Market 1		Market 2	
Submarkets	Meetings	Submarkets	Meetings
(1,1)	(1,1) ↔ (2,1)	(2,1)	(1,1) ↔ (2,2)
(1,2)	(1,2) ↔ (2,2)	(2,2)	(1,2) ↔ (2,1)

2. From now on, assume that utility functions are strictly quasi-concave and increasing, differentiable and that allocations are interior (as Y. Balasko and K. Shell (1980) do always). Consider the first example of O. Hart (1975) : $m = 2$, $n = 2$ and $r = 4$.

$Z^1 = \left\{ z \in \mathbb{R}^4 \mid z_3 = z_4 = 0 \right\}$

$Z^2 = \left\{ z \in \mathbb{R}^4 \mid z_1 = z_2 = 0 \right\}$

Utility functions are differentiable, strictly quasi-concave and strictly increasing and initial endowments are interior to consumption sets $(\mathbb{R}^r_+)$.

$u_1(x_1) = u_1^1(x_1^1) + \beta_1 u_1^2(x_1^2)$

$u_2(x_2) = u_2^1(x_2^1) + \beta_2 u_2^2(x_2^2)$

where $x_1^1 \in Z^1$ and $x_1^2 \in Z^2$.

Assume $\beta_1 > 1$ and $0 < \beta_2 < 1$.

Consider the initial allocation. Assume that it is efficient but not Pareto optimal i.e. :

$u_1^1(w_1^1)$ and $u_2^1(w_2^1)$ are colinear, $u_1^2(w_1^2)$ and $u_2^2(w_2^2)$ are also colinear but there is a direction $t = (t_1,..,t_4) \in \mathbb{R}_4$ such that $\sum_h u'_{1h}(w_1)\, t_h > 0$ and $\sum_h u'_{2h}(w_2)\, t_h < 0$. Obviously, this example can be cast in an overlapping generations framework. There is one agent in each generation and each agent lives for two periods. An agent born at an even period is the same as agent 2 and an agent born at an odd period is the same as agent 1. One has then the following pattern of meetings.

Period	Markets(goods)	Meetings of agents
	⋮	⋮
0	(1,2)	- 1 0
1	(3,4)	0 1
2	(1,2)	1 2
3	(3,4)	2 3
	⋮	⋮

The sequence of initial endowments is a competitive equilibrium for which each agent maximizes his utility subject to <u>one</u> budgetary constraint. How to explain that there are two budgetary constraints in the incomplete markets version and one budgetary constraint in the overlapping generations version ? It is simply because, in the overlapping generations version, one can use "<u>rates of interest</u>" in order to make the price vector faced by an agent born at an odd period different from the price-vector faced by an agent born at an even period. Obviously, when using the Paradigm, one can define a P.E. for which each agent maximizes his utility function subject to one budgetary constraint. Consider $M = [Z^1,..,Z^n]$ for which A4(i) holds. Choose a basis Λ^α for each Z^α and let π^α be a linear form on Z^α, relative to the basis Λ^α.

<u>Definition 3'</u> : Given $M = [Z^1,..,Z^n]$, a pseudo equilibrium is a pair $\left[(a^*_{i,\gamma}),\ (\pi^{*\alpha,\beta})\right]$ such that

(i) $(a^*_{i,\gamma})_{i,\gamma}$ is feasible.

(ii) $\forall(i,\gamma)$: $a^*_{i,\gamma}$ maximizes $V_i(a)$ in the set

$$\left\{a \in P_i \;\middle|\; \sum_{(\alpha,\beta)\in N(i,\gamma)} \pi^{*\alpha,\beta}\, a^{\alpha,\beta} \leq 0\right\}.$$

3. As soon as there is more than one agent for each generattion and/or agents live for more than two periods, it seems that one cannot assume anymore that if two agents meet on some market, they cannot meet on another one. However, it is straightforward to show that one can redefine the problem in order to get that the set of agents of generation θ can meet the set of agents of generation θ′(≠ θ) on one market at most. (Recall that at least two goods must appear on each market).

Index members of a generation by their common date of birth θ, and for each couple (θ,θ′), define a "market" by $Z^{\theta,\theta'} \equiv Z^{\theta',\theta}$. For instance, if there are two goods at each time period θ(θ1 and θ2) and if each generation lives for three periods, one gets :

Periods	Goods θ1,θ2	Meetings of generations
1	11,12	G-1 G0 G1
2	21,22	G0 G1 G2
3	31,32	G1 G2 G3
4	41,42	G2 G3 G4
5	51,52	G3 G4
6	61,62	G4

Consider generation 1. Members of generation 1 meet on market $Z^{1,1}$ which includes the following list of goods {11,12,21,22,31,32} . Members of generation 1 meet members of generation -1 on market $Z^{1,-1}$ which includes the following list of goods {11,12}. Members of generation 1 meet members of generation 0 on market $Z^{1,0}$ which includes the following list of goods : {11,12,21,22} etc...

Thus the general overlapping generation model can be cast in our framework. Using methods of Capital theory, it is straightforward to check that efficiency fully characterizes competitive equilibria of the overlapping generations model, when differentiability and interiority obtain. The analysis of the case for which differentiability and/or interiority do not hold requires a separate study, because, inter alia, the hyperplane which supports, for each agent, the set of his preferred trades is no more unique. □

Since Example 6, we have relaxed the assumption of symmetry or "stationarity" and that allows us to face the last point. In the incomplete markets approach, one assumes not only that all agents of the same type are treated identically, but also that the same price vector prevails on all submarkets of a given market. In order to begin the analysis, let us compare definitions 3 and 3′ (of Example 6) with the following one :

<u>Definition 3"</u> : Given $M = [Z^1,..,Z^n]$, a pseudo-equilibrium* (PE*) is a pair $\left[(a^{**}_{i,\gamma}),(\pi^{**\alpha,\beta})\right]$ such that

(i) $(a^{**}_{i,\gamma})_{i,\gamma}$ is feasible

(ii) $\forall(i,\gamma)$: $a^{**}_{i,\gamma}$ maximizes $V_i(a)$ in the set $\left\{a \in P_i \mid \pi^{**\alpha,\beta} a^{\alpha,\beta} \leq 0 \quad \forall(\alpha,\beta) \in N(i,\gamma)\right\}$.

Assume, that in these definitions, for each (i,α) :

$$a^{\alpha,\beta}_{i,\gamma} = a^{\alpha}_i \qquad \forall(\alpha,\beta) \in N(i,\gamma).$$

If $a^{\alpha}_i = 0 \quad \forall(i,\alpha)$, then if $\left[(a^{*}_{i,\gamma}),\ (\pi^{*\alpha,\beta})\right]$ is a P.E. according to definition 3′, it is a P.E.* (according to Definition 3"). When $a^{\alpha}_i \neq 0$ for some (i,α), if $\left[(a^{*}_{i,\gamma}),\ (\pi^{*\alpha,\beta})\right]$ is a P.E. according to definition 3′, it is a P.E.* (according to definition 3") with transfers (this is implied by $\sum_{(i,\gamma)\in M(\alpha,\beta)} a^{\alpha,\beta}_{i,\gamma} = 0 \quad \forall(\alpha,\beta)$).

Conversely, consider a P.E.* (according to definition 3") $\left[(a^{**}_{i,\gamma}),(\pi^{**\alpha,\beta})\right]$. Then there is $(\pi^{*\alpha,\beta})_{\alpha,\beta}$ such that $\left[(a^{**}_{i,\gamma}),(\pi^{*\alpha,\beta})\right]$ is a P.E. (according to definition 3′). Let us now come to the interesting point which is the comparison between Definitions 3 and 3". Given the assumption of identical treatment of agents of the same type, definition 3 is definition 3" for which, for each α :

$$\pi^{**\alpha,\beta} = \pi^{*\alpha} \qquad \forall\beta$$

This is related with what happens in the theory of capital for the characterization of stationary programs (See E. Malinvaud (1953) and the subsequent litterature). When differentiability and interiority obtain, definitions 3 and 3" are equivalent. This is no more the case when these assumptions are relaxed. It seems that Example 5 is an argument in this direction. Actually, the no trade allocation <u>is an equilibrium</u> according to

Definition 3". Let us make the argument on the basis of Figure 2d. Consider agent (1, 21-3) who is of type 1 and who attends submarkets (1, 21-3), (2, 21-3) and (3, 21-3). Choose $\pi^{1.21-3} > 0$, $\pi^{2.21-3} > 0$ and $\pi^{3.21-3} < 0$, i.e. demand is rationed on submarkets (1,21-3), (2,21-3) and supply is rationed on submarket (3,21-3). Thus, the initial endowment maximizes utility subject to the three budgetary constraints. Consider now agent (2,11-3) who is of type 2, who meets agent (1,21-3) on submarket (1,21-3) and who participates also to submarkets (2,12-3) and (3,11-2). As one has already $\pi^{1.21-3} > 0$, one chooses $\pi^{2.12-3} < 0$ and $\pi^{3.11-2} < 0$. Thus supply is rationed on submarkets (2,12-3) and (3,11-2). Thus, for agent (2,11-3), the initial allocation maximizes utility subject to the three budgetary constraints. It is straightforward to check that the induction can always be made. However, one cannot choose, e.g. :

$$\pi^{2.21-3} = \pi^{2.12-3} \text{ and } \pi^{3.21-3} = \pi^{3.11-2}$$

This is a "non symmetric" D.G.E.

