OPTIMAL PRICE DYNAMICS AND SPECULATION WITH A STORABLE GOOD

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DYNAMIQUE DE PRIX ET SPECULATION OPTIMALES
AVEC UN BIEN STOCKABLE

RESUME

On analyse l'équilibre d'un marché confrontant d'un côté le vendeur monopolistique d'un bien stockable, qui doit adapter son prix à l'inflation environnante mais pour lequel les changements de prix sont coûteux, et de l'autre ses clients, qui spéculent constamment sur les dates d'ajustement du prix, cherchant à stocker juste avant ces augmentations. Le problème est modélisé comme un jeu dynamique à horizon infini entre vendeur et spéculateurs. On montre qu'il existe un seul équilibre Markovien parfait et l'on caractérise complètement les dynamiques de prix et de stockage qui en résultent. Celles-ci comprennent en général une phase de stratégies mixtes, pendant laquelle le vendeur essaye de déjouer la spéculation en introduisant de l'incertitude dans son prix, tandis qu'un nombre croissant de spéculateurs stocke, culminant parfois en "ruée" spéculative généralisée sur le bien. On examine ensuite les conséquences macroéconomiques de ce type d'équilibre, calculant les coûts sociaux du tandem inflation-spéculation et établissant d'autre part un résultat d'agrégation des stratégies de prix aléatoires d'un grand nombre de vendeurs identiques. Les résultats du modèle montrent en particulier qu'en situation de concurrence imparfaite, la spéculation peut être déstabilisante, et surtout ils fournissent un fondement théorique à l'idée fréquemment rencontrée - et confirmée empiriquement - que l'inflation, même anticipée, engendre de l'incertitude sur les prix.

<u>Mots clef</u> : inflation, spéculation, incertitude des prix, jeu dynamique.
Codes J.E.L. : 020, 130

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ABSTRACT

This paper analyses the optimal price and storage strategies on the part of, respectively, the seller of a storable good, who must keep pace with inflation but incurs a cost to changing his price, and his customers, who speculate on the timing of price adjustments to buy and store just before. The problem is modelled as a game with infinite horizon between firm and speculators. A unique Markov perfect equilibrium is shown to exist, and the resulting price and storage dynamics are fully characterized. They generally involve a phase of mixed strategies, during which the firm tries to elude speculation by injecting uncertainty into its price dynamics, while speculators store in increasing numbers, with possibly a final generalized "run" on the good. The welfare costs of inflation-generated speculation are analyzed, and macroeconomic conclusions are drawn. In particular, the stochastic price policies of a large number of such firms are shown to aggregate back to a price index growing at the rate of the environing inflation in response to which they arose. Thus, a constant rate of inflation at the macroeconomic level can at the same time generate and cover up significant uncertainty and social costs at the microeconomic level. The results of the model establish that speculation can be destabilizing, even in a context of perfect information ; most importantly, they provide a theoretical foundation for the often mentioned (and empirically verified) claim that inflation causes price uncertainty.

<u>Keywords</u> : inflation, speculation, price uncertainty, dynamic game.
J.E.L. codes : 020, 130.

INTRODUCTION

> " Very soon, nobody knew how much things cost any more. Prices were jumping up in a completely arbitrary manner; a box of matches cost, in a shop which had increased its price at the right moment, twenty times as much as in another one, where a decent fellow was still selling is merchandise at the previous day's price. As a reward for his honesty, his shop was emptied within an hour, because the word was passed on, everyone rushed and bought what was for sale, whether they needed it or not." (Zweig,[1944])

This account of the Austrian hyperinflation of 1921 dramatically illustrates two important aspects of inflationary economies: sellers face a crucial and repeated problem of when to adjust their prices, while buyers speculate on the timing of these increases to go on a buying-for-storage spree just before. Less extreme situations, such as oil shocks, the removal of subsidies, or even steady but high enough inflation also provide ample evidence that most goods can, and indeed will be stored if buyers expect their price to go up significantly (or shortages).[1]

The quotation also confirms that most prices are adjusted at discrete intervals and not continuously; Mussa [1981] also found that the frequency of price increases for four selected commodities during the German hyperinflation of 1923 was, in a sense, small in comparison to the speed of inflation. Indeed, changing prices entails some costs: new information must be gathered, price tags, lists and catalogues updated, contracts and collusive agreements renegociated, etc.; price changes may also trigger search by customers.

The optimal price policy for a monopolistic seller of a non-storable good who faces a fixed cost of changing his price was characterized by Barro [1972] when the price must adapt to demand shocks, and by Sheshinski and Weiss [1977, 1979, 1983] when it must keep pace with environing inflation. The solution was shown to be an (S,s) rule, according to which (in the latter case) the real price is readjusted to some ceiling level S every time inflation has eroded it below some floor level s; with constant inflation, such adjustments occur with fixed periodicity. But storability endows the price adjustment problem with a new, speculative dimension, which generally renders such deterministic price policies suboptimal: if consumers knew that their supermarket or gas station adjusted its prices every Friday morning, they would store just the day before, thereby depriving the seller of his sales at the peak and increasing them at the lowest point of the real price cycle. This prospect would in turn give him an incentive to advance the price increase to Thursday morning; buyers would then try to store on Wednesday instead, etc.

This paper solves the problem of the optimal price and storage strategies for a firm selling a storable good and its customers as the Markov perfect equilibrium of a dynamic game with infinite horizon. It is shown in particular that the firm may find it profitable to inject randomness into its price dynamics, while, as time goes by before the price adjustment occurs, buyers store in increasing numbers, with possibly a final generalized "run" on the good. The first result provides a theoretical foundation for the often-mentioned link between inflation and price uncertainty, and the second an accurate description of buyer behaviour in inflationary situations, as well as a proof that speculation may be destabilizing, even in a context of perfect information.

The model is presented in section I, which also establishes a result restricting the existence possibilities of deterministic price adjustments in equilibrium. Section II fully characterizes the possible types of equilibria, while section III establishes existence and uniqueness. Section IV relates the equilibrium to the no-storage case, and examines its comparative dynamics with respect to the rate of inflation. It draws several important welfare and macroeconomic consequences from the results, examining in particular the cross-sectional distribution and aggregate price indices resulting from the optimal price strategies of a large number of firms. Proofs are gathered at the end of the paper, in appendices corresponding to the different sections. The main results are illustrated on Figures 1 to 2.3, in Section II.

I-THE MODEL

I.1 Description of the market

The firm: A monopolistic firm sells a storable good [2] which lasts for two periods. The firm faces an inflationary environment: all other prices, in particular its costs and the aggregate price level, increase at a constant rate of π per period. With all nominal prices deflated by the aggregate price index, the firm operates with constant real costs, which are: a production cost of c per unit and a fixed cost of changing prices (so-called "menu cost") of β.[3] The firm is risk-neutral, infinitely-lived and maximizes in each period the expected present value of its profits, with a discount factor of $\delta=1/(1+r)<1$.

Buyers: There is a unit continuum of infinitely lived, risk-neutral buyers (consumers, retailers, or other industries), who maximize in each period the expected present value of their instantaneous utilities, with the same discount factor δ as the firm. Each of them requires one unit of the good per period, provided the real price is below a common reservation value $S > c$.[4] In each period, buyers consume any previous inventories, then buy from the firm to satify their current needs (if inventories were inexistent or insufficient) as well as for storage until the next period, if they so desire.[5] Storage is costly because of foregone interest on the value of goods stored, and of a constant real storage cost per unit.

The degree of speculativeness of the market will be parametrized as follows: there is a fraction 1-x ($0<x<1$) of buyers who can store at a cost $\alpha < \delta S$ (hereafter referred to as speculators, or speculating customers), while the remaining x (non-speculators) face a storage cost $\alpha' > \delta S$, which renders storage always unprofitable.[6] A perhaps more familiar interpretation of the same model is that consumers cannot store but competitive speculators are active on the market, storing and reselling the good with a total capacity equal to a fraction 1-x of demand. This latter structure is very common in models of speculation (for instance Hart-Kreps [1986]), and the former one is easily seen to be equivalent: because of satiation (and assuming that transactions costs prohibit resales by low storage cost customers to high storage cost ones), when some customers store in anticipation of a price increase in the next period, they never want to store more than one unit each; thus total storage is bounded by 1-x. Customer heterogeneity (or equivalently, the capacity constraint on storage) is introduced to encompass these more general market structures, but also to allow an assumption on x (cf. section I.3) which yields an important simplification of the problem.

I.2 The game between firm and customers

In every period, buyers observe the current price, then make their purchasing and storage decisions; in the following period, the firm, having observed inventories, sets a new nominal price or keeps the same one; buyers then come back, etc.[7] If the fixed cost β prevents the firm from inflating its nominal price by π every period, inflation will gradually erode its real price below the monopoly level S, until an adjustment occurs. The possibility of storage then transforms the problem from one of optimization of the frequency of price changes, into a game between firm and speculators: the latter will store if they know that a price adjustment is likely, while the former will try to take advantage of periods when inventories are low to implement its price increase.

I.2.1 Necessity and meaning of mixed strategies

As discussed above, deterministic price and storage strategies will generally not be optimal. More likely, the firm will adjust its price at random intervals (or by random amounts, but such is not the case here) to try and elude speculation, while only a fraction of speculators will store (one unit each) in every period. The usual criticism of mixed strategies of course applies here: in a situation of indifference, the firm must randomize with the right probabilities, and buyers must store in the right proportions. It is, however, subtantially weakened by the results of Harsanyi [1973] and Milgrom and Weber [1985], showing that a mixed strategy equilibrium in a game of perfect information can be interpreted, and formally justified, as the limit of pure strategy equilibria in the same game perturbed by an infinitesimal amount of incomplete information.

I.2.2 The equilibrium concept

The equilibrium should be subgame perfect: whatever the previous history of price and storage decisions, firm and speculators must maximize their respective objective functions from there on, given the other side's strategy. This perfection requirement rules out non-credible threats, but the set of admissible strategies must be further restricted. One will follow here Maskin and Tirole [1982], [1985] by retaining the concept of Markov perfect equilibrium, which allows each player's strategy to depend only on those historical (or state) variables which are payoff relevant, i.e. those which "physically" matter to him because they directly affect the current and future payoffs from his decision.[8] One essential reason for this choice is of course simplicity, not in the analytical sense (Markov strategies are typically much more complicated than the usual "trigger" strategies of supergames) but in the sense of eliminating the plethora of equilibria (Folk theorem) which arise when all history-dependant strategies are allowed, and among which a necessarily arbitrary selection then has to be made.[9] Of equal importance, however, is a concern for robustness: Markov perfect equilibria are more robust to renegotiation among players, and most importantly, to the choice of a finite or infinite horizon, than equilibria based on threats of punishment for deviant behaviour.[10] These desirable properties arise precisely because players react only to those variables which constitute a physical intertemporal link in the game and not to "immaterial" past behaviour (for detailed discussions, cf. Maskin and Tirole [1982] and Gertner [1986]).

I.2.3 The state variable and strategies

The payoff to customer i in period k is:

$$G_i(k) = \max(S-P(k),0) + q^i(k-1)\min(S,P(k)) - q^i(k)(P(k)+\alpha)$$

where P(k) is the current real price, $q^i(k-1)$ and $q^i(k)$ repectively his previous and current storage decisions. The only state variable which is payoff-relevant for the choice of $q^i(k)$ is therefore P(k). In particular, previously accumulated inventories $q^i(k-1)$ are not payoff-relevant for the storage decision (although they are for the current consumption decision), whether or not they are completely consumed when $q^i(k)$ is chosen. In the following period (k+1), the firm's payoff is:

$$G_F(k+1) = P(k+1)[1 + (1-x)(q'(k+1)-q'(k))] - \beta\Delta(P(k),P(k+1))$$

where $q'(k)=\int_0^1 q^i(k)di$ and $q'(k+1)=\int_0^1 q^i(k+1)di$ are the quantities stored by speculators (on average) in the previous and current periods respectively; $1 - \Delta(y,z)$ denotes the Kronecker function, equal to 1 if y=z, and to zero otherwise. The only payoff-relevant state variables for the choice of P(k+1) are therefore P(k) (because of the adjustment cost) and q'(k) (the stock of inventories to be consumed at the expense of new sales). According to the Markov restriction, the firm's strategy P(k+1) can only depend P(k) and q'(k). Its strategy space can even be further restricted, by virtue of the following remark. In order to find the Markov perfect equilibrium paths of the game, it is sufficient to restrict attention to subgames which exclude simultaneous deviations by a positive mass of buyers in the preceding period [11]; in any such subgame, each $q^i(k)$, hence also q'(k), is a function of P(k) only, therefore P(k+1) can be specified as function of P(k) only. The market is thus completely described by -and players' strategies conditioned on- a single state variable, the current real price, or more conveniently, its logarithm.

Definition I.1: For any $t \in \mathbb{R}$, the market is said to be in state t if the real price charged by the firm is $P_t \equiv S(1+\pi)^{-t} \equiv S\theta^{-t}$.

Note the distinction between P(k) (real price in period k) and P_t (real price in state t). Along the equilibrium path, only integer states $t \in \mathbb{N}$ will be observed, and t will be the number of periods elapsed since the real price was last adjusted to S; to establish perfection, however, one must also consider real prices above S (states t<0) or ouside the grid $\{S, S/(1+\pi), S/(1+\pi)^2, \ldots\}$ (states $t \notin \mathbb{N}$), since the firm could always choose such a price, or it could be the initial condition of a subgame.

I.3 The complete and simplified versions of the game:

The most general strategy space for the firm it the set of functions associating to any state $t \in \mathbb{R}$ a probability distribution F_t over $(\mathbb{R}_+) x \{0,1\}$, (real prices and the action of closing down, which brings the game to an end when future expected profits are negative). Bénabou [1986a] treats the game in this general form, and shows in particular that, as long as the cost of price adjustment is smaller than the maximum revenue from non-speculating customers $(x(S-c) > \beta)$, attention can be restricted to a simpler game, where in every period the firm only decides between adjusting its real price back to the reservation level S and not adjusting it. Specifically, the unique equilibrium of the simplified game is then also an equilibrium of the general one; it is the only equilibrium when $\alpha = c = 0$, and always the only one involving adjustment to a constant level; if other equilibria of the larger game exist, they must involve adjustments to (variable) real prices which are close to S. For expositional clarity and brevity, only the simplified version of the game will be presented here, and throughout the paper it will be assumed:

Assumption A: $x(S-c) > \beta$.

Attention will thus be restricted to states $t \in \mathbb{R}_+$ [12] and strategies:

* <u>for the firm</u>: $q: \mathbb{R}_+ \dashrightarrow [0,1]$, specifying the probability q_t of a price adjustment following each state t.

* <u>for buyers</u>: $q': \mathbb{R}_+ \dashrightarrow [0,1]$, specifying the proportion q'_t of speculators storing in each state t.

If no adjustment takes place, a transition occurs to state t+1; after each adjustment, the game is back in state 0 and starts a new "cycle". In each state t, let $p_t \equiv P_t - c$ denote the real price net of production costs. The set of functions from $\mathbb{R}_+$ into [0,1] will be denoted by $[0,1]^{\mathbb{R}+}$. For all $y \in \mathbb{R}$, let Int[y] denote its integer part, and $K[y] \equiv -\text{Int}[-y] = \min\{k \in \mathbb{N} | k \geq y\}$.

<u>I.4 The equilibrium conditions</u>

Let $(q,q') \in ([0,1]^{\mathbb{R}+})^2$ be an equilibrium, and consider the market in some state $t \in \mathbb{R}_+$, at some date $k \in \mathbb{N}$; the real price is $P(k)=P_t$, speculators store $(1-x)q'(k)=(1-x)q'_t$, and in period k+1, the firm sets a new real price P(k+1), equal to S or P_{t+1} with probabilities q_t and $1-q_t$. As explained above, both speculators' and the firm's decisions are independant of q'(k-1) (which is not payoff relevant for anyone), and can therefore be computed as if q'(k-1) were zero. Let V_t (resp. W_{ti}) denote the expected present value of the firm's profits (resp. of speculator i's utility), given that the current state is t, that it was entered with zero inventories, and that all players play their equilibrium strategies from there on. The function $[t \dashrightarrow V_t]$ thus satifies the following dynamic programming equation, which also determines the optimal q_t:

$$V_t = \text{Max}\{p_t(1+(1-x)q'_t)+\delta[q(V_0-\beta-(1-x)q'_t p_0)+(1-q)(V_{t+1}-(1-x)q'_t p_{t+1})]\}$$

$$\text{over } q \in [0,1] \qquad (1)$$

The first term represents net total sales, for consumption and storage, in state t. The last two are also simple to interpret: with probability q, the firm adjusts its real price to $S=P_0$, achieving a valuation equal to V_0 minus the adjustment cost β and the net value $(1-x)q'_t p_0$ of sales at the new price lost because of storage; with probability 1-q, it lets its real price fall to P_{t+1}, and achieves a valuation of V_{t+1} minus the net value $(1-x)q_t^e p_{t+1}$ of lost sales.

