February 1987

ON THE STRUCTURE OF PARETO-OPTIMA IN AN INFINITE HORIZON ECONOMY WHERE AGENTS HAVE RECURSIVE PREFERENCES

Rose-Anne DANA, Université Paris-VI

Cuong LE VAN, CNRS-CEPREMAP

N°8711

ABSTRACT

ON THE STRUCTURE OF PARETO-OPTIMA IN AN INFINITE HORIZO ECONOMY WHERE AGENTS HAVE RECURSIVE PREFERENCES

This article generalizes the one-agents growth theory with discounting to the case of several agents with recursive preferences. In a multi-consumption goods world, we show that, under some regularity conditions, any Pareto-optimum can be viewed as the trajectory of a dynamical system. The state space can be chosen to be either the product of the space of capitals by the (n-1)-simplex or the state of couples, capital-utilities achievable by (n-1) agents from that capital. We define and study the properties of generalized value functions. A generalized Euler's equation is introduced. It is then being used to give uniqueness and local stability conditions for a steady state.

RESUME

STRUCTURE DES OPTIMA DE PARETO DANS UNE ECONOMIE A HORIZON INFINI OU LES AGENTS ONT DES PREFERENCES RECURSIVES

Cet article généralise la théorie de la croissance à un agent qui utilise un critère de choix avec taux d'escompte au cas de plusieurs agents avec préférences récursives. Dans un monde à plusieurs biens de consommation, on montre que sous certaines conditions on peut représenter tout optimum de Pareto comme trajectoire d'un système dynamique. On peut prendre comme espace des états, soit le produit de l'espace des biens de capital par le (n-1)-simplexe, soit l'espace des couples capital et utilités réalisables par (n-1) agents à partir de ce stock de capital.On introduit des fonctions-valeurs généralisées ainsi qu'une équation d'Euler généralisée. On l'utilise par la suite pour donner des conditions d'unicité et de stabilité locale de l'état stationnaire.

Journal of Economic Literature : 020, 110, 210

Mots Clés : Préférences récursives, optimum de Paréto, dynamique, fonctions valeurs, équations de Bellman, équations d'Euler.

INTRODUCTION

These last years attention has focused on the dynamics of equilibrium models where agents are infinitely lived. Becker [1980], Bewley [1982], Yano [1984], [1985], Coles [1985] [1986] have studied the dynamics of intertemporal equilibrium allocations when agents have separable preferences. The last three authors have shown that if agents have the same discount factor and if it is sufficiently close to one then in equilibrium the economy has the well known turnpike property of optimal growth.

On the other hand, following Koopmans' work [1960], [1969], Iwai [1972], Benhabib et ali [1985] for the one agent case, Lucas and Stokey [1984] and Benhabib et ali [1986] studied the dynamics of Pareto-optimal allocations in models where agents have recursive preferences. They all assumed basically the existence of one consumption good only. The purpose of this paper is to carry on this last work one step further in a more general setting. (We use many consumption goods) and to generalize well-known results of growth theory.

In order to do so, we introduce in section one an economy with m infinitely lived agents and a producer at each date. An initial stock is given. Agents have recursive preferences, the technology is markovian and time invariant. In section two, we show that under strict concavity and differentiability assumptions on utilities and strict convexity assumptions on production any Pareto optimal sequence may be viewed as the trajectory of a dynamical system. The state space can be chosen to be the product of the capital space by the simplex (capital-utility weights assigned to agents) or the space of capital and utilities achievable by $n-1$ agents from that capital stock. In section three, we generalize the one-agent value function. This can be done in two ways according to the state space chosen, as shown by Lucas and Stokey [1984] and Benhabib et ali [1986]. We extend their definitions to the multi-consumption goods case and study the properties of the generalised value functions.

In part four, we generalize "Euler's equations".

We then first use these equations to give sufficient conditions of existence and uniqueness of stationary states in the multi-consumption goods case (our approach is based on Brock [1973] and Burmeister [1980]). We then extend Mangasarian's result [1966] that any bounded sequence that satisfies Euler's generalised equations is in fact optimal.

We conclude with a couple of examples. The first one extends slightly Benhabib et ali's [1986].

In the second one agents have separable utilities and same discount factor, we thus get a turnpike theorem for discount factors close to one.

I - THE MODEL

I.1 - *Generalities* :

We consider an economy with n consumers each of whom lives for an infinite number of periods $t=1,2\dots$. There are m consumption goods and p capital goods. The commodity space is then $(R_+^m)^\infty \times (R_+^p)^\infty$.

We shall use in R^h and $(R^h)^\infty$ ($h = m$ or p) the following conventional notations : in R^h, $z' \geq z \iff \forall j = 1 \dots h \quad z'_j \geq z_j$

$$z' > z \iff z' \geq z \text{ and } z' \neq z$$

$$z' \gg z \iff \forall j = 1 \dots h \quad z'_j > z_j$$

in $(R^h)^\infty$, $\underset{\sim}{z}' \geq \underset{\sim}{z} \iff \forall t \geq 0 \; z'_t \geq z_t$; $\underset{\sim}{z}' > \underset{\sim}{z} \iff \underset{\sim}{z}' \geq \underset{\sim}{z}$ and $\underset{\sim}{z}' \neq \underset{\sim}{z}$

T will denote the shift on sequences. For $\underset{\sim}{x} = (x_o, x_1, \dots)$, $T\underset{\sim}{x} = (x_1, x_2, \dots)$.

The economy is described by the list

$E = \left((R_+^m)^\infty , W^i , i \in I = \{1,\dots,n\}, (R_+^p)^\infty, B, k_o \right)$. We shall define below each element of this list.

I.2 - *Recursive preferences* :

Let $x_t^i = (x_{t1}^i, \dots, x_{tm}^i) \in R_+^m$ denote the quantity agent i consumes at date t. Let $x^i = (x_0^i, x_1^i \dots)$ denote the infinite sequence of agent i consumptions. Let X be the space of consumption sequences $(R_+^m)^\infty$ endowed with the product topology. Let us recall that this topology is metrisable and that one can for example define $\tilde{d}(\underset{\sim}{x},\underset{\sim}{y})$ by

$$\tilde{d}(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i=0}^{\infty} \left[\frac{d(x_i, y_i)}{1+d(x_i,y_i)} \right] \frac{1}{2^i}$$ where d is any distance in R^m. Since

$d(Tx,Ty) \leq 2\, d(x,y)$, T is a continuous map from X into X .

Let S be the space of bounded continuous functions from X into R_+ endowed with the sup norm $\| u \| = \sup\limits_{x \in X} u(x)$.

Following Beals and Koopmans [1969] and Lucas and Stokey [1984] we shall assume that preferences are representable by a utility function which belongs to a class that we next define.

A function $W : R_+^m \times R_+ \to R_+$ is an <u>aggregator</u> function if it satisfies the following properties :

W1 : continuous and satisfies $W(x,0) \leq M, \forall x$

W2 : concave

W3 : $W(0,0) = 0$

W4 : $(x,z) \leq (x',z')$ and $(x,z) \neq (x',z')$ implies $W(x,z) < W(x',z')$;

and for some $0 \leq \beta < 1$

W5 : $|W(x,z) - W(x,z')| \leq \beta \; |z-z'|$ for all $x \in R_+^m$ and all $z, z' \in R$.

We have the following theorem :

<u>*THEOREM I.1*</u> :

Every aggregator W defines an operator on S as follows $T_W u(\underset{\sim}{x}) = W(x_0, u(T(\underset{\sim}{x})))$. T_W is a contraction. There exists a unique $u \in S$ such that $T_W u = u$ with the following properties :

(i) u is concave (strictly concave whenever W is strictly concave in x)

(ii) Non decreasing

(iii) $u(0) = 0$

<u>*Proof*</u> : Let us first prove that T_W defines an operator on S . Note that $T_W u$ is continuous on S as composite of continuous maps. $T_W u$ is bounded since $|T_W u(\underset{\sim}{x}) - W(x_0,0)| \leq \beta \; u(T\underset{\sim}{x}) \leq \beta \|u\|$ by W_5 and therefore

$$\sup_{\underset{\sim}{x} \in X} | \, T_W u(\underset{\sim}{x}) \, | < M + \beta \, \| u \| \quad .$$

It follows from W_5 that T_W is a β-contraction on S since $|T_W u(\underset{\sim}{x}) - T_W v(\underset{\sim}{x})| \leq \beta \; |u(T\underset{\sim}{x})) - v(T \, \underset{\sim}{x})| \leq \beta \; \|u-v\| \; , \forall \underset{\sim}{x}$.

As T_W maps concave, non decreasing maps into themselves, T_W has a unique fixed point which is concave and non decreasing. As $u(0) = T_W\, u(0) = W(0,u(0))$ and is unique, $u(0) = 0$.

If W is strictly concave in x, then u is also strictly concave. This can be proved by induction. Let $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ be such that $x_0 \neq x'_0$ then

$$u(\lambda \underset{\sim}{x} + (1-\lambda)\underset{\sim}{x}') = W(\lambda\, x_0 + (1-\lambda)\, x'_0,\ u(\lambda\, T(\underset{\sim}{x}) + (1-\lambda)\, T(\underset{\sim}{x}')))$$

$$> \lambda\, W(x_0, u(T(\underset{\sim}{x})) + (1-\lambda)\, W(x'_0, u(T(\underset{\sim}{x}')) = \lambda u(\underset{\sim}{x}) + (1-\lambda)\, u(\underset{\sim}{x}')$$

Now let $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ be such that $x_i = x'_i\ \forall i < j$ and $x_j \neq x'_j$

Then $u(\lambda\, T^j(\underset{\sim}{x}) + (1-\lambda)T^j(\underset{\sim}{x}')) > \lambda\, u(T^j(\underset{\sim}{x})) + (1-\lambda)\ u(T^j \underset{\sim}{x}')$

as $u(T^{j-1}(\underset{\sim}{x})) = W(x_{j-1}, u(T^j(\underset{\sim}{x})))$ and since W satisfies W_4 we have

$u(\lambda\, T^{j-1}(\underset{\sim}{x}) + (1-\lambda)\quad T^{j-1}\ (\underset{\sim}{x}')) > \lambda\ u(T^{j-1}\ (\underset{\sim}{x})) + (1-\lambda)\ u(T^{j-1}\ (\underset{\sim}{x}'))$

and therefore $u(\lambda \underset{\sim}{x} + (1-\lambda)\underset{\sim}{x}') > \lambda\ u(\underset{\sim}{x}) + (1-\lambda)\ u\ (\underset{\sim}{x}')$, by the induction hypothesis.

Example I.1 : *The discounted case.*

Let $v : R^m_+ \to R^+$ be a continuous, concave, uniformly bounded function such that $v(0) = 0$. Let $W(x,z) = v(x) + \beta z$. Then it can be easily verified that all the properties $W_1 - W_5$ are fulfilled.

Let $u(\underset{\sim}{x}) = \sum_0^\infty \beta^t\ v(x_t)$. Since v is uniformly bounded u is well defined and satisfies $u(\underset{\sim}{x}) = W(x_0, u(T(\underset{\sim}{x})))$. It is therefore the unique solution to the functional equation $T_W\ u = u$.

Therefore the classical growth theory and intertemporal general equilibrium theory with discounted preferences is a particular case of the theory we shall develop.

Remark I.1 : Note that we work with a weaker topology then Lucas and Stokey [1984] who use ℓ^m_+ . All proofs concerning compactness and continuity are thus much easier. Moreover notice that we have weakened their hypotheses that $W(x,y) \leq M\ \ \forall x, \forall y$ into $W(x,0) \leq M, \forall x$ in order to be able to apply it to linearly separable preferences.

Remark I.2 : If W is assumed to be continuous and satisfies $W_2 - W_5$ it is still possible to associate with it a utility function (not necesseraly bounded). Indeed let $X = \ell_+^m$ and S' be the space of continuous function f on X such that $\sup \frac{|f(x)|}{1+\|x\|_\infty} < \infty$. Then it can easily be shown that S' endowed with the norm $\|f\|_{S'} = \sup \frac{|f(x)|}{1+\|x\|_\infty}$ is a Banach space. As W is concave, there exists a $C > 0$ such that $W(x,0) \leq C(1+\|x\|)$. Therefore with W we can associate an operator on S' as follows $T_W u(\underset{\sim}{x}) = W(x_0, uT(\underset{\sim}{x}))$. Indeed from W_5 we have that $|W(x_0,u(T(x)) - W(x_0,0)|$ $\leq \beta\ (1+\|x\|_\infty)\ \|u\|_{S'}$ and therefore $|W(x_0,u(T(x)))| \leq (\beta\ \|u\|_{S'} + C)(1+\|x\|_\infty)$. Moreover $T_W u$ is continuous on ℓ_+^m therefore $T_W(u) \in S'$. However the counterpart of this weakened hypotheses is as in Montrucchio [1984] a stronger assumption on the production set so as to get a bounded utility set.

Remark I.3 : The aggregator W defined above can be viewed as an extension of the one defined in Koopmans [1960] and Koopmans et ali [1964]. Indeed consider a function $\mathcal{V} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies $W_1 - W_5$. Let $C : \mathbb{R}_+^m \to \mathbb{R}_+$ be any concave, strictly increasing function that verifies $C(0) = 0$. Then $W(x,z) = \mathcal{V}(C(x),z)$ is an aggregator function in the sense defined above.

I.3 - *Stationary Markovian technology* :

The technology at date t will be assumed to depend only on the vector of capital stocks $K_t = (K_{t1},\dots,K_{tp}) \in R_+^p$ on hand at the beginning of t and the production possibilities will be assumed to be invariant in time.

Given a stock k we shall assume the existence of a set of pairs (x,y) of current consumption goods and beginning of next period capital stocks that are jointly producible $B(k) \subseteq R_+^m \times R_+^p$, so that the technology is characterized by a correspondence $B : R_+^p \to R_+^m \times R_+^p$ with the following properties :

B0 : B is continuous

B1 : for each k, B(k) is compact and convex

B2 : $(x,y) \in B(k)$ and $0 \leq (x',y') \leq (x,y)$ implies $(x',y') \in B(k)$

B3 : $k' \leq k$ implies $B(k') \subseteq B(k)$

B4 : if $(x,y) \in B(k)$ and $(x',y') \in B(k')$, then

$$\Big((\theta x + (1-\theta)x'),\ (\theta y + (1-\theta)\ y')\Big) \in B\ (\theta k + (1-\theta)k')$$

for $\theta \in [0,1]$

B5 : $(x,y) \in B(0) \Rightarrow y = 0$ and there exists $x > 0$ $(x,0) \notin B(0)$.