It follows that the allocation considered in Example 5 satisfies the requirements of Definition 9′ also with an infinite number of ${}^{2}D^{\alpha,\beta}_{i,\gamma}$ containing strictly $\tilde{\tilde{z}}^{\alpha,\beta}_{i,\gamma} = 0$. On Figure 2d, an arrow indicates a set of offers different from $\{0\}$ made by the agent from whom the arrow starts to the agent towards whom the arrow points. When an arrow starts from an agent of type 1, it corresponds to a set of purchase offers containing strictly $\{0\}$ and when an arrow starts from an agent of type 2, it corresponds to a set of sale offers containing strictly $\{0\}$. Thus this new system of offers contains strictly (from the point of view of the inclusion of sets) the previous one described in Example 5 with $D^{\alpha,\beta}_{i,\gamma} = \{0\}\ \forall(\alpha,\beta)$ and $\forall(i,\gamma)$. But this system of offers is not "symmetric" in the sense that one has not, $\forall(i,\alpha)$:

$$ {}^{2}D^{\alpha,\beta}_{i,\gamma} = {}^{2}D^{\alpha}_{i} \quad \forall\beta. $$

On the contrary, in Example 4 (with m = n = 2) there is another system of offers $\left\{{}^{2}D^{\alpha,\beta}_{i,\gamma}\right\}$ satisfying Definition 9′, with an infinite number of ${}^{2}D^{\alpha,\beta}_{i,\gamma}$ containing strictly $\tilde{\tilde{z}}^{\alpha,\beta}_{i,\gamma} = 0$, and which is "stationary" or symmetric. Thus, part of the strenght of stability w.r.t. outward induction comes from the fact that it forces to consider always a system of offers large enough. This fact has to be seen in the perspective of the other approach that it would have been possible to follow. Consider finite optimality as described in Part II and consider a net $\{L^{\nu}\}$. For getting a price system for the set of submarkets L^{ν}, one does not consider as fixed the trades on submarkets not belonging to L^{ν} but only the trades of agents who do not attend any submarket of L^{ν}. From the point of view of optimality, this is the same, but not for getting sets of preferred points with a non empty interior. Thus, with this approach, using the methods of Capital Theory, it is hopefully possible to get a price system $\{\pi^{\alpha,\beta}\}$. But this price system is not necessarily symmetric if differentiability and interiority do not hold. Notice that the methods of Capital Theory for getting a stationary price system cannot a priori he carried over directly, even if $m = n = 2$, because stationarity in the theory of capital is a little different from symmetry as defined here : stationarity in the theory of Capital means stationarity for each pair of consecutive periods.

Thus the force of the Definition of outward induction comes also partly from the fact that it is applied to Definition 10 which requires "stationarity", Definition 10' being only an introduction to Definition 10. More work is needed to locate exactly the meaning of the comparison of Example 4 and 5. From this point of view, notice that Pareto optimality which is here the strongest concept of "collective rationality" entails automatically the symmetry of the price system when the symmetry of the allocation is assumed (See proofs of Proposition 5 and 6). It corresponds to the Golden Rule. In this perspective, the extensive form, which the Paradigm is, constitutes a very natural tool for studying the phenomena related to sunspots (D. Cass and K. Shell (1983)) and "symmetry breaking" (Y. Balasko (1986)) in models with incomplete markets.

IV

Clearly, the notion of collective rationality is a weak one, in Game Theory. For instance, the concepts of Core and Bargaining Set imply collective rationality. Cross-fertilization of the theory of games-even of its "cooperative" branch only-and the theory of incomplete markets is beyond the scope of this article. This last part has mainly one objective : To show how, when the economy is large, it is possible to deduce the axiom of voluntary trade which allows us to define a D.G.E. in general (and no more at 0) and more generally to fix the links between the theory of incomplete markets and that of equilibria with possible rationing. In this part, one assumes convexity.

When $M = [Z^1, \ldots, Z^n]$, the Paradigm married with the assumption of identical treatment of agents of the same type leads to the following argument. Given a collectively rational $\approx{z}^* = \left[\approx{z}_1^*, \ldots, \approx{z}_m^*\right]$, a coalition S^α can guarantee itself on market α :

$$v^\alpha(S^\alpha) \equiv \left\{ (\bar{U}_i)_{i\in S^\alpha} \;\middle|\; \exists\, (\approx{z}_i)_{i\in S^\alpha} \text{ with} \right.$$

$$\approx{z}_i \in B_i,\ \sum_{i\in S^\alpha} z_i^\alpha = 0,\ z_i^{\alpha'} \in -\sum_{j\in S^{\alpha'}\setminus\{i\}} \approx{z}_j \text{ for some } S^{\alpha'} \subset \{1,\ldots,m\}\ \forall \alpha' \neq \alpha$$

$$\left. \text{and } U_i(\approx{z}_i) \geq \bar{U}_i\ \forall i \in S^\alpha \right\}.$$

Notice that, given the Paradigm, it is legitimate to make the coalition depending on α (See Figures 2). $\approx{z}^*$ is not blocked if for each α, there is no S^α and $(\bar{U}_i)_{i\in S^\alpha} \in v^\alpha(S^\alpha)$ with $\bar{U}_i > U_i(\approx{z}_i^*)$ $\forall i \in S^\alpha$.

In this perspective, an additional degree of freedom is gained if one assumes a number of submarkets different in each market. As the number of agents is given, this device will induce, when translated at the level of the reduced model, a different degree of competition in each market. When the economy is replicated and the number of agents of each type goes to infinity, it can be proved that a variant of the Core or of the Bargaining Set imply the following characterization :

Axiom of voluntary trade :For each α, each agent i may always reduce his trade.

This axiom allows us to define a D.G.E. generally and no more at O. Recall that when M satisfies A4(ii) it is possible to argue in terms of $a = [a^1,..,a^\alpha,..,a^n]$ where $z^\alpha = a^\alpha . t^\alpha$.

$$Z^\alpha = \left\{ z \in \mathbb{R}^r \mid \exists\, a^\alpha \in \mathbb{R} \text{ such that } z^\alpha = a^\alpha . t^\alpha \right\}.$$

Given a^*, define for each (i,α) :

$(a_i^{*\alpha}) = \left\{ a^\alpha \in \mathbb{R} \;\middle|\; a^\alpha = \mu\, a_i^{*\alpha},\ \mu \in [0,1] \right\}$. This set is also noted $O(a_i^{*\alpha})$.

Definition 11 : Given M satisfying A4(ii), a feasible a^* is a D.G.E. w.r.t. M if there is $\pi^* = [\pi^{*1},..,\pi^{*n}] \in \mathbb{R}^n$ such that $\forall i$:
a_i^* maximizes $V_i(a_i)$ subject to $a_i \in P_i$ and $\forall \alpha$:

$$a^\alpha \in \left\{ (a_i^{*\alpha}) + \left\{ {}^1a^\alpha \mid \pi^{*\alpha}\, {}^1a^\alpha \leq 0 \right\} \right\}$$

When M satisfies A4(i), by using T^α associated with Z^α and $T = \underset{\alpha}{\cup} T^\alpha$, one defines a D.G.E. by using the notion of signed measure, because T is then a continuum. Consider then a net of finite $\{T^\nu\}$ such that $T^{\nu'} \supset T^\nu$ if $\nu' > \nu$ and $\underset{\nu}{\cup} T^\nu = T$. The following proposition asserts that, for every finite T, there exists a D.G.E. An accumulation point of the net is then a C.E..

However, an accumulation point of the net does not always exist and this creates well known inexistence problems when the market structure is incomplete.

Proposition 7 : Under A1-A3, there exists a D.G.E. if T is finite.

Proof : See annex.

In duality with the notion of D.G.E., it is possible to combine directly collective rationality and the axiom of voluntary trade to get the notion of Strong Non Cooperative Equilibrium (S.N.C.E.).

Definition 12 : Given M satisfying A4(i) or A4(ii), a feasible $\tilde{\tilde{z}}^*$ is a S.N.C.E. iff there are m.n. convex sets $\tilde{\tilde{Z}}_i^\alpha$ and m.n. sets $\bar{Z}_i^\alpha$ such that

(i) $\forall (i,\alpha)$

$$\tilde{\tilde{Z}}_i^{\alpha} = \left\{ z^{\alpha} \in Z^{\alpha} \;\middle|\; \exists\, \tilde{\tilde{z}} = [z^{\alpha}, z^{-\alpha}] \in B_i \text{ with } U_i(\tilde{\tilde{z}}) \geq U_i(\bar{\tilde{\tilde{z}}})\ \forall\, \bar{\tilde{\tilde{z}}} \text{ such that} \right.$$

$$\left. \bar{z}^{\alpha} \in O(-\sum_{j \neq i} \tilde{\tilde{Z}}_j^{\alpha})\ \forall \alpha \text{ and } z^{\alpha'} \in O\left(-\left[\sum_{j \neq i} \tilde{\tilde{Z}}_j^{\alpha'} \cup \bar{Z}_i^{\alpha'}\right]\right)\ \forall \alpha' \neq \alpha \right\}.$$

(ii)
$\bar{Z}_i^{\alpha}$ satisfies conditions (iia) and (iib) of Definition 10.

(iii) $\forall (i,\alpha)$
$\tilde{\tilde{z}}_i^{*\alpha} \in \tilde{\tilde{Z}}_i^{\alpha}$

Remark : Definition 12 incorporates the axiom of voluntary trade.
As $\tilde{\tilde{z}}^{*}$ is feasible, for each (i,α) :

$$z_i^{*\alpha} = \sum_{j \neq i} z_j^{*\alpha}$$

Thus (i) and (iii) of Definition 12 imply that when he make offers on market α, each agent i considers that be can reduce his trades on all markets $\alpha' \neq \alpha$.

Remark : It is straightforward to show that, under A1-A3 and if M satisfies A4(i), there is identify between the set of S.N.C.E.'s and the set of C.E.'s with respect to M.

Remark : Competitive equilibrium with price taking behaviour implies a high degree of multilateralism. Arguing in terms of sets of trade offers, as we did, gives the same result. However, if one assumes that some agents consider as fixed the trades of some other agents and restricts the degree of multilateralism, other phenomena of coordination failures appear. An instance of this statement is the analysis of the coordination failures which appear at the limit of a sequence of monopolistically competitive equilibria à la Cournot-Nash (J. Gabszewicz and J. Vial (1972)), for which there is no residual market power.