Similarly, customer i chooses his pure strategy $q_{ti}^e \in \{0,1\}$, given q_t, (over which, being negligible, he has no influence) by solving the dynamic programming problem:

(2) $W_{ti} = \text{Max}\{S-P_t-q_t^e(P_t+\alpha)+\delta[q_t(W_{0i}+q_t^e P_0)+(1-q_t)(W_{t+1i}+q_t^e P_{t+1})]\}$
over $q_t^e \in \{0,1\}$

The term $S-P_t$ represents current utility and the following one is the cost of storing one unit. If the firm raises the price in the next period, this customer achieves a valuation W_{0i}, plus savings of P_0 if he had stored; if the real price falls to P_{t+1}, he achieves a valuation W_{t+1i}, plus savings of P_{t+1} if he had stored; hence the third term.

The necessary and sufficient conditions for the linear maximization problems in (1) and (2) are:

* <u>Buyers</u>:

$$(3)\begin{cases} \text{if } P_t+\alpha > \delta(q_t P_0+(1-q_t)P_{t+1}) & \text{then } q_{ti}^e=0 \ (\forall i), \text{ or: } q_t^e=0 \\ \text{if } P_t+\alpha < \delta(q_t P_0+(1-q_t)P_{t+1}) & \text{then } q_{ti}^e=1 \ (\forall i), \text{ or: } q_t^e=1 \\ \text{if } P_t+\alpha = \delta(q_t P_0+(1-q_t)P_{t+1}) & \text{then } q_{ti}^e\in\{0,1\} \ (\forall i), \text{ or: } q_t^e\in[0,1] \end{cases}$$

Thus, speculators simply compare the cost of buying an extra unit today at the price P_t and storing it at a cost α, with the discounted value of the price expected to prevail in the following period. In case of equality, each of them is indifferent between storing and not storing, so any proportion may decide to do so.

* <u>Firm</u>:

$$(4)\begin{cases} \text{if } q_t^e(1-x)(p_0-p_{t+1}) < V_0-\beta-V_{t+1} & \text{then } q_t=1 \\ \text{if } q_t^e(1-x)(p_0-p_{t+1}) > V_0-\beta-V_{t+1} & \text{then } q_t=0 \\ \text{if } q_t^e(1-x)(p_0-p_{t+1}) = V_0-\beta-V_{t+1} & \text{then } q_t\in[0,1] \end{cases}$$

The firm thus reaches its decision by comparing the increment in valuation $V_0-\beta-V_{t+1}$ which results from adjusting the net price from p_t to p_0 instead of letting it fall further to p_{t+1}, with the increment in expected lost revenues resulting from this decision (p_0 is lost rather than p_{t+1}, on a fraction $q_t^i(1-x)$ of customers). When equality prevails, it is indifferent and this decision can therefore be randomized.

I.5 Equilibrium and continuation value equilibrium

A Markov perfect equilibrium of the game is a pair of strategies (q,q') satisfying conditions (3) and (4), where $[t \dashrightarrow V_t]$ is the firm's valuation function, itself generated by q' as the solution to the functional fixed-point equation (1). The proofs of existence and characterization of the equilibrium proceed in two stages.

First, one will treat V_0, which appears in the right-hand side of (1) and in (4), as an exogenous parameter $V \in \mathbb{R}_+$; this is equivalent to replacing the original game by one which terminates when the firm adjusts its real price back to S, at which time it receives an exogenously given (continuation) value V, but must buy back all inventories at the real price S. The advantage of this method is that the latter game can be shown to end in a stochastic but bounded time (in contrast to the original one which is cyclical) and that its equilibria, henceforth termed continuation value equilibria, can be solved for backwards and fully characterized.

Definition I.2: A continuation value equilibrium is a triplet $(q,q',V) \in ([0,1]^{\mathbb{R}_+})^2 \times \mathbb{R}_+$ such that (q,q') satisfy equations (3) and (4), where V_0 is replaced by V and $[t \dashrightarrow V_t]$ is a solution to the functional equation (1), where V_0 is replaced by V on the right-hand side.

In the second stage of the construction, equilibria of the original game are derived as fixed points. Note first that if a pair (q,q') is an equilibrium of the original game, then (q,q',V_0) is a trivially a continuation value equilibrium. Conversely, to every continuation value V is associated a continuation value equilibrium (q_V, q'_V, V), under which the firm's valuation in state zero (i.e. V_0, given recursively by the modified equation (1)) can be computed as a function $f(V)$; then, by construction, (q_V, q'_V) is an equilibrium of the full game if and only if $V_0=f(V)$ coincides with V. The method of proof adopted here thus allows to replace an infinite-horizon game by a family of finite horizon ones, and a fixed-point problem in $([0,1]^{\mathbb{R}+})^2$ by one in $\mathbb{R}$.

I.6 The inexistence of deterministic price adjustments

The firm's strategy will be said to involve a deterministic price adjustment if the probability of adjustment jumps from 0 to 1 in some state t: $q_{t-1}=0$ and $q_t=1$ ($q_{-1}\equiv 0$).

Let $q_t=1$; the condition required by (3) for buyers' indifference in state t becomes: $P_t+\alpha=\delta P_0$ or $\theta^{-t}=\delta-\alpha/S$. Since $\alpha<\delta S$, define:

(5) $$\tau = \mathrm{Log}[1/(\delta-\alpha/S)]/\mathrm{Log}(\theta)$$

When faced with a sure price adjustment in the following period, all speculators store if $t>\tau$, none do if $t<\tau$, and they are indifferent if $t=\tau$. Indeed, only after the real price has fallen sufficiently (below P_τ), do the savings realized by storing justify the necessary costs, even if it is certain that the real price is about to increase back to S. If storage costs or the real interest rate are too high (i.e. if the assumption $\alpha(1+r)<S$ is not satisfied), the price differential never justifies storage ($\tau\equiv+\infty$) and the game reduces to the optimization by the firm of the frequency of price adjustments (Sheshinski and Weiss [1977]). This result also justifies the interpretation of x as the fraction of customers whose storage cost is some $\alpha'\geq\delta S$.

Consider now whether a deterministic price adjustment giving rise to storage by all speculators can be optimal for the firm: let $q_t=1$ and $t>\tau$, so $q'_t=1$, inflicting on the firm a sure loss of $(1-x)(\delta p_0-p_t)$. If it is too large, the firm will try to avoid reaching state t by implementing the price increase earlier, with positive probability ($q_{t-1}>0$, if $t\geq 1$), thereby precluding a deterministic adjustment in state t. If it is small, on the contrary -say if there are few speculators ($x\approx 1$)- the firm could be willing to forfeit this loss in order to economize on adjustment costs. Since $(\delta p_0 - p_t)$ increases with t, intuition suggests the existence of some threshold μ such that deterministic price adjustments with storage by all speculators can be sustained by the firm only in states $t \leq \mu$. The following theorem indeed establishes that:

(6) $$\mu = \text{Log}[(1-z/\theta)/(1-z)]/\text{Log}(\theta), \text{ where } z = 1/(2-x).$$

<u>Theorem I</u>: There exists $\tau(a/S, \delta, \pi) > 0$ and $\mu(x, \pi) > 0$ such that, in any continuation value equilibrium:

i) $(\forall t,\ 0 \leq t < \tau)\ (q_t^{\prime} = 0)$

ii) $(\forall t, t > \tau)\ (q_t = 1 \Rightarrow q_t^{\prime} = 1)$

iii) $(\forall t \geq 1)\ (t > \max(\tau, \mu)$ and $q_t = 1 \Rightarrow q_{t-1} > 0)$

iv) $\partial\tau/\partial a > 0;\ \partial\tau/\partial r > 0;\ \partial\tau/\partial\pi < 0;\ \partial\mu/\partial x > 0;\ \partial\mu/\partial\pi < 0$ *and* $\mu < 1/(1-x)$.

<u>Proof</u>: <u>cf</u>. Appendix I.

These results can be interpreted as follows. There is an upper bound $\tau+1$ on the period of nominal rigidity which can exist between two consecutive deterministic price adjustments without storage (à la Sheshinski and Weiss); this upper bound is shorter, the lower storage costs and the real interest rate, and the higher the rate of inflation. Moreover, that same $\tau+1$ is a lower bound, and there exists an upper bound $\mu+1$ on the period which can exist between two deterministic price adjustments with storage by all 1-x speculators: this upper bound is shorter, the higher 1-x and the rate of inflation. Finally, the most important result is that there can be no deterministic price adjustment in a state $t > \max(\tau, \mu)$, and in particular, no periodic price adjustment of frequency less than $1/[\max(\tau, \mu)+1]$, whatever the value of the adjustment cost β. It will be assumed from here on that $\tau \notin \mathbb{N}$ and $\mu \notin \mathbb{N}$, or $\text{Int}[\tau] < \tau$, $\text{Int}[\mu] < \mu$, which holds generically.

II-CHARACTERISATION OF THE EQUILIBRIUM

Equilibria of the game (and more generally, continuation value equilibria) will now be fully characterized; in particular, they will be shown to consist of three distinct phases, separated by threshold states $\underline{T}$ and $\overline{T}$: pure strategies during $[0,\underline{T})$, then mixed during $[\underline{T},\overline{T})$, then again pure strategies during $[\overline{T},+\infty)$. The chain of reasonning proceeds along the following three main stages. First, the phases of pure strategies are examined; it is shown in particular that there exist T^* and $\overline{T}$, $T^*<\overline{T}<+\infty$, such that $q_t=0$ for $t<\min(T^*,\tau-1)$ and $q_t=1$ for $t>\overline{T}$. The intermediate mixed strategy phase is then analyzed; q_t is shown to be a given function Q_t of the state t, derived from (3), while q_t^f is obtained as the solution $Q_{t,V}^f$ to a linear difference equation with variable (state-dependant) coefficients, derived from (4) and (1). Finally, this system is solved backwards from $\overline{T}$ to determine $\underline{T}\leq T^*$.

II.1 The phases of pure strategies

Assume first that buyers never store. The opportunity cost of postponing the price adjustment for one period is then $(1-\delta)(V-\beta)$, while the gain from that postponement is next period's net real revenue p_{t+1}. The firm therefore adjusts its price, with probability one, in the state T^* such that these two quantities are equal;[13] it thus follows an (S,s) rule, with $s-c=p_{T^*+1}=(1-\delta)(V-\beta)$ (as in Sheshinski and Weiss [1977]). Similarly, if the 1-x speculators always store, the loss from postponing the adjustment for one period is $(1-\delta)(V-\beta-(1-x)p_0)$, and the optimal policy is an (S,s) rule, with s-c equated to that expression.

Definition II.1: Let $\Gamma=[(p_0-\delta\beta)/(1-\delta),\ p_0/(1-\delta)])$. For all $V\in\Gamma$, define $T^*(V)$ and $\overline{T}(V)$ by:

(7) $$p_{T^*(V)+1} = (1-\delta)(V-\beta);\quad p_{\overline{T}(V)+1} = (1-\delta)(V-\beta-(1-x)p_0).$$

For all $V\in\Gamma$: $(1-\delta)(V-\beta-(1-x)p_0) \geq p_0[1-(1-\delta)(1-x)]-\beta > xp_0-\beta > 0$ by

Assumption (A); hence (7) is always licit. Moreover, $0 \leq T^* + 1 < \bar{T} + 1$ (the dependance on V will not be explicited when no confusion arises). These threshold states, which determine adjustment in the above benchmark cases can also be shown to provide bounds on the true time of of adjustment:

<u>Lemma II.1</u>: *Let (q, q', V) be a continuation value equilibrium, with $V \in \Gamma$. For all $t \in R_+$:*

i) if $t < \min(T^*, \tau - 1)$, then $q_t = 0$.

ii) if $t > \bar{T}$ or $t = \bar{T} \leq \tau$, then $q_t = 1$.

iii) if $t = \bar{T} > \tau$, then $q_t \geq (P_t + \alpha - \delta P_{t+1})/[\delta(P_0 - P_{t+1})]$ and $q'_t = 1$.

<u>Proof</u>: <u>cf</u>. Appendix II.

Thus, the game starts with a (possibly empty) phase of pure, inactive, strategies ($q_t = q'_t = 0$ for $t < \min(T^*, \tau - 1)$) and ends with a phase of pure, active strategies ($q_t = q'_t = 1$ for $t > \max(\bar{T}, \tau)$). In particular, if the price has not been adjusted by $\bar{T}$, the adjustment occurs with certainty in the following period, even though all speculators have stored: the firm cannot wait any longer and gives up its attempts at a surprise adjustment.

<u>II.2 The phase of mixed strategies</u>

By (3), the probability q_t of a price increase in the next period which leaves speculators indifferent between storing at the price P_t ($t > \tau$) and not storing is defined by: $P_t + \alpha = \delta(q_t P_0 + (1 - q_t) P_{t+1})$ or:

(8) $\quad q_t = (P_t + \alpha - \delta P_{t+1})/[\delta(P_0 - P_{t+1})] = [\theta^{-t}(1/\delta - 1/\theta) + \alpha/\delta S]/[1 - \theta^{-(t+1)}]$

which is less than one, since $t > \tau$, or $P_t + \alpha < \delta P_0$. Similarly, the fraction q'_t of speculating customers storing in state t which leaves the firm indifferent between the real prices P_0 and P_{t+1} in the next period is given by (4): $q'_t = (V - \beta - V_{t+1})/[(1-x)(p_0 - p_{t+1})]$. But by (1):

$V_{t+1} \geq p_{t+1}(1 + (1-x)q'_{t+1}) + \delta(V - \beta - (1-x)q'_{t+1} p_0)$, with equality if $q_{t+1} > 0$.

Hence, during a mixed strategy phase, q'_t obeys the difference equation:

(9) $\quad q'_t = [(\delta p_0 - p_{t+1}) q'_{t+1} + ((1-\delta)(V - \beta) - p_{t+1})/(1-x)]/(p_0 - p_{t+1})$.

Definition II.2: Define the following functions:

i) For all $t\in \mathbb{R}_+$, $Q_t = \min(1, [\theta^{-t}(1/\delta-1/\theta) + \alpha/\delta S]/[1-\theta^{-(t+1)}])$

ii) For all $V\in\Gamma$, $[(t,y)\dashrightarrow \Psi_{t,V}(y)]$ defined on $[\tau-1,+\infty)\times\mathbb{R}$ by:

$$\Psi_{t,V}(y) = [(\delta p_0-p_{t+1})y + ((1-\delta)(V-\beta)-p_{t+1})/(1-x)]/(p_0-p_{t+1}).$$

$$\equiv a_{t+1}y + b_{t+1,V}$$

During a mixed strategy phase, $q_t=Q_t$ and $q'_t=\Psi_t(q'_{t+1})$ by (8) and (9). The following result generalizes (9) to the boundaries of such a phase.

Lemma II.2: For any continuation value equilibrium (q,q',V) and any $t\in\mathbb{R}_+$:

i) $(V-\beta-V_{t+1})/[(1-x)(p_0-p_{t+1})]\le \Psi_t(q'_{t+1})$, *with equality when* $q_{t+1}>0$.

ii) If $q_t\in(0,1)$ *and* $q_{t+1}>0$, *then* $q'_t = \Psi_t(q'_{t+1})$.

iii) If $q_t=1$, *then* $q'_t \le \Psi_t(q'_{t+1})$.

If $q_t=0$ *and* $q_{t+1}>0$, *then* $q'_t \ge \Psi_t(q'_{t+1})$.

Proof: cf. Appendix II.