B6 : $k > 0$ implies that there exist $x > 0$, $y > 0$, $(x,y) \in B(k)$.

This last hypothesis implies that from a non zero initial capital stock one can generate capital and consumptions sequences that are not zero at each date. All others hypotheses are standard.

Example I.2 : Let $F(k,x,y)$ from $\mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^p_+ \to \mathbb{R}$ be a continuous convex function, increasing in x and y and strictly decreasing in k . Let $F(0,x,y) \leq 0$ imply $y = 0$ and $F(0,0,0) \leq 0$. Then let $B(k) = \{(x,y),\ F(k,x,y) \leq 0\}$ then B satisfies $B_0 \leq B_6$.

I.4 - *Feasible consumption paths. Utility set* :

In what follows let us denote by $\hat{x} = \sum_i x^i$, $\hat{x}_t = \sum_i x^i_t$, $\hat{\underset{\sim}{x}} = \sum_i \underset{\sim}{x}^i$.

For x and x' in $\mathbb{R}^h$, $h=1,m,n$ or p, let $x^\lambda = \lambda x + (1-\lambda)x'$, $\lambda \in]0,1[$.

Given k_0 the initial stock a feasible allocation path is a couple $(x,k) \in X^m \times (R^p_+)^\infty$ such that $(\hat{x}_t\ ,\ k_{t+1}) \in B(k_t)$ $\forall t \geq 0$. We denote by $X(k_0)$ the set of all $\underset{\sim}{x} \in X^m$ which are feasible. Let us also define feasible consumption paths as follows

$$C(k_0) = \{\underset{\sim}{x} \in X,\ (x_t,\ k_{t+1}) \in B(k_t)\ \forall t \geq 0 \text{ for some } \underset{\sim}{k} \in (R^p_+)^\infty, k_0 \text{ given}\}$$

It can be rewritten as follows :

$$C(k_0) = \{\underset{\sim}{x} \in X,\ \exists\ k_1 \in R^p_+,\ (x_0,\ k_1) \in B(k_0) \text{ and } T\ \underset{\sim}{x} \in C(k_1)\}$$

The correspondence $C : R^p_+ \to X$ has the following properties :

THEOREM I.2 :

(1) $\forall k_0$, $C(k_0)$ *is convex compact. If* $x \in C(k_0)$ *and* $x' \leq x$ *then* $x' \in C(k_0)$

(2) *The correspondence* $k \to C(k)$ *has a closed graph and is lower semi-continuous*

(3) $\forall \lambda \in [0,1], \lambda\, C(k_0) + (1-\lambda)\, C(k'_0) \subseteq C(\lambda\, k_0 + (1-\lambda)\, k'_0)$

(4) $k' \leq k$ *implies* $C(k') \subseteq C(k)$.

Proof : (1) The convexity of $C(k)$ follows from B_1 and B_4 . To prove that $C(k_0)$ is compact let $\Gamma(k_0) = \{(\underset{\sim}{x},\underset{\sim}{k}) \in X \times (R^p_+)^\infty,\ \underset{\sim}{x} \in C(k_0)\}$, let $\Pi^i, i = 1,2$ denote the i^{th} projection of $R^m_+ \times R^p_+$ on the i^{th} factor. Let $\Pi^1 B(k_0) = C_0(k_0)$, $\Pi^2\, B(k_0) = K_1(k_0)$, and define by induction $\Pi^1\, B(K_t(k_o)) = C_{t+1}(k_0)$, $\Pi^2\, B(K_t(k_0)) = K_{t+1}\,(k_0)$. As Π^i and B are continuous so are the correspondences C_t . Moreover they are compact valued as B is compact valued.

As $\Gamma(k_0) \subseteq \overset{\infty}{\underset{0}{\pi}} C_t(k_0) \times \overset{\infty}{\underset{0}{\pi}} K_{t+1}(k_0)$, it follows from Tychonov's theorem that $\Gamma(k_0)$ is relatively compact. Moreover $\Gamma(k_0)$ is closed since $B(k_0)$ is closed and B is upper semi-continuous. Therefore $\Gamma(k_0)$ is compact and so is $C(k_0)$ its projection on X .

(2) Since $B(k)$ is closed for every k and the correspondence $k \to B(k)$ has a closed graph, the correspondence $k \to C(k)$ has a closed graph. To prove its lower semi-continuity let $k^n_0 \to k_0$ and $(\underset{\sim}{x}) \in C(k_0)$. Let $(\underset{\sim}{x}, \underset{\sim}{k}) \in \Gamma(k_0)$. Then $(x_0, k_1) \in B(k_0)$. As B is continuous there exists a sequence $(x^n_0, k^n_1) \in B(k^n_0)$ converging towards (x_0, k_1). Similarly there exists a sequence $(x^n_1, k^n_2) \in B(k^n_1)$, $(x^n_1, k^n_2) \to (x_1, k_2)$. One constructs by induction a sequence $(x^n_t, k^n_{t+1}) \in B(k^n_t)$, such that $(x^n_t, k^n_{t+1}) \to (x_t, k_{t+1})$. Therefore $\underset{\sim}{x}^n \in C(k^n_0)$ and $\underset{\sim}{x}^n$ converges towards $\underset{\sim}{x}$.

(3) and (4) that follow from B4 and B3 respectively are omitted. □

Following Negishi' [1960] we now introduce the utility set $U(k_0)$ the set of utility vectors which can be reached by attainable allocations :

$$U(k_0) = \left\{ (u^i(\underset{\sim}{x}^i)) \ :\ \hat{\underset{\sim}{x}} \in C(k_0) \right\}$$

Let $\Phi : X \to X^m$ be defined as follows : $\Phi(\underset{\sim}{x}') = \{(\underset{\sim}{x}^i) \in X^m,\ \hat{\underset{\sim}{x}} = \underset{\sim}{x}'\}$

It will be shown in appendix one that Φ is continuous.

Let $\tilde{U} : X^m \to R^n_+$ denote the function $\tilde{U}(\underset{\sim}{x}) = (u^1(\underset{\sim}{x}^1), u^2(\underset{\sim}{x}^2), \dots u^n(\underset{\sim}{x}^n))$

$\tilde{U}$ is clearly continuous and $U(k) = \tilde{U}(\Phi(C(k)))$

$U(k)$ has the following properties :

THEOREM I.3 :

(1) For every k, $U(k)$ is compact, convex (strictly if all u^i are strictly concave) and satisfies free disposal : $\forall u \in U(k)$, $0 \leq u' \leq u$ implies $u' \in U(k)$.

(2) $U(\lambda k + (1-\lambda)\, k') \supseteq \lambda\, U(k) + (1-\lambda)\, U(k')$. (The inclusion is strict if all the W^i are strictly concave).

(3) $\forall k > 0$ there exists $z >> 0$ such that $z \in U(k)$

(4) The correspondence from $R^p_+ \to R^n_+$: $k \to U(k)$ is continuous.

Proof : (1) The (strict) convexity of $U(k)$ follows from the convexity of $B(k)$ and the (strict) concavity of the u^i. Compactness of $U(k)$ follows from the compactness of $C(k)$ and the continuity of Φ and U. Free disposal follows from the fact that $x \in C(k)$ implies $x' \in C(k)$ for every $x' \leq x$.

(2) Let $\underset{\sim}{x} \in C(k)$ and $\underset{\sim}{x}' \in C(k')$ be such that $\sum_i \underset{\sim}{x}^i = \underset{\sim}{x}$ and $\sum_i \underset{\sim}{x}'^i = \underset{\sim}{x}'$. Then $\lambda\, \underset{\sim}{x} + (1-\lambda)\, \underset{\sim}{x}' \in C(\lambda\, k + (1-\lambda)\, k')$ by theorem I.2 (c). For some i, $x^i \neq x'^i$, $u^i\, (\lambda \underset{\sim}{x}^i + (1-\lambda)\, \underset{\sim}{x}'^i) > \lambda u^i\, (\underset{\sim}{x}^i) + (1-\lambda)\, u^i(\underset{\sim}{x}'^i)$ and for $j \neq i$ $u^j\, (\lambda \underset{\sim}{x}^j + (1-\lambda)\, \underset{\sim}{x}'^j) \geq \lambda u^j(\underset{\sim}{x}^j) + (1-\lambda)\, u^j(\underset{\sim}{x}'^j)$ which proves 2.

(3) By assumptions B_5 and B_6, there exists $x > 0$, $y \geq 0$ such that $(x,y) \in B(k)$ therefore $x_j > 0$ for some j. Let us consider the following consumption sequences : $x^i_{0j} = \frac{1}{n}\, x_j$ and $x^i_{0\ell} = 0$ $\forall i$, $\forall \ell \neq j$, $x^i_{tj} = 0$, $\forall t \geq 1$, $\forall i$, $\forall j$. By W_3, $u^i\, (\underset{\sim}{x}^i) > 0$ $\forall i$.

(4) U is lower semi-continuous and has a closed graph as composite of lower semi-continuous correspondences and maps with a closed graph. The u^i being uniformly bounded, $U(k)$ belongs to some fixed compact set for every k. U is therefore upper semi-continuous.

II - PARETO-OPTIMALITY

$\underset{\sim}{x} \in X(k_0)$ is Pareto-optimal if there exists no $\underset{\sim}{x}' \in X(k_0)$ with $\tilde{U}(\underset{\sim}{x}') > \tilde{U}(\underset{\sim}{x})$. We shall denote by $\hat{U}(k_0)$ the image by U of the Pareto-optima corresponding to the initial stock k. It is a classical result that $\hat{U}(k)$ is compact for every k and homeomorphic to the unit simplex Δ^{n-1} of R^n, the homeomorphism being the "radial projection" on Δ^{n-1} (see Mas-Colell [1985] page 154).

For further use the rest of this section is devoted to show that under some further regularity conditions on the technology and on preferences, for every k, the Pareto frontier is homeomorphic to Δ^{n-1} the homeomorphism being the map from $\Delta^{n-1} \to \hat{U}(k)$, $\theta \to \arg\max \sum_{i=1}^{n} \theta^i u^i$, $(u^i) \in \hat{U}(k)$.

Let us first quote the following result

LEMMA II.1 :

The correspondence $k \to \hat{U}(k)$ *has a closed graph.*

Proof : Let (k_ν, z_ν), $z_\nu \in \hat{U}(k_\nu)$ converge to (k, z). $z \in \hat{U}(k)$ iff there exists a $\theta \in \Delta^{n-1}$ such that $\Sigma\, \theta^i z^i$ maximizes $\sum_i \theta^i u^i, (u^i) \in U(k)$ (see Mas-Collel [1985]). Thus there exists a sequence θ^i_ν such that $\sum_i \theta^i_\nu z^i_\nu \geq \sum_i \theta^i_\nu u^i_\nu$, $\forall (u^i_\nu) \in U(k_\nu)$. Let $v \in U(k)$. Since U is l.s.c. there exists a sequence $v_\nu \in U(k_\nu), v_\nu \to v$. Therefore $\sum_i \theta^i_\nu z^i_\nu \geq \sum_i \theta^i_\nu v^i_\nu$. Let θ be a limit point of the sequence θ_ν. Then we get $\sum_i \theta^i z^i \geq \sum_i \theta^i v^i$ for every $v \in U(k)$. Therefore $z \in \hat{U}(k)$.

We shall now on assume the following.

B_4 Biss For every $k \geq 0$, $k' \geq 0$, $k \neq k'$ if $(x,y) \in B(k)$ and $(x,y') \in B(k')$ then there exists $x' \geq x$, $x' \neq x$ such that $(x', y^\lambda) \in B(k^\lambda)$.

W_2 Biss For every i, W^i is strictly concave in x.

W_6 For every i, W^i is C^1 on $R^m_{++} \times R_+$ and satisfies the following boundary conditions.

W_7 (1) Inada conditions $\frac{\partial W^i}{\partial x^i_j}(x^i, z^i) \to \infty$

$x^i_j \to 0$

(2) If $x^i_j = 0$ for all $j = 1 \dots h$ then $W^i(x^i, z^i)$ is a C^1 function of x^i_p, $p > h$ and z^i.

We first prove the following lemma :

<u>*LEMMA II.2* :</u>

For every $u \in \hat{U}(k)$ there exists a unique triple $((x^i), y, (z^i))$ in $R^{mn}_+ \times R^p_+ \times R^n_+$ such that $W^i(x^i, z^i) = u^i$, $\forall i$ $(\hat{x}, y) \in B(k)$ and $z \in U(y)$. The map e from graph $\hat{U}$ into $R^{mn}_+ \times R^p_+ \times R^n_+$, $(k,u) \to ((x^i), y, (z^i))$ is continuous.

<u>*Proof*</u> : Let k be fixed and $u \in \hat{U}(k)$. Since the W^i are strictly concave there exist unique (x^i), such that $u^i = W^i(x^i, z^i)$ $\forall i$. Uniqueness of the associated y follows from B4 biss. To prove uniqueness of (z^i) assume $(x,y) \in B(k)$ and z and $z' \in U(y)$. Since $U(y)$ is strictly convex, z^λ belongs to $\overset{o}{\widehat{U(y)}}$. Thus there exists $z'' > z^\lambda$, $z'' \in U(y)$. Therefore $u_i < W^i(x^i, z''^i)$ $\forall i$, which contredicts the pareto optimality of u.