It is well known that in a world with a finite number of commodities and a finite number of types of firms (production sets), the limit of a sequence of monopolistic competitive equilibria à la Cournot-Nash, when the number of firms of each type tends to infinity, is not necessarily a competitive equilibrium (See A. Mas Colell's (1982)- Economically, this is due to two very different reasons.

(i) It may happen that the market power does not disappear when the number of firms is very large. This feature is shared by many examples provided by K. Roberts (1980) and we are not interested in it.

(ii) Even if the market power disappears, it may happen that the limit, when allocation are not interior and/or differentiability is not assumed, is not a competitive equilibrium. This feature is shared by many examples provided by O. Hart (1980). There is a coordination failure, at the limit, which is not due to the presence of a residual market power.

Conclusion

Intuitively, we have tried to explain the fact that each agent faces many budgetary constraints, when the market structure is incomplete, in terms of a pattern of meetings for which two (generations of) agents cannot meet on more than one market. We think that this is a natural idea and that it captures some part of the economic reality : we do not buy goods at the firm we are working in. This idea gives substance to the notion of markets. It can be used also in models with search or with random meetings of agents. In this case, one assumes that the probability that two agents meets on more than one market is 0. The same idea pervades the three fields we have studied and we must not be trapped by the differences of presentation of the notions of equilibrium used in these fields. On the one hand, rationing is equivalent to the use of a separation theorem in a vector space of dimension 1. On the other hand, if each agent, in overlapping generations models, faces one budgetary constraint, it is because the use of (implicit or explicit) rates of interest entails that all generations do not face the same price vector.

- Annex -

The mathematical background can be found in R. Rockafellar (1970).

Proof of Proposition 1 :

The first part is obvious. Consider a C.E. $(\approx{z}^*, (\pi^{*\alpha})_\alpha)$ w.r.t. M which satisfies A5. There is, $\forall i$, $\approx{z} \in B_i$ with $\pi^{*\alpha} z^\alpha \leq 0 \quad \forall\alpha$ and $\pi^{*\alpha_i} z^{\alpha_i} < 0$. Ad absurdum, there is $_1\approx{z} \in B_i$ with $U_i(_1\approx{z}) > U_i(\approx{z}^*_i)$ and $\pi^{*\alpha}\ {}_1z^\alpha \leq 0\ \forall\alpha$. Define $^\beta\approx{z} = \beta_1\approx{z} + (1-\beta)\ \approx{z}$ for $\beta \in]1,0]$.
Thus $\pi^{*\alpha_i}\ {}^\beta z^{\alpha_i} < 0$ and $\pi^{*\alpha}\ {}^\beta z^\alpha \leq 0\ \forall\alpha$. By continuity, for β large enough, $U_i(^\beta\approx{z}) > U_i(z^*_i)$: a contradiction.
Proving that, under A6, a P.E. is a Q.E. involves a no-arbitrage argument. Choose a basis Λ^α for each Z^α. Thus $\pi^\alpha = \left[\pi^\alpha_1, \ldots, \pi^\alpha_k, \ldots, \pi^\alpha_{q(\alpha)}\right]$ and $\pi = [\pi^1, \ldots, \pi^\alpha, \ldots, \pi^n] \in \mathbb{R}^q \setminus \{0\}$ are given (* is suppressed). From A3 and A6 $\pi^\alpha \neq 0\ \forall\alpha$. $D \subset \mathbb{R}^q$ is defined by

$$D = \left\{ d = \{d^1, \ldots, d^\alpha, \ldots, d^n\} \in \mathbb{R}^q \;\middle|\; \exists\, p \in \mathbb{R}^r_+ \text{ and } b = [b^1, \ldots, b^\alpha, \ldots, b^n] \in \mathbb{R}^n_+ \right.$$

$$\left. \text{such that } \forall\alpha : d^\alpha = p'\Lambda^\alpha - b^\alpha \pi^\alpha \right\}.$$

D is a convex polyedral cone, the relative interior of which is $)D($ where

$$)D(= \left\{ d = \{d^1, \ldots, d^n\} \in \mathbb{R}^q \;\middle|\; \exists\, p \in \mathbb{R}^r_{++} \text{ and } b \in \mathbb{R}^n_{++} \text{ such that} \right.$$

$$\left. d^\alpha = p'\ \Lambda^\alpha - b^\alpha \pi^\alpha\ \forall\alpha \right\}.$$

Ad absurdum, assume that there is no $p \in \mathbb{R}^r_{++}$ and $b \in \mathbb{R}^n_{++}$ such that $d^\alpha = p'\ \Lambda^\alpha - b^\alpha \pi^\alpha = 0\ \forall\alpha$. Thus $)D($ does not contain $0 \in \mathbb{R}^q$. There is then a hyperplane $\bar{a} \neq 0$ such that

(i) $\bar{a}.\ d \geq 0 \quad \forall\, d \in D$.

(ii) $\bar{a}.\ d > 0 \quad \forall\, d \in)D($.

(ii) can be written

$$\sum_{\alpha}\sum_{k}^{q(\alpha)}\left[p'.\lambda_k^{\alpha} - b^{\alpha}\pi^{\alpha}\right]\bar{a}^{\alpha} > 0 \quad \forall\, p \in \mathbb{R}^{r}_{++} \text{ and } \forall\, b \in \mathbb{R}^{n}_{++}$$

First : $\pi^{\alpha}\bar{a}^{\alpha} \leqslant 0\ \forall\alpha$. Ad absurdum, if $\pi^{\alpha_1} a^{\alpha_1} > 0$ for α_1, one chooses b^{α_1} large and $b^{\alpha} = 0$ for $\alpha \neq \alpha_1$, and $p = 0$. A contradiction with (i).

Second : $\sum_{\alpha}\sum_{k}\lambda_{kh}^{\alpha}\bar{a}^{\alpha} \geqslant 0\ \forall h$. Ad absurdum, if for $h = h_1$, one has $\sum_{\alpha}\sum_{k}\lambda_{kh_1}\bar{a}^{\alpha} \geqslant 0$, then by choosing p_{h_1} large and $p_h = 0\ \forall h \neq h_1$ and $b = 0$, one gets a contradiction with (i).

Third, let us show that we have

either $\pi^{\alpha}\bar{a}^{\alpha} < 0$ for at least one α : Case 1

and/or $\sum_{\alpha}\sum_{k}\lambda_{kh}^{\alpha}\bar{a}^{\alpha} > 0$ for at least one h : Case 2

Assume that $\pi^{\alpha}\bar{a}^{\alpha} = 0\ \forall\alpha$. Then for each $b \in \mathbb{R}^{n}_{++}$ and $p \in \mathbb{R}^{n}_{++}$, one has

$$\sum_{h} p_h \sum_{\alpha}\sum_{k}\lambda_{kh}^{\alpha}\bar{a}_k^{\alpha} > 0$$

and thus $\sum_{\alpha}\sum_{k}\lambda_{kh}^{\alpha}\bar{a}_k^{\alpha} > 0$ for at least one h.

Assume that $\sum_{\alpha}\sum_{k} t_{kh}^{\alpha}\bar{a}^{\alpha} = 0\ \forall h$. Then for each $b \in \mathbb{R}^{n}_{++}$, one has

$$-\sum_{\alpha} b^{\alpha}(\pi^{\alpha}\bar{a}^{\alpha}) > 0$$

and thus $\pi^{\alpha}\bar{a}^{\alpha} < 0$ for at least one α.

Let us show that we get a contradiction in both cases. Given the basis which are chosen, let be $a^* = \left[a_1^*,..,a_m^*\right]$ such that for each i , a_i^*maximizes $V_i(a)$ subject to $a \in P_i$ and $\pi^{\alpha}a^{\alpha} \leqslant 0\ \forall\alpha$.

Case 1 : If $\pi^{\alpha_1}\bar{a}^{\alpha_1} < 0$ for $\alpha = \alpha_1$ and $\pi^{\alpha}\bar{a}^{\alpha} \leqslant 0\ \forall\alpha \neq \alpha_1$ then by A3, there is an agent i_1 and $\bar{\bar{a}} = [0,..,0,\bar{\bar{a}}^{\alpha},0,..,0]$ such that

$${}_1a \equiv a_i^* + \bar{a} + \bar{\bar{a}} \in P_i,\ V_{i_1}({}_1a) > V_{i_1}(a_i^*) \text{ and } \pi^{\alpha}\,{}_1a^{\alpha} \leqslant 0\ \forall\alpha.$$

This is contradiction

Case 2 : If $\sum_{\alpha}\sum_{k}\lambda_k^{\alpha} \geqslant 0$ for each h with strict inequality for some h, by assumption A6, there is an agent i_2 for whom $V_{i_2}(a^*_{i_2} + \bar{a}) > V_{i_2}(a^*_{i_2})$. This is a contradiction. Q.E.D.

Proof of Proposition 3 :

Choose p^* colinear with $u_1'(w_1)$, the gradient of u_1 at w_1. 0 maximizes $U_1(\tilde{\tilde{z}})$ subject to $\tilde{\tilde{z}} \in B_1$ and $p^* z^\alpha \leq 0$ $\forall\alpha$. Ad absurdum, assume that for some $i \neq 1$, there is $\tilde{\tilde{z}} \in B_i$ with $p^* \leq 0$ and $U_i(\tilde{\tilde{z}}) > U_i(0)$. From diferentiability, the directional derivative $u_i'(w_i ; \bar{t})$ of u_i at w_i along the direction $\bar{t} = z / //z//$, where $z = \sum_\alpha z^\alpha$, is strictly positive. As $w_i \in \mathring{X}_i$, there is α such that $u_i(w_i + \lambda z^\alpha) > u_i(w_i)$, for some $\lambda > 0$ with $w_i + \lambda z^\alpha \in X_i$. Thus $\lambda z^\alpha \in {}^\alpha W_i{}^\alpha$ with $p^*. \lambda z^\alpha \leq 0$. It follows that there is $\bar{z} \in^\alpha W_i$ and $-\bar{z} \in^\alpha W_1$, a contradiction with efficiency of zero net trade. Q.E.D.