For $t > \max(\bar{T}(V),\tau)$, $q'_t=1$ by Lemma II.1 and Theorem I. From this terminal condition, one can construct, by backwards induction, a solution $Q'_{t,V}$ on $[\tau-1,\bar{T}(V)]$ (when it is not empty) to the difference equation:

(10) $$Q'_{t,V} = a_{t+1}Q'_{t+1,V} + b_{t+1,V}$$

Definition II.3: For all $V\in\Gamma$, define $[t \dashrightarrow Q'_{t,V}]$ on $[\tau-1,+\infty)$ by:[14]

i) $Q'_{t,V} = 1$ on $[\max(\tau-1,\bar{T}(V)),+\infty)$;

ii) $Q'_{t,V} = \Psi_{t,V}(Q'_{t+1,V})$ on $[\tau-1,\bar{T}(V))$, or more explicitly:

$$Q'_{t,V} = \Psi_{t,V}\circ\ldots\circ\Psi_{t+k,V}(1),\ \text{where } k \equiv \min\{j\in\mathbb{N}\,|\,t+j+1\ge T\}.$$

II.3 The form of the optimal mixed strategies

Unlike Q_t, the functions $b_{t+1,V}$, $\Psi_{t,V}$ and $Q'_{t,V}$ depend on V; this index will be dropped for notational simplicity, when no confusion is possible. Anticipating slightly on the proof that $q_t=Q_t$ and $q'_t=Q'_t$ during a unique, continuous, mixed strategy phase of the equilibrium, the following lemma describes the dynamics of Q_t and Q'_t.

Lemma II.3: i) The function Q_t is continuous, equal to 1 for $t \leq \tau$ and then decreasing to its limit $Q_\infty = \alpha/\delta S > 0$.

ii) For all $V \in \Gamma$ such that $\overline{T}(V) > \tau-1$, the function $Q'_{t,V}$ restricted to $[\tau-1, \overline{T}(V)]$ is continuous and increasing.

Proof: cf. Appendix II.

During the mixed strategies phase, the (conditional) probability Q_t of a price adjustment in state t is decreasing. This somewhat surprising result can be explained as follows. Not only does the real gain $\delta P_0 - P_t - \alpha$ realized by storing before a price increase become larger over time, but the loss $P_t + \alpha - \delta P_{t+1}$ incurred if the adjustment does not materialize becomes smaller; to keep speculators indifferent between storing and not storing, the probability of realizing the gain must decrease over time.[15]

The increasing fraction $(1-x)Q'_t$ of buyers who store during the phase of mixed strategies, on the other hand, accords well with intuition. However, this does not occur because successful storage becomes increasingly profitable, but again because Q'_t must keep the firm indifferent between adjusting and not adjusting its price. The incremental benefit from adjustment (with respect to doing nothing) is $V_0 - \beta - V_{t+1}$, while the corresponding loss is $p_0 - p_{t+1}$ per storing customer. Both quantities increase with t (because V_{t+1} and p_{t+1} decrease with t), but the first one increases faster (intuitively, adjustement grows more urgent), so that the total gain rises faster than the loss per customer. To achieve indifference on part of the firm, the number $(1-x)Q'_t$ of storing customers must be increasing.

II.4 Characterization

The time $\underline{T}$ at which the active phase of the game (during which $q_t > 0$ and $q'_t > 0$ if $t > \tau$) effectively starts can now be computed, by moving backwards from $\overline{T}$, and looking for the unique zero (if any) of the decreasing and continuous function Q'_t.

<u>Definition II.4</u>: For all $V \in \Gamma$, define $\underline{T}(V)$ as follows:

i) If $T^*(V) \le \tau-1$, $\underline{T}(V) = T^*(V)$.

ii) If $T^*(V) > \tau-1$, $\underline{T}(V) = \min \{t \in [\tau-1, \overline{T}(V)] \mid Q'_{t,V} \ge 0\}$.

Note that $0 \le \underline{T}+1 < \overline{T}+1$. The three phases of the equilibrium, separated by $\underline{T}$ and $\overline{T}$, can now be linked together through the following definition.

<u>Definition II.5</u>: For all $V \in \Gamma$, define the following non-empty sets:

i) $\Omega_F(V) = \{q \in [0,1]^{\mathbb{R}_+} \mid \forall t \ge 0\colon q_t = 0 \text{ if } t \in [0,\underline{T});\ q_t \in [0,Q_t] \text{ if } t=\underline{T};$

$q_t = Q_t \text{ if } t \in (\underline{T},\overline{T});\ q_t \in [Q_t,1] \text{ if } t=\overline{T};\ q_t = 1 \text{ if } t \in (\overline{T},+\infty)\}$.

ii) $\Omega_C(V) = \{q' \in [0,1]^{\mathbb{R}_+} \mid \forall t \ge 0\colon q'_t = 0 \text{ if } t \in [0,\max(\tau,\underline{T}));\ q'_t = 0 \text{ if } t=\underline{T}\ge\tau;$

$q'_t \in [0,Q'_t] \text{ if } t=\tau>\underline{T};\ q'_t = Q'_t \text{ if } t \in (\max(\tau,\underline{T}),+\infty)\}$.

iii)

$$\Omega(V) = \begin{cases} \Omega_1(V) = (\Omega_F(V) \cap \{q \mid q_{\underline{T}} = Q_{\underline{T}}\}) \times \Omega_C(V) & \text{if } \underline{T}=\tau-1\ge 0 \text{ and } \P_{\tau-1}(q'_\tau)>0 \\ \Omega_2(V) = (\Omega_F(V) \cap \{q \mid q_{\underline{T}} = 0\}) \times \Omega_C(V) & \text{if } \underline{T}=\tau-1\ge 0 \text{ and } \P_{\tau-1}(q'_\tau)<0 \\ \Omega_0(V) = \Omega_F(V) \times \Omega_C(V) & \text{in all other cases.} \end{cases}$$

The correspondance Ω_F uniquely determines the firm's strategy q_t in all states $t \in \mathbb{R}_+$ except possibly $\underline{T}$ and $\overline{T}$, while Ω_C uniquely determines customers' (aggregate) strategy q'_t in all states except possibly τ.
The following characterization result is central to the paper.

<u>Theorem II.1</u>: For any $V \in \Gamma$ and any strategy pair (q,q'), the triplet (q,q',V) is a continuation value equilibrium if and only if $(q,q') \in \Omega(V)$. Moreover, $\underline{T}(V) \le T^*(V)$, with strict inequality if and only if $T^*(V) > \tau-1$.
<u>Proof</u>: <u>cf</u>. Appendix II.

The fist part of the theorem, together with Definition II.5, confirms that equilibrium strategies are first equal to zero, then mixed according to Q_t for the firm and Q'_t for speculators, then equal to one. The second part is quite intuitive: if $T^*+1\leq\tau$, the firm can adjust its price at T^*+1 before any speculation becomes profitable ($\underline{T}+1=T^*+1$); if $T^*+1>\tau$, on the contrary, speculators try to store before the price increase, while the firm tries to adjust its price before too many of them store; as in many models of asset price determination (foreign exchange, gold, etc.), this results in part of the price change's (here, the total change but with probability less than one) taking place earlier than it would in the absence of speculation ($\underline{T}+1<T^*+1$). In addition to advancing the potential price increase, the firm randomizes it, when $\tau<\underline{T}<\overline{T}$; but since both sides play at discrete intervals $t\in\mathbb{N}$, mixed strategies will not be efffectively implemented along the equilibrium path unless $[\underline{T},\overline{T})$ contains an integer, or $K[\underline{T}(V)]< K[\overline{T}(V)]$ (recall that $K[y]\equiv\min\{k\in\mathbb{N}|k\geq y\}$, $\forall y\in\mathbb{R}$). The following result gives a sufficient[16] condition for this to occur:

Proposition II.2: If $\max(\tau,\mu)\leq K[\overline{T}(V)]$, then $K[\underline{T}(V)] < K[\overline{T}(V)]$.
Proof: cf. Appendix II.

II.5 The four possible forms of equilibrium

Using the results of section II.4, one can completely describe continuation value equilibria[17] -and in particular any full equilibrium- of the game: they can take one of the four basic forms illustrated in Figures 1 to 2.3 (corresponding to the paragraphs below), on which the solid dots indicate the discrete states $t\in\mathbb{N}$ in which players act.

1-Pure Strategy Equilibrium: When $\overline{T}\leq\tau$, an equilibrium involves only pure strategies: the firm increases its price if the time elapsed since the last adjustment is greater than $\underline{T}$, and speculators store if it is greater than τ. The firm in fact adopts an (S,s) rule, resulting in price adjustments of periodicity $K[\underline{T}]+1$.

1.1: when $K[\underline{T}]\leq\mathrm{Int}[\tau]$, this adjustment occurs without any storage; it is the discrete time analog of the adjustment of Sheshinski and Weiss [1977], who deal with the limiting case $\tau=+\infty$ (so that $\underline{T}=T^*=\overline{T}$); cf. Figure 1.

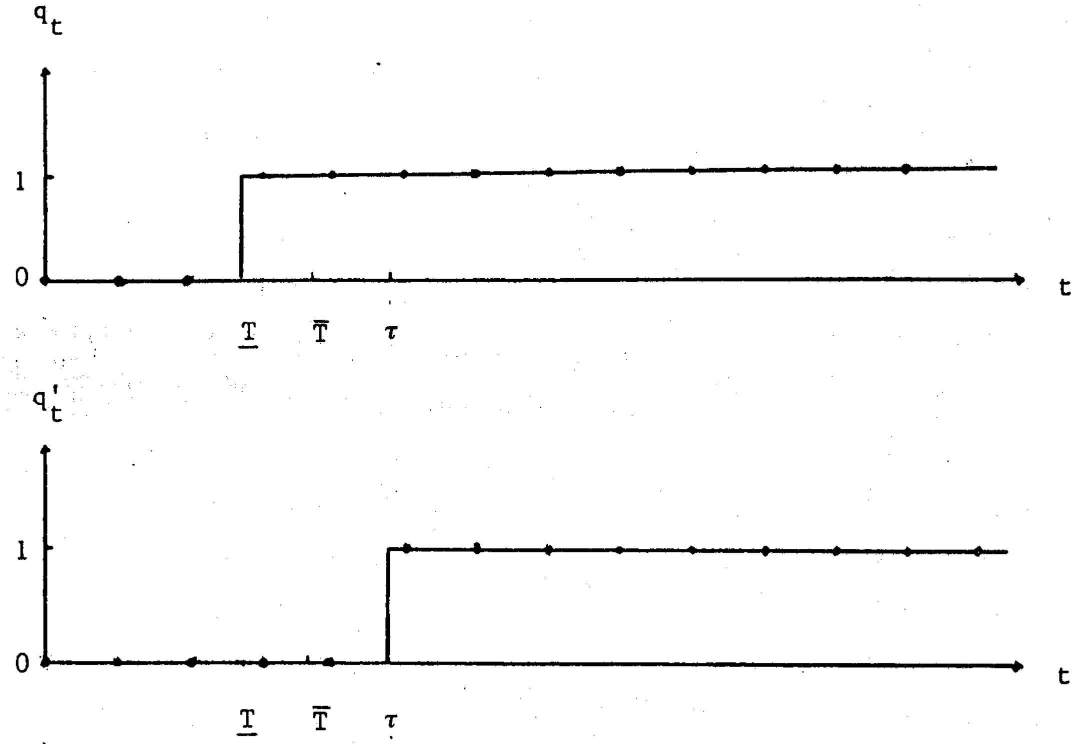

Figure 1: Pure Strategy Equilibrium with no Storage

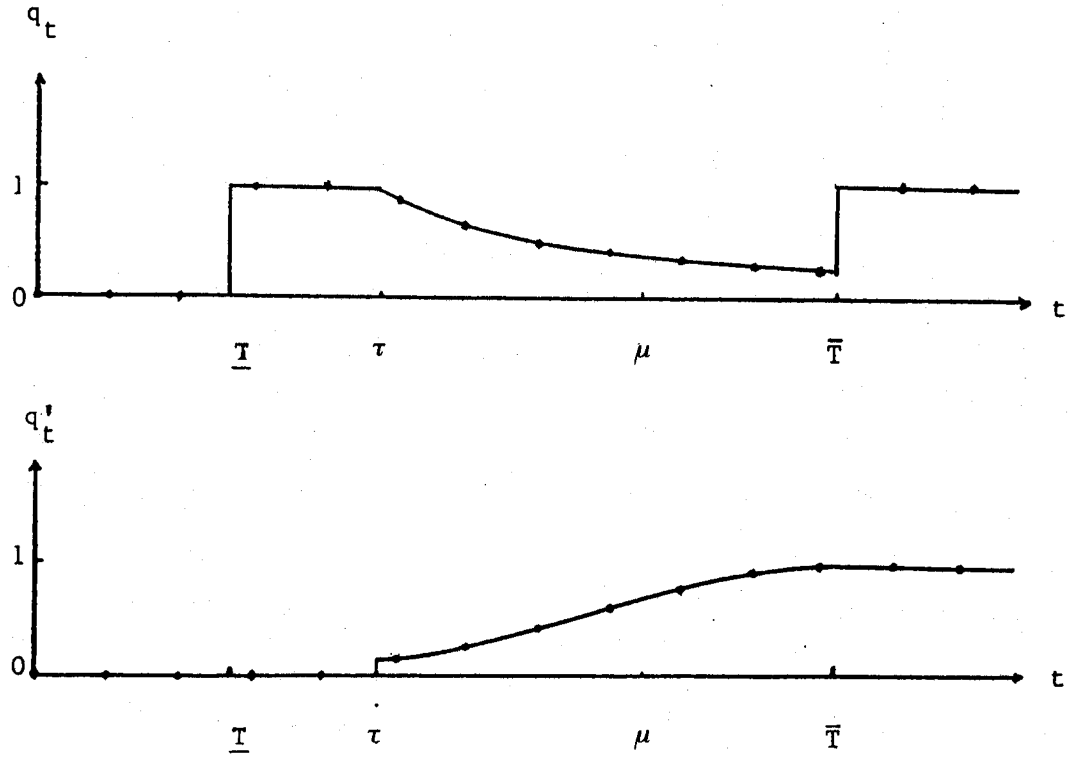

Figure 2.1: Mixed Strategy Equilibrium, Deterministic Outcome with no Storage

1.2: when $K[\underline{T}]=Int[\tau]+1$ (which requires $K[\bar{T}(V)]\leq Int[\mu]$), the adjustment occurs with all speculators storing; this case is identical, for all practical purposes, to 2.2 below, and is therefore not illustrated.

2-Mixed strategy equilibrium: When $\bar{T}>\tau$, there is a phase $(\max(\tau,\underline{T}),\bar{T})$ of mixed strategies on both sides; given that players act at discrete intervals, three types of outcome are possible.

2.1-Mixed strategy equilibrium with deterministic outcome and no storage: When $K[\underline{T}]\leq Int[\tau]$, the outcome is again an adjustment of periodicity $K[\underline{T}]+1\leq Int[\tau]+1$ and no storage. Only if the firm deviated -voluntarily or by mistake- so that the real price dropped below P_τ, would mixed strategies be implemented (cf. Figure 2.1).

2.2-Mixed strategy equilibrium with deterministic outcome and full storage: When $K[\underline{T}]>Int[\tau]$ and the interval $(\underline{T},\bar{T})$ contains no integer (which requires that $K[\bar{T}]\leq Int[\mu]<1/(1-x)$), the phase of mixed strategies is so short that players' actual moves "skip over it" to the final phase of pure strategies, and the outcome consists of price adjustments of periodicity $K[\underline{T}]+1$, with storage by all 1-x speculators. This case occurs when 1-x is small (so that μ is large): the firm maintains a deterministic (S,s) rule, forfeiting the small loss from storage for the benefit of charging the maximum price to non-speculators (cf. Figure 2.2).