To show the continuity of e, let $k_\nu \to k$ and $u_\nu \to u$. Let $((x^i_\nu), y_\nu, (z^i_\nu)) = e(k_\nu, u_\nu)$. As $\hat{U}$ has a closed graph, $(k,u) \in$ graph $\hat{U}$. Let K_o be any fixed compact set such that k_ν and k belong to K_o. Let $B(K_o) = \bigcup_{k \in K_o} B(k)$. Then $e(k_\nu, u_\nu) \in (\Pi^1 B(Ko))^n \times \Pi^1 B(Ko) \times U(\Pi^2 B(Ko))$, $\forall \nu$

Let $((\bar{\bar{x}}^i), \bar{\bar{y}}, (\bar{\bar{z}}^i))$ be a limit point of the sequence. Then $W^i(\bar{\bar{x}}^i, \bar{\bar{z}}^i) = u^i$ by continuity of W^i, $(\bar{\bar{x}}, \bar{\bar{y}}) \in B(k)$ since B has a closed graph and $\bar{\bar{z}} \in U(\bar{\bar{y}})$ since U has a closed graph. Therefore $((\bar{\bar{x}}^i), \bar{\bar{y}}, (\bar{\bar{z}}^i)) = e(k,u)$ $\bar{\bar{x}}^i = x^i$, $\bar{\bar{y}} = y$, $\bar{\bar{z}}^i = z^i$. Therefore $x^i_\nu \to x^i$, $z^i_\nu \to z^i$ $\forall i$ and $y_\nu \to y$. □

We next have the following basic proposition :

PROPOSITION II.1 : *Consider the problem (P)*

$$\max \left\{ \Sigma\, \theta^i W^i(x^i, z^i),\ (\hat{x}, y) \in B(k),\ z \in U(y) \right\}.$$ *Let* $((\bar{x}^i), \bar{y}, (\bar{z}^i))$ *denote the corresponding optimal solution* $(\bar{x}^i = x^i(k,\theta), \bar{z}^i = z^i(k,\theta), \forall i)$. *Then :*

a) $\theta^\ell = 0$ *implies* $\bar{x}^\ell = 0$ *and* $\bar{z}^\ell = 0$

b) *If there exists* i_o *and* j *such that* $\bar{x}^{i_o}_j > 0$ *then for all* i *such that* $\theta^i > 0$, $\bar{x}^i_j > 0$.

c) $\theta^\ell > 0$ *implies* $\bar{x}^\ell > 0$

d) $\theta^i \dfrac{\partial W^i}{\partial x^i_h}(\bar{x}^i, \bar{z}^i) = \theta^j \dfrac{\partial W^j}{\partial x^j_h}(\bar{x}^j, \bar{z}^j)$ *for all* $i, j \in J = \{i, \theta^i > 0\}$, *for all* $h \in H = \{h, x^i_h > 0\}$

e) *The map* f *from* $\mathbb{R}^p_+ \times \Delta^{n-1}$ *into* $\mathbb{R}^{mn}_+ \times \mathbb{R}^p_+ \times \mathbb{R}^n_+$

$(k, \theta) \to ((\bar{x}^i), \bar{y}, (\bar{z}^i))$ *is continuous*

Proof : Let k et θ be fixed. The maximisation problem stated above is equivalent to $\max\limits_{\xi \in U(k)} \sum \theta^i \xi^i$. The solution $\hat{\xi}$ is unique since $U(k)$ is strictly convex and belongs to $\hat{U}(k)$. By lemma II.1, there exist a unique triple $((x^i)\ y, (z^i))$ which solve the initial problem P.

Proof of a : Suppose that $\theta^\ell = 0$ and $\bar{x}^\ell_{i_0} > 0$ for some good i_0. Choose j such that $\theta^j > 0$. Let all consumptions and all z^i be unchanged except the consumption of good i_0 for agents ℓ and j. Let $x'^\ell_{i_0} = \dfrac{\bar{x}^\ell_{i_0}}{2}$ and $x'^j_{i_0} = \bar{x}^j_{i_0} + \dfrac{\bar{x}^\ell_{i_0}}{2}$, then $\sum\limits_i \bar{x}^i = \sum\limits_i x'^i$ and by W_4, $\sum\limits_i \theta^i W^i(x'^i, \bar{z}^i) > \sum\limits_i \theta^i W^i(\bar{x}^i, \bar{z}^i)$. This contredicts the definition of $((\bar{x}_i), \bar{y}, (\bar{z}_i))$.

Suppose that $\bar{z}^\ell \neq 0$. Let $0 < \chi < \bar{z}^\ell$. $U(k)$ being strictly convex and satisfying free disposal $(\bar{z}^1, \bar{z}^2, \ldots \chi, \bar{z}^n) \in \overset{o}{\widehat{U(k)}}$. Therefore there exists $z' \gg (\bar{z}^1\ \bar{z}^2, \chi, \bar{z}^n)$ in $U(k)$. The W^i being strictly monotone in z, $\sum\limits_i \theta^i W^i(\bar{x}^i, z'^i) > \sum\limits_i \theta^i W^i(\bar{x}^i, \bar{z}^i)$, again a contradiction.

Proof of b : Suppose that there exists ℓ such that $\bar{x}^{\ell}_j > 0$ and i with $\theta^i > 0$ and $\bar{x}^i_j = 0$. Let all quantities, production and utilities be unchanged except the consumption of the j^{th} good by agents ℓ and i. Let $\varepsilon > 0$ be such that $\bar{x}^{\ell}_j > \varepsilon$. Let $x'^{\ell}_j = \bar{x}^{\ell}_j - \varepsilon$ and $x'^{i}_j = \varepsilon$.

Let $\Delta W = \theta^i W^i(x'^i,\bar{z}^i) + \theta^{\ell} W^{\ell}(x'^{\ell}, \bar{z}^{\ell}) - \theta^i\ W^i(\bar{x}^i,\bar{z}^i) - \theta^{\ell} W^{\ell}(\bar{x}^{\ell},\bar{z}^{\ell})$.

By W_7 $\forall A, \exists\ \varepsilon' > 0$ $x''^{i}_j < \varepsilon$ implies $W^i(x''^i,\bar{z}^i) - W^i(\bar{x}^i,\bar{z}^i) > A\ \varepsilon'$. By W_6 and W_7 $\exists\ c > 0$ $|x'^{\ell}_j - \bar{x}^{\ell}_j| < \varepsilon' \Rightarrow W^{\ell}\ (x'^{\ell},\bar{z}^{\ell}) - W^{\ell}(x^{\ell}, \bar{z}^{\ell}) > -\varepsilon'\ c$,

then $\Delta W > (\theta^i\ A\ -\ \theta^{\ell} c)\ \varepsilon'$. Choose A such that $A > \frac{\theta^{\ell}}{\theta^i} c$ and ε', so that $x^{\ell}_j - \varepsilon' > 0$, then $\Delta W > 0$.

Proof of c : Suppose that $\theta^{\ell} > 0$ and $\bar{x}^{\ell} = 0$. It then follows from b) that $\bar{x}^j = 0\ \forall j$. Let $(x,y) \in B(k)$ be such that $x_r > 0$ for some good r and $x_q = 0\ \ \forall q \neq r$, (existence of (x,y) follows from B_6 and B_2).

Let $x^{\lambda} = \lambda x + (1-\lambda) \sum_i \bar{x}^i = \lambda x$, $y^{\lambda} = \lambda y + (1-\lambda)\ \bar{y}$ and

$z^{\lambda} = \lambda z + (1-\lambda)\ \bar{z}$. By B_4, $(x^{\lambda},y^{\lambda}) \in B(k)$ and $z^{\lambda} \in U(y^{\lambda})$ by theorem I.3. Let agent ℓ consume $x'^{\ell} = x^{\lambda}$ and let $x'^j = 0\ \forall j \neq \ell$ and let agents pick z^{λ} as future utilities. Then by W_5, $\Delta W = \sum_i \theta^i\ [W^i(x'^i,z^{\ \lambda}) - W^i(\bar{x}^i,\bar{z}^i)] =$

$\sum_{j\neq\ell} \theta^j\ [W^j(0,z^{j\lambda}) - W^j(0,\ \bar{z}^j)] + \theta^{\ell}[W^{\ell}(x^{\lambda},z^{\ell\lambda}) - W^{\ell}(0,\bar{z}^i)] \geq$

$-\ \beta\ \lambda \sum_i \theta^i\ |\bar{z}^i - \tilde{z}^i| + \theta^{\ell}[W^{\ell}(x^{\lambda},\bar{z}^{\ell}) - W^{\ell}(0,\bar{z}^{\ell})]$.

By W_7 for every A, $\exists\,\varepsilon$ such that $|x^{\lambda}| = \lambda|x| < \varepsilon$ implies
$W^{\ell}(x^{\lambda},\bar{z}^{\ell}) - W^{\ell}(0,\bar{z}^{\ell}) > A\,\lambda|x|$.

Thus $\Delta W \geq \lambda[A|x|\theta^{\ell} - \beta \sum_i \theta^i\ |\bar{z}^i - \tilde{z}^i|]$. Choose A so that this last quantity is positive and $\lambda < \frac{\varepsilon}{|x|}$. Then $\Delta W > 0$ a contradiction.

Proof of d : Assume that $\theta^{\ell} > 0$ and $\bar{x}^{\ell}_j = 0\ \ \forall j = 1 \ldots h$, $h < m$. It then follows from b) that $\bar{x}^i_j = 0\ \ \forall i \in I$, $\forall j = 1 \ldots h$.

Let $B'(k) = B(k) \cap \{\{0\} \times \ldots \{0\} \times \mathbb{R}^{m-h}_+\} \times \mathbb{R}^p_+$. Let $\widetilde{B}'(k)$ be its projection on $\mathbb{R}^{m-h}_+ \times \mathbb{R}^p_+$.

Let $P(k) = \{(x,z),\ \exists\, y,(x,y) \in B'(k), z \in U(y)\}$ and $\widetilde{P}(k)$ be its projection on $\mathbb{R}_+^{m-h} \times \mathbb{R}_+^n$. Define $\widetilde{W}^i(\widetilde{x},z^i) = W^i((0,\widetilde{x}),z^i)$ for $\widetilde{x} \in \mathbb{R}_+^{m-h}$, $(0,\widetilde{x}) \in \mathbb{R}_+^m$. Consider the restricted problem

$$\max \{ \sum_i \theta^i W^i(x^i,z^i),\ (\hat{x},y) \in B'(k),\ z \in U(y) \}$$

which is equivalent to

$$\max \{ \sum_i \theta^i \widetilde{W}^i(\widetilde{x}^i,z^i),\ (\hat{\widetilde{x}},y) \in \widetilde{B}(k),\ z \in U(y) \}$$

Clearly $((\bar{\widetilde{x}}^i), \bar{y}, (\bar{z}^i))$ is a solution to that problem. We shall apply Aubin-Ekeland [1984] lemma p.223. Let A be the linear operator from $\mathbb{R}_+^{mn} \times \mathbb{R}_+^n \to \mathbb{R}_+^{m-h} \times \mathbb{R}_+^n$ defined as follows $A((x^i),(z^i)) = (\sum_i \widetilde{x}^i,(z^i))$. The problem we study is

$$\max \{ \sum_i \theta^i \widetilde{W}^i(\widetilde{x}^i,z^i),\ A((x^i),(z^i)) \in \widetilde{P}(k) \}.$$

We need to show that the Slater condition $0 \in \mathrm{Int}[A\mathrm{dom}(\sum_i \theta^i \widetilde{W}^i) - \widetilde{P}(k)]$ is satisfied or $0 \in \mathrm{Int}(\mathbb{R}_+^{m-h} \times \mathbb{R}_+^n - \widetilde{P}(k))$. It thus suffices to show that there exists a strictly positive element in $\widetilde{P}(k)$. Under B_6 and by theorem I.3 there exists an element $(x,y) \in B'(k)$ such that $y > 0$ and $z \in U(y)$, $z >> 0$. Let $x^\lambda = \lambda\bar{\widetilde{x}} + (1-\lambda)x$, y^λ and z^λ. Then $(x^\lambda, y^\lambda) \in \widetilde{B}(k)$ by B_4, $x^\lambda >> 0$ in R_+^{m-h} and $z^\lambda >> 0$. So the slater condition is verified. Let A^* denote the adjoint of A. By Aubin-Ekeland [1984] lemma there exists a $\bar{q} \in \mathbb{R}_+^{m-h} \times \mathbb{R}_+^m$ such that $\partial(\sum_i \theta^i W^i(\bar{x}^i,\bar{z}^i)) = A^*\bar{q}$. We thus have

$$\theta^i \frac{\partial W^i}{\partial x_h^i}(\bar{x}^i, \bar{z}^i) = \theta^j \frac{\partial W^j}{\partial x_h^j}(\bar{x}^j, \bar{z}^j) \quad \forall\, i,j \in J,\ h \in H. \ \square$$

<u>Proof of e</u> : Define the following correspondences from $\mathbb{R}_+^p$ into $\mathbb{R}_+^{mn} \times \mathbb{R}_+^p \times \mathbb{R}_+^n$

$$\Gamma_1(k) = \{((x^i),\ y,(z^i)),\ (\hat{x},y) \in B(k)\}$$

$$\Gamma_2(k) = \{((x^i),\ y,(z^i)),\ z \in U(y)\}$$

Then Γ_1 and Γ_2 are continuous and the optimisation problem is

$$\max \{ \sum_i \theta^i W^i(x^i,z^i),\ ((x^i),\ y,(z^i)) \in \Gamma_1(k) \cap \Gamma_2(k) \}.$$

It follows from the maximum principle that the unique optimal solution is a continuous function of (k,θ). □

Let us next recall the following definition :

Definition : Let $U \subseteq \mathbb{R}^n_+$. A vector $\bar{u} \in U$ is supported by $\theta \in \Delta^{n-1}$ if $\bar{u}$ maximizes $\sum_i \theta^i u^i$ subject to $u \in U$. We have the following result :

THEOREM II.1 :

Let $(k,u) \in$ graph $\hat{U}$. Then there exists a unique θ that supports u. The map φ from graph $\hat{U}$ onto Δ^{n-1}, $(k,u) \xrightarrow{\varphi} \theta$ is continuous.

Proof : Let $u \in \hat{U}(k)$ be given. Then there exists a θ that supports u (see Mas-Colell 1985 p.125). Let $(\bar{x}^i)$, $\bar{y}, (\bar{z}^i)$ be the unique triple associated to u by lemma II.2.