Proof of Proposition 4 :

As $\tilde{\tilde{z}}^*$ is Pareto optimal, there are $[Y_1,..,Y_m]$ such that :

(i) $\forall i$: $Y_i = Cl\left[\left\{\tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}^*_i)\right\} \cup \tilde{\tilde{z}}^*_i\right]$

(ii) there is no balanced $\tilde{\tilde{z}}$ with $\forall i$: $\tilde{\tilde{z}}_i \in Y_i$ and $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}^*_i)$

$\forall(i,\alpha)$: let Y^α_i be the projection of Y_i on Z^α and define :

$Z^\alpha_i = Y^\alpha_i \setminus \sum_{j\neq i} Y^\alpha_j.$

In order to take into account (iib) of Definition 10, define $\bar{Z}^\alpha_i = \bigcup_i Z^\alpha_i$ $\forall(i,\alpha)$. Set for each (i,α) :

$${}^\alpha\tilde{\tilde{Z}}_i = Cl\left[\left\{\tilde{\tilde{z}} \in B_i \;\middle|\; U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}^*_i) \text{ and } z^{\alpha'} \in -\left[\sum_{j\neq i} Y^{\alpha'}_j \cup \bar{Z}^{\alpha'}_i\right] \forall\alpha' \neq \alpha\right\} \cup \tilde{\tilde{z}}^*_i\right].$$

Thus $\forall(i,\alpha)$: $\tilde{\tilde{Z}}^\alpha_i \equiv {}^\alpha\tilde{\tilde{Z}}^\alpha_i = Y^\alpha_i$ and $\tilde{\tilde{z}}^*$ is collectively rational. In case of convexity, one defines $\bar{Z}^\alpha_i$ large enough in order that $\sum_{j\neq i} Z^\alpha_j \cup \bar{Z}^\alpha_i$ is convex. Q.E.D.

<u>Proof of Proposition 5</u> :

Assume that 0 is a D.G.E. For each (i,α), define $^{\alpha}\tilde{\tilde{A}}_i = \mathrm{Cl}\left[\left\{a \in P_i \middle| V_i(a) > V_i(0) \text{ and } \pi^{*\alpha'}a^{\alpha'} \leq 0 \ \forall\alpha \neq \alpha'\right\} \cup \tilde{\tilde{z}}_i^*\right]$ and $\tilde{\tilde{A}}_i^\alpha \equiv {}^{\alpha}\tilde{\tilde{A}}_i^\alpha$ as the projection of $^{\alpha}\tilde{\tilde{A}}_i$ on market α. Define also :

$$\bar{A}_i^\alpha = \left\{a^\alpha \mid \pi^{*\alpha}a^\alpha \geq 0\right\} \setminus \sum_{j\neq i} \tilde{\tilde{A}}_j^\alpha$$

It is easy to theck that the 2 m.n sets $\tilde{\tilde{A}}_i^\alpha$ and $\bar{A}_i^\alpha$ support 0 as a collectively rational allocation. Particularly, condition (iia) of Definition 10 is satisfied because $V_j(a) > V_j(0)$ implies $\pi^\alpha a^\alpha > 0$ for at least one α.

Conversely, assume that A3 holds and that 0 is collectively rational and supported by 2 m.n sets $\tilde{\tilde{A}}_i^\alpha$ and $\bar{A}_i^\alpha$. One shall define a partition of $\{1,..,n\}$ into three subsets (some of them being, possibly, empty).

(i) $N_1 = \left\{\alpha \middle| \sum_i \tilde{\tilde{A}}_i^\alpha \cup \sum_i \bar{A}_i^\alpha \text{ is not contained in a half space of } \mathbb{R} \text{ bounded by } 0\right\}$.

(ii) $N_2 = \left\{\alpha \middle| \sum_i \tilde{\tilde{A}}_i^\alpha \cup \sum_i \bar{A}_i^\alpha \text{ is contained in a half space of } \mathbb{R} \text{ bounded by } 0 \text{ and it is different from } \{0\}\right\}$.

(iii) $N_3 = \left\{\alpha \middle| \sum_i \tilde{\tilde{A}}_i^\alpha \cup \sum_i \bar{A}_i^\alpha = \{0\}\right\}$.

For $\alpha \in N_2$, define $\pi^\alpha \in \mathbb{R}\setminus\{0\}$ in such a way that $\pi^\alpha a^\alpha \geq 0$ for $a^\alpha \in \sum_i \tilde{\tilde{A}}_i^\alpha \cup \sum_i \bar{A}_i^\alpha$ .Notice that for $\alpha \in N_1$, for each i : $0 \in \mathrm{int} -\left[\sum_{j\neq i} \tilde{\tilde{A}}_j \cup \bar{A}_i^\alpha\right]$ from (ii) of Definition 10.

Let us check that for each i, there is no $a \in P_i, V_i(a) > V_i(0)$ with $\pi^\alpha a^\alpha \leq 0 \ \forall \alpha \in N_2$ and $a^\alpha \neq 0$ for at least one $\alpha \in N_3$. Ad absurdum, assume that there is such an act $_1a$. By A3(semi strict concavity), one gets that there is $a = \lambda\, _1a (\lambda \in]0,1])$ inheriting the properties of $_1a$ and with the property that $a^\alpha \in -\left[\sum_{j\neq i} \tilde{\tilde{A}}_j \cup \bar{A}_i^\alpha\right] \forall \alpha \in N_1$.

However from (iib) of Definition 10, $\forall \alpha \in N_2$, there is a segment ($[0,e]$ or $[e,0]$) belonging simultaneously to $-\sum_{j=i} \tilde{\tilde{A}}_j^\alpha \cup \bar{A}_i^\alpha$ and $\left\{a^\alpha \mid \pi^\alpha a^\alpha \leq 0\right\}$.

From (iia) of Definition 10, it follows that for some $\alpha \in N_3$, $\sum_i \tilde{\tilde{A}}_i^{\alpha} \cup \sum_i \bar{A}_i^{\alpha}$ contains another element than 0, a contradiction.
Without loss of generality, one can assume that $N_1 = [1,..,n_1]$ if N_1 is not empty.
Case 1 : If N_1 is empty, choose then $\pi^{\alpha} \in \mathbb{R}$ for $\alpha \in N_3$. We have shown that if 0 is collectively rational, it is a D.G.E., because then, there is no $a_i \in P_i$ with $V_i(a_i) > V_i(0)$ and $\pi^{\alpha} a^{\alpha} \leqslant 0 \quad \forall\alpha$.

Case 2 : If N_2 is empty, then 0 is Pareto optimal. Ad absurdum, assume that there is a feasible $a = [a_1,..,a_m]$ with $V_i(a_i) > V_i(0)$ $\forall i$. From semi strict concavity, it follows that for each $\mu \in]0,1]$, $V_i(\mu\, a_i) > V_i(0)$ and $\mu\, a$ is feasible. However, for each $\alpha \in N_1$, there is (Defintion 10) $\mu(\alpha)$ such that $\mu(\alpha) a^{\alpha} \varepsilon\, \tilde{\tilde{A}}_i^{\alpha}$, a contradiction. Consider then
$A_i\,(1) = \{a \in P_i | V_i(a) > V_i(0)\}$.
From A3, $\sum_i A_i(1)$ is convex. If its relative interior is empty, that means that N_1 is also empty and we are home. If the relative interior of $\sum_i A_i(1)$ is not empty, there is $\pi(1) \in \mathbb{R}^n \setminus \{0\}$ such that $\forall i$:
$\pi(1) a \geqslant 0 \quad \forall\, a \in A_i(1)$. Let $N(1)$ be the set of α's for which $\pi^{\alpha}(1) \neq 0$ and consider $\forall i$:

$$A_i(2) = \left\{ c \in {}^{n-n(1)} \;\middle|\; \exists\, a = [b,c] \in P_i,\ b \in \mathbb{R}^{n(1)},\ \text{such that } V_i(a) > V_i(0) \text{ and } \pi^{\alpha}(1)\, b^{\alpha} \leqslant 0 \quad \forall\alpha \in N(1) \right\}.$$

$\forall i$, notice that $b^{\alpha} = 0$ for each b such that $(b,c) \in P_i$ and $c \in A_i(2)$, because otherwise, we would have $\pi^{\alpha}(1)\, b^{\alpha} < 0$ for some $\alpha \in N(1)$ and then there would exist $a \in P_i$ with $V_i(a) > V_i(0)$ and $\pi^{\alpha}(1)\, a^{\alpha} \leqslant 0 \quad \forall\alpha$ with strict inequality for some α. Thus $A_i(2)$ can be written
$A_i(2) = \left\{c \in \mathbb{R}^{n-n(1)} \middle| \exists\, a = [0,c] \in P_i,\ 0 \in \mathbb{R}^{n(1)},\ \text{s.t. } V_i(a) > V_i(0)\right\}$.
If the relative interior of $\sum_i A_i(2)$ is empty, then by semi-strict quasi-concavity, $\forall i$, there is no $a = (b,c)$ with $c \in A_i(2)$ such that $V_i(a) > V_i(0)$ and 0 is a D.G.E. . Assume thus that the relative interior of $\sum_i A_i(2)$ is not empty. $\sum_i A_i(2)$ is convex and from Pareto Optimality, there is no $[0,c_i]$ and $c_i \in A_i(2)$ such that
$\sum_i c_i = 0$ and $V_i([0,c_i]) > V_i(0)$ $\forall i$.
There is then $\pi(2) \in \mathbb{R}^{n-n(1)} \setminus \{0\}$ such that $\forall i$:

$$\pi(2)\ c \geqslant 0 \quad \forall\, c \in A_i(2)$$

For at least one $\alpha \in N \setminus N(1)$ one has $\pi^{\alpha}(2) \neq 0$. Assume that the set $N(2)$

of α's for which $\pi^{\alpha}(2) \neq 0$ is constituted of $\alpha = n(1) + 1,.., n(2)$. Define $\bar{\pi}(2) = [\pi^{1}(1),..,\pi^{n(1)}(1),\pi^{n(1)+1}(2),..,\pi^{n(2)}(2), 0,..,0]$. Thus for every i and every $a \in A_i(1)$, one cannot have $\bar{\pi}^{\alpha}(2)\ a^{\alpha} \leq 0\ \ \forall\alpha$ with strict inequality for at least one α. Continuing in this way in a finite number of steps, one gets $\bar{\pi} \in \mathbb{R}^n\setminus\{0\}$ such that for each i, there is no $a \in P_i$ with $V_i(a) > V_i(0)$ and $\bar{\pi}^{\alpha}\ a^{\alpha} \leq 0\ \ \forall\alpha$ and strict inequality for at least one α. That implies that for each i : 0 maximizes $V_i(a)$ subject to $a \in P_i$ and $\bar{\pi}^{\alpha}\ a^{\alpha} \leq 0\ \ \forall\alpha$.