2.3-Mixed strategy equilibrium with stochastic outcome and increasing storage: The case where $K[\underline{T}]>Int[\tau]$ and the interval $[\underline{T},\bar{T})$ contains at least one integer (which is guaranteed by $K[\bar{T}]>\max(Int[\tau],Int[\mu])$), gives rise to a radically new type of outcome. The firm might be said to follow an $(S,\tilde{s})$ real price rule, where the tilde indicates a random variable, which here has support in $[P_{\underline{T}+1}, P_{\bar{T}+1}]$.[18] The nominal price remains pegged for $K[\underline{T}]$ periods; in every following period there is a probability q_t that a price increase is about to take place; if it still has not occured after $K[\bar{T}]$ periods, it then takes place with probability one in the next period. As to buyers, they store in increasing numbers until the adjustment takes

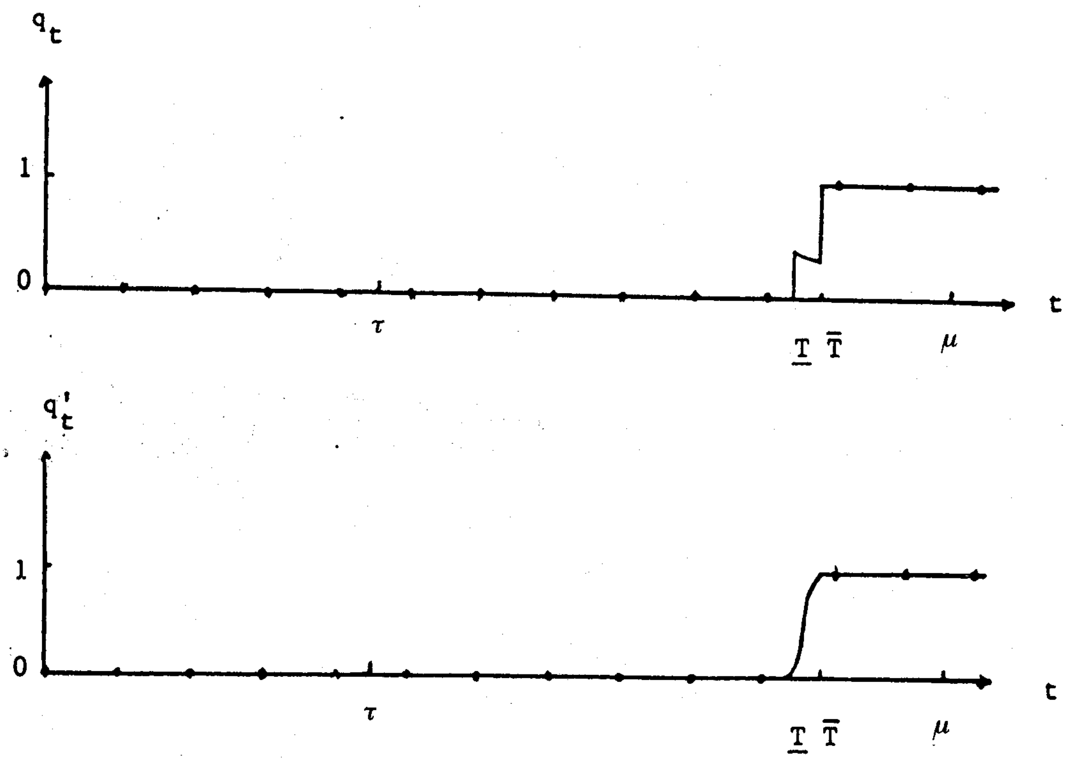

Figure 2.2: Mixed Strategy Equilibrium, Deterministic Outcome with Full Storage

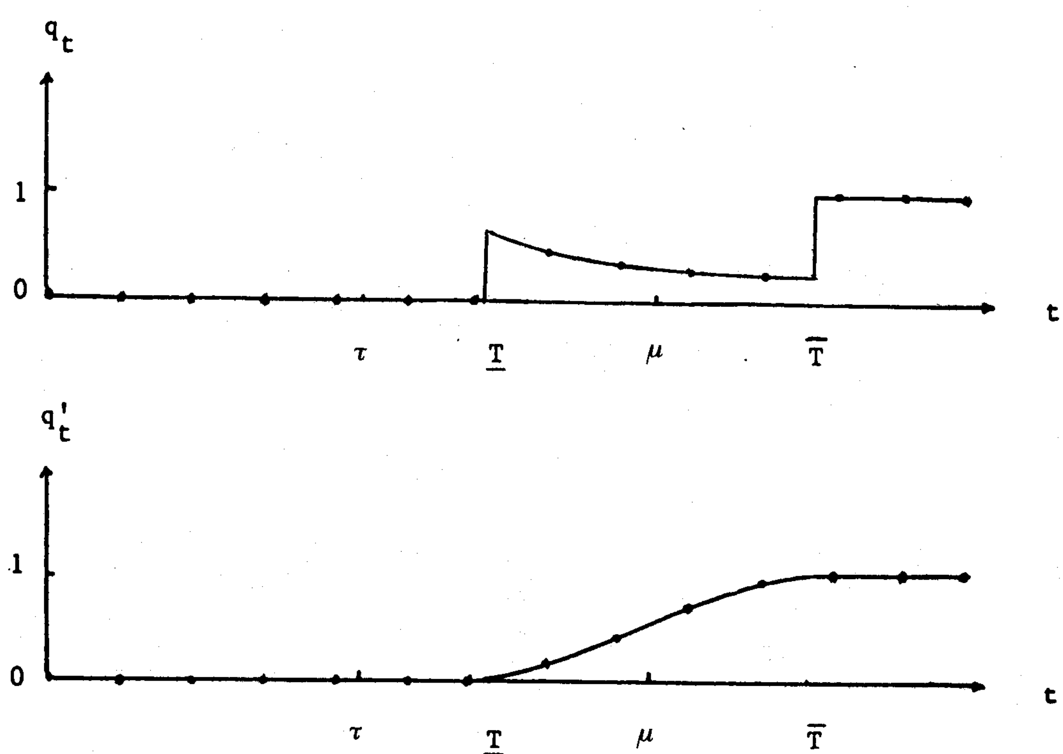

Figure 2.3: Mixed Strategy Equilibrium, Stochastic Outcome with Storage

place; if it has not occured after $K[\overline{T}]$ periods, there is a generalized "run" on the good (cf. Figure 2.3).[19]

III-EXISTENCE AND UNIQUENESS

III.1 Existence

An equilibrium of the game is a pair of strategies (q,q') such that (q,q',V_0) is a continuation value equilibrium, where V_0 is the expected present value of profits in state zero generated by strategies (q,q') -which will now be computed. For all $V\in\Gamma$ and any $q'\in\Omega_c(V)$, define from here on $\underline{K}=K[\underline{T}]$, $K^*=K[T^*]$ and $\overline{K}=K[\overline{T}]$. Since $\tau\notin N$[20], $\underline{K}\neq\tau$ hence $q'_{\underline{K}}$ is the same for all $q'\in\Omega_c(V)$, and it is legitimate to define:

$$(11) \qquad f(V) = \Sigma_{k=0}^{\underline{K}} \delta^k p_k + \delta^{\underline{K}} q'_{\underline{K}}(1-x)(p_{\underline{K}}-\delta p_0) + \delta^{\underline{K}+1}(V-\beta)$$

Moreover, Definition II.5 indicates that $q_k=q'_k=0$ for $k<\underline{T}(V)$ and that two cases are possible in state $\underline{K}\geq\underline{T}$: either $\underline{T}\notin \mathbb{N}$, so $\underline{K}>\underline{T}$ and $q_{\underline{K}}=Q_{\underline{K}}>0$, or $\underline{K}=\underline{T}\in \mathbb{N}$, hence $\underline{T}\neq\tau-1$, so $q_{\underline{K}}\in[0,Q_{\underline{K}}]\neq\{0\}$. In both cases, adjusting the price is one of the firm's preferred actions following state $\underline{K}$ (and never before), and its valuation in state zero is therefore $V_0=f(V)$. Thus, for (q,q') to be an equilibrium, V_0 must be a fixed point of f.

Lemma III.1: Let $V\in\Gamma$; for any sequence $(V^n)_{n\in N}$ converging to V, the sequence of functions $(Q'_{t,V^n})_{n\in N}$ converges to $Q'_{t,V}$ on $[\tau-1,+\infty)$ for the norm of uniform convergence.

Lemma III.2: The function f is continuous and has a fixed point in Γ.

Proofs: cf. Appendix III.

Let V_0 be such a fixed point. For any $(q,q')\in\Omega(V_0)\neq\emptyset$, (q,q',V_0) is a continuation value equilibrium, satisfying conditions (3) and (4) in all states (by Theorem II.1). Moreover, $V_0=f(V_0)$ is the initial expected present value of the firm's profits under the strategies (q,q'). Hence:

Theorem III.1: There exists a Markov perfect equilibrium of the game.

Proof: cf. Appendix III.

III.2 Uniqueness

By Theorem II.1, there is only one equilibrium (except for indeterminacies at threshold points) corresponding to a given value V_0; in fact, two equilibria with different valuations cannot exist either.

Theorem III.2: The Markov perfect equilibrium of the game is unique[21]*, up to possible indeterminacies of speculators' strategy at their threshold point τ, and of the firm's strategy at its threshold points $\underline{T}$ and $\overline{T}$.*

Proof: cf. Appendix III.

IV-WELFARE AND MACROECONOMIC IMPLICATIONS

IV.1 Destabilizing speculation:

The effects of speculation on price dynamics are best undestood by comparing the equilibrium outcome to the optimal periodic adjustment in the absence of storage (Sheshinski and Weiss [1977]). Let therefore K^{ns} be the state in which the price is adjusted with probability one when no customer can store, i.e. the value of $\underline{K}=K^*$ when x=1 or when $\alpha\geq\delta S$.

Proposition IV.1: In equilibrium:

i) if $K^{ns}\leq Int[\tau]$, then $\underline{K}=K^*=K^{ns}\leq\overline{K}$

ii) if $K^{ns}>Int[\tau]$, then $Int[\tau]\leq\underline{K}\leq K^{ns}\leq K^*\leq\overline{K}$

iii) if $K^{ns}>max(Int[\tau],Int[\mu])$ then $\underline{K}\leq\overline{K}-1$

Proof: cf. Appendix IV.

These results have a simple interpretation:

1) If, at the time $K^{ns}+1$ when the firm would adjust its price in the absence of storage, the magnitude of the price increase does not justify

storage by speculators ($K^{ns}+1 \leq Int[\tau]+1$), this deterministic policy remains optimal (Figure 1 applies).

2) If the price increase at $K^{ns}+1$ is sufficient to induce storage by speculators but the total loss which they inflict on the firm is not too large ($Int[\tau]+1 < K^{ns}+1 \leq Int[\mu]+1$) -because for instance there are few of them- adjustment at $K^{ns}+1$ remains optimal[22], although all speculators have stored (Figure 2.2 applies).

3) If, on the contrary, the threat of speculation is effective ($K^{ns}+1 > \max(Int[\tau], Int[\mu])+1$), the firm must implement a different strategy, leading to either a deterministic adjustment of shorter period $Int[\tau]+1$ (Figure 2.1 with $\underline{K}=Int[\tau]$), or a randomized adjustment which also attaches more weight to earlier dates (Figure 2.3). The price increase will thus generally occur before $K^{ns}+1$, as the outcome of a phase of price uncertainty and increasing amounts of storage by buyers, which may culminate in a generalized storing spree.

In this last case, speculation is destabilizing, of both prices and quantities, in any sense that one can think of; section IV.3 will show that it reduces social welfare as well. Moreover, these striking results arise in the absence of any stochastic shocks or private information, from the sole optimal dynamics of imperfect competition. Agents were assumed risk-neutral, for analytical tractability, but the essential results of the model would remain with risk-aversion, since even risk-averse buyers will store when faced with a certain and large enough price increase. Randomisation is therefore still required to elude speculation, in spite of the welfare costs it may entail for the firm and its customers.

IV.2 Inflation causes price uncertainty:

Inflation lies here at the origin of speculative storage. The comparative dynamics of the equilibrium with respect to the rate of inflation π are therefore of particular interest, both to ascertain whether speculation increases with π, and to shed some light on the positive relationship between inflation and relative price uncertainty which features prominently in macroeconomic discussions of the cost of inflation (cf. Modigliani and Fischer [1978], Fischer [1981a,b], [1984]). Such a correlation is confirmed by many empirical studies (surveyed in Fischer [1981a] and Taylor [1981]), but no theoretical basis has been offered for it.[23] The mechanism analyzed here, by which an optimal type and level of noise are injected into price dynamics in order to elude speculation, provides such a foundation. It is worth noting, moreover, that price randomness is here endogenous, in contrast to the models of Sheshinski and Weiss [1983] or Caplin and Spulber [1985] where inflation is an exogenous stochastic process, and to models which rely on a combination of misperceptions or staggered contracts with exogenous shocks to generate attenuated noise in the price system.[24]

The equilibrium unfortunately depends on π in too complex a manner to allow comparative dynamics to be performed analytically, as can be done with β or α (cf. Bénabou [1986a]). The problem must be solved numerically, and Tables 1 and 2 report the results of some of these computations, which point to the following characteristics.

Results of simulations: As the rate of inflation increases:

i) The support $\{\underline{K}+1,..,\bar{K}+1\}$ of the random period $\tilde{T}$ of price rigidity separating consecutive price adjustments shifts down by steps; its expectation $E(\tilde{T})$ decreases with large enough increases in π, but may increase with small ones (when the support remains unchanged).

ii) The amount of speculation increases in every period.

iii) The variance of the following period's price increases with π in all periods preceding the occurrence of the adjustment; thus, more inflation causes more uncertainty.[25]

It is also interesting to note that, for any given inflation rate, the

Table 1 (1.1 to 1.3): Variation of the equilibrium outcome with the rate of inflation. For each value of π, the upper line gives the unconditional probability f_k of a price change in each period ($f_{k+1}=(1-q_0)..(1-q_{k-1})q_k$ for all $k\geq1$), while the lower line gives the proportion q_k' of speculators who store. The symbol "-" stands for zero. The basic period is a week, but π is given here as an annual rate. The following parameters are fixed: $\beta/S=.25$, $c/S=.0$, $\alpha/S=.02$, $x=.5$, $r=.05$ per year.

Table 1.1

$\pi \backslash k$	13	14	15	16	17	18	19	20	21	22	23	24	25	26	$I[\tau]+1$	$K^{ns}+1$	$E(\tilde{T})$
4	-	-	-	-	-	-	-	-	-	-	-	-	-	1.00	29	26	26.00
	-	-	-	-	-	-	-	-	-	-	-	-	-	-			
5	-	-	-	-	-	-	-	-	-	-	1.00	-	-	-	23	23	23.00
	-	-	-	-	-	-	-	-	-	-	-	-	-	-			
7	-	-	-	-	1.00	-	-	-	-	-	-	-	-	-	17	20	17.00
	-	-	-	-	-	-	-	-	-	-	-	-	-	-			
8	-	-	-	.96	.04	.004	.001	-	-	-	-	-	-	-	15	19	16.05
	-	-	.37	.70	.91	1.00	-	-	-	-	-	-	-	-			
10	-	.90	.09	.01	.004	-	-	-	-	-	-	-	-	-	12	17	14.13
	.26	.65	.89	1.00	-	-	-	-	-	-	-	-	-	-			

Table 1.2

$\pi \backslash k$	4	5	6	7	8	9	10	11	12	13	$I[\tau]+1$	$K^{ns}+1$	$E(\tilde{T})$
20	-	-	-	-	-	-	.71	.19	.06	.04	7	13	10.44
	-	-	-	-	-	.36	.77	.98	1.00				
30	-	-	-	-	.65	.20	.08	.01	-	-	5	11	8.56
	-	-	-	.23	.74	.98	1.00	-	-	-			
50	-	-	-	.53	.22	.25	-	-	-	-	3	9	7.72
	-	-	.66	.97	1.00	-	-	-	-	-			
100	-	.52	.21	.27	-	-	-	-	-	-	2	7	5.75
	.49	.94	1.00	-	-	-	-	-	-	-			

Table 1.3 (Table 1 continued)

$\pi\backslash k$	2	3	4	5	6	7	$I[\tau]+1$	$K^{ns}+1$	$E(\tilde{T})$
200	-	-	.50	.50	-	-	2	5	4.50
	-	.57	1.00	-	-	-			
300	-	-	.45	.55	-	-	1	5	4.55
	-	.83	1.00	-	-	-			
500	-	.53	.47	-	-	-	1	4	3.47
	.50	1.00	-	-	-	-			
1000	-	.48	.52	-	-	-	1	4	3.52
	.83	1.00	-	-	-	-			

Table 2: The one-period ahead variance of the real price (x10^4) and the rate of inflation (annual rate, in %)

Table 2.1

$\pi\backslash k$	7	8	9	10	11	12	13	14	15	16	17	18
8	-	-	-	-	-	-	-	-	.23	.55	.88	1.20
10	-	-	-	-	-	-	.59	1.00	1.41	1.81	-	
20	-	-	2.46	3.29	4.11	4.92	-	-	-	-	-	-
30	3.55	4.80	6.04	7.72	-	-	-	-	-	-	-	-

Table 2.2

$\pi\backslash k$	2	3	4	5	6	7	8
40	-	-	-	-	4.65	6.33	7.99
50	-	-	-	-	7.02	9.10	11.16
100	-	-	10.38	14.51	18.53	-	-
200	-	16.42	24.12	-	-	-	-
300	-	25.29	35.73	-	-	-	-
500	24.00	39.76	-	-	-	-	-
1000	41.66	65.10	-	-	-	-	-

uncertainty faced by buyers <u>increases</u> over time, until it is suddenly resolved by the occurence of the price adjustment. Inflation thus generates -and when it increases, exacerbates- growing price uncertainty, a shortening of the price cycle ($E(\tilde{T}) < K^{ns}+1$ in general, <u>cf</u>. Table 1)), and mounting speculative storage.[26] These results confirm and give precise meaning to the following description by Buchanan and Wagner [1977] (quoted in Fischer [1984]):

> "Inflation detroys expectations and creates uncertainty;...it prompts behavioural responses that reflect a general shortening of the time horizon".