We then have $\sum_i \theta^i W^i(\bar{x}^i, \bar{z}^i) = \sum_i \theta^i u^i = \max \{ \sum_i \theta^i W^i(x^i,y^i), (\hat{x},y) \in B(k)$ $z \in U(y) \}$. Let us show that θ is unique. It follows from proposition II.1.c that $\bar{x}^i = 0$ implies $\theta^i = 0$. Let us then consider $J = \{i, \bar{x}^i \neq 0\}$ and $H = \{h, \bar{x}^i_h \neq 0 \ \forall i \in J\}$. Let h be fixed in H. Then (θ^i) is uniquely determined by the linear system $\theta^i \frac{\partial W^i}{\partial x^i_h}(\bar{x}^i, \bar{z}^i) = \theta^j \frac{\partial W^j}{\partial x^j_h}(\bar{x}^j, \bar{z}^i) \ \forall i, j \in J$

To prove the continuity of φ,

let (k_ν, u_ν) converge to $(\bar{k},\bar{u})$. Then by lemma II.1, $(\bar{k},\bar{u}) \in$ graph $\hat{U}$. There exists a sequence θ_ν that supports u_ν, therefore $\sum_i \theta^i_\nu u^i_\nu \geq \sum_i \theta^i_\nu u^i \ \forall u \in U(k_\nu)$, $\forall \nu$. Pick $u' \in U(\bar{k})$ then since U is continuous, there exists a sequence $u'_\nu \in U(k_\nu)$ such that $u'_\nu \to u'$. Let θ be a limit point of θ_ν. Then one has $\sum_i \theta^i_\nu u^i_\nu \geq \sum_i \theta^i_\nu u'^i_\nu$ for every ν and therefore $\sum_i \theta^i \bar{u}^i \geq \sum_i \theta^i u'^i$. This implies that θ supports $\bar{u}$. As it is unique, θ_ν converge towards θ. □

Remark II.1 : For every fixed k, the map $u \to \theta$ is a continuous bijection, it is therefore a homeomorphism. We thus have another homeomorphism of the set of Pareto-optima and the unit simplex besides the radial projection.

THEOREM II.2 :

All Pareto-optima are described by a trajectory of the dynamical system $\tau(k,\theta)$ on $\mathbb{R}^p_+ \times \Delta^{n-1}$ obtained as follows :

let $(x^i(k,\theta))$, $y(k,\theta)$, $(z^i(k,\theta))$ be the optimal solution of P. Then

$$\tau(k,\theta) = (y(k,\theta), \varphi(y(k,\theta), (z^i(k,\theta)))$$

Proof : let k and $v \in \hat{U}(k)$ be given. By remark II.1, $v \in \Delta^{n-1}$ corresponds to a unique $\theta \in \Delta^{n-1}$. By solving (P), one gets $((x^i(k,\theta)), y(k,\theta), (z^i(k,\theta)))$. Let $w(k,\theta) = \varphi(y(k,\theta), (z^i(k,\theta)))$.

Let the decision maker solve a similar problem with initial data (y,w) . Then the whole trajectory corresponding to the Pareto-optima can thus be reconstructed.

Let us compute $\tau(k,\theta)$ in the case of separable agregators with different discount factor.

Example II.1 : Let $W^i(x^i,z^i) = u^i(x^i) + \beta^i z^i$.

Let k and θ be given. We need to solve

$$\max \{ \sum_i \theta^i u^i(x^i) + \sum_i \beta^i \theta^i z^i , \quad (\hat{x},y) \in B(k), \ z \in U(y) \} .$$

Let $\theta'^i = \dfrac{\theta^i \beta^i}{\sum_j \theta^j \beta^j}$. Then it is equivalent to :

$$\max \{ \sum_i \theta^i u^i(x^i) + (\sum_i \beta^i \theta^i)(\sum_i \theta'^i z^i), (\hat{x},y) \in B(k), \ z \in U(y) \}$$

The problem being separable, one chooses first z , y given so as to $\max \{ \sum_i \theta'^i z^i , \ z \in U(y) \}$. Let $v(y,\theta')$ denote the maximum value. Then one solve :

$$\max \{ \sum_i \theta^i u^i(x^i) + (\sum_i \beta^i \theta^i) \ v(y,\theta'), \quad (\hat{x},y) \in B(k) \} .$$

Thus θ'^i supports z^i , $\forall y$ which implies $\varphi(y,z) = \theta'$, $\forall y$. Therefore $\tau(k,\theta) = (y(k,\theta),\theta')$. The dynamics of θ is independent of that of y . It can easily be shown that the map $\theta \to \theta'$ has all vertex of the simplex as fixed points and an extra point $\theta^i = \frac{1}{q} \ \forall i$ such that $\beta^i = \max_j \beta^j$ and q equals the number of such i and $\theta^j = 0$ otherwise. The vertex are unstable fixed point and the last fixed point is globally attractive. Consider now an agent i with $\beta^i < \max \beta^j$, let $B(k)$ belong to a fixed compact set K_o , then $x^i(k_t,\theta_t) \to 0$ (since $x^i(k,\theta)$ is continuous and $x^i(k,\theta) = 0 \ \forall k$ if $\theta^i = 0$) , where $(k_t,\theta_t) = \tau^t(k,\theta)$.

One can therefore easily rediscover Ramsey's result but one cannot say anything about the path of capital stocks.

An alternative way of describing a trajectory.

As it has been shown by Benhabib and ali [1986] one can parametrise the trajectory in a different way. Let Π^{-1} be the map from $\mathbb{R}^n \to \mathbb{R}^{n-1}$ defined by $\Pi^{-1}(x_1 \ldots x_n) = (x_2 \ldots x_n)$.

Let $(k, z_0^2, \ldots, z_0^n)$ belong to the graph of $\Pi^{-1}U(k)$.

Consider the following problem

$$\max W^1(x^1, z_1^1),\ W^i(x^i, z_1^i) \geq z_0^i\ ,\ i \geq 2,\ (\sum_i x^i, y) \in B(k), (z_1^i)_{i \geq 1} \in U(y)\ .$$

It can be easily shown that under W_2 biss and B_4 biss there is a unique solution $x^i(k, (z_0^i)_{i \geq 2})$, $y(k, (z_0^i)_{i \geq 2})$, $z_1^i(k, (z_0^i)_{i \geq 2})$ and the map from Graph $\Pi^{-1}U$ into $(\mathbb{R}_+^{mn}) \times \mathbb{R}_+^p \times \mathbb{R}^n$, $(k, (z_o^i)_{i \geq 2}) \to (x^i_{i \geq 1}(k,z), y(k,z), z^i_{i \geq 1}(k,z))$ is continuous.

The trajectory can therefore be described step by step. This definition does not require any differentiability assumption on the W^i .

In some cases, Lucas and Stokey's parametrisation may be more useful (as in the case of separable utilities) in other cases Benhabib and ali's one may be more interesting.

III - VALUE FUNCTIONS

In this section, we introduce two generalized versions of the value function of optimal growth, one is due to Lucas and Stokey [1984], the other to Benhabib et ali [1986]. We study their properties, continuity, concavity and differentiability.

Lucas and Stokey's value function

Let $V(k,\theta) = \sup\{\sum_i \theta^i z^i, z \in U(k)\} = \sup\{\sum_i \theta^i W^i(x^i,z^i), (\hat{x},y) \in B(k), z \in U(y)\}$

PROPOSITION III.1 :

a) The map from $\mathbb{R}^p_+ \times \Delta^{n-1} \to \mathbb{R}$, $(k,\theta) \to V(k,\theta)$ *is continuous*

b) Under B_4 *and* W_2, *for every* θ , *the map from* $\mathbb{R}^p_+ \to \mathbb{R}^+$ $k \to V(k,\theta)$ *is strictly concave in* k .

Under W_2, W_6, W_7 *for every fixed* k , *the map from* Δ^{n-1} *into* $\mathbb{R}^+$, $\theta \to V(k,\theta)$ *is strictly convex.*

Proof : The proofs of a) and first statement of b) are omitted. Let us prove the strict convexity in θ . Let $(x^i(k,\theta)),(z^i(k,\theta))$ denote the optimal solution conrresponding to θ . Let θ and θ' be given, $\theta^\lambda = \lambda\theta + (1-\lambda)\theta'$. It follows from lemma II.1 and proposition II.1 that $\theta \neq \theta'$ implies $(x^i(k,\theta)) \neq (x^i(k,\theta')) \neq (x^i(k,\theta^\lambda))$. Therefore $V(k,\theta^\lambda) = \sum_i \theta^{i\lambda} W^i(x^{i\lambda},z^{i\lambda}) = \lambda \sum_i \theta^i W^i(x^{i\lambda},z^{i\lambda}) + (1-\lambda) \sum_i \theta'^i W^i(x^{i\lambda},z^{i\lambda})$ $< \lambda V(k,\theta) + (1-\lambda) V(k,\theta')$ □

In order to study the differentiability properties of $v(k,\theta)$ we introduce the following auxiliary function. Let

$D = \{(k,y,\theta,z)\ k \in \mathbb{R}^p_+ ,\ y \in \Pi^2 B(k),\ \theta \in \Delta^{n-1},\ z \in U(y)\}$. D is a convex set of $\mathbb{R}^p_+ \times \mathbb{R}^p_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$. Let $G : D \to \mathbb{R}_+$ be defined as follows :

$$G(k,y,\theta,z) = \max_x \{\sum_i \theta^i W^i(x^i,z^i),\ (\hat{x},y) \in B(k)\}$$

Then $V(k,\theta) = \max_{y,z} \{G(k,y,\theta,z),\ y \in \Pi^2B(k),\ z \in \hat{U}(y)\}$

We first need the following lemma.

LEMMA III.1 :

Let Φ be a continuous, compact convex-valued correspondence from $\mathbb{R}^m$ into itself. Let $x_0 \in \overset{\circ}{\widehat{\Phi(k_0)}}$. Then there exists a neighbourhood $V(k_0)$ such that $x_0 \in \Phi(k)$ for every k in $V(k_0)$.

The proof is given in appendix one. □

Let D^2 be the projection of D on $\mathbb{R}^p_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

LEMMA III.2 :

a) Let (y,θ,z) be fixed in D^2 , then the map

$k \to G(k,y,\Theta,z)$ is concave in k .

b) If y is interior to $\Pi^2B(k)$. If G is differentiable with respect to k then V is differentiable with respect to k and

$\mathrm{grad}_k V(k,\theta) = \mathrm{grad}_k\, G(k,y(k,\theta),\theta,z(k,\theta))$

Proof : The proof of a) is omitted. Let (k_0,y_0,θ_0,z_0) be fixed with $y_0 = y(k_0,\theta_0)$, $z_0 = z(k_0,\theta_0)$. The correspondence $k \to \Pi^2B(k)$ is compact convex valued. Let $y \in \overset{\circ}{\widehat{\Pi^2B(k)}}$. By lemma III.I, there exists a neighbourhood $V(k_0)$ such that $y_0 \in \Pi^2B(k)$, $\forall k \in V(k_0)$. $\forall k \in V(k_0)$, $G(k,y_0,\theta_0,z_0) \leq$ $\leq V(k,\theta_0)$ and $G(k_0,y_0,\theta_0,z_0) = V(k_0,\theta_0)$. By Benveniste and Sheinkman's [1979] theorem I, if G is differentiable with respect to k so is V and $\mathrm{grad}_k\ V(k,\theta) = \mathrm{grad}_k\ G(k,y(k,\theta),\theta\ ,\ z(k,\theta))$ □

In order to prove the differentiability of G , we shall restrict the technology. Let $F : \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^p_+ \to \mathbb{R}^p$ $(k,x,y) \to F(k,x,y)$ be a strictly convex, C^2 function strictly increasing in x , that satisfies moreover properties of example I.2. Let us assume

B7 : $B(k) = \{(x,y),\ F(k,x,y) \leq 0\}$

PROPOSITION III.2 :

Assume $B_1 \to B_7$ and W^i of class C^2 for every i . Let $\theta \in \overset{o}{\widehat{\Delta^{n-1}}}$ and $k \in \mathbb{R}^p_{++}$.
Then G is C^1 with respect to k in $\mathbb{R}^p_{++}$ and

$$grad_k\, G(k,y,\theta,z) = -\lambda\, grad_k\, F(k,\hat{x}(k,\theta),\ y(k,\theta))$$

$$\text{with } \lambda = \frac{\theta^i\ W^i_j(x^i(k,\theta),z^i(k,\theta))}{F_{2j}(k,\hat{x}(k,\theta),y(k,\theta))}$$

Proof : Let z be fixed in $\hat{U}(y)$. Then as F is increasing in x

$$G(k,y,\theta,z) = \max_{x^i}\left\{ \sum_i \theta^i\ W^i(x^i,z^i),\ F(k,\hat{x},y) = 0 \right\}$$

Let us first assume that $x^i \gg 0$ for some i. As $\theta \in \overset{o}{\widehat{\Delta^{n-1}}}$, by proposition II.1, this implies $x^j \gg 0$ for all j .
The first order condition implies since F is stricly increasing in x .

$$\theta^i\ grad_x\ W^i(x^i,z^i) = \lambda\ grad_x\ F(k,\hat{x},y) \quad \text{for some} \quad \lambda > 0$$

Differentiating we get :

$$\begin{cases} \theta^i \dfrac{\partial^2 W^i}{\partial x^{i2}}(x^i,z^i)dx^i - \lambda \dfrac{\partial^2 F}{\partial x^2}(k,\hat{x},y)(\sum\limits_j dx^j) - grad_x\ F(k,\hat{x},y)d\lambda = \lambda \dfrac{\partial^2 F}{\partial x \partial k}(k,\hat{x},y)dk \\ grad_k\ F(k,\hat{x},y)dk + grad_x\ F(k,\hat{x},y)(\sum\limits_j dx^j) = 0 \end{cases}$$

This can be written in matrix form as follows : let $\dfrac{\partial^2 W^i}{\partial x^2}$ and $\dfrac{\partial^2 F}{\partial x^2}$ denote the hessians of W^i and F with respect to x.

$$\begin{bmatrix} \theta^1 \dfrac{\partial^2 W^1}{\partial x^2} - \lambda \dfrac{\partial^2 F}{\partial x^2}, & -\lambda \dfrac{\partial^2 F}{\partial x^2}, & \ldots\ldots & -\lambda \dfrac{\partial^2 F}{\partial x^2}, & -grad_x\ F \\ -\lambda \dfrac{\partial^2 F}{\partial x^2} & \theta^n \dfrac{\partial^2 W^2}{\partial x^2} - \lambda \dfrac{\partial^2 F}{\partial x^2} & & & -grad_x\ F \\ & & & & \\ -\lambda \dfrac{\partial^2 F}{\partial x^2} & & & \theta^n \dfrac{\partial^2 W^n}{\partial x^2} - \lambda \dfrac{\partial^2 F}{\partial x^2} & -grad_x\ F \\ {}^t grad_x\ F & {}^t grad_x\ F & & {}^t grad_x\ F & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ \\ dx^n \\ d\lambda \end{bmatrix} = A\ dk$$

where $A \in L(\mathbb{R}^p, \mathbb{R}^{nm+1})$.