<u>Case 3</u> : N_1 and N_2 are not empty.
The same argument as in Case 2 can then be made. For each $\alpha \in N_2$, choose $\pi^{*\alpha}(1) \neq 0$ such that $\pi^{*\alpha}(1)\ a^{\alpha} > 0\ \ \ \forall\ a^{\alpha} \neq 0$ belonging to $\sum_i \tilde{\tilde{A}}{}_i^{\alpha} \cup \sum_i \bar{A}_i^{\alpha}$.
Define

$$A_i(2) = \left\{ b \in \mathbb{R}^{n-n_2}\ \middle|\ \exists\ a = (b,c) \in P_i,\ c \in \mathbb{R}^{n_2} \text{ such that } V_i(a) > V_i(0) \text{ and } \pi^{*\alpha}(1)\ c^{\alpha} \leq 0\ \ \forall\alpha \in N_2 \right\}.$$

As for each $\alpha \in N_1$, one has $0 \in \operatorname{int}\left[\sum_i \tilde{\tilde{A}}{}_i^{\alpha} \cup \bar{A}_i^{\alpha}\right]$ the same argument as in case 2 can be made. Q.E.D.

<u>Proof of Proposition 6</u> :

Assume that $(\tilde{\tilde{z}}^{*},p^{*})$ is a Q.E. w.r.t. M for which A4(i) holds. Define for each (i,α) :

$$\tilde{\tilde{Z}}{}_i^{\alpha} = \mathrm{Cl}\left[\left\{ z^{\alpha} \in Z^{\alpha}\ \middle|\ \exists\ \tilde{\tilde{z}} = [z^{1},..,z^{\alpha},..,z^{n}] \in B_i \text{ with } U_i(\tilde{\tilde{z}}) > U_i(\tilde{\tilde{z}}{}_i^{*}) \text{ and } p^{*} z^{\alpha'} \leq 0\ \ \forall\alpha' \neq \alpha \right\} \cup \left\{ z_i^{*\alpha} \right\}\right].$$

and

$$\bar{Z}_i^{\alpha} = \left\{ z^{\alpha} \in Z^{\alpha}\ \middle|\ p^{*} z^{\alpha} \geq 0 \text{ and } z^{\alpha} \notin \sum_{j\neq i} \tilde{\tilde{Z}}{}_j^{\alpha} \right\}.$$

It is straightforward to verify that $\tilde{\tilde{z}}^{*}$ is collectively rational. Conversely, assume that $\tilde{\tilde{z}}^{*} = 0$ is collectively rational and supported by 2 m.n sets $\left\{\tilde{\tilde{Z}}_i^\alpha\right\}_{(i,\alpha)}$ and $\left\{\bar{Z}_i^\alpha\right\}_{(i,\alpha)}$. From A3, $\forall\alpha$: $\sum_i \tilde{\tilde{Z}}_i^\alpha$ has a non empty relative interior.

Define

$N_1 = \left\{\alpha \;\middle|\; \sum_i \tilde{\tilde{Z}}_i^\alpha \cup \sum_i \bar{Z}_i^\alpha\right.$ is not contained in a half space of Z^α bounded by an hyperplane through $\left. 0 \in Z^\alpha \right\}$.

and then

$N_1^1 = \left\{\alpha \in N_1 \;\middle|\; \sum_i \tilde{\tilde{Z}}_i^\alpha\right.$ is contained in a half space of Z^α bounded by an hyperplane through $\left. 0 \in Z^\alpha \right\}$.

$N_1^2 = \left\{\alpha \in N_1 \;\middle|\; \sum_i \tilde{\tilde{Z}}_i^\alpha\right.$ is not contained in a half space of Z^α bounded by an hyperplane through $\left. 0 \in Z^\alpha \right\}$.

and finally

$N_2 = \{1,\ldots,n\} \setminus N_1$.

Case 1 : $N_1^1 \cup N_2$ is not empty.

For each $\alpha \in N_1^1$, there is a half-space of Z^α bounded by the linear form $\pi^{*\alpha}$, which contains $\sum_i \tilde{\tilde{Z}}_i^\alpha$ and such that for each i : for each $t \in T^\alpha$, if $\pi^{*\alpha}(z_i^{*\alpha} + a\, t) > 0$, then $z_i^{*\alpha} + \bar{a}\, t \in \sum_{j\neq i} \tilde{\tilde{Z}}_j^\alpha \cup \bar{Z}_i^\alpha$ for $\bar{a} \neq 0$ with sign $\bar{a}$ = sign a ((ii) of Definition 10).

For each $\alpha \in N_2$, there is a half-space of Z^α bounded by the linear form $\pi^{*\alpha}$, which contains $\sum_i \tilde{\tilde{Z}}_i^\alpha \cup \sum_i \bar{Z}_i^\alpha$.

For $\alpha \in N_1^2$, choose $\pi^{*\alpha} = 0$. Definition 10 implies that $\forall i$: there is ball of Z^α, $W(z_i^{*\alpha})$, contained in $\sum_{j\neq i} \tilde{\tilde{Z}}_i^\alpha \cup \bar{Z}_i^\alpha$ $\forall\, \alpha \in N_1^2$.

First, if N_2 is empty, $\tilde{\tilde{z}}^*$ is a C.E. Ad absurdum, assume that for some i, there is $\tilde{\tilde{z}}_i \in B_i$ with $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}_i^*)$, $\pi^{*\alpha} z_i^\alpha \leq 0$ $\forall\alpha$, with strict inequality for $\alpha = \alpha_1 \in N_1^1$. By construction, that implies that $\tilde{\tilde{z}}_i \in {}^{\alpha_1}\tilde{\tilde{Z}}_i$ and thus that $z_i^{\alpha_1} \in {}^{\alpha_1}\tilde{\tilde{Z}}_i^{\alpha_1} \equiv \tilde{\tilde{Z}}_i^{\alpha_1}$, because $\forall i$, $\forall\, \alpha \in N_1$, $\sum_{j\neq i} \tilde{\tilde{Z}}_i^\alpha \cup \bar{Z}_i^\alpha$ contains a ball (in Z^α) which contains $z_i^{*\alpha}$. This is a contradiction.

Second, assume that N_2 is not empty and let us show that $\tilde{\tilde{z}}^*$ is a C.E. .

Ad absurdum, assume that for some i, there is $\tilde{\tilde{z}}_i \in B_i$ with $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}_i^*)$, $\pi^{*\alpha} z_i^\alpha \leqslant 0$ $\forall \alpha$, with strict inequality for $\alpha \in \bar{N}_2 \cup \bar{N}_1^1 \subset N_2 \cup N_1^1$.

From semi strict concavity, one can assume for each α : if $\exists \lambda \in]0,1]$ s.t. $\lambda z_i^\alpha \in - \sum_{j \neq i} \tilde{\tilde{Z}}_j^\alpha \cup \bar{Z}_i$, then $z_i^\alpha \in - \sum_{j \neq i} \tilde{\tilde{Z}}_j^\alpha \cup \bar{Z}_i$.

For each $\alpha \in N_1^1$, $z_i^\alpha \in - \sum_{j \neq i} \tilde{\tilde{Z}}_j^\alpha \cup \bar{Z}_i^\alpha$ and thus one cannot have $z_i^\alpha \in - \sum_{j \neq i} \tilde{\tilde{Z}}_j^\alpha \cup \bar{Z}_i^\alpha$ $\forall \alpha \in N_2$, because then one would have $z_i^\alpha \in \tilde{\tilde{Z}}_i^\alpha$, $\forall \alpha \in \bar{N}_2 \cup \bar{N}_1^1$, a contradiction. Thus, for at least one $\alpha \in N_2$, one has $z_i^\alpha \notin - \sum_{j \neq i} \tilde{\tilde{Z}}_j \cup \bar{Z}_i^\alpha$. But, this is a contradiction with (ii) of Definition 10 (See Figure A).

<u>F I G U R E   A</u>

Case 2 : $N_1^1 \cup N_2$ is empty.

Choose for each α a basis $\Lambda^\alpha = \left[\lambda_1^\alpha, .., \lambda_k^\alpha, .., \lambda_{q(\alpha)}^\alpha\right]$ for Z^α.

The argument is now made in $\mathbb{R}^q$ where $q = \sum_\alpha q(\alpha)$ and define (uniquely) $a_i^{*\alpha} = 0$ by $z_i^{*\alpha} = \Lambda^\alpha a_i^{*\alpha}$. In the same may, one defines $^\circ\tilde{\tilde{A}}_i$ and $\bar{A}_i^\alpha$. Now, from A3, 0 is Pareto optimal. Assume not. There exists then $a = [a_1, .., a_m] \in \mathbb{R}^{m \cdot q}$ with $\sum_i a_i = 0$ and $a_i \in P_i$ $\forall i$ such that $V_i(a_i) > V_i(0)$ $\forall i$. Then for each $\beta \in]0,1]$, $V_i(\beta a_i) > V_i(0)$ $\forall i$. Then $\beta a_i \in {^\circ\tilde{\tilde{A}}_i}$ $\forall i$, for every α : a contradiction.