<u>IV.3 Some "new" welfare costs of inflation</u>:

The storable nature of most commodities provides each price-maker with an incentive to inject some uncertainty into the price system (price competition between oligopolists may have similar but weaker effects; <u>cf</u>. Maskin and Tirole [1985]). Such a private incentive to make one's price noisy could be the source of a price uncertainty externality, which no ones likes to experience but all contribute to generate. Even in the absence of risk-aversion, price uncertainty has a cost, because it prevents the synchronization of price decisions between suppliers and their customers (synchronization of output price adjustment with wage contracting, for instance); as a result, inflationary pressures may propagate relative price distortions and misallocations throughout the economy (Blanchard [1983]). While an analysis of these phenomena would require a multisectorial model, the present one already explicitly identifies several other costs of inflation, to be added to the lists drawn by Modigliani and Fischer [1978] and Fischer [1981]. Indeed:

<u>*Proposition IV.2*</u>*: Expected intertemporal social welfare is:*

$$(12) \qquad SW_0 = \frac{S-c}{1-\delta} - \frac{[\beta + (1-x)(\alpha+c(1-\delta))\ \Sigma_{k=\underline{K}}^{\bar{K}}\ \delta^k(1-q_{\underline{K}})\ldots(1-q_{k-1})q_k']}{[1-\delta\ \Sigma_{k=\underline{K}}^{\bar{K}}\ \delta^k(1-q_{\underline{K}})\ldots(1-q_{k-1})q_k]}$$

<u>Proof</u>: <u>cf</u>. Appendix IV.

The first term is welfare in the absence of inflation; the second one is therefore the total social cost of inflation (or of the speculation it induces), which has three components: price adjustment costs, storage costs and the intertemporal misallocation of production due to speculative purchases. These three effects are linked together by the stochastic process governing the date of price adjustment, so that a change, say in β or x, induces changes in the whole sequence (q,q') and therefore in all components of the welfare loss. Apart from this social cost, a stochastic equilibrium gives rise in each period to a significant amount of income redistribution between firm and customers, depending on whether the price increase materializes or not.

IV.4-Long run equilibrium and aggregation

The preceding sections showed how discontinuous and even randomized price dynamics arose at the level of individual price-setters in response to a constant rate of increase in the general price level. This raises two important and related questions, which were addressed by Caplin [1985] in the context of (S,s) inventory policies and Caplin and Spulber [1985] in that of (S,s) real price rules. First, are these individual strategies consistent with the assumed general inflationary process, both individually (a given firm's price should increase, on average, at the rate π) and in the aggregate (an index of many such firms' prices should increase at the rate π)? Secondly, what is the cross-sectional price distribution generated by the individual strategies of many such sellers?

IV.4.1 The steady state distribution of real prices

Consider a sector of an economy, consisting of a continuum of identical monopolistic sellers of (non-substitutable[27]) storable goods, which engage in optimally randomized $(S,\tilde{s})$ real price policies with respect to some common aggregate price index P^* (for instance the cost of labor).[28] These firms will be indexed by $i\in[0,1]$. For all $t\in \mathbb{N}$, define $h(t) \equiv (h_0(t),..,h_{\bar{K}}(t))$, where $h_k(t)$ is the proportion of firms in state k, i.e. with a real price of $P_k=S(1+\pi)^{-k}$, in period t (h(t) can also be interpreted as an unconditional prior on the state of a single firm at time t). Because firms form a continuum, if $h_k(t)>0$ there is an infinity of them in state k; by the law of large numbers, the fraction of these who implement a price adjustment in period t+1 is equal to the probability q_k of such an adjustment for an individual firm among them; the remaining fraction $1-q_k$ let their real price be eroded to P_{k+1}. Thus:

$$(13)\qquad \begin{cases} h_0(t+1) = \Sigma_{k=0}^{\bar{K}} q_k h_k(t) \\ h_j(t+1) = (1-q_{j-1})h_{j-1}(t) \qquad (\forall j\in\{1,..\bar{K}\}) \end{cases} \quad \text{or:}$$

$$(14)\qquad h(t+1) = h(t)\underline{M},$$

where:

$$(15)\qquad \underline{M} = \begin{bmatrix} q_0 & 1-q_0 & 0 & . & . & . & . & 0 \\ q_1 & 0 & 1-q_1 & 0 & . & . & . & 0 \\ . & & & & & & . & . \\ q_k & 0 & . & 0 & 1-q_k & 0 & . & 0 \\ . & & . & . & . & . & . & . \\ q_{\bar{K}-1} & 0 & . & . & . & . & . & 1-q_{\bar{K}-1} \\ 1 & 0 & . & . & . & . & . & 0 \end{bmatrix}$$

The dynamics of the cross-sectional distribution of real prices are therefore characterized by a Markov chain with transition matrix $\underline{M}$, which clearly also governs the evolution of any individual firm's real price. Under certain conditions (examined below) the long-run behaviour of the system (single firm or continuum) can be precisely described by the invariant, or stationary, probability distribution of this Markov chain (cf. Feller [1968]). By (14), a distribution h is invariant over time if and only if $h=h\underline{M}$, i.e. if it is a left-eigenvector of $\underline{M}$ with eigenvalue 1.

Proposition IV.3: The Markov chain governed by $\underline{M}$ has a unique invariant probability distribution $h^* = (h_0^*, h_1^*, .., h_{\bar{K}}^*)$, defined by:

$$(16)\qquad (\forall k\in\{0,..,\bar{K}\}):\ h_k^* = (1-q_0)\dots(1-q_{k-1})/H,$$

where: $H = \Sigma_{k=0}^{\bar{K}} (1-q_0)\dots(1-q_{k-1}) = E[\tilde{T}]$ *(and* $q_{-1}\equiv 0$*).*

Proof: cf. Appendix IV.

This invariant, or steady-state, distribution is uniform ($h_k^*=1/H$) over the real prices $\{P_0,..P_{\underline{K}}\}$ belonging to the phase of non-stochastic adjustment (as in Caplin and Spulber [1985]), then decreasing (approximately geometrically) over the real prices $\{P_{\underline{K}+1},..P_{\bar{K}}\}$ which are reached through randomized adjustments. The analysis of individual and aggregate price behaviour will be focused mainly on the long run, by assuming that real prices or priors are initially -and remain- distributed according to h. When price strategies are non-stochastic (S,s) rules, this restriction is somewhat arbitrary, because the distribution $h(t)=h(0)\underline{M}^t$ does not generally converge to h^* as t tends to infinity, since $\underline{M}$ is then cyclical (of index $\bar{K}+1$). In particular, a non-negligible group of firms starting with the same real price will remain synchronized forever, generating a component of the cross-sectional distribution which cycles through all states and causes any aggregate index to be discontinuous. In the case of effectively randomized $(S,\tilde{s})$ rules, on the contrary, firms sort themselves out through different random drawings, so that:

<u>Theorem IV.4</u>: If firms' common $(S,\tilde{s})$ price strategy is randomized, i.e. if $q_{\underline{K}}<1$, the cross-sectional distribution h(t) of their real prices converges to the invariant distribution h^*, for any initial h(0).

<u>Proof</u>: <u>cf</u>. Appendix IV.

Similarly, when $q_{\underline{K}}<1$, any unconditional prior over the state of an individual firm converges to h^*. With either interpretation, one is thus justified in identifying the long-run with the steady-state distribution.

<u>IV.4.2 Individual price strategies and general inflation</u>

Denoting by $E^*[\cdot]$ the expectation operator with respect to the distribution h^*, one can compute the average (unconditional expectation) inflation rate of an individual firm's nominal price:

Proposition IV.5: Let $P_i(t)$ denote firm i's nominal price at time t. Then, for all $i \in [0,1]$: $E^*[Log(P_i(t+1)/P_i(t))] = Log(1+\pi)$.

Proof: cf. Appendix IV.

As in Caplin and Spulber [1985], who deal with the case of a deterministic (S,s) policy in a given stochastic inflationary environment, $P_i(t+1)/P_i(t)$ is therefore a geometric mean-preserving spread of the aggregate inflation rate. Models of fixed costs of price adjustment thus seem to possess the general property of being noise-amplifying (and even here, endogenously noise-generating): the resulting price dynamics are more noisy than the inflationary process in response to which they arise. This feature stands in sharp contrast to the noise-dampening characteristic of models based on misperceptions of nominal and real signals (à la Lucas) or on any type of convex adjustment costs, where the only uncertainty in the price system is a fraction of the exogenous noise injected into the model. The consistency of individual pricing strategies with the initial assumption of smoothly inflating aggregate prices will now be established.

Proposition IV.6: If firms' real prices are distributed according to the invariant distribution h^*, any index of their nominal prices of the form $\bar{P}(t) = G[\int_0^1 w(P_i(t))di]$ which is homogeneous of degree one, grows at the rate π. If adjustments are randomized ($q_K < 1$), any such index converges over time to an exponential trend of rate π, for any initial distribution of real prices.

Proof: cf. Appendix IV.

The indices covered by this proposition include in particular the arithmetic and geometric averages. The individual randomized $(S,\check{s})$ price policies of a large number of monopolistic firms thus aggregate back to a price index inflating at the same rate π as the one in response to which they arose, and this result has important macroeconomic implications. First, even a constant aggregate rate of inflation can at the same time generate and "cover up" a large amount of uncertainty and misallocations at the microeconomic level; thus, potentially large social costs are incurred even in a smoothly inflating economy, such as one where the growth rate of the money supply is constant. As was seen in sections IV.1 to IV.3, these include menu costs, storage costs, distorsions in the timing of production and sales, and price uncertainty. Secondly, although price-setters keep pace with inflation in a growth rate sense (increasing their prices, on average and in the aggregate, at the same rate as the rest of the economy), inflation alters the relative prices of a sector where (S,s) or $(S,\tilde{s})$ rules prevail and the rest of the economy; indeed, both $P^*(t)$ and any appropriate aggregate index of $(S,\tilde{s})$ firms' prices, such as for instance the arithmetic average P^a, grow at the rate π, but the ratio:

(17) $$P^a(t)/P^*(t) = \int_0^1 P_i(t)di = S[\Sigma_{k=0}^{K} h_k^*(1+\pi)^{-k}]$$

clearly depends on π, and is not equal to 1 except by coincidence. This non-neutrality of the inflation rate with respect to relative prices can serve as a basis for a model of the Phillips curve (Naish [1985]) or of search market equilibrium with optimal price dynamics (Bénabou [1986b]). Finally, it is worth noting that on the transition path to the steady state (following for instance an unanticipated general inflationary shock), the time varying cross-sectional distribution of real prices will generate a (dampened) cycle in aggregate inventories.

IV.5 A related topic: exchange rates

A country which tries to peg its exchange rate, but has a positive inflation differential with its trading partners, will have to devalue repeatedly in order to maintain purchasing power parity or trade balance in the long run. Since devaluations are costly, politically or because they require international bargaining (as in the case of the European Monetary System), and cannot take place at predictable dates because of speculation, the situation is very similar to the price adjustment problem treated here. A variant of the model could therefore be applied to the game opposing speculators to a central bank which tries to peg the exchange rate between occasional devaluations, and is able to use partial capital controls to limit the total amount of speculation (only allowing, for instance, speculation with trade receipts and payements) as well as the adjustment of the interest rate. The model predicts in particular that the central bank will generally inject randomness into its exchange rate policy; this could explain endogenously the time pattern of interest rates which Giavazzi and Pagano [1985] observe within the European Monetary System and attribute to the realization of exogenous shocks on certain parameters.

CONCLUSION

The optimal price and storage strategies for a firm selling a storable good in an inflationary environement and its speculating customers were derived as the Markov perfect equilibrium of a dynamic game with infinite horizon. It was shown that the firm generally introduces randomness into its price dynamics, while customers store in increasing numbers, with possibly a final generalized "run" on the good. These results establish that speculation can be destabilizing, even in a context of perfect information, and provide a theoretical foundation for the often-mentioned claim that inflation causes price uncertainty.

APPENDIX I:

Proof of Theorem I:

i,ii) Proved in the text.

iii) The following result will prove useful many times:

(A1) $(\forall t \geq \tau - 1)(p_{t+1} \leq p_{\tau} = \delta S - \alpha - c < \delta(S-c) = \delta p_0)$

Let now $t \geq 1$ such that $t > \tau$, $q_{t-1}=0$ and $q_t=1$; then $q^{\ell}_{t-1}=0$ and $q^{\ell}_t=1$ from (ii). Moreover, as $q_{t+1}=1$ is always feasible, (1) implies:

$$V_{t+1} \geq p_{t+1} + \delta(V-\beta) + q^{\ell}_{t+1}(1-x)(p_{t+1} - \delta p_0).$$

Since $q_t = q^{\ell}_t = 1$, (4) requires:

$$(1-x)(p_0 - p_{t+1}) = q^{\ell}_t(1-x)(p_0 - p_{t+1}) \leq V - \beta - V_{t+1} \Rightarrow$$

$$(1-x)(p_0 - p_{t+1}) \leq (1-\delta)(V-\beta) - p_{t+1} + q^{\ell}_{t+1}(1-x)(\delta p_0 - p_{t+1}) \Leftrightarrow$$

$$(1-\delta)(V-\beta) \geq (1-\delta q^{\ell}_{t+1})(1-x)p_0 + (1-(1-x)(1-q^{\ell}_{t+1}))p_{t+1}.$$

Since $q_t = q^{\ell}_t = 1$, $V_t = (2-x)p_t + \delta(V - \beta - p_0(1-x))$; since $q_{t-1} = q^{\ell}_{t-1} = 0$, (4) requires:

$$V - \beta - V_t \leq 0(p_0 - p_t) \text{ or: } (1-\delta)(V-\beta) \leq (2-x)p_t - (1-x)\delta p_0, \Rightarrow$$

$$(2-x)p_t - (1-x)\delta p_0 \geq (1-\delta q^{\ell}_{t+1})(1-x)p_0 + (1-(1-x)(1-q^{\ell}_{t+1}))p_{t+1} \Leftrightarrow$$

$$(2-x)p_t - (1-x)p_0 \geq (1-q^{\ell}_{t+1})(1-x)\delta p_0 + (1-(1-x)(1-q^{\ell}_{t+1}))p_{t+1} > p_{t+1}$$

by (A1). Hence: $F(t;\theta,x) \equiv p_t - (z p_{t+1} + (1-z)p_0) > 0$,

where $z \equiv 1/(2-x)$. Equivalently, since $p_k = S\theta^{-k} - c$:

$$\mu(\pi,x) \equiv \text{Log}[(1-z/\theta)/(1-z)]/\text{Log}(\theta) > t.$$

Therefore, $q_t=1$ and $q_{t-1}=0$ is impossible for $t > \max(\tau,\mu)$. q.e.d.

iv) The sign of the derivatives of τ are staightforward algebra, and so is that of $\partial\mu/\partial x$. As to $\partial\mu/\partial\pi = \partial\mu/\partial\theta$:

$$\theta(1-z/\theta)(\text{Log}(\theta))^2 \partial\mu/\partial\theta = \text{Log}(\theta)z/\theta + \text{Log}[(1-z)/(1-z/\theta)](1-z/\theta)$$

$$< \text{Log}\{z\theta/\theta + [(1-z)(1-z/\theta)/(1-z/\theta)]\} = 0$$

because $z/\theta \in (0,1/2)$ and the Log function is concave. As a consequence:

$$\mu(\theta,x) < \lim_{\theta \to 1^+} (\mu(\theta,x)) = z/(1-z) = 1/(1-x)$$ q.e.d.