As the matrices $\theta^i \dfrac{\partial^2 W^i}{\partial x^2}$ and $-\dfrac{\partial^2 F}{\partial x^2}$ are definite negative, the upper matrix which belongs to $L(\mathbb{R}^{nm}, \mathbb{R}^{nm})$ is definite negative. It then follows from Pallu de la Barrière [1966] Corollary p.295 that the matrix above has a determinant different from zero. The implicit function theorem can therefore be applied and

$$\text{Grad}_k\, G(k,y,\theta,y)\,dk = \sum_i \theta^i\, \text{grad}_x\, W^i(x^i(k,\theta),\, z^i(k,\theta))\,dx^i =$$

$$\lambda\, \text{grad}_x\, F(k,\hat{x}(k,\theta),\, y(k,\theta))(\sum_i dx^i) = -\lambda\, \text{grad}_k\, F(k,\hat{x}(y,\theta),y(k,\theta))\,dk$$

If $x^i \geq 0$ for some i, let $x^i_j = 0 \quad \forall j = 1 \dots h,\ k < m$.

From proposition II.1, $x^\ell_j = 0$ for every ℓ since $\theta^\ell > 0$.

One can consider a restricted optimisation problem with variables of the form $x^i = (0,\tilde{x}^i),\ \tilde{x}^i \in \mathbb{R}^{m-h}_+$.

Then $G(k,y,\theta,z) = \max_{\tilde{x}^i}\left\{\sum_i \theta^i\, W^i(0,\tilde{x}^i,z^i),\ F(k,o,\hat{\tilde{x}},y) = 0\right\}$.

The proof given above carries over to that case.

Let us summarize our results in the following theorem.

THEOREM III.1 :

Assume $B_1 - B_7$ and W^i of class C^2 for every i. Then V is C_1 in $\overset{o}{\mathbb{R}}{}^p_+ \times \overset{o}{\mathbb{R}}{}^n_+$.

$$grad_k\, V(k,\theta) = -\lambda\, grad_k\, F(k,\hat{x}(k,\theta),y(k,\theta)) \text{ with } \lambda = \frac{\theta^i W^i_j(x^i(\theta,k)z^i(k,\theta))}{F_{2j}(k,\hat{x}(k,\theta),y(k,\theta))}$$

$$grad_\theta\, V(k,\theta) = W^i(x^i(k,\theta),z^i(k,\theta))$$

We now give Benhabib et ali's value function

Benhabib et ali's value function

Let $\bar{V}\left(k,(z^i_o)_{i\geq 2}\right) = \max\left\{W^1(x^1,z^1),\ W^i(x^i,z^i) \geq z^i_0\ \forall i \geq 2,\ (\hat{x},y) \in B(k) \text{ and } (z^i)_{i\geq 1} \in U(y)\right\}$

PROPOSITION III.3 : *a) The map from graph* $\Pi^{-1} U \to \mathbb{R}^+, (k, (z_o^i)_{i\geq 2}) \to \bar{V}(k, (z_o^i)_{i\geq 2})$ *is continuous and strictly concave*

b) $\bar{V}$ *satisfies the generalized Bellman's equation* .

$$\bar{V}\left(k, (z_o^i)_{i\geq 2}\right) = \max \left\{W^1(x^1, \bar{V}\left(y, (z_1^i)_{i\geq 2}\right)), (\hat{x}, y) \in B(k),\ W^i(x^i, z_1^i) \geq z_o^i \ \forall i \geq 2 \right\}$$

The proof of a) is straightforward. Strict concavity of $\bar{V}$ follows from B_4 biss, W_2 biss and theorem I.3.

To prove b), let $\tilde{T}$ be the operator defined on the set of continuous functions from graph of $\Pi^{-1} U$ into $\mathbb{R}^+$ endowed with the sup-norm :

$$\tilde{T} f(k, (z_o^i)_{i\geq 2}) = \max \left\{W^1(x^1, f(y, z_1^i)_{i\geq 2}), (\hat{x}, y) \in B(k), W^i(x^i, z_1^i) \geq z_o^i, \forall i \geq 2\right\}$$

One easily shows as in dynamic programming that T is a contraction that maps concave functions onto concave functions. It thus have a fixed point which is concave.

We next prove a theorem which is similar to theorem III.1 .

THEOREM III.2 :

Assume $B_1 - B_7$ *and* W_o^i *of class* C^2 *for every* i

Then $\bar{V}$ *is* C_1 *in* $\widehat{\text{graph } \Pi^{-1} U}$

Moreover we have $\text{grad}_k \bar{V}(k, (z_o^i)_{i>2}) = - \lambda^1 \text{grad}_k F(k, \hat{x}(k, z_o^i)_{i\geq 2}, y(k, z_o^i)_{i\geq 2})$

$$\text{with } \lambda^1 = \frac{W_j^1(x^1(k, (z_o^i)_{i\geq 2})}{F_{2j}(k, \hat{x}(k, (z_o^i)_{i\geq 2}), y(k, (z_o^i)_{i\geq 2}))}$$

$$\text{and } \frac{\partial \bar{V}}{\partial z^i}(k, (z_o^i)_{i\geq 2}) = - \frac{W_j^1(x^1(k, (z_o^i)), z^1(k, (z_o^i)))}{W_j^i(x^i(k, (z_o^i)), z^i(k, (z_o^i)))} \ \forall j$$

Proof : The proof of differentiability of $\bar{V}$ is analogous to that of V . One introduces the auxiliary function defined on the convex subset of $\mathbb{R}_+^p \times \mathbb{R}_+^{n-1} \times \mathbb{R}_+^p \times \mathbb{R}_+^n$.

$$\tilde{D} = \left\{(k, (z_o^i)_{i\geq 2}, y, (z_1^i)_{i\geq 1}),\ k \in \mathbb{R}_+^p,\ (z_o^i)_{i\geq 2} \in \Pi^{-1} U(k), y \in \Pi^2 B(k), (z_1^i)_{i\geq 1} \in U(y)\right\}$$

$$\tilde{V}(k, (z_o^i)_{i\geq 2}, y, (z_1^i)_{i>1}) = \max \left\{W^1(x^1, z_1^1), W^i(x^i, z_1^i) \geq z_o^i, \forall i \geq 2 \quad (\hat{x}, y) \in B(k)\right\}$$

As in lemma III.2, $\tilde{V}$ differentiable implies $\bar{V}$ differentiable and

$$\mathrm{grad}_{(k,z)}\ \bar{V}(k,(z_o^i)_{i\geq 2}) = \mathrm{grad}_{(k,z)}\ \tilde{V}(k,(z_o^i)_{i\geq 2},\ y(k,(z_o^i)_{i\geq 2}),z_1^i,(k,(z_o^i)_{i\geq 2})).$$

In order to prove the differentiability of $\tilde{V}$ let us assume B_7.

As in proposition III.2 the first order conditions lead to the following system. (One first assume $x' \gg 0$ and then one can consider a restricted problem).

$$\begin{cases} \mathrm{grad}_x\ W^1(x^1,z^1) = \lambda^1\ \mathrm{grad}_x\ F(k,\hat{x},y) \\ \lambda^i\ \mathrm{grad}_x\ W^i(x^i,z^i) = \lambda^1\ \mathrm{grad}_x\ F(k,\hat{x},y)\ . \end{cases}$$

Let us add the relations : $W^i(x^i,z_1^i) = z_o^i \quad \forall\, i \geq 2$ and $F(k,\hat{x},y) = 0$.

In order to solve for the x^i and λ^i we have to show that the following matrix E is invertible.

$$
\begin{array}{c}
nm\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \\
1\left\{\begin{array}{c} \\ \end{array}\right. \\
n-1\left\{\begin{array}{c} \\ \\ \\ \end{array}\right.
\end{array}
\begin{array}{cccc|ccc}
\frac{\partial^2 W^1}{\partial x^2} - \lambda^1 \frac{\partial^2 F}{\partial x^2} & -\lambda^1 \frac{\partial^2 F}{\partial x^2} & \cdots & -\lambda^1 \frac{\partial^2 F}{\partial x^2} & -\mathrm{grad}_x F & 0 & 0 \\
-\lambda^1 \frac{\partial^2 F}{\partial x^2} & \lambda^2 \frac{\partial^2 W^2}{\partial x^2} - \lambda^1 \frac{\partial^2 F}{\partial x^2} & & -\lambda^1 \frac{\partial^2 F}{\partial x^2} & -\mathrm{grad}_x F & \mathrm{grad}_x W^2 & \\
\vdots & & & & & & \ddots \\
-\lambda^1 \frac{\partial^2 F}{\partial x^2} & & & \lambda^n \frac{\partial^2 W^n}{\partial x^2} - \lambda^1 \frac{\partial^2 F}{\partial x^2} & -\mathrm{grad}_x F & & \mathrm{grad}_x W^n \\
\hline
{}^t\mathrm{grad}_x F & & & {}^t\mathrm{grad}_x F & 0 & 0 & 0 \\
\hline
0 & {}^t\mathrm{grad}_x W^2 & & 0 & & & \\
0 & & & & & 0 & \\
 & 0 & & {}^t\mathrm{grad}_x W^n & & & \\
\hline
\multicolumn{4}{c|}{nm} & \multicolumn{3}{c}{n}
\end{array}
$$

Clearly $E = \begin{bmatrix} A & B \\ {}^tB & 0 \end{bmatrix}$ where $A \in L(\mathbb{R}^{nm}, \mathbb{R}^{nm})$ and is negative definite and $B \in L(\mathbb{R}^{n}, \mathbb{R}^{nm})$ and has rank n

Let us show that $\operatorname{Ker} E = \{0\}$. Assume $E\begin{pmatrix} X \\ Y \end{pmatrix} = 0$ where $X \in \mathbb{R}^{nm}$ and $Y \in \mathbb{R}^n$. Then $AX + BY = 0$ and ${}^tBX = 0$. This implies ${}^tY\,{}^tB\,A^{-1}BY = 0$ and $X = -A^{-1}BY$. Thus $BY = 0$ and as B is of rank n, $Y = 0$ and $X = 0$

Therefore $dV = \operatorname{grad}_x W^1(x^1,z^1)dx^1 = \lambda^1 \operatorname{grad}_x F(k,\hat{x},y)\, dx^1 =$
$-\lambda^1(\operatorname{grad}_k F\, dk + \operatorname{grad}_x F \sum_{i\geq 2} dx^i)$. As

$\frac{\lambda^i}{\lambda^i} \operatorname{grad}_x F\, dx^i = \operatorname{grad}_x W^i(x^i,z^i)dx^i = dz_o^i \quad \forall\, i \geq 2$, we get

$dV = -\lambda^1 \operatorname{grad}_k F\, dk - \sum_{i\geq 2} \lambda^i\, dz_o^i$. □

<u>*Remark III.1*</u> : It can easily be shown that $\lambda^i = \frac{\theta^i}{\theta^1}$, $\forall\, i \geq 2$, so that $\theta^1 = \frac{1}{\sum_{i\geq 2} \lambda^i + 1}$

and $\theta^i = \frac{\lambda^i}{\sum_{j\geq 2} \lambda^j + 1}$, $i \geq 2$.

IV - <u>EULER'S EQUATIONS - STATIONARY POINTS</u>

— <u>*Euler's equation*</u> :

Let us now reconsider the problem $\max \left\{ \sum_i \theta^i W^i(x^i,z^i),\ F(k,\sum_i x^i, y) \leq 0\quad z \in \hat{U}(y) \right\}$

Let $w = \varphi(y,z)$. Let us recall that $z \in \hat{U}(y)$ iff $V(y,w) = wz$.

Wherever interior, the optimal solution satisfies the following equations obtained by differentiating with respect to x, y, z, w and using theorem III.I.

(1) $\theta_t^i \operatorname{grad}_x W^i(x_t^i, z_{t+1}^i) = \lambda_t \operatorname{grad}_x F(k_t, \hat{x}_t, k_{t+1})$

(2) $\theta_t^i \frac{\partial W^i}{\partial z}(x_t^i, z_{t+1}^i) = \mu_t\, \theta_{t+1}^i$

(3) $F(k_t, \hat{x}_t, k_{t+1}) = 0$

(4) $\sum_i \theta_t^i = 1$

(5) $\lambda_t \operatorname{grad}_y F(k_t,\hat{x}_t,k_{t+1}) + \mu_t\, \lambda_{t+1} \operatorname{grad}_k F(k_{t+1},\hat{x}_{t+1},k_{t+2}) = 0$

(6) $z^i_{t+1} = W^i(x^i_{t+1}, z^i_{t+2})$

Let us assume that k_t and θ^i_t are given. The unknowns are $x^i_t, k_{t+1}, z^i_{t+1}, \theta^i_{t+1}, \lambda_t, \mu_t, \lambda_{t+1}, k_{t+2}, x^i_{t+1}, z^i_{t+2}$.

There are $2mn + 2p + 3n + 3$ unknowns but only $mn + 2n + p$ equations.

Let us add the following equations

(7) $\theta^i_{t+1} \text{ grad}_w W^i(x^i_{t+1}, z^i_{t+2}) = \lambda_{t+1} \text{ grad}_x F(k_{t+1}, \hat{x}_{t+1}, k_{t+2})$

(8) $\theta^i_{t+1} \frac{\partial W^i}{\partial z}(x^i_{t+1}, z^i_{t+2}) = \mu_{t+1}\, \theta^i_{t+2}$

(9) $F(k_{t+1}, \hat{x}_{t+1}, k_{t+2}) = 0$

(10) $\sum_i \theta^i_{t+1} = 1$

Given $\theta^i_t, k_t, \theta^i_{t+1}, k_{t+1}$, we thus have $2mn + 3n + p + 4$ unknowns (add μ_{t+1} to the previous) with the adequate number of equations.

This body of equations constitute Euler's generalized equations E_t and in principle unables us to write k_{t+2}, θ^i_{t+2} as functions of $\theta^i_t, k_t, k_{t+1}, \theta^i_{t+1}$.

One can also differentiate with respect to x^i, y and z^i Bellman's generalized equation and use theorem III.2. For further use let us use a direct approach. The solution maximizes $W^1(x^1_o, W^1(x^1_1, \ldots, W^1(x^1_2, \ldots, W^1(x^1_t, \ldots,))))$ under the constraints $F(k_t, \hat{x}_t, k_{t+1}) = 0$ and $z^i_t = W^i(x^i_t, z^i_{t+1})\ \forall i \geq 2,\ \forall t \geq 0$.