Define then for each i

$A_i = \{a_i \in P_i \mid V_i(a) > V_i(0)\}$

$\sum_i A_i$ is convex. From A3, its relative interior is not empty. There is then $\pi \in \mathbb{R}^q \setminus \{0\}$ such that $\forall i$:

$$\pi . a_i \geqslant 0 \quad \forall \; a_i \in A_i$$

It follows that 0 is a C.E.

When $\bar{z}^{*} \neq 0$ is collectively rational, the same kind of argument is made, while transfers are introduced. The fact that $\sum_i z_i^{*\alpha} = 0 \ \forall\alpha$ implies that one can define $d_i^{*\alpha} \equiv \pi^{*\alpha} z_i^{*\alpha}$ with $\sum_i d_i^{*\alpha} = 0 \quad \forall\alpha.$ Q.E.D.

Proof of Proposition 7 :

As T is finite, let us denote by [T] the matrix formed from the t's $\in$ T. With this formalism, for each i : $z_i = [T]\, a_i$.

The proof rests on the following lemma.

Lemma : Given T containing $\bar{n}$ directions of trade, there is a bounded cube $\bar{B} \subset \mathbb{R}^{\bar{n}}$ such that no agent i looses anything in computing an optimal program subject to the additional constraint $\bar{a} \in \bar{B}$, given the fact that $w_i + [T]\, \bar{a} \leqq \sum_i w_i \ \forall_i$.

Proof : Each i faces essentially constraints of the form $a_i(t) \leqq v(t)$ (if v(t) is positive) or $a_i(t) \geqq v(t)$ (if v(t) is negative). Let be

$v = [v(1),..,v(\bar{n})]$. As T contains $\bar{n}$ t's, consider the compact set :

$$\bar{Y}_i \equiv \left\{ z \in \mathbb{R}^r \;\middle|\; w_i + z \in X_i \text{ and } w_i + z \leqq \sum_i w_i \right\}.$$

Consider a compact cube Y containing the closure of $\bigcup_i \bar{Y}_i$ in its interior. Without loss of generality, we can consider separately the cases for which for each t, $a_i(t)$ is either bounded from below or from above. There is a finite number of such cases because $\bar{n}$ is finite. Consider, e.g., the case for which a(t) is bounded from above for the first n_1 t's and bounded from below for the remaining ones.

For each $v \in \mathbb{R}^{\bar{n}}$, let be

$$\bar{A}(v) = \left\{ \bar{a} \in \mathbb{R}^{\bar{n}} \;\middle|\; 0 \leqq a(t) \leqq v(t) \text{ for the first } n_1 \text{ t's and } 0 \geqq a(t) \geqq v(t) \text{ for the remaining ones} \right\}.$$

Let be $\mathcal{N} \subset \mathbb{R}^{\bar{n}}$ the null space of [T] and $\mathcal{R} \subset \mathbb{R}^r$ the range of [T]. There is a linear homeomorphism between the orthogonal complement in $\bar{\mathbb{R}}^n$ of $\mathcal{N}(\mathcal{N}^+)$ and $\mathcal{R}$. We shall confuse $\mathcal{N}^+$ and $\mathcal{R}$. As $\mathcal{R} \cap Y$ is compact, the set $\left\{a \in \mathcal{N}^+ \mid [T]\, a = z \text{ for some } z \in \mathcal{R} \cap Y\right\}$ is also compact. Moreover each $a \in \bar{\mathbb{R}}^n$ can be expressed in a unique way as $a = {}^1a + {}^2a$ with ${}^1a \in \mathcal{N}^+$ and ${}^2a \in \mathcal{N}$. It follows that, for each v, $[T]\, \bar{A}(v)$ is compact.

For each v and $z \in Y \cap \left\{[T]\, \bar{A}(v)\right\}$, let be $A(v,z)$ the set of a's which <u>minimize</u> $//a//$ subject to the constraints $[T]\bar{a} = z$ and $\bar{a} \in \bar{A}(v)$. We want to show that $\bigcup_v \bigcup_{z \in Y \cap \left\{[T]\bar{A}(v)\right\}} A(v,z)$ is bounded and thus can be embedded in a cube $\bar{B}$ of $\bar{\mathbb{R}}^n$. Assume that it is false. There would exist a sequence $\{v^\gamma\}$ with $\|v^\gamma\| \longrightarrow \infty$, $\bar{a}^\gamma \in \bar{A}(v^\gamma), z^\gamma = [T]\, \bar{a}^\gamma \in Y$ such that it is impossible to find another sequence $\{\tilde{a}^\gamma\}$ with $\tilde{a}^\gamma \in \bar{A}(v^\gamma)$ with $[T]\, \tilde{a}^\gamma = z^\gamma$ and which is bounded. Without loss of generality we can extract-retain the same notation – a converging subsequence $\{z^\gamma\} \longrightarrow z^0 \in \mathcal{R} \cap Y$ and $\{{}^1a^\gamma\} \longrightarrow {}^1a^\gamma \in \mathcal{N}^+$ and $[T]\, {}^1a^\gamma = z^\gamma$. Thus $\{{}^2a^\gamma\}$ with ${}^2a^\gamma = a^\gamma - {}^1a^\gamma \in \mathcal{N}$ is such that $\|{}^2a^\gamma\| \longrightarrow \infty$. Let T^0 be the set of t's for which $a^\gamma(t)$ is uniformly bounded and $T \setminus T^0$ the set of t's for which $\|a^\gamma(t)\| \longrightarrow \infty$. We can assume also that $\{a^\gamma(t)\} \longrightarrow a^0(t)$ for $t \in T^0$. Consider $\dfrac{{}^2a^\gamma}{\|{}^2a^\gamma\|}$ which is bounded. There is a subsequence converging towards ${}^2a^0 \in \mathcal{N}$, which is of norm equal to 1. Consider the resulting subsequence $a^\gamma(1) \equiv {}^1a^\gamma + {}^2a^\gamma - //{}^2a^\gamma//\, {}^2a^0$. We have $[T]\, a^\gamma(1) = z^\gamma\ \forall\gamma$ and $a^\gamma(1) \in \bar{A}(v^\gamma)$ for γ large enough. Le be $T^1 \supset T^0$ the set of $t's \in T$ for which the sequence $a^\gamma(1)$ converges. Repeating the same operation at most $|T \setminus T^0|$ times, one gets $\{\approx{a}^\gamma\}$ which is bounded and with $[T]\approx{a}^\gamma = z^\gamma, \forall\gamma$, and $\approx{a}^\gamma \in \bar{A}(v^\gamma)$ for γ large enough : a contradiction.
Q.E.D.

Thus we can bound a_i, for each i, in such a way that $|a_i(\alpha)| \leqslant \bar{\gamma}\forall\alpha$. Let be $\bar{P}_i$ the set of possible a's for i which satisfy this bound. Define also $R(v) = \left\{a \in \bar{\mathbb{R}}^n \,\middle|\, 0 \leqq a^\alpha \leqq v^\alpha \text{ if } v^\alpha > 0 \text{ and } 0 \geqq a^\alpha \geqq v^\alpha \text{ if } v^\alpha < 0\ \forall\alpha\right\}$.

Define for each $t^\alpha \in T$

$\mu(\alpha) = \sum_i a_i^\alpha$ and $\mu = [\mu(1),\ldots,\mu(\bar{n})]$

Choose $\varepsilon > 0$ and define

$$\Delta = \left\{ \delta \in R^n \;\middle|\; -\varepsilon \leqq \delta(\alpha) \leqq \varepsilon \;\forall\varepsilon \right\}$$

Define two continous and non-increasing functions $\bar{v}^\alpha$ and $\underline{v}^\alpha : \Delta \longrightarrow R_+$ such that :

$$\underline{v}^\alpha = -\bar{\gamma} \text{ if } \delta(\alpha) \geqq 0 \qquad \bar{v}^\alpha = \bar{\gamma} \text{ if } \delta(\alpha) \leqq 0$$

$$\underline{v}^\alpha = 0 \text{ if } \delta(\alpha) = -\varepsilon \qquad \bar{v}^\alpha = 0 \text{ if } \delta(\alpha) = \varepsilon$$

With each δ - and thus for each $\bar{v}^\alpha(\delta)$ and $\underline{v}^\alpha(\delta)$ - we associate, for each i, the set of acts which maximize $V_i(a_i)$ in $R(v(\delta)) \cap \bar{P}_i$. We get thus a correspondence g which associates with each δ a m-uple of sets of optimal acts. g is compact and convex valued and u.h.c. There exists then a compact convex set S in $R^{m\cdot\alpha}$ which contains $g(\Delta)$. With each $a \in S$, we associate the set of all $\delta \in \Delta$ which maximize $\delta.\mu$. Let f be this correspondence $S \longrightarrow \Delta$. f is compact and convex valued and u.h.c.. Consider the following diagram :

$$\begin{tikzcd} \Delta \arrow[drr, "g"] & & \Delta \\ S \arrow[urr, "f"'] & & S \end{tikzcd}$$

There is thus a fixed point (δ^*,a^*). Let us show that it is a D.G.E. Let us show that $\sum_i a_i^* = 0$. If for some α, $\mu^*(\alpha) > 0$, then $\delta^{*\alpha} = \varepsilon$ because $\delta^*.\mu^* \geqq \delta.\mu^* \;\forall\, \delta \in \Delta$. It follows that $\bar{v}^\alpha(\delta^*) = 0$ and we get a contradiction. Conversely, if for some $\alpha, \mu^{*\alpha} < 0$, then $\delta^*(\alpha) = -\varepsilon$ because $\delta^*\mu^* \geqq \delta\mu^*$, $\forall\, \delta \in \Delta$. It follows that $\underline{v}^\alpha = 0$ and we get a contradiction Q.E.D..

Footnotes

* I benefited from helpful discussions with D. Cass, G. Demange, E. Malinvaud and particularly M. Jerison.
I benefited also greatly from referee reports.

(1) See E. Malinvaud and Y. Younès (1977a) and (1977b) and Y. Younès (1982).