APPENDIX II

Notations: from here on, let $\sigma(V)\equiv(1-\delta)(V-\beta)$ and $\Phi(V)\equiv(1-\delta)(V-\beta-(1-x)p_0)$; the dependance of σ and Φ on V will be omitted when no confusion results.

Proof of Lemma II.1:

i) Since $t+1<\tau$: $q'_{t+1}=0$ and: $V_{t+1}\geq p_{t+1}+\delta(V-\beta)+0$. Hence:

$V-\beta-V_{t+1}\leq(1-\delta)(V-\beta)-p_{t+1}=\sigma-p_{t+1}=p_{T^*+1}-p_{t+1}<0$, so $q_t=0$ by (4). q.e.d.

ii) Claim 1: $(\forall t\in\mathbb{R}_+)$ $(t\geq\overline{T} \Rightarrow q_t>0)$. Indeed, let $q_t=0$ for such a t.

* One cannot have $q_{t+1}>0$, otherwise:

$V_{t+1}=p_{t+1}+\delta(V-\beta)+q'_{t+1}(1-x)(p_{t+1}-\delta p_0)$ and since it is always the case that: $q'_{t+1}(p_{t+1}-\delta p_0)\leq 0$ (the first term is zero when $t<\tau$ and the second is negative when $t\geq\tau$ by (A1)), this would imply: $V_{t+1}\leq p_{t+1}+\delta(V-\beta)$, hence: $V-\beta-V_{t+1}\geq\sigma-p_{t+1}>\Phi-p_{\overline{T}+1}=0$, contradicting (4) with $q_t=0$.

* One cannot have $(\forall n\in\mathbb{N},\ q_{t+n}=0)$, otherwise:

$(\forall n)$ $V_{t+n}=p_{t+n}+\delta V_{t+n+1}$ so: $V_{t+1}=\Sigma_{k=0}^{\infty}\ \delta^k p_{t+1+k}<p_{t+1}/(1-\delta)<V-\beta$ since $(1-\delta)(V-\beta)=\sigma>\Phi=p_{\overline{T}+1}\geq p_{t+1}$. Thus, again: $V-\beta-V_{t+1}>0$, which contradicts $q_t=0$. Thus Claim 1 is established.

Claim 2: $(\forall t\in R_+)$ $(t>\overline{T}$ and $q_t\in(0,1) \Rightarrow q_{t+1}\in(0,1))$. Indeed, if $q_t\in(0,1)$ and $q_{t+1}=1$ for such a t, then:

$$V_{t+1}=p_{t+1}+\delta(V-\beta)+q'_{t+1}(1-x)(p_{t+1}-\delta p_0),\ \text{hence:}$$

$$V-\beta-V_{t+1}=\sigma-p_{t+1}-q'_{t+1}(1-x)(p_{t+1}-\delta p_0)>0,$$

because the first term is positive since $t+1\geq\overline{T}+1>T^*+1$, and the second is non-negative due to (A1). If $t<\tau$, $q'=0$, so (4) and the above inequality imply $q_t=1$, a contradiction. If $t\geq\tau$, $q_{t+1}=1$ implies $q'_{t+1}=1$, hence:

$$V-\beta-V_{t+1}=\sigma-(2-x)p_{t+1}+(1-x)\delta p_0=q'_t(1-x)(p_0-p_{t+1})\leq(1-x)(p_0-p_{t+1})$$

by (4), hence: $p_{t+1}\geq\sigma-(1-\delta)(1-x)p_0=\Phi=p_{\overline{T}+1}$, contradicting $t>\overline{T}$. Thus Claim 2 is established. To prove the first part of (ii) by contradiction, it is therefore sufficient to show that $q_t\in(0,1)$ is impossible for $t>\max(\overline{T},\tau)$. Indeed, $q_t\in(0,1)$ would imply: $q_{t+1}\in(0,1)$, $q'_{t+1}=1$ and:

$$V-\beta-V_{t+1}=\sigma-\delta p_0(1-x)-p_{t+1}(2-x)=\Phi-p_{t+1}+(1-x)(p_0-p_{t+1})>(1-x)(p_0-p_{t+1})$$

so $q_t=1$ by (4), a contradiction. Therefore, for all $t>\overline{T}$, $q_t=1$ is the only possible case. We now prove the second part of (ii). For $0 \le t = \bar{T} \le \tau$, $q_{t+1} = 1$ from what precedes, so: $V_{t+1} \le p_{t+1} + \delta(V-\beta)$ hence:

$$V-\beta-V_{t+1} \ge \sigma - p_{t+1} > \Phi - p_{t+1} = 0.$$

If $t<\tau$, $q_t' = 0$ so $q_t = 1$ by (4); if $t=\tau$, $q_t < 1 = Q_t$ would imply $q_t' = 0$, hence $q_t = 1$, a contradiction. q.e.d.

iii) By (ii), $q_{t+1} = 1$, and by Claim 1 above, $q_t > 0$. Thus either $q_t = 1$, and then $q_t' = 1$, or $q_t \in (0,1)$, so:

$$V-\beta-V_{t+1} = \sigma-(2-x)p_{t+1}+(1-x)\delta p_0 = q_t'(1-x)(p_0-p_{t+1}).$$

Therefore $q_t' < 1$ would imply: $\Phi < p_{t+1}$, a contradiction. It is thus necessary that $q_t' = 1$, hence by (3): $q_t \ge (P_t + \alpha - \delta P_{t+1})/[\delta(P_0 - P_{t+1})]$. q.e.d.

Proof of Lemma II.2:

Since: $V_{t+1} \ge p_{t+1} + \delta(V-\beta) + q_{t+1}'(1-x)(p_{t+1} - \delta p_0)$, with equality when $q_{t+1} > 0$, then:

$$V-\beta-V_{t+1} \le \sigma - p_{t+1} - q_{t+1}'(1-x)(p_{t+1}-\delta p_0) = (1-x)(p_0-p_{t+1})\Psi_t(q_{t+1}')$$

by definition, hence (i); (ii,iii) result from (i) and (4). q.e.d.

Proof of Lemma II.3:

i) Continuity is straightforward. For $t \ge \tau$, $-(\partial Q_t/\partial t)$ has the sign of:

$$\begin{vmatrix} 1/\delta - 1/\theta & \alpha/\delta S \\ -1/\theta & 1 \end{vmatrix} = 1/\delta - 1/\theta(1-\alpha/\delta S) > 1/\delta - 1/\theta > 0$$ q.e.d.

ii) The proof of monotonicity and continuity proceeds in three steps.

First it is shown by backward induction that:

(A2) $(\forall t \in [\tau-1, \bar{T}+1))\ (Q_t' \ge (\sigma - p_t)/[(1-\delta)(1-x)p_0]$,

with strict inequality on $[\tau-1, \bar{T})$.

Indeed, for $t \in [\bar{T}, \bar{T}+1]$: $Q_t' = 1$, and the inequality is equivalent to $p_t \ge \Phi = p_{\bar{T}+1}$, so the property is true. Suppose now that it holds for $t \ge \tau$:

$$(1-x)Q_{t-1}' = [(\delta p_0 - p_t)(1-x)Q_t' + \sigma - p_t]/(p_0 - p_t)$$

$$\ge [(\delta p_0 - p_t)(\sigma - p_t)/(p_0 - \delta p_0) + \sigma - p_t]/(p_0 - p_t)$$

$$= [(\sigma - p_t)(\delta p_0 - p_t + p_0 - \delta p_0)]/[(p_0 - p_t)(p_0 - \delta p_0)]$$

$$= (\sigma - p_t)/(p_0 - \delta p_0) > (\sigma - p_{t-1})/(p_0 - \delta p_0).$$

hence the property holds for t-1, and (A2) is established. It implies:

(A3) $\quad (\forall t\in[\tau-1,\bar{T}])\ (Q'_t > (\sigma-p_0)/[(1-\delta)(1-x)p_0] \equiv \underline{y})$.

Secondly, the function $\Psi:(t,y) \dashrightarrow \Psi_t(y)$ is not only clearly continuous, but also increasing in both arguments on $[\tau-1,+\infty)\mathrm{x}(\underline{y},+\infty)$. Indeed, it is affine in y, with a positive coefficient by (A1), and decreasing in p_{t+1} because the determinant:

$$\begin{vmatrix} -1-(1-x)y & \sigma+(1-x)y\delta p_0 \\ -1 & p_0 \end{vmatrix} = \sigma-p_0(1+(1-x)(1-\delta)y)=p_0(1-x)(1-\delta)(\underline{y}-y).$$

is negative for $y>\underline{y}$.

Finally, (ii) can now be established by backward induction on successive intervals $I_k \equiv [\max(\tau-1,\bar{T}-k),\bar{T}-(k-1)]$. On the semi-open interval $[\max(\tau-1,\bar{T}-1),\bar{T})$, by definition: $Q'_t=\Psi_t(1)$ which is continuous and increasing due to the above properties of Ψ (note that $\underline{y}<1$). As the left limit $\Psi_{\bar{T}}(1)$ of Q'_t at $\bar{T}$ is easily seen to equal $1=Q'_{\bar{T}}$, these two properties are also true on the closed interval I_1. Suppose now that the proposition holds up to rank k. On $I_{k+1}=[\max(\tau-1,\bar{T}-k-1),\bar{T}-k]$, $Q'_t=\Psi_t(Q'_{t+1})$; since Q'_{t+1} is continuous and increasing on I_k, and Ψ_t continuous and increasing in both arguments (by (A3) $Q'_{t+1}>\underline{y}$), Q'_t also possesses these two properties. q.e.d.

<u>Proof of Theorem II.1</u>:

Since $Q_t=1$ for $t\leq\tau$, Lemma II.1.ii,iii and Definition II.2 imply:

(A4) $\quad (\forall t\geq 0)$ (If $t\in(\bar{T},+\infty)$, $q_t=1$; if $t=\bar{T}$, $q_t\in[Q_t,1]$).

(A5) $\quad (\forall t\geq 0)$ (If $t\in(\max(\tau,\bar{T}),+\infty)$ or $t=\bar{T}>\tau$, $q'_t=1=Q'_t$).

Thus, there only remains to characterize q_t on $[0,\bar{T})$ and q'_t on $[\tau,\bar{T})$ when $\bar{T}>\tau$, on $\{\tau\}$ when $\bar{T}\leq\tau$ (by definition, $[a,b)\equiv\emptyset$ if $a\geq b$).

<u>Case A</u>: $\bar{T}+1\leq\tau$

This implies $T^*<\tau-1$, so $\underline{T}=T^*$ by definition and $q_t=0$ on $[0,T^*)$ by Lemma II.1.i. Moreover, by (A4), $q_\tau=1$, so by definition of τ, q'_τ can take any value in $[0,1]=[0,Q'_\tau]$. It remains to examine q_t for $t\in[T^*,\bar{T}]\cap\mathbb{R}_+$.

On $(\max(T^*,\bar{T}-1),\bar{T})\cap\mathbb{R}_+$: $q_{t+1}=1$ and $q'_{t+1}=0$ imply $V_{t+1}=p_{t+1}+\delta(V-\beta)$, hence $V-\beta-V_{t+1}=\sigma-p_{t+1}>0$; thus $q_t=1$ by (4), since $q'_t=0$. Applying this proof to successive intervals $(\max(T^*,\overline{T}-k),\max(T^*,\overline{T}-k+1)) \cap \mathbb{R}_+$ yields by induction: $(\forall t\in(T^*,\overline{T}) \cap \mathbb{R}_+,\ q_t=1)$. Finally, when $T^*=\underline{T}\geq 0$: $q_{T^*+1}=1$, $q'_{T^*+1}=0$ so $V-\beta-V_{T^*+1}=\sigma-p_{T^*+1}=0$ and thus $q_{T^*}\in[0,1]=[0,Q_{\underline{T}}]$ is the only restriction imposed on $q_{\underline{T}}=q_{T^*}$ by (3)-(4). This finishes to establish that $(q,q')\in \Omega_0(V)$, with $\underline{T}=T^*<\tau-1$.

Case B: $T+1>\tau$

Note first that: $[0,\overline{T}]=[\tau,\overline{T}] \cup \{(\tau-1,\min(\tau,\overline{T})] \cap \mathbb{R}_+\} \cup [0,\tau-1]$, and that all the intervals on the right-hand-side are disjoint. Depending on whether the function Q'_t has a zero on $I\equiv[\tau-1,\overline{T}]$ or not, (cf. Lemma II.3.ii), two subcases are possible.

Subcase B1: $(\forall t\in[\tau-1,\overline{T}))\ (Q'_t\in(0,1))$

By Definition II.4, $\underline{T}$ is in this case equal to $\min(\tau-1,T^*)$.

Claim 1: (A6) $\quad(\forall t\in(\tau,\overline{T}))(q_t=Q_t,\ q'_t=Q'_t)$.

The proof is by induction (assuming $(\tau,\overline{T})\neq\emptyset$). For any t in $I_1\equiv(\max(\tau,\overline{T}-1),\overline{T})\neq\emptyset$, $Q'_{t+1}=1$ by Definition II.3, therefore one must have $q_t>0$, or else : $0=q'_t\geq\Psi_t(q'_{t+1})=\Psi_t(1)=Q'_t$ by Lemma II.2.iii, a contradiction of hypothesis B1. Similarly, $q_t<1$ or else by the same lemma: $1=q'_t\leq\Psi_t(1)=Q'_{t+1}$, another contradiction. Therefore $q_t\in(0,1)$, which by Lemma II.2.ii requires: $q'_t=\Psi_t(q'_{t+1})=Q'_t\in(0,1)$; this in turn implies $q_t=Q_t$, by (3). Suppose it has been established that:

$$(\forall t\in I_k\equiv(\max(\tau,\overline{T}-k),\overline{T}))\ (q_t=Q_t,\ q'_t=\Psi_t(q'_{t+1})=Q'_t).$$

Let $t\in I_{k+1}$; since $q_{t+1}=Q_{t+1}>0$, if q_t were zero, Lemma II.2.iii would imply: $0=q'_t\geq\Psi_t(q'_{t+1})=\Psi_t(Q'_{t+1})=Q'_t$, a contradiction; similarly, if $q_t=1$, then $1=q'_t\leq\Psi_t(q'_{t+1})=\Psi_t(Q'_{t+1})=Q'_t$, again a contradiction. So $q_t\in(0,1)$, which implies $q'_t=Q'_t\in(0,1)$ by Lemma II.2.ii, and thus $q_t=Q_t$ by (3); this finishes to establishes (A6).

Claim 2: (A7) $\quad(\forall t\in(\tau-1,\min(\tau,\overline{T})] \cap \mathbb{R}_+,\ q_t=Q_t=1;\ q'_t\leq Q'_t)$

Let t belong to the above interval. If $t+1\geq\overline{T}$, $q_{t+1}>0$ (by (A4); if $t+1<\overline{T}$, $q_{t+1}>0$ (by (A6). Thus in both cases, by Lemma II.1.ii:

$$V-\beta-V_{t+1}=(p_0-p_{t+1})(1-x)\Psi_t(q'_{t+1})=(p_0-p_{t+1})(1-x)Q'_t>0.$$

Thus $q_t<1$, implying $q'_t=0$, would also require $q_t=1$ by (4), a contradiction. Moreover, from $q_\tau=1$ and Lemma II.2.iii: $q'_\tau\le\Psi_\tau(q'_{\tau+1})=Q'_\tau$, i.e. q'_τ can take any value in $[0,Q'_\tau]$.

Finally, there only remains to examine q_t for $t\in[0,\tau-1]$, in the case $\tau\ge1$.

B1.1 If $T^*<\tau-1$: By Lemma II.1.i: $(\forall t<T^*,\ q_t=0)$. Let us now show:

Claim 3: (A8) $\quad(\forall t\in(T^*,\tau-1]\cap\mathbb{R}_+,\ q_t=1)$.