Let L denote the Lagrangian of this problem :

$$L(\underset{\sim}{x}, \underset{\sim}{k}, \underset{\sim}{z}) = W^1(x^1_o, z^1_1) - \sum_o^\infty \lambda_t F(k_t, \hat{x}_t, k_{t+1}) + \sum_{t=o}^\infty \sum_{i=2}^n \mu^i_t \left[W^i(x^i_t, z^i_{t+1}) - z^i_t\right].$$

Whenever the optimal solution is interior, it satisfies the following system $(\tilde{E}_t)$

$$\begin{cases} \left(\prod_{h=o}^{t-1} \frac{\partial W^1}{\partial z}(x^1_h, z^1_{h+1})\right) \text{grad}_x W^1(x^1_t, z^1_{t+1}) = \lambda_t \text{ grad}_x F(k_t, \hat{x}_t, k_{t+1}) & (11) \\ \mu^i_t \text{ grad}_x W^i(x^i_t, z^i_{t+1}) = \lambda_t \text{ grad}_x F(k_t, \hat{x}_t, k_{t+1})\ \forall i \geq 2 & (11)\text{ biss} \end{cases}$$

(12) $$\mu^i_t \frac{\partial W^i}{\partial z}(x^i_t, z^i_{t+1}) = \mu^i_{t+1} \qquad \forall i \geq 2$$

(13) $F(k_t, \hat{x}_t, k_{t+1}) = 0$

(14) $\lambda_t \operatorname{grad}_k F(k_t, \hat{x}_t, k_{t+1}) + \lambda_{t-1} \operatorname{grad}_y F(k_{t-1}, \hat{x}_{t-1}, k_t) = 0$

(15) $z_t^i = W^i(x_t^i, z_{t+1}^i) \qquad \forall i \geq 2$

As in the previous case we express $\left(\underset{i\geq 2}{z_{t+2}^i}, k_{t+2}\right)$ as functions of $\left(k_t, k_{t+1}, \underset{i\geq 2}{z_t^i}, \underset{i\geq 2}{z_{t+1}^i}\right)$.

Remark IV.1 : Assume that $\underset{\sim}{k}$, $\underset{\sim}{x}$, $\underset{\sim}{z}$ are bounded. Let

$b = \inf\limits_t \min\limits_j \frac{\partial F}{\partial x_j}(k_t, x_t, k_{t+1})$. Then $b > 0$ and $b\lambda_t \leq \beta^{t-1} \sup_t \|\operatorname{grad}_x W^1(x_t^1, z_{t+1}^1)\|$.

Therefore $\lambda_t \to 0$ and consequently $\mu_t^i \to 0 \quad \forall i \geq 2$.

On the uniqueness of the steady state :

Besides the assumptions of theorem III.1, let us assume that $B(\mathbb{R}_+^p)$ is a convex compact subset of $\mathbb{R}_+^m \times \mathbb{R}_+^p$. Then a fixed point argument ensures that τ which is a continuous map defined on a compact set, has a fixed point (or steady state). Let us assume moreover that it is an interior point. Then Euler's equations lead to the following system of equations :

(16) $$\theta^i \frac{\partial W^i}{\partial x_j^i}(x^i, z^i) - \lambda \frac{\partial F}{\partial x_j}(y, \hat{x}, y) = 0, \quad i = 1,\dots,n, \quad j = 1,\dots,m$$

(17) $$\frac{\partial W^i}{\partial z}(x^i, z^i) = \mu$$

(18) $$\frac{\partial F}{\partial y_j}(y, \hat{x}, y) + \mu \frac{\partial F}{\partial k_j}(y, \hat{x}, y) = 0, \quad j = 1,\dots,p$$

(19) $$F(y, \hat{x}, y) = 0$$

(20) $$\sum_i \theta_i = 1$$

(21) $$z^i = W^i(x^i, z^i), \quad i = 1,\dots,n$$

Let us assume for the remainder of this sub-section that :

B_7 biss; $F(k,x,y) = \Phi_1(x) - \Phi_2(k,y)$ where Φ_2 is strictly concave in (k,y) and increasing in k , decreasing in y and twice continuously differentiable and Φ_1 is convex, increasing in x and C^1 .

In order to prove uniqueness of the steady state, following Lucas and Stokey [1984] we shall show that given μ equations (16), (17), (20), (21) uniquely determine (x,z,θ,λ) for μ near one. When the production function is separable, given μ, equation (18) uniquely determines y for μ near one. Finally we shall use (19) in order to show that μ is uniquely determined.

We prove first that given μ near one, (16), (17), (20) and (21) have a unique solution (x,z,θ,λ). It is sufficient to prove that the determinant of the Jacobian matrix of this system has a constant sign. This matrix is a follows :

$$
\begin{array}{cccc}
\overbrace{\hphantom{xxxxxxxxxxxxxxxx}}^{nm \text{ terms}} & \overbrace{\hphantom{xxxxxxxxxxxxxx}}^{n \text{ terms}} & \overbrace{\hphantom{xxxxxxxxxxxxxx}}^{n \text{ terms}} & \\
\end{array}
$$

$$
\left[\begin{array}{c|c|c|c}
\left[\theta^i \dfrac{\partial^2 W^i}{\partial x^{i2}} - \dfrac{\partial^2 \Phi_1}{\partial x^2}\right] &
\begin{matrix} \theta^1 \dfrac{\partial^2 W^1}{\partial x^1 \partial z^1} & 0 \ldots & 0 \\ & \ddots & \\ 0 & & \theta^n \dfrac{\partial^2 W^n}{\partial x^n \partial z^n} \end{matrix} &
\begin{matrix} \dfrac{\lambda}{\theta^1} \operatorname{grad} \Phi_1 & 0 \;.\; 0 \\ 0 & \\ \vdots & \\ 0 & \dfrac{\lambda}{\theta^n} \operatorname{grad} \Phi_1 \end{matrix} &
\begin{matrix} -\operatorname{grad}\Phi_1 \\ \\ \\ -\operatorname{grad}\Phi_1 \end{matrix} \\
\hline
\begin{matrix} \dfrac{\partial^2 W^1}{\partial x^1 \partial z^1} & 0 \ldots 0 \\ 0 & \dfrac{\partial^2 W^n}{\partial x^n \partial z^n} \end{matrix} &
\begin{matrix} \dfrac{\partial^2 W^1}{\partial z^2} & \\ 0 & \dfrac{\partial^2 W^n}{\partial z^2} \end{matrix} &
0 & 0 \\
\hline
\begin{matrix} -\dfrac{\lambda}{\theta^1} \operatorname{grad} \Phi_1 & 0 \ldots\ldots \\ 0 & -\dfrac{\lambda}{\theta^n} \operatorname{grad} \Phi_1 \end{matrix} &
\begin{matrix} 1-\mu & & 0 \\ & \ddots & \\ 0 & & 1-\mu \end{matrix} &
0 & 0 \\
\hline
0 & 0 & 1 \ldots\ldots\ldots\ldots 1 & 0
\end{array}\right]
$$

Multiplying every $(nm + n + j)$ column, for $j = 1,\dots,n$, by $\frac{\theta^j}{\lambda}$, and adding it to the last column, we get a determinant which last column and last row are zero except the $(nm+2n+1)^{th}$ element of the $(nm+2n+1)^{th}$ column which is equal to $\frac{1}{\lambda}$. Hence the determinant of this matrix is $\frac{D}{\lambda}$ where D is the determinant of the $(nm \times 2n) \times (nm + 2n)$ principal matrix obtained by deleting the last column and last line. For $\mu = 1$, an argument similar to that of proposition III.2 shows that the principal matrix has a determinant different from zero. As this determinant is a continuous function of $(x^i)_{i\in I}$, $(z^i)_{i\in I}$, $(\theta^i)_{i\in I}$ and λ, it keeps a constant sign on $\mathring{\mathbb{R}}^{mn}_+ \times \mathring{\mathbb{R}}^n_+ \times \mathring{\Delta}^n \times \mathring{\mathbb{R}}_+$. By a standard argument based on degree theory, $((x^i), (z^i), (\theta^i))$ and λ are uniquely determined for $\mu = 1$. By the implicit function theorem, they are also uniquely determined in a neighbourhood of $\mu = 1$.

From B_7 biss (18) simplifies into :

(18) biss $\quad \frac{\partial\Phi_2}{\partial y_j}(y,y) + \mu \frac{\partial\Phi_2}{\partial k_j}(y,y) = 0$.

Let $J(\mu)_{j\ell} = \left[\frac{\partial^2\Phi_2}{\partial y_j \partial y_\ell} + \frac{\partial^2\Phi_2}{\partial y_j \partial k_\ell} + \mu \left(\frac{\partial^2\Phi_2}{\partial k_j \partial y_\ell} + \frac{\partial^2\Phi_2}{\theta k_j \partial k_\ell} \right)\right]$

be the jacobian matrix.

We shall prove that if Φ_2 is strictly concave in (y,k) then $J(\mu)$ is definite negative for μ near one. Indeed, in that case, the matrix :

$$\Phi_2''(y,y) = \begin{pmatrix} \frac{\partial^2\Phi_2}{\partial y_1^2} \cdots & \frac{\partial^2\Phi_2}{\partial y_1 \partial y_p} & \frac{\partial^2\Phi_2}{\partial y_1 \partial k_1} & \cdots \frac{\partial^2\Phi_2}{\partial y_1 \partial k_p} \\ \frac{\partial^2\Phi_2}{\partial y_p \partial y_1} \cdots & \frac{\partial^2\Phi_2}{\partial y_p^2} & \frac{\partial^2\Phi_2}{\partial y_p \partial k_1} & \cdots \frac{\partial^2\Phi_2}{\partial y_p \Phi k_p} \\ \frac{\partial^2\Phi_2}{\partial k_1 \partial y_1} \cdots & \frac{\partial^2\Phi_2}{\partial k_1 \partial y_p} & \frac{\partial^2\Phi_2}{\partial k_1^2} & \cdots \frac{\partial^2\Phi_2}{\partial k_1 \partial k_p} \\ \frac{\partial^2\Phi_2}{\partial k_p \partial y_1} \cdots & \frac{\partial^2\Phi_2}{\partial k_p \partial y_p} & \frac{\partial^2\Phi_2}{\partial k_p \partial k_1} & \cdots \frac{\partial^2\Phi_2}{\partial k_p^2} \end{pmatrix}$$

is negative definite and the expression ${}^t\tilde{h}\ \Phi_2''\ \tilde{h}$, with ${}^t\tilde{h} = (h_1,\ldots,h_p,h_1,\ldots,h_p)$, is strictly negative if $\tilde{h} \neq 0$. Tedious computations give :

$$0 > {}^t\tilde{h}'\ F_y''\ \tilde{h} = -\sum_{j,\ell} h_j\ h_\ell \left(\frac{\partial^2 F}{\partial y_j \partial y_\ell} + \frac{\partial^2 F}{\partial y_j \partial k_\ell} + \frac{\partial^2 F}{\partial k_j \partial y_\ell} + \frac{\partial^2 F}{\partial k_j \partial k_\ell}\right) = {}^t h\ J(1) h\ .$$

Since $J(\mu)$ is negative for μ near one its determinant is also strictly negative. Hence, using again an argument based on degree theory, one can conclude that (18) determines y uniquely as function of μ for μ sufficiently near one.

Let us finally solve equation (19) or $\Phi_1(x(\mu)) = \Phi_2(y(\mu),y(\mu))$

The first assumption one can make, as in Brock [1973], is that the Hessian of $\Phi_2(y,y)$ has a negative inverse. Hence the Jacobian matrix of the system $\frac{\partial \Phi_2}{\partial y_j} + \mu \frac{\partial \Phi_2}{\partial k_j} = 0$ has also a negative inverse if μ is near one (see III.1).

Then one has $\frac{dy_j}{d\mu} > 0$, $\forall j$, since $\frac{\partial \Phi_2}{\partial k_j} > 0$, $\forall j$, and $dy = -J^{-1}_{(\mu)} \left[\frac{\partial \Phi_2}{\partial k_j}\right] d\mu$

Hence $\tilde{\Phi}_2'(\mu) = \sum_j \left(\frac{\partial \Phi_2}{\partial y_j} + \frac{\partial \Phi_2}{\partial k_j}\right) \frac{dy_j}{d\mu} = (1-\mu) \sum_j \frac{\partial \Phi_2}{\partial k_j} \frac{\partial y_j}{\partial \mu} > 0$ since $\mu \leq \beta < 1$

The assumption made seems to be too strong. One way to relax it is to suppose, as in Burmester [1980], that the technology is regular in the sense of Burmester-Turnovsky.

B_7 ter : The technology is assumed to be regular in the sense of Burmester-Turnovsky, i.e., for every $0 \leq \mu \leq 1$, the solution of

$$\frac{\partial \Phi_2}{\partial y_j} + \mu \frac{\partial \Phi_2}{\partial k_j} = 0\ ;\quad j = 1,\ldots,p \quad \text{verifies } \sum_j \frac{\partial \Phi_2}{\partial k_j} \frac{dy_j}{d\mu} > 0$$

Hence, as previously, $\tilde{\Phi}_2'(\mu) > 0$

Remark IV.2 : If the technology is regular in the sense of Burmeister-Turnovsky, then the value of the variations of the capital is always negative at the equilibrium rate of interest. Indeed $\frac{\partial \Phi_2}{\partial k_j}$ can be viewed as the equilibrium price of capital k_j and μ is related to the equilibrium rate of interest r by the expression $\mu = \frac{1}{1+r}$.

Let us know consider the consumption side and the function $\Phi_1(x(\mu)) = \tilde{\Phi}_1(\mu)$. As Lucas and Stokey [1984] we shall show that $\tilde{\Phi}_1$ is a decreasing function of μ.

Let us try to motivate the "increasing marginal impatience" assumption that we next make.