(2) One argues in an exchange economy only for avoiding well known problems of shareholders' behaviour in case of incomplete market structures. However, it is possible to assume that each firm is owned by one household, without modifying anything in our analysis.

(3) $x^1 > x^2$ means that $x^1_h \geqslant x^2_h$; $x^1 > x^2$ means that $x^1_h \geqslant x^2_h$ $\forall h$ with strict inequality for some h ; $x^1 \gg x^2$ means that $x^1_h > x^2_h$ $\forall h$.

(4) Semi strict quasi concavity allows to deduce local no satiation from satiation.

(5) The differences between a Dreze Equilibrium (D.E.) and a D.G.E. are the following. In his 1975 paper, J. Dreze assumes that the "market structure" is such that each good $h \neq r$ is exchanged against good r (numeraire or quasi-money) and that the price of each good $h \neq r$ in terms of numeraire varies in a convex set of $\mathbb{R}$: P^h. "Price taking" behaviour is assumed and there is a condition in the definition of a D.E. that if supply (demand) is rationed on market $h(\neq r)$ then the price of good h is on the lower (upper) bound of P^h. The first generalization consists in assuming that the two bundles with non negative elements which are exchanged on each market are not anymore constituted of some good h and good r. Then one can use Dreze's condition when $\pi^\alpha \in \mathbb{R}$, varies in a convex set P^α. It would have been possible to substitute for A4(ii) such a condition and to eliminate

also the assumption that each Z^{α} is a vector subspace. Balance, in such a framework could be written without logical problem, knowing that each agent has to choose

$z^{\alpha} \in Z^{\alpha} = \left\{ z \in \mathbb{R}^r \;\middle|\; \exists\, a^{\alpha} \in \mathbb{R}^2,\ \pi^{\alpha} \in P^{\alpha}\colon\ a_1^{\alpha}\pi^{\alpha} + a_2^{\alpha} = 0 \text{ and } \Lambda^{\alpha} a^{\alpha} = z \right\}$.

The problem with this approach is that it is not obvious how to generalize Dreze's condition when there are more than two bundles on some market. The other possibility is to stick to Dreze's basic ideas but to generalize farther, i.e. dropping Dreze's condition, as it is done in this parper, while getting it, only through agents'behaviour, under the above assumptions. From this point of new, it is the notion of direction of exchange t which is relevant (rationing can have meaning only in a naturally totally ordered set like $\mathbb{R}$) and the set T, also. However, here, in order to avoid a heavy machinery, I assumed that T is finite

$$T = \{t^1,..,t^{\alpha},..,t^n\}.$$

Thus each $Z^{\alpha} = \left\{ z \in \mathbb{R}^r \mid \exists\, a^{\alpha} \in \mathbb{R} \text{ s.t. } t^{\alpha}.a^{\alpha} = z \right\}$, a one dimensional vector subspace, embodies simultaneously price constraints and constraints on the way exchanges are made. I argue that this approach is legitimate and that is coherent with the paradigm.

(i) Two agents can trade only at one "price". It would be strange to assume that two agents trade at more than one price, because it would mean that they trade at an intermediate one.

(ii) When T can be decomposed in $[T^1,..,T^{\alpha},..,T^n]$ with $T = \bigcup_{\alpha} T^{\alpha}$, where each T^{α} is such that

$Z^{\alpha} = \left\{ z \in \mathbb{R}^r \mid \exists\, t \in T^{\alpha} \text{ and } a \in \mathbb{R} \text{ such that } z = a\,t \right\}$ is a vector subspace, one gets the same result as if we began to use the Walrasian approach, by using a D.G.E. as a S.N.C.E., as it is indicated in Part IV.

(iii) When T can be decomposed in $[T^1,..,T^{\alpha},..,T^n]$ with $T = \bigcup_{\alpha} T^{\alpha}$ where each T^{α} is such that it gives rise to

$Z^{\alpha} = \left\{ z \in \mathbb{R}^r \;\middle|\; \exists\, a^{\alpha} \in \mathbb{R}^2,\ \pi^{\alpha} \in P^{\alpha} :\ a_1^{\alpha}\pi^{\alpha} + a_2^{\alpha} = 0 \text{ and } \Lambda^{\alpha} a^{\alpha} = z \right\}$ then one gets Dreze's condition, when Z^{α} is two-dimensional.

Thus, in some sense, the approach in terms Dreze Equilibrium (or S.N.C.E.) is more general than the Walrasian approach, contrary to what appears when one uses a finite T in order to avoid the use of a heavy mathematical machinery.

(6) All relevant definitions are "weak" ones – e.g. "weak" Pareto optimum – because one wants to get characterization theorems for definitions which are written in the same way for A4(i) and A4(ii). Then one encounters a well known purely technical problem. Assume $r = m = 2$. The price ratio is fixed and there is one market which is one dimensional. Consider the initial allocation. Agent 1 is indifferent between buying, selling or remaining at his initial allocation. Agent 2's utility increases if he buys at this price ratio. The initial allocation is a Dreze's equilibrium at which demand is rationed. However, at this price ratio, there is an allocation for which only agent 2's utility has increased. Notice that the same analysis could have been made if we had chosen the following criterion (in all relevant definitions) :

There is no balanced $\tilde{\tilde{z}}$ with $U_i(\tilde{\tilde{z}}_i) \geq U_i(\tilde{\tilde{z}}_i^*)$ with strict inequality for at least one i and $U_i(\tilde{\tilde{z}}_i) > U_i(\tilde{\tilde{z}}_i^*)$ for each i for which $\tilde{\tilde{z}}_i \neq \tilde{\tilde{z}}_i^*$. This criterion would have been more intuitively linked with the use of the Paradigm.

(7) Actually, one requires a property which is weaker than the convexity of $F_i^\alpha \equiv \sum_{j \neq i} {}^\alpha\tilde{\tilde{Z}}_j^\alpha \cup \bar{Z}_i^\alpha$, $\forall (i,\alpha)$, when A3 holds. As, $z_i^{*\alpha} \in F_i^\alpha$, one requires only :

(i) If $z^\alpha \in F_i^\alpha$ then the segment $\left[z_i^{*\alpha}, z^\alpha\right] \subset F_i^\alpha$.

(ii) If ${}^1z^\alpha \in F_i^\alpha$ and ${}^2z^\alpha \in F_i^\alpha$, then for each $\lambda \in [0,1]$, there are ${}^3z^\alpha \in \left[z_i^{*\alpha}, {}^1z^\alpha\right]$ and ${}^4z \in \left[z_i^{*\alpha}, {}^2z^\alpha\right]$ with $\lambda\, {}^3z^\alpha + (1-\lambda)\; {}^4z^\alpha \in F_i^\alpha$.

(8) In view of the proof of proposition 6, the definition of a collectively rational $\tilde{\tilde{z}}^*$ can be written, if A3, A6 and A7 are made : Given M, a feasible $\tilde{\tilde{z}}^*$ is "collectively rational" if there are 2 m.n sets $(\tilde{\tilde{Z}}_i^\alpha)_{(i,\alpha)}$ and $(\bar{Z}_i^\alpha)_{(i,\alpha)}$ such that :

(i) $\forall(i,\alpha)$:

$$\tilde{\tilde{Z}}_i^{\alpha} = \left\{ z^{\alpha} \in Z^{\alpha} \;\middle|\; \exists\, \tilde{\tilde{z}}_i = [z^{\alpha}, z^{-\alpha}] \in B_i \text{ such that } U_i(\tilde{\tilde{z}}) \geq U_i(\bar{\tilde{\tilde{z}}}) \right.$$

$$\left. \forall\, \bar{\tilde{\tilde{z}}} \text{ with } \bar{z}^{\alpha} \in -\sum_{j \neq i} \tilde{\tilde{Z}}_j^{\alpha}\ \forall\alpha \text{ and } z^{\alpha'} \in -\left[\sum_{j \neq i} \tilde{\tilde{Z}}_j^{\alpha'} \cup \bar{Z}_i^{\alpha'}\right] \forall\alpha' \neq \alpha \right\}.$$

where $\left[\sum_{j \neq i} \tilde{\tilde{Z}}_j^{\alpha'} \cup \bar{Z}_i^{\alpha'}\right]$ is convex, $\forall\alpha'$.

(ii) It is the same as in Definition 10.

(iii) $\forall(i,\alpha)$

$z_i^{*\alpha} \in \tilde{\tilde{Z}}_i^{\alpha}$

(9) See J.M. Grandmont, G. Laroque and Y. Younès (1978).

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F I G U R E S

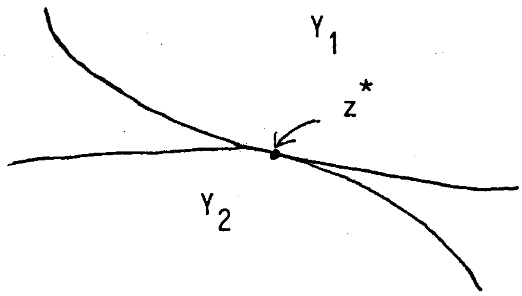
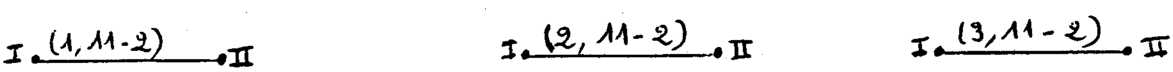
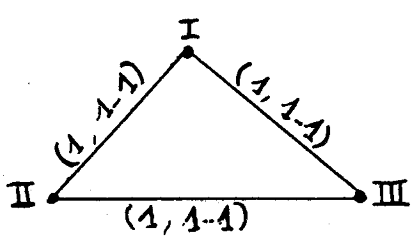