Indeed, for $t\in(\max(T^*,\tau-2),\tau-1]\cap\mathbb{R}_+$, either $t+1>\overline{T}$, so $q_{t+1}=1$ by (A4), or $t+1\in(\tau-1,\min(\tau,\overline{T})]\cap\mathbb{R}_+$, so $q_{t+1}=1$ by (A7). Hence:

$$V_{t+1}=p_{t+1}+\delta(V-\beta)+q'_{t+1}(1-x)(p_{t+1}-\delta p_0)\le p_{t+1}+\delta(V-\beta)$$

by (A1), because the last term is zero unless $t+1=\tau$. Therefore: $V-\beta-V_{t+1}\ge\sigma-p_{t+1}>0$. But $q'_t=0$, so by (4): $q_t=1=Q_t$. An induction identical to that of Case A above completes the proof of (A8). Finally, $q_{T^*+1}=1$ and $q'_{T^*+1}=0$ imply: $V-\beta-V_{T^*+1}=\sigma-p_{T^*+1}=0$, so (when $T^*\ge0$) q_{T^*} can take any value in $[0,1]=[0,Q_{T^*}]$. Thus we have shown: $(q,q')\in\Omega_0(V)$, with $\underline{T}=T^*<\tau-1$.

B1.2 If $T^*=\tau-1\ge0$: then $q_t=0$ on $[0,T^*)=[0,\tau-1)$ by Lemma II.1.i. If $\tau>\overline{T}$, $q_\tau=1$ by (A4); if $\tau\le\overline{T}$, $q_\tau=1$ by (A7), hence:

$V-\beta-V_\tau=(1-x)(\delta p_0-p_\tau)\Psi_{\tau-1}(q'_\tau)=\sigma-p_\tau+(1-x)q'_\tau(\delta p_0-p_\tau)=(1-x)q'_\tau(\delta p_0-p_\tau)$.

Therefore, either $\Psi_{\tau-1}(q'_\tau)=0$, i.e. $q'_\tau=0$, and then $(q,q')\in\Omega_0(V)$; or else $\Psi_{\tau-1}(q'_\tau)>0$, i.e. $q'_\tau>0$, and then $q_{\underline{T}}=q_{\tau-1}=1$ by (4), so $(q,q')\in\Omega_1(V)$. In both cases, $\underline{T}=T^*=\tau-1$.

B1.3 If $T^*>\tau-1\ge0$: by Lemma II.1.i, $q_t=0$ on $[0,\tau-1))$. Moreover, by (A7):

$$V-\beta-V_\tau=(1-x)(\delta p_0-p_\tau)\Psi_{\tau-1}(q'_\tau)=\sigma-p_\tau+(1-x)q'_\tau(\delta p_0-p_\tau).$$

Since q'_τ can take any value in $[0,Q'_\tau]$, $V-\beta-V_\tau$ can take any value between $\sigma-p_\tau<0$ and $(1-x)(p_0-p_\tau)Q'_{\tau-1}>0$ (by hypothesis B1). Therefore, by (4), (q,q') belongs to $\Omega_0(V)$ if $\Psi_{\tau-1}(q'_\tau)=0$, to $\Omega_1(V)$ if $\Psi_{\tau-1}(q'_\tau)>0$, and to $\Omega_2(V)$ if $\Psi_{\tau-1}(q'_\tau)<0$, with $\underline{T}=\tau-1<T^*$ always.

Subcase B2: $(\exists!\underline{t}\in[\tau-1,\overline{T}))(Q'_{\underline{t}}=0)$.

This requires: $\tau-1\le\underline{t}<T^*$. Otherwise, $m\equiv\max(T^*,\tau-1)$ satisfies $\delta p_0-p_{m+1}>0$, $\sigma-p_{m+1}\ge0$ and $Q'_m\le0$; but $(1-x)(p_0-p_m)Q'_m=\sigma-p_{m+1}+(\delta p_0-p_{m+1})(1-x)Q'_{m+1}$, requiring $Q'_{m+1}\le0$. By induction, this implies: $Q'_{m+2}\le0$, $Q'_{m+3}\le0$,..

$\ldots, Q^{\ell}_{m+k} \leq 0$, where $k \equiv \min\{n \in \mathbb{N} \mid m+n \geq \overline{T}\}$. But $Q^{\ell}_{m+n}=1$ by definition, hence a contradiction. Since $T^{*} \geq \tau-1$, Definition II.4 then states that $\underline{T}=\underline{t} \in [\tau-1, T^{*})$. The function Q^{ℓ} is negative on $[\tau-1,\underline{T})$ and takes values in $(0,1)$ on $(\underline{T},\overline{T})$.

a) Let us first examine $(\underline{T},\overline{T})$. The induction used in Case B.1 to prove (A6) can be applied to the intervals $J_k \equiv (\max(\tau,\underline{T},\overline{T}-k),\overline{T})$ to show:

(A9) $\quad (\forall t \in (\max(\tau,\underline{T}),\overline{T}))\ (q_t = Q_t\ ;\ q^{\ell}_t = Q^{\ell}_t)$.

Similarly, one proves, exactly as for (A7):

(A10) $\quad (\forall t \in (\underline{T},\tau] \cap \mathbb{R}_+,\ q_t = 1 = Q_t\ ;\ q^{\ell}_{\tau} \leq Q^{\ell}_{\tau})$.

b) Let us now examine $t=\underline{T}$ (when $\underline{T} \geq 0$). If $\underline{T} > \tau-1$, $q^{\ell}_{\underline{T}+1} = Q^{\ell}_{\underline{T}+1}$ by (A4), (A5) and (A9), so: $\Psi_{\underline{T}}(q^{\ell}_{\underline{T}+1}) = Q^{\ell}_{\underline{T}} = 0$, hence by Lemma II.2.i: $V-\beta-V_{\underline{T}+1} \leq 0$. Therefore, $q^{\ell}_{\underline{T}}=0$, and by (3): $q_{\underline{T}} \in [0,Q_{\underline{T}}]$. If $\underline{T}=\tau-1$, $q^{\ell}_{\tau} \leq Q^{\ell}_{\tau}$ (by (A10)) implies: $\Psi_{\tau-1}(q^{\ell}_{\tau}) \leq \Psi_{\tau-1}(Q^{\ell}_{\tau}) = Q^{\ell}_{\tau-1} = 0$. Therefore, by Lemma II.2.i:

* if $q^{\ell}_{\tau} = Q^{\ell}_{\tau}$, i.e. $\Psi_{\tau-1}(q^{\ell}_{\tau})=0$, then $q_{\underline{T}} \in [0,Q_{\underline{T}}]$;
* if $q^{\ell}_{\tau} < Q^{\ell}_{\tau}$, i.e. $\Psi_{\tau-1}(q^{\ell}_{\tau})<0$, then $q_{\underline{T}}=0$.

c) Finally, it will be established by induction that $q_t=0$ on $[0,\underline{T})$. For $t \in [\underline{T}-1,\underline{T}) \cap \mathbb{R}_+$, by (A9) and (A10): $q_{t+1}=Q_{t+1}>0$ and $q^{\ell}_{t+1} \leq Q^{\ell}_{t+1}$ (with equality except perhaps at $\tau-1$). So by Lemma II.2.i:

$$V-\beta-V_{t+1} = (p_0-p_{t+1})(1-x)\Psi_t(q^{\ell}_{t+1}) \leq (p_0-p_{t+1})(1-x)Q^{\ell}_t < 0$$

hence $q_t=0$ by (4). Assume that the proposition holds on $[\underline{T}-k,\underline{T}) \cap \mathbb{R}_+$, and let $t \in [\underline{T}-k-1,\underline{T}-k) \cap \mathbb{R}_+$. Then $q_{t+1}=q^{\ell}_{t+1}=0$, and by Lemma II.2.i:

$$V-\beta-V_{t+1} \leq (1-x)(p_0-p_{t+1})\Psi_t(q^{\ell}_{t+1}) = (1-x)(p_0-p_{t+1})\Psi_t(0) = \sigma-p_{t+1} < 0$$

since $t+1 < \underline{T}+1 < T^{*}+1$. Therefore $q_t=0$, which finishes to prove that, for all $t \in \mathbb{R}_+$:

* If $t \in [0,\underline{T})$, $q_t=0$; if $t \in (\underline{T},\overline{T})$, $q_t=Q_t$; if $t \in (\overline{T},+\infty)$, $q_t=1$.
* If $t=\underline{T}$, $q_t \in [0,Q_t]$, with the additional restrictions that $q_t=0$ when $t=\underline{T}=\tau-1$ and $\Psi_{\tau-1}(q^{\ell}_{\tau})<0$, or $q_t=Q_t$ when $t=\underline{T}=\tau-1$ and $\Psi_{\tau-1}(q^{\ell}_{\tau})>0$;
* If $t=\overline{T}$, $q_t \in [Q_t,1]$.
* If $t \in [0,\max(\tau,\underline{T}))$, $q^{\ell}_t=0$; if $t \in (\max(\tau,\underline{T}),+\infty)$, $q^{\ell}_t=Q^{\ell}_t$; if $\underline{T}<\tau$, $q^{\ell}_{\tau} \in [0,Q^{\ell}_{\tau}]$; if $\underline{T} \geq \tau$, $q^{\ell}_{\tau}=0$.

Equivalently: (q,q') is in $\Omega_1(V)$ if $\underline{T}=\tau-1\geq 0$ with $\Psi_{\tau-1}(q'_\tau)>0$, in $\Omega_2(V)$ if $\underline{T}=\tau-1\geq 0$ with $\Psi_{\tau-1}(q'_\tau)<0$, and in $\Omega_0(V)$ otherwise. The condition $(q,q')\in\Omega(V)$ is therefore necessary for (q,q',V) to be a continuation value equilibrium. Since the requirements of (1), (3) and (4) have been used and exhausted state by state in this proof, this condition is sufficient as well (this is also easy to check directly). Finally, the only cases where $\underline{T}=T^*$ are Cases A, B1.1 and B1.2, where $T^*\leq\tau-1$.

<u>Proof of Proposition II.2</u>:

Let $0<\max(\tau,\mu)\leq\bar{K}\equiv K[\bar{T}]$; then $Q'_{\bar{K}}=1$ and $p_{\bar{K}+1}\leq p_{\bar{T}+1}=\Phi$, so:

$$(1-x)(p_0-p_{\bar{K}})Q'_{\bar{K}-1} = (\delta p_0-p_{\bar{K}})(1-x) + \sigma - p_{\bar{K}}$$

$$\geq (\delta p_0-p_{\bar{K}})(1-x) + p_{\bar{K}+1} + (1-\delta)(1-x)p_0 - p_{\bar{K}}$$

$$= (1-x)p_0 + p_{\bar{K}+1} - (2-x)p_{\bar{K}} = -(2-x)F(\bar{K};\theta,x) \geq 0$$

because $\bar{K}\geq\mu$ (<u>cf</u>. proof of Theorem I.iii). Therefore, $\underline{T}\leq\bar{K}-1$ by Definition II.4, hence the result. q.e.d.

<u>APPENDIX III</u>:

<u>Proof of Lemma III.1</u>:

Define: $b^n_{t+1}\equiv b_{t+1,V^n}$, $b_{t+1}=b_{t+1,V}$, $\bar{T}^n\equiv\bar{T}(V^n)$, $\Psi^n_t\equiv\Psi_{t,V^n}$, $\Psi_t\equiv\Psi_{t,V}$, $Q'^n_t\equiv Q'_{t,V^n}$, and $Q'_t\equiv Q'_{t,V}$. By Definition II.3, Q'^n_t and Q'_t are equal on $[\max(\tau-1,\bar{T},\bar{T}^n),+\infty)$. Let us compare them on

$$[\tau-1,\max(\bar{T},\bar{T}^n))=[\tau-1,\min(\bar{T},\bar{T}^n)) \cup [\min(\bar{T},\bar{T}^n),\max(\bar{T},\bar{T}^n))$$

(when non empty), for n large enough to have $|\bar{T}^n-\bar{T}|<1$.

a) For all $t[\tau-1,\min(\bar{T}^n,\bar{T}))$, by Definition II.3:

$$Q'_t=\Psi_t(Q'_{t+1})=\ldots=\Psi_t\circ\ldots\circ\Psi_{t+k}(1),$$

where $k\in N$ is defined by $\bar{T}-1\leq t+k<\bar{T}$. Similarly:

$$Q'^n_t=\Psi^n_t(Q'^n_{t+1})=\ldots=\Psi^n_t\circ\ldots\circ\Psi^n_{t+k^n}(1),$$

where $k^n\in N$ is defined by: $\bar{T}^n-1\leq t+k^n<\bar{T}^n$. Moreover: $|\bar{T}^n-\bar{T}|<1$ implies that $|k-k^n|<2$, so three cases are possible:

<u>1)</u> $k^n=k$: $|Q'^n_t-Q'_t| \leq a_{t+1}|Q'^n_{t+1}-Q'_{t+1}| + |b^n_{t+1}-b_{t+1}|$

$$\leq a_\infty|Q'^n_{t+1}-Q'_{t+1}| + |b^n_{t+1}-b_{t+1}| \leq\ldots$$

$$\ldots\leq (a_\infty)^{k+1}|Q'^n_{t+1+k}-Q'_{t+1+k}| + \Sigma^k_{j=0}\ (a_\infty)^j|b^n_{t+1+j}-b_{t+1+j}|$$

by induction, where $a_\infty \equiv (\delta p_0 - p_\infty)/(p_0 - p_\infty) \in (0,1)$ is the limit, reached from below, of the function a_t (cf. Definition II.2) at $t=+\infty$. But since $k=k^n$, $Q'^n_{t+1+k} = Q'_{t+1+k} = 1$ so the first term is zero, while for the second:

$$|b^n_{t+1+j} - b_{t+1+j}| = (1-\delta)|V^n - V|/[(1-x)(p_0 - p_{t+1+j})] \leq |V^n - V|/[(1-x)p_0]$$

for all $j \in \mathbb{N}$ and $t \geq \tau - 1$, by (A1). Therefore:

(A11) $\quad |Q'^n_t - Q'_t| \leq |V^n - V|/[(1-x)(1-a_\infty)p_0]$

2) $k^n = k+1$: The same formula applies, but now $Q'_{t+1+k} = 1$, $Q'^n_{t+1+k} = \psi^n_{t+1+k}(1)$, so:

$$|Q'^n_{t+1+k} - Q'_{t+1+k}| = |(1-\delta)p_0 - (\sigma(V^n) - p_{t+k+2})/(1-x)|/(p_0 - p_{t+k+2})$$

$$= |\Phi(V^n) - p_{t+k+2}|/[(1-x)(p_0 - p_{t+k+2})] = |p_{\bar{T}^n+1} - p_{t+k+2}|/[(1-x)(p_0 - p_{t+k+2})]$$

But since $k^n = k+1$, $\bar{T}^n \leq t+k+2 \leq \bar{T}^n + 1$, and $t+k+2 \geq \bar{T} + 1$ so:

$$0 \leq \bar{T}^n + 1 - (t+k+2) \leq \bar{T}^n - \bar{T} < 1, \text{ and:}$$

$$|Q'^n_{t+1+k} - Q'_{t+1+k}| \leq |p_{\bar{T}^n+1} - p_{\bar{T}+1}|/[(1-x)(p_0 - p_{t+k+2})]$$

$$= (1-\delta)|V^n - V|/[(1-x)(p_0 - p_{t+k+2})] < |V^n - V|/[(1-x)p_0],$$

which leads, with (A11), to:

(A12) $\quad |Q'^n_t - Q'_t| \leq (1 + 1/(1-a_\infty))|V^n - V|/[(1-x)p_0]$

3) $k^n = k-1$: The induction of case (1) still holds up to rank $k-1$, so:

$$|Q'^n_t - Q'_t| \leq (a_\infty)^k |Q'^n_{t+k} - Q'_{t+k}| + \Sigma_{j=0}^{k} (a_\infty)^j |b^n_{t+j} - b_{t+j}|$$

and $|Q'^n_{t+k} - Q'_{t+k}| = |1 - \psi_{t+k}(1)| = |p_{\bar{T}+1} - p_{t+k+1}|/[(1-x)(p_0 - p_{t+k+1})]$

$$\leq |p_{\bar{T}+1} - p_{\bar{T}^n+1}|/[(1-x)(p_0 - p_{t+k+1})] < |V^n - V|/[(1-x)p_0],$$

because $k^n = k-1$ requires: $\bar{T}^n + 1 \leq t+k+1 \leq \bar{T} + 1$; so once again (A12) holds.

b) For all $t \in [\min(\bar{T}^n, \bar{T}, \max(\bar{T}^n, \bar{T}))$, either:

$$(Q'_t = 1,\ Q'^n_t = \psi^n_t(1)), \text{ or } (Q'^n_t = 1,\ Q'_t = \psi_t(1)),$$

according to whether $\bar{T}^n \geq \bar{T}$ or $\bar{T} \geq \bar{T}^n$; the proofs of (a), Cases 2 and 3 respectively, can then be replicated to finish establishing that (A12) holds for all $t \in [\tau - 1, +\infty)$. q.e.d.