Consider a stationary path $\underset{\sim}{x}^i = (x^i, x^i, \ldots)$. Then $z^i = u^i(\underset{\sim}{x}^i) = W^i(x^i, u^i(\underset{\sim}{x}^i))$. Let $\mu^i = \left(\frac{\partial W^i}{\partial z^i}(x^i, z^i)\right)_{z^i = W^i(x^i, z^i)}$. An easy computation shows that

$$\mu^i = \left[\frac{\frac{du^i}{dh}(x^i, x^i + h\xi, x^i, \ldots)}{\frac{du^i}{dh}(x^i + h\xi, x^i, x^i, \ldots)}\right]_{h=0}.$$

By the implicit function theorem μ^i is a function of x^i. However in the case considered by Koopmans and ali [1964], $W(x^i, z^i) = \mathcal{V}(c(x^i), z^i)$ for some well chosen V and C (see remark I.3). In that case μ^i is a function of z^i. We define marginal impatience by the condition $\frac{d\mu^i}{dz^i} \leq 0$, i.e. when the utility level of a constant path increases, the ratio between the marginal variations of utility due to changes in the same amount in second and first period decreases. (Examples of $\mathcal{V}$ functions that satisfy that property can be found in Koopmans and Ali [1964]. The following condition can be viewed as its generalisation :

W_8 *Increasing marginal impatience* :

Let $x^i(\mu)$, $z^i(\mu)$ be the unique solution for consumer i of (16),(17),(18),(19) then $\frac{dW^i}{d\mu}(x^i(\mu),z^i(\mu)) < 0$ for every i .

Then under W_8 , $\tilde{\Phi}_1$ is decreasing function. Indeed $\Phi_1'(\mu) = \sum_j \frac{\partial\Phi_1}{\partial x_j}\frac{dx_j}{d\mu}$ with $x_j = \sum_i x_j^i \frac{dW^i}{d\mu}(x^i(\mu), z^i(\mu)) = \sum_j \frac{\partial W^i}{\partial x_j^i}\frac{dx_j^i}{d\mu} + \frac{\partial W^i}{\partial z^i}\frac{dz^i}{d\mu}$. From (17) and (18) $dz^i = \frac{1}{1-\mu}\sum_j \frac{\partial W^i}{\partial x_j^i} dx_j^i$, therefore $\frac{dW^i}{d\mu}(x^i(\mu),z^i(\mu))=(1+\frac{\mu}{1-\mu})\sum_j \frac{\partial W^i}{\partial x_j^i}\frac{dx_j^i}{d\mu} =$ $\frac{\lambda}{(1-\mu)\theta^i}\sum_j \frac{\partial\Phi_1}{\partial x_j}\frac{dx_j^i}{d\mu}$ hence, $\Phi_1'(\mu) = \frac{1-\mu}{\lambda}\sum_i \theta^i \frac{dW^i}{d\mu} < 0$.

As $\tilde{\Phi}_2(\mu)$ is increasing in μ and $\tilde{\Phi}_1(\mu)$ is decreasing in μ , $\tilde{\Phi}_1(\mu) = \tilde{\Phi}_2(\mu)$ has a unique solution.

From W_5 and the implicit functions theorem, relation (21) defines z^i as a continuously differentiable function ζ^i of x^i . Define

$$\hat{\mu} = \min_i \inf_{x^i} \left\{\frac{\partial W^i}{\partial z}(x^i, \zeta^i(x^i))\right\}$$

One can then summarize the previous results in the following theorem :

THEOREM IV.1 :

Under $W_1 - W_8$, $B_1 - B_7$ ter and for $\hat{\mu}$ near one, if there exists an interior steady state $(\bar{x},\bar{y},\bar{z},\bar{\theta})$, then it is unique.

We next generalize Mangasarian's proof [1966].

THEOREM IV.2 :

Let k_o and (z_o^i), $i \geq 2$, be given. Let $(\underset{\sim}{x}, \underset{\sim}{k}, \underset{\sim}{z})$ with initial datas k_o and (z_o^i) $i \geq 2$ satisfy $\tilde{E}_t$. Assume $(\underset{\sim}{k},\underset{\sim}{z})$ uniformly bounded. Then $(\underset{\sim}{x},\underset{\sim}{k},\underset{\sim}{z})$ is optimal.

Proof : Let $(\underset{\sim}{\bar{x}} , \underset{\sim}{\bar{k}} , \underset{\sim}{\bar{z}})$ denote the optimal solution from $k_o,(z_o^i)$, $i \geq 2$. $F(\bar{k}_t,\hat{\bar{x}}_t,\bar{k}_{t+1}) = 0$ $\forall t$ and $\bar{z}_i^t = W^i(\bar{x}_t^i,\bar{z}_{t+1}^i)$, $\forall i \geq 2$ $\forall t$. By theorem 1.3, $(\underset{\sim}{\bar{z}})$ is bounded. Let $(\underset{\sim}{x} ,\underset{\sim}{k} ,\underset{\sim}{z})$ satisfy $\tilde{E}_t$ for every t . If $(\underset{\sim}{k})$ is bounded then $(\underset{\sim}{x}^i)$ are bounded since $\hat{x}_t \in \Pi^1 B(k_t)$. $(\underset{\sim}{z})$ being bounded and verifying $z_t^i = W^i(x_t^i,z_{t+1}^i)$ $\forall i$, then $z_t^i = W^i(x_t^i,W^i(x_{t+1}^i,\ldots)$ $\forall i$. Let L be as in page 23. Let us compute $\Delta W = W^1(x_o^1,z_1^1) - W^1(\bar{x}_o^1,\bar{z}_1^1) = L(\underset{\sim}{x},\underset{\sim}{k},\underset{\sim}{z}) - L(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{k}},\underset{\sim}{\bar{z}})$

Since the constraints are binding, we have :

$$W^1(x_o^1,z_1^1) - W^1(\bar{x}_o^1,\bar{z}_1^1) \geq \mathrm{grad}_x W^1(x_o^1,z_1^1)(x_o^1 - \bar{x}_o^1) + \frac{\partial W^1}{\partial z}(x_o^1,z_1^1)(z_1^1 - \bar{z}_1^1) .$$

$$- \lambda_o \left[F(k_o,\hat{x}_o,k_1) - F(k_o,\hat{\bar{x}}_o,\bar{k}_1)\right] \geq \lambda_o \ \mathrm{grad}_x F(k_o,\hat{x}_o,k_1)(\hat{\bar{x}}_o^i-\hat{x}_o^i) +$$

$$\lambda_o \ \mathrm{grad}_y F(k_o,\hat{x}_o,k_1)(\bar{k}_1 - k_1) .$$

and $\forall i \geq 2$

$$\mu_o^i \left[W^i(x_o^i,z_1^i) - W^i(\bar{x}_o^i,\bar{z}_1^i)\right] \geq \mu_o^i \ \mathrm{grad}_x W^i(x_o^i,z_1^i)(x_o^i - \bar{x}_o^i)$$

$$+ \mu_o^i \frac{\partial W^i}{\partial z}(x_o^i,z_1^i)(z_1^i - \bar{z}_1^i) .$$

Similar inequalities can be written at date t. As $\bar{z}_i^t=W^i(\bar{x}_t^i,\bar{z}_{t+1}^i)$ and $z_i^t=W^i(x_t^i,z_{t+1}^i)$ and using the fact that (x_t, k_t, z_t) satisfies $\widetilde{E}_t$ we get :

$$\Delta W \geq \prod_{h=o}^{t-1} \frac{\partial W^1}{\partial z}(x_h^1, z_{h+1}^1)(z_t^1 - \bar{z}_t^1) + \lambda_t \operatorname{grad}_x F(k_t, \hat{x}_t, k_{t+1})(\bar{x}_t^1 - x_t^1)$$

$$+ \lambda_t \operatorname{grad}_y F(k_t, \hat{x}_t, k_{t+1})(\bar{k}_{t+1} - k_{t+1}) + \sum_{i \geq 2} \mu_{t+1}^i (z_{t+1}^i - \bar{z}_{t+1}^i) .$$

By remark IV.1, $(\underset{\sim}{k}, \underset{\sim}{x}, \underset{\sim}{z})$ bounded imply that $\lambda_t \to 0$ and $\mu_t^i \to 0$ $\forall i \geq 2$ so that $\Delta W \geq \limsup_t (\lambda_t \operatorname{grad}_x F(k_t, \hat{x}_t, k_{t+1})(\bar{x}_t^1) + \lambda_t \operatorname{grad}_y F(k_t, \hat{x}_t, k_{t+1})\bar{k}_{t+1}) \geq 0.$

Therefore $(\underset{\sim}{x}, \underset{\sim}{k}, \underset{\sim}{z})$ is optimal.

Remark IV.3 : The result proved above is typically used in the following case : Let us assume that $(x_t^i)_{i \geq 1}$, λ_t and μ_t^i, $i \geq 2$ can be eliminated and that $\left(k_{t+2}, (z_{t+2}^i)_{i \geq 2}, k_{t+1}, (z_{t+1}^i)_{i \geq 2}\right)$ can be expressed as a function of $\left(k_{t+1}, (z_{t+1}^i)_{i \geq 2}, k_t, (z_t^i)_{i \geq 2}\right)$. Let us furthermore assume that the jacobian of that map at the steady state $\left(k^*, (z^i)^*_{i \geq 2}, k^*, z^{i*}_{i \geq 2}\right)$ is a hyperbolic isomorphism of $\mathbb{R}^{p+m-1} \times \mathbb{R}^{p+m-1}$ with $2(p+m-1)$ eigenvalues, $|\lambda_i| < 1$ for $i \leq p+m-1$ and $|\lambda_i| > 1$ for $i > p+m-1$. Let $E_1 \oplus E_2$ be a decomposition of $\mathbb{R}^{2(p+m-1)}$ such that $T_1 = T_{/E_1}$ has eigenvalues λ_i, $i = 1 \ldots p+m-1$ and similarly for E_2 . Assume moreover the following regularity condition :

"<u>Regularity condition</u>"

The projection of E_1 on $\mathbb{R}^{p+m-1} \times \{0\}$ is an isomorphism.

Then optimal solutions corresponding to initial condition $(k_o, (z_o^i)_{i \geq 2})$ close to the stationary state $(k^*, (z^i)^*_{i \geq 2})$ converge towards it. The argument is Scheinkman's [1976] page 25. Given any $(k_o (z_o^i)_{i \geq 2})$ sufficiently close to $(k^*, (z^i)^*_{i \geq 2})$, one can find by the regularity condition a unique $(k_1, (z_1^i)_{i > 2})$ such that $(k_o, (z_o^i)_{1 \geq 2}, k_1, (z_1^i)_{i \geq 2})$ is on the stable manifold of the steady state. The path generated by Euler's system with these initial conditions is locally stable. By theorem IV.2, it is optimal.

Example 1 :

Suppose that we have one consumption good and one capital good and a production function $\hat{x} = G(k,y)$, which is stricly concave, twice continuously differentiable, and verifies $\frac{\partial G}{\partial k} < 0$, $\frac{\partial G}{\partial y} > 0$. Assume to simplify that the steady state is unique and that the utility functions W^i verify the normality condition for future utility at the steady state :

$$W_9 \quad : \quad \frac{\partial W^i}{\partial z^i} \frac{\partial^2 W^i}{\partial x^{i2}} - \frac{\partial W^i}{\partial x^i} \frac{\partial^2 W^i}{\partial x^i \partial z^i} < 0$$

Then we have :

COROLLARY IV.1 :

Assume $W_1 - W_9$ and $B_1 - B_7$, then the steady state is unique and there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that if $\left|\frac{\partial^2 G}{\partial k \partial y}\right| \leq \varepsilon_1$, and $\frac{\partial G}{\partial k} + 1 \geq - \varepsilon_2$ at the steady state of $(\tilde{E}_t)$, then it is locally stable.

Proof : By remark IV.3, it suffices to prove that, the steady state is a saddle point of the linearised Euler's equation and that the regularity condition is fulfilled. The proof that the steady state is a saddle point is omitted since it is tedious and similar to theorem 1 in Benhabib and Ali [1985]. The proof of the regularity condition can be found in the Apendix.

Example 2 :

Let us consider now the case where the aggregator function W^i are linear with the same discount factor β , i.e.,

$$W^i(x^i,z^i) = u^i(x^i) + \beta z^i \;, \quad \forall i$$

From equation (2) of E_t , one has : $\mu_t = \beta \;\; \forall t$ and $\theta_t^i = \theta_o^i \;\; \forall i , \forall t$

One can easily check that (x_t^i) depend continuously on (θ_o^i) and (k_t, k_{t+1}), since the Jacobian matrix of equations (1) and (3) is the following :

$$\left[\begin{array}{cc|c} \theta_o^i \dfrac{\partial^2 u^i}{\partial x^2} & - \quad \lambda \dfrac{\partial^2 F}{\partial x^2} & \begin{array}{c} -\operatorname{grad}_x F \\ \\ -\operatorname{grad}_x F \end{array} \\ \hline {}^t\operatorname{grad}_x F \;\ldots\; & {}^t\operatorname{grad}_x F & 0 \end{array}\right]$$

and is obviously non-singular (see proposition III.2).

Let us assume :

$$B_8 : \forall j\,,\ \forall \ell\,,\quad \frac{\partial^2 F}{\partial x_j \partial y_\ell} = \frac{\partial^2 F}{\partial x_j \partial k_\ell} = 0$$

By differentiating equations (1) and (3) of E_t , one gets :

$$d\lambda_t = \frac{1}{H}\left(\frac{\partial F}{\partial k}\,dk_t + \frac{\partial F}{\partial y}\,dk_{t+1}\right)$$

where, if V denotes the vector : $\left(\dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_m}, \ldots, \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_m}\right)$

and J the negative definite matrix $\left[\theta_o^i \dfrac{\partial^2 u^i}{\partial x^2} - \lambda \dfrac{\partial^2 F}{\partial x^2}\right]$, $H = -{}^tV\,J^{-1}\,V > 0$.