Figure 1



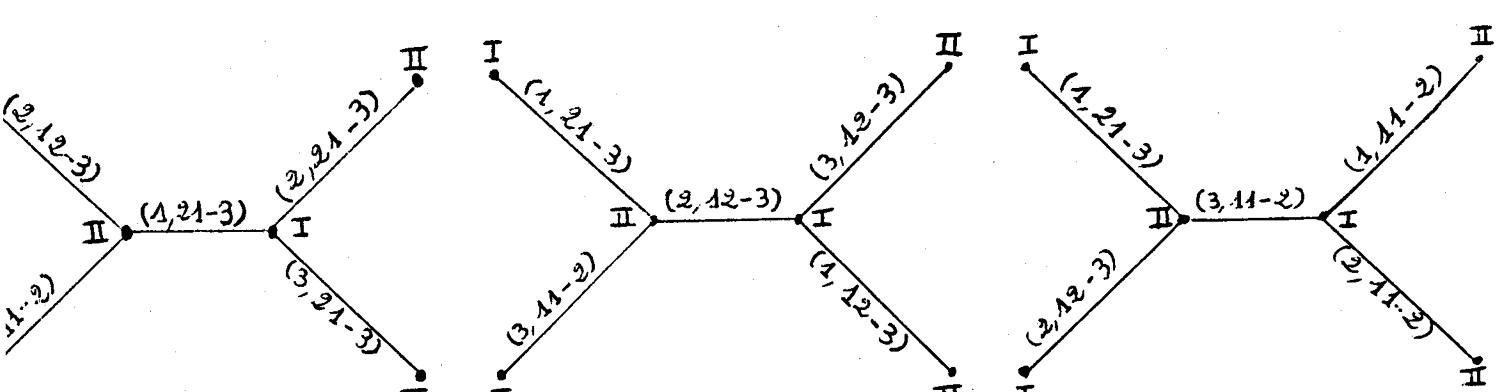
$L^1(1)$ $L^2(1)$ $L^3(1)$

Figure 2a (m = 2 and n = 3)



$L^1(1)$

Figure 2b (m = 3 and n = 2)



$L^1(2)$ $L^2(2)$ $L^3(2)$

Figure 2c (m = 2 and n = 3)

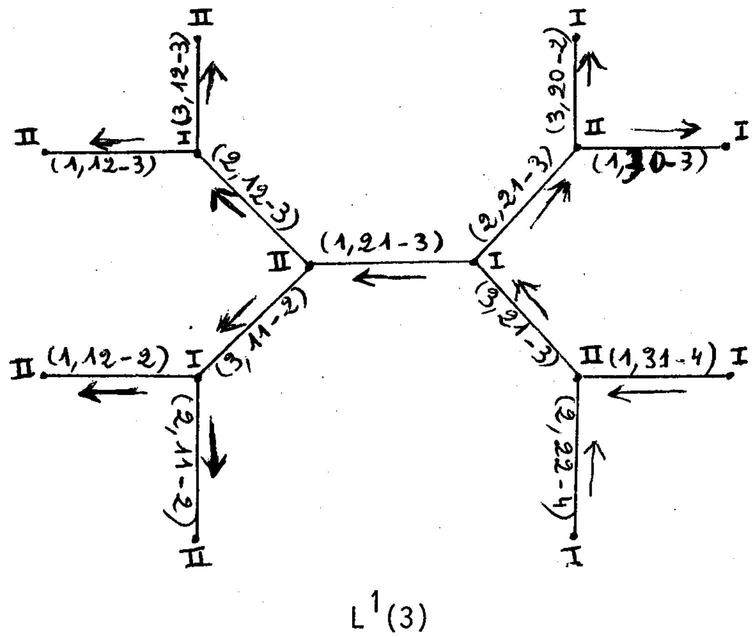


Figure 2d (m = 2 and n = 3)

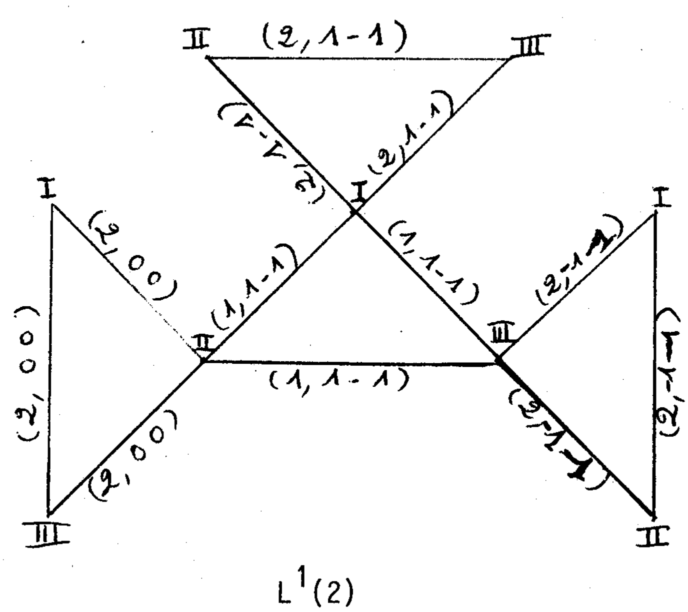


Figure 2e (m = 3 and n = 2)

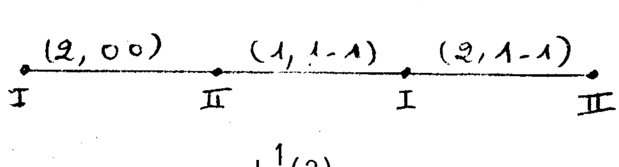


Figure 2 f (m = n = 2)

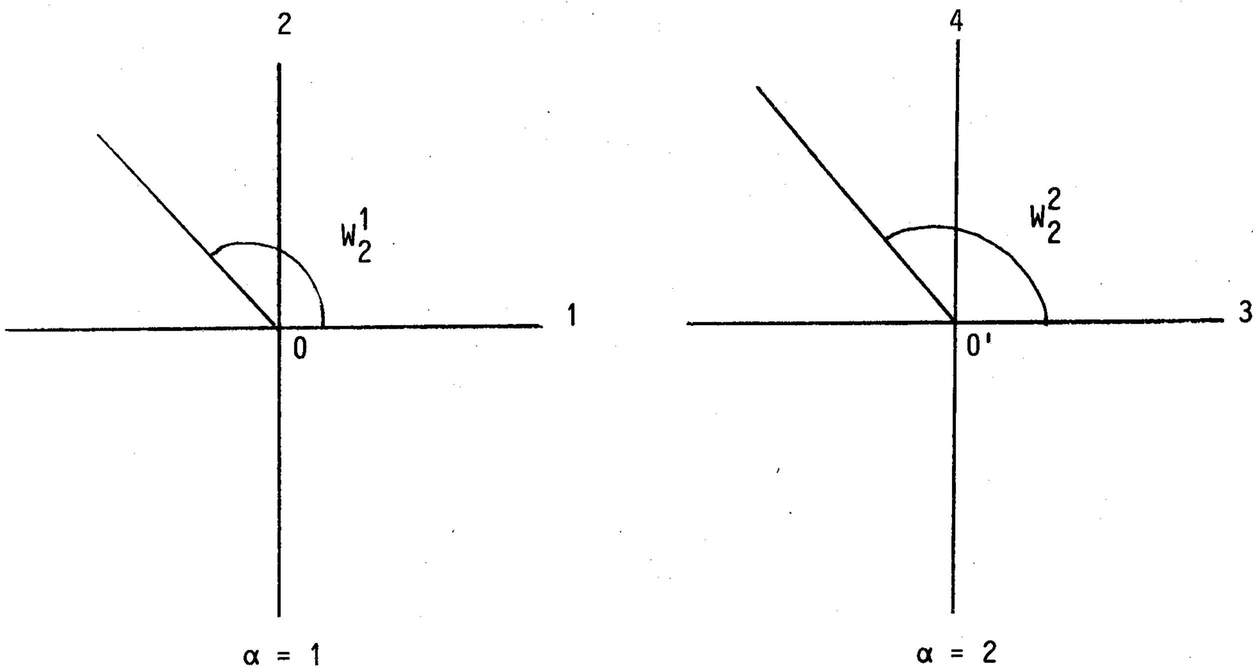


Figure 3

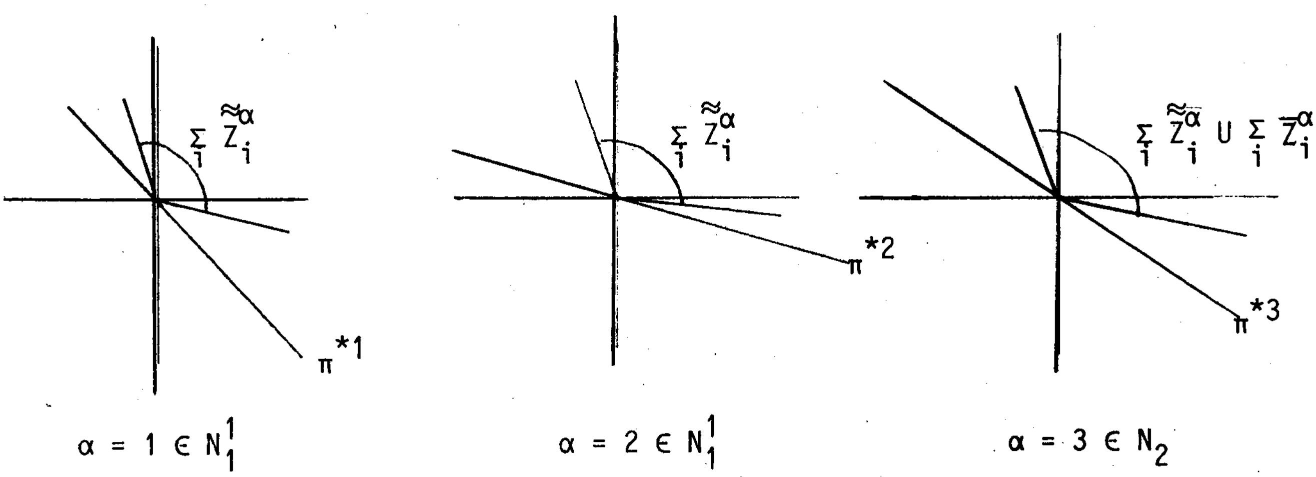


Figure A