By differentiating equation (5) , and replacing $d\lambda_t$ and $d\lambda_{t+1}$ by the expression obtained above, one gets the following system :

$$\sum_\ell \left(\frac{\partial F}{\partial y_j}\frac{\partial F}{\partial k_\ell} + \lambda H \frac{\partial^2 F}{\partial y_j \partial k_\ell}\right) dk_{\ell,t}$$

$$+ \sum_\ell \left(\left[\lambda H \frac{\partial^2 F}{\partial y_j \partial y_\ell} + \frac{\partial F}{\partial y_j}\frac{\partial F}{\partial y_\ell}\right] + \beta\left[\frac{\partial F}{\partial k_j}\frac{\partial F}{\partial k_\ell} + \lambda H \frac{\partial^2 F}{\partial k_j \partial k_\ell}\right]\right) dk_{\ell,t+1}$$

$$+ \beta \sum_\ell \left(\frac{\partial F}{\partial k_j}\frac{\partial F}{\partial y_\ell} + \lambda H \frac{\partial^2 F}{\partial k_j \partial y_\ell}\right) dk_{\ell,t+2} = 0$$

Obviously, this system can be rewritten in the following matrix form :

$B\,dk_t + (A + \beta C)\,dk_{t+1} + \beta\,{}^tB\,dk_{t+2} = 0$ where A and C are symmetric, It is well-known (Levhari and Leviatan [1972]) that if α is a characteristic root of $\det [\alpha^2 B + (A + \beta\cdot C)\alpha + \beta\, {}^tB] = 0$ then $\frac{1}{\alpha\beta}$ is another one, if one has the following assumption :

B_9 : $\det B \neq 0$

Moreover if the matrix $\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$ is positive definite, then, when $\beta = 1$, the corresponding characteristic roots are not on the unit-circle (see Levhari and Leviatan [1972]); therefore for β near one the steady-state is a saddle-point.

We prove now that the matrix

$\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$ is positive definite.

Tedious computations show that

$$\left({}^tx, {}^ty\right)\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \left(\sum_j x_j \frac{\partial F}{\partial k_j}\right)^2 + \left(\sum_j x_j \frac{\partial F}{\partial y_j}\right)\left(\sum_j y_j \frac{\partial F}{\partial k_j}\right)$$

$$+ \left(\sum_j y_j \frac{\partial F}{\partial k_j}\right)\left(\sum_\ell x_\ell \frac{\partial F}{\partial y_\ell}\right) + \left(\sum_j y_j \frac{\partial F}{\partial y_j}\right)^2 + \left({}^tx, {}^tt\right) L \begin{pmatrix} x \\ y \end{pmatrix},$$

where L is the hessian of F with respect to (k,y) at a steady state. Since $\frac{\partial F}{\partial y_j} = -\beta \frac{\partial F}{\partial k_j}$ at a steady state, the L.H.S. of the previous relation is equal to

$$\left(\sum_j x_j \frac{\partial F}{\partial k_j}\right)^2 - 2\beta\left(\sum_j x_j \frac{\partial F}{\partial k_j}\right)\left(\sum_j y_j \frac{\partial F}{\partial k_j}\right) + \beta^2\left(\sum y_j \frac{\partial F}{\partial k_j}\right)^2 + \left({}^tx, {}^ty\right) L \begin{pmatrix} x \\ y \end{pmatrix}$$

and, hence, is strictly positive, if $\left({}^tx, {}^ty\right) \neq 0$.

A method similair to that of Scheinkman's [1976] can be used to show that the regularity condition is fulfilled for β sufficiently close to one.

One can summarize the results established above in the following corollary :

COROLLARY IV.2 :

Under assumptions $W_1 - W_4$, those of Theorem III.1 and B_9, there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that, if one has :

$$\forall j\ ,\ \forall \ell\ ,\quad \left\| \frac{\partial F}{\partial x_j \partial y_\ell} \right\| \leq \varepsilon_1\ ,\ \left\| \frac{\partial F}{\partial x_j \partial k_\ell} \right\| \leq \varepsilon_1$$

and $1 - \varepsilon_2 \leq \beta < 1$

then the optimal path $((x_t^i), k_t)$ converges when $t \to \infty$.

Remark IV.4 :

In the case of linear aggregator function with the same discount factor, one has a continuum of stationary points which depend on (θ_o^i) except when the production function is separable. In that case the steady state of the path (k_t) given by Euler's equations is independent of (θ_o^i) .

Appendix 1

Lemma A.1.

Let G be a continuous, convex compact valued correspondence form $\mathbb{R}^m$ into $\mathbb{R}^m$. Assume $x^o \in \text{int } G(k^o)$, then there exist a neighbourhood $V(k^o)$ of k^o such that for every k in $V(k^o)$, x^o belongs to $G(k)$.

Proof : Let $x^o \in \text{int } G(k^o)$. Then there exists a ball $B(x^o, \rho)$ centered at x^o and with radius ρ included in $G(k^o)$. Since G is lower semi-continuous, there exists a neighbourhood of k^o such that for every k in $V_1(k^o)$, $G(k) \cap B(x^o, \rho) \neq \Phi$.

Assume that the conclusion is false. Then there exists a sequence k^n, $k^n \to k^o$ such that $G(k^n) \cap B(x^o, \rho) \neq \Phi$ but $x^o \notin G(k^n)$.

Let $\bar{x}^n$ denote the projection of x^o on $G(k^n)$ and let y^n be diametrically opposed to $\bar{x}^n$ in $B(x^o, \rho)$, so that $\bar{x}^n$ is also the projection of y^n on $G(k^n)$. Then, $\| \bar{x}^n - x^o \| = \min \{ \| z - x^o \| , z \in G(k^n) \} \to 0$ as $n \to \infty$.

Thus $\bar{x}^n$ converges to x. On the other hand, let y be a cluster point of the sequence y^n. We have $d(y, G(k^o)) = \lim_n d(y_n, G(k^n)) \geq \rho$ so that $y \notin G(k^o)$. On the contrary by construction $y \in S(x^o, \rho) \subset G(k^o)$ a contradiction.

APPENDIX 2

Proof of the "regularity condition" in example I, p.32

Define $\xi_t = (k_t, (z_t^i)_{i\geq 2}) \in R \times R^{n-1}$

It has been shown in Benhabib et ali [1985] that the Euler system in a neighbourhood of the steady state is equivalent to :

$$(\xi_{t+1}, \xi_{t+2}) = F(\xi_t, \xi_{t+1})$$

The Jacobian matrix DF calculated at the steady state has the following form :

$$DF = \begin{bmatrix} 0 & I \\ -f'I & A \end{bmatrix}$$

Where A and I (identity matrix) are (n x n) - matrices and f' is a positive scalar. It has been proven in Benhabib et ali [1985] that :

i) DF has 2 n eigenvalues r_i, $i=1,\dots,2n$; $0 < |r_i| < 1$ for $i \leq n$, and $|r_i| > 1$ for $i > n$.

ii) A is similar to a positive definite matrix with positive eigenvalues γ_i strictly greater than $1 + f'$.

Let E_1 denote the stable subspace of DF.

It can be defined as follows :

$$E_1 \left\{ (Q_1 v, Q_2 v), v \in R^n \right\}$$

where Q_1 and Q_2 are (n x n)-matrices.

One has to prove this assertion :

$$\forall \xi \in R^n , \exists v \in R^n , \xi = Q_1 v,$$

or in other words $\qquad \det Q_1 \neq 0.$

If it is not the case, and since E_1 is n-dimensional, then there exists $(0, v)$, with $v \neq 0$, which belongs to E_1. That means that $DF^t \begin{pmatrix} 0 \\ v \end{pmatrix} \to 0$ when $t \to \infty$.

Since there exists P such that :

$$P^{-1} A P = \Gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix}$$

Then $$DF^t \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 0 & I \\ -f'I & \Gamma \end{pmatrix}^t \begin{pmatrix} 0 \\ W \end{pmatrix}, \quad W = P^{-1} v \neq 0.$$

Let (W_i) be the coordinates of W.

One can easily check that

$$\begin{pmatrix} 0 & I \\ -f'\,I & \Gamma \end{pmatrix}^t \begin{pmatrix} 0 \\ W \end{pmatrix} = \left((\beta_{i,t}\, W_i)_{i=1,\dots,n} \, , \, (\beta_{i,t+1}\, W_i)_{i=1,\dots,n} \right)$$

where the sequence $\{\beta_{i,t}\}$ verifies the induction relation :

$$\beta_{i,t+1} = \gamma_i \,\beta_{i,t} - f' \,\beta_{i,t-1}$$

with initial values $\beta_{i,0} = 0$, $\beta_{i,1} = 1$

Thus : $\beta_{i,t} = \lambda_{i,1}\, r_{i,1}^t + \lambda_{i,2}\, r_{i,2}^t$

where $r_{i,1}$, $r_{i,2}$ are the positive roots of the characteristic equation $r^2 - \gamma_i r + f' = 0$.

Furthermore : $r_{i,1} < 1$, $r_{i,2} > 1$

Since, $\beta_{i,0} = 0$, and $\beta_{i,1} = 1$ one has $\lambda_{i,2} \neq 0$

As $W \neq 0$ there exists i_0 such that $\left|\beta_{i_0,t}\, W_{i_0}\right| \to +\infty$ when $t \to +\infty$, yielding a contradiction.

$\square$

Lemma A2 - Let Σ be the correspondence from R_+^m into $(R_+^m)^n$ defined as follows : $\Sigma(x) = \left\{(x^i) \in (R_+^m)^n,\ \sum_{i=1}^{n} x^i = x\right\}$. Then Σ is compact convex valued and continuous.

Proof Σ is trivially compact convex valued and has a closed graph.

Let us show that it is lower semi-continuous : let $\hat{x}$ be fixed in R_+^m. Assume $\hat{x}_1 = \ldots = \hat{x}_l = 0$ and $\hat{x}_{l+1} > 0 \ldots \hat{x}_m > 0$.

Let $(x^i) \in \Sigma(\hat{x})$ ie $\sum_i x^i = \hat{x}$. This implies $x_h^i = 0$, $\forall i$, $\forall h = 1 \ldots l$.

Let $\hat{x}_v \rightarrow \hat{x}$. En particular $\hat{x}_{vh} \rightarrow 0$ for $h = 1 \ldots l$.

Let $1 \leqslant h \leqslant l$, define $x'^{i}_{vh} = \dfrac{\hat{x}_{vh}}{n}$ then $x'^{i}_{vh} \rightarrow 0$ and $\sum_i x'^{i}_{vh} = \hat{x}_{vh}$

Let $h \geqslant l + 1$, then there exist $j(h)$, $\varepsilon > 0$ such that $x_h^j > \varepsilon$.

There exists v_0 such that $v > v_0$ implies $|\hat{x}_{vh} - \hat{x}_h| < \varepsilon$.

Let $x'^{j}_{vh} = \hat{x}_{vh} - \hat{x}_h + x_h^j > 0$ and for $i \neq j(h)$, $x'^{i}_{vh} = x_h^i$.

Then $\sum_i x'^{i}_{vh} = \hat{x}_{vh}$ and $x'^{i}_{vh} \rightarrow x_h^i$ $\forall i$. $\square$

Proposition - The correspondence from X into X^m

$\phi(\underset{\sim}{x}) = \left\{ (\underset{\sim}{x}^i) \in X^m,\ \sum_i \underset{\sim}{x}^i = \underset{\sim}{x} \right\}$ is continuous.

Proof - X^m being endowed with the product topology, it suffices to show that each "coordinate map", $\underset{\sim}{x} \rightarrow \left\{(x_t^i) \in (R_+^m)^n,\ \sum_i x_t^i = x_t\right\}$ is continuous. This last map is the composite of the ""th projection" with Σ defined above and is therefore continuous.

R E F E R E N C E S

Aubin J.-P., Ekeland I. (1984) - Applied Nonlinear Analysis, Wiley - Interscience.

Beals R. and Koopman T.(1969 -"Maximising Stationary Utility in a Constant Technology"- S.I.A.M. Journal of Applied Mathematics, 17, 1001-1015.

Becker R. (1980) -"On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households"- Quarterly Journal of Economics, 95, 375-382.

Benhabib J., Majumdar M. and Nishimura K. (1985) -"Global Equilibrium Dynamics with Stationary Recursive Preferences"- Working paper.

Benhabib J., Jafarey S. and Nishimura K. (1985) -"The dynamics of efficient intertemporal allocations with many agents recursive preferences and production"- Working paper.

Benveniste L.M. and Scheinkman J.A. -"On the differentiability of the value function in dynamic model of economics"- Econometrica vol. 47 n°3, May 79.

Bewley T. (1982) -"An Integration of Equilibrium Theory and turnpike theory"- Journal of Economic Theory 10, p.233-267.

Brock, W. (1973) -"Some results on the uniqueness of steady states in multisector models of optimum growth when future utilities are discounted"- International Economic Review - 14, n°3, 535-559.

Burmeister E. (1980) - Capital Theory and dynamics Cambridge University Press.

Coles J. (1985) -"Equilibrium turnpike theory with constant returns to scale and possibly heterogenous discount factors"- International Economic Review 26, 671-679.

Coles J. (1986) -"Equilibrium turnpike theory with time-separable utility", Journal of Economic Dynamics and Control 10 (1986) 367-394.

Koopmans T. (1960) -"Stationary Ordinal Utility and Impatience"- Econometrica, 28, 287-309.

Iwai K. (1972) -"Optimal Economic Growth and Stationary Ordinal Utility, A Fisherian Approach"- Journal of Economic Theory, 5, 121-51.

Levhari D. and Liviatan N. (1972) -"On Stability in the saddle-point sense"- Journal of Economic Theory, 4, p.88-93.

Lucas R. and Stokey N. (1984) -"Optimal Growth with Many Consumers"- Journal of Economic Theory, 32, 139-171.

Magill M.J.P. (1981) -"An equilibrium existence theorem"- Journal of Mathematical Analysis and Applications, 84, 1, 162-169.

Mangasarian O.L. (1966) -"Sufficient conditions for the optimal control of nonlinear systems"- S.I.A.M., Journal on Control, 4, n°1, p

Mas-Colell A. (1983) -"The price equilibrium existence problem in Banach lattices"- Working paper, Harvard University.

Mas-Colell A. (1985) - The theory of General Economic Equilibrium : A Differentiable Approach - Econometric Society Monograph, Cambridge University Press.

Montrucchio L. (1984) - "Optimal decisions over time and Strange Attractors" - Rapporto interno, n° 9, Politecnico di Torino.

Negishi T. (1960) -"Welfare economics and existence of an equilibrium for a competitive economy" -Metroeconomica, 92-97.

Pallu de la Barrière R. (1966) - Cours d'automatique théorique - Dunod.

Scheinkman J. (1976) -"On optimal steady states of n-sector growth models when utility is discounted"- Journal of Economic Theory, 12, p. 11-20.

Yano M. (1984) -"Competitive equilibria on turnpikes in a Mc Kenzie economy I. A neighbourhood turnpike theorem"- International economic review, 25, n°3, p. 695-717.

Yano M. (1985) -"Competitive equilibria on turnpikes in a Mc Kenzie economy II. An asymptotic turnpike theorem"- International economic review, 26, n°23, p. 661-669.