Proof of Lemma III.2:

a) Continuity: Note that it suffices to show separately that $f(V^n)$ converges to $f(V)$ for sequences (V^n) converging to $V \in \Gamma$ from above and from below. The following cases must be distinguished.

Case 1) $\underline{T}<\tau-1$. It must be that $\underline{T}=T^*<\tau-1$, so $\underline{K}=K^*\leq Int[\tau]<\tau$, $q'_{\underline{K}}=0$ and:

$$f(V)=\Sigma_{j=0}^{\underline{K}}\ \delta^j p_j+\delta^{\underline{K}+1}(V-\beta).$$

1.1) $p_{K^*}>\sigma(V)>p_{K^*+1}$; for n large enough, $p_{K^*}>\sigma(V^n)>p_{K^*+1}$ so K^{*n} is equal to K^*; hence $\underline{K}^n=\underline{K}$, so $f(V^n)-f(V)=\delta^{\underline{K}+1}(V^n-V)$ which yields the result.

1.2) $p_{K^*}>\sigma(V)=p_{K^*+1}$; equivalently, $K^*=T^*<\tau-1$. If (V^n) converges to V from above, $K^{*n}=K^*$, and one is back in Case 1.1. From here on in the rest of Case 1 it will therefore be assumed that $V^n<V$ for all n, which implies that $K^{*n}=K^*+1$ for n large enough.

1.2.1) $K^*+1\leq Int[\tau]$; $f(V^n)=\Sigma_{j=0}^{\underline{K}+1}\ \delta^j p_j+\delta^{\underline{K}+2}(V^n-\beta)$ converges, as $n\to+\infty$, to: $\Sigma_{j=0}^{\underline{K}}\ \delta^j p_j+\delta^{\underline{K}+1}[p_{K^*+1}+\delta(V-\beta)]$

$$=\Sigma_{j=0}^{\underline{K}}\ \delta^j p_j+\delta^{\underline{K}+1}[(1-\delta)(V-\beta)+\delta(V-\beta)]=f(V).$$

1.2.2) $K^*=Int[\tau]$, so $K^{*n}=Int[\tau]+1$; since $\underline{T}<\tau-1$, it must be (cf. proof of Theorem II.1, subcase B2) that: $(\forall t\in[\tau-1,\bar{T}]$: $Q'_t>0)$. Lemma III.1 then implies that, for large enough n: $((\forall t\in[\tau-1,\bar{T}]$, $Q'^n_t>0)$. Therefore, $\underline{T}^n\leq\tau-1$, while on the other hand: $V^n<V$, so $T^{*n}>T^*=K^*=Int[\tau]>\tau-1$. Definition II.4 then requires $\underline{T}^n=\tau-1$, $\underline{K}^n=Int[\tau]=K^*=\underline{K}$, and convergence is again immediate.

Case 2) $\tau-1\leq\underline{T}<Int[\tau]$; This implies that $\underline{K}=Int[\tau]>\tau-1$, and therefore: $(\forall t\in[Int[\tau],\bar{T}]$, $Q'_t>0)$. Then by Lemma III.1, for n large enough: $(\forall t\in[Int[\tau],\bar{T}]$, $Q'^n_t>0)$, therefore $\underline{K}^n\leq Int[\tau]$. On the other hand, $\underline{T}\geq\tau-1>Int[\tau]-1$, so for n large enough, $\underline{T}^n>Int[\tau]-1$, $\underline{K}^n>Int[\tau]-1$; hence $\underline{K}^n=Int[\tau]=\underline{K}$, and convergence is again immediate.

Case 3) $\underline{T}>Int[\tau]$. This requires: $\underline{K}\geq1$ and: $(\exists!\underline{T}$, $Int[\tau]<\underline{T}<\bar{T})(Q'_{\underline{T}}=0)$.

3.1) If $\underline{T}\notin N$; or equivalently: $Q'_{\underline{K}-1}<0$, $Q'_{\underline{K}}>0$. By Lemma III.1, for n large enough, $Q'^n_{\underline{K}-1}<0$ and $Q'^n_{\underline{K}}>0$, hence $\underline{K}^n=\underline{K}$ and:

$$f(V^n)=\Sigma_{j=0}^{\underline{K}}\ \delta^j p_j\ +\ \delta^{\underline{K}}(p_{\underline{K}}-\delta p_0)(1-x)Q'^n_{\underline{K}}+\ \delta^{\underline{K}+1}(V^n-\beta)$$

which converges to:

$$\Sigma_{j=0}^{\underline{K}}\ \delta^j p_j\ +\ \delta^{\underline{K}}(p_{\underline{K}}-\delta p_0)(1-x)Q'_{\underline{K}}\ +\delta^{\underline{K}+1}(V-\beta)\ =\ f(V).$$

3.2) If $\underline{T}\in N$; equivalently: $Q'_{\underline{K}}=0$; if (V^n) converges from above, it is easy to verify (by induction) that: $(\forall t\in[\tau-1,\bar{T})$, $Q'^n_t>Q'_t)$; in particular: $Q^{\tau n}_{\underline{K}} > Q^{\tau}_{\underline{K}} = 0$. Moreover, Lemma III.1 still implies, for n large enough: $Q^{\tau n}_{\underline{K}-1} < 0$; therefore $\underline{K}^n = \underline{K}$, which brings back to Case 3.1. From here on, it will be assumed that $(\forall n, V^n < V)$, implying that $Q^{\tau n}_{\underline{K}} < 0$. But since $\underline{K}+1 > \tau$: $Q^{\tau}_{\underline{K}+1} > Q^{\tau}_{\underline{K}} = 0$, hence by Lemma III.1, for n large enough: $Q^{\tau n}_{\underline{K}+1} > 0$. Thus $\underline{K}^n = \underline{K}+1$, and:

$$f(V^n) = \Sigma^{\underline{K}+1}_{j=0}\ \delta^j p_j\ +\ \delta^{\underline{K}+1} Q^{\tau n}_{\underline{K}+1}(1-x)(p_{\underline{K}+1}-\delta p_0)\ +\ \delta^{\underline{K}+2}(V^n-\beta)$$

which converges to:

$$\Sigma^{\underline{K}}_{j=0}\ \delta^j p_j\ +\ \delta^{\underline{K}+1}[p_{\underline{K}+1}\ +\ Q^{\tau}_{\underline{K}+1}(1-x)(p_{\underline{K}+1}-\delta p_0)\ +\ \delta(V-\beta)]$$

$$= \Sigma^{\underline{K}}_{j=0}\ \delta^j p_j\ +\ \delta^{\underline{K}+1}[V-\beta\ -(1-x)(p_0-p_{\underline{K}+1})\Psi_{\underline{K}}(Q^{\tau}_{\underline{K}+1})].$$

Since $\Psi_{\underline{K}}(Q^{\tau}_{\underline{K}+1}) = Q^{\tau}_{\underline{K}} = 0$, this last expression is equal to f(V).

<u>Case 4)</u> $\underline{T} = \mathrm{Int}[\tau]$. Equivalently: $Q^{\tau}_{\mathrm{Int}[\tau]} = 0$. If V^n converges to V from above, $Q^{\tau n}_{\mathrm{Int}[\tau]} > 0$, so $\underline{K}^n \le \mathrm{Int}[\tau]$; moreover, $\underline{T} = \mathrm{Int}[\tau] > \tau-1$, which requires: $T^* > \tau-1$, hence $T^{*n} > \tau-1$, $\underline{T}^n \ge \tau-1$, $\underline{K}^n \ge \mathrm{Int}[\tau]$. Thus $\underline{K}^n = \mathrm{Int}[\tau] = \underline{K}$, and the result is immediate. Now if V^n converges from below, then for n large enough, $\underline{K}^n = \mathrm{Int}[\tau]+1 = \underline{K}+1$, and the proof is the same as in Case 3.2. q.e.d.

b) <u>Fixed point</u>: By (11), for all $V \in \Gamma = [(p_0-\delta\beta)/(1-\delta),\ p_0/(1-\delta)]$:

$$f(V)\ \le\ [(1-\delta^{\underline{K}+1})/(1-\delta)]p_0\ +\ \delta^{\underline{K}+1}V\ \le\ p_0/(1-\delta).$$

By construction, f(V) is the payoff obtained by the firm in the continuation value game under its optimal strategy $\{q_t \mid t \in \mathbb{R}_+\}$ (given that customers play $\{q^{\tau}_t \mid t \in \mathbb{R}_+\}$). It is therefore at least equal to the payoff received by adjusting in state 0, given $\{q' \mid t \in \mathbb{R}_+\}$ (note that $q^{\tau}_0 = 0$):

$$f(V)\ \ge\ p_0+\delta(V-\beta)\ \ge\ p_0\ +\ \delta(p_0-\beta)/(1-\delta)\ =\ (p_0-\delta\beta)/(1-\delta).$$

Thus f is continuous and maps Γ into itself, hence the result. q.e.d.

<u>Proof of Theorem III.2</u>: Let there be two equilibria with initial firm valuations V^1_0 and V^2_0 -in short V^1, V^2- with $V^1 > V^2$. Then $\sigma(V^2) > \sigma(V^1)$, $\Phi(V^2) > \Phi(V^1)$ hence $\overline{T}(V^2) > \overline{T}(V^1)$, and by a straightforward induction: $(\forall t \in [\tau-1,+\infty)$: $Q^{\tau}_{t,V^1} \ge Q^{\tau}_{t,V^2})$, from which follows: $(\forall t \in R_+ : q^{\tau}_{t,V^1} \ge q^{\tau}_{t,V^2})$ where $q^{\tau}_{t,Vj}$, $j \in \{1,2\}$, denotes speculators' strategy in the equilibrium with firm valuation V^j. This implies in turn: $\underline{T}^1 \equiv \underline{T}(V^1) \le \underline{T}(V_0) \equiv \underline{T}^2$, and thus $\underline{K}^1 \le \underline{K}^2$. In particular:

$(\forall k<\underline{K}^1:\ 0 = q_k^{t,v1} = q_k^{t,v2})$ and: $q_{\underline{K}1}^{t,v1} \geq q_{\underline{K}1}^{t,v2} \geq 0$.

Hence: $q_{\underline{K}1}^{t,v1}(\delta p_0 - p_{\underline{K}^1}) \geq q_{\underline{K}1}^{t,v2}(\delta p_0 - p_{\underline{K}^1})$, because $\delta p_0 - p_{\underline{K}^1} \geq 0$ unless $\underline{K}^1 < \tau$, but then $q_{\underline{K}1}^{t,v1}=0$. Therefore, (11) and $V^1 = f(V^1)$ imply:

$$V^1 - \beta = [\Sigma_{k=0}^{\underline{K}^1}\ \delta^k p_k + \delta^{\underline{K}^1} q_{\underline{K}1}^{t,v1}(1-x)(p_{\underline{K}^1} - \delta p_0) - \beta]/[1-\delta^{\underline{K}^1+1})]$$

$$\leq [\Sigma_{k=0}^{\underline{K}^1}\ \delta^k p_k + \delta^{\underline{K}^1} q_{\underline{K}1}^{t,v2}(1-x)(p_{\underline{K}^1} - \delta p_0) - \beta]/[1-\delta^{\underline{K}^1+1})]$$

The last term is the firm's payoff (minus β) under a strategy of periodic adjustments in state $\underline{K}^1$ (with probability one), given speculators' strategy $\{q_t^{t,v2} | t\in \mathbb{R}_+\}$. By definition, it is no greater than its payoff from the optimal strategy given $\{q_t^{t,v2} | t\in \mathbb{R}_+\}$, i.e. the equilibrium payoff (minus β) $V^2-\beta$. Hence $V^1-\beta \leq V^2-\beta$, a contradiction. q.e.d.

APPENDIX IV

Proof of Proposition IV.1:

Some preliminary results on the firm's intertemporal payoff under various strategies -in the presence and in the absence of speculation- must first be established. For all $k\in N$, let $M(k)$ denote this payoff when speculators play their equilibrium strategies $\{q_t^t | t\in \mathbb{R}_+\}$ but the firm adjusts its price periodically (with probability one) when state k is reached:

(A13) $$M(k) \equiv [\Sigma_{j=0}^{k-1}\ \delta^j (p_j + q_j^j(1-x)(p_j - \delta p_{j+1})) + \delta^k (p_k + q_k^k(1-x)(p_k - \delta p_0)) - \beta]/[1-\delta^{k+1})]$$

Since $q_j^j=0$ for $j<\underline{K}$ and $V_0=f(V_0)$, (11) can be rewritten: $V_0-\beta=M(\underline{K})$. Thus periodic adjustment at $\underline{K}$ is optimal (given $\{q_t^t | t\in \mathbb{R}_+\}$). Moreover:

(A14) $(\forall k<\underline{K})\ (M(k) \leq M(\underline{K})$, with strict inequality for $k<\underline{K})$,

because adjustment in a state $k<\underline{K}$ is strictly suboptimal (given $\{q_t^t | t\in \mathbb{R}_+\}$) since: $V_0-\beta-V_{k+1} \leq (1-x)(p_0-p_{k+1})Q_k^k < 0$ by Lemma II.2.i and Definition II.5. Consider now the limiting no-storage case ($x=1$), with all variables superscripted by "ns". The system (3)-(4) is then identical to what it would be with $x\in(0,1)$ but $\alpha\geq\delta S$ (i.e. $\tau=+\infty$):

(3') $(\forall t\geq 0,\ q_t^t=0)$.

(4') $(\forall t\geq 0,\ q_t=0,\ q_t=1$ or $q_t\in[0,1]$, according to whether $V_0-\beta-V_{t+1}<0,\ V_0-\beta-V_{t+1}>0$ or $V_0-\beta-V_{t+1}=0)$.

Indeed, $x=1$ or $\alpha\geq\delta S$ both mean that no one can ever store profitably. Therefore, when $x=1$, the game still has a unique solution, which is the same as when $\alpha\geq\delta S$, and yields an equilibrium payoff V_0^{ns} to the firm. It is easily seen, from Definitions II.1 and II.4, that this equilibrium is also the limit of the solutions with $\alpha<\delta S$ and x tending to 1 from below, and that: $\underline{T}^{ns}=\overline{T}^{ns}\equiv T^{ns}$, $\underline{K}^{ns}=\overline{K}^{ns}\equiv K^{ns}$. For this equilibrium, (A14) yields:

(A15) $(\forall k<K^{ns})\ (M^{ns}(k)\ \leq M^{ns}(K^{ns})$, with strict inequality for $k<K^{ns})$,

where $M^{ns}(k)$ is the firm's payoff to periodic adjustment in state k, given that customers never store. Note that $M^{ns}(k)$ is given by (A13) with all (q_j^s)'s replaced by zero; in particular, $M(\underline{K})\leq M^{ns}(\underline{K})$, with equality if and only if $q_{\underline{K}}^s=0$. It will now be shown that:

(A16) $\underline{K}\ \leq K^{ns}\ \leq K^*$.

Indeed, if $K^{ns}<\underline{K}$: $M^{ns}(K^{ns}) = M(K^{ns}) < M(\underline{K}) \leq M^{ns}(\underline{K})$, contradicting (A15).

Moreover: $p_{T^{ns}+1} = (1-\delta)(V_0^{ns}-\beta) = (1-\delta)M^{ns}(K^{ns})$

$$\geq (1-\delta)M^{ns}(\underline{K}) \geq (1-\delta)M(\underline{K}) = \sigma = p_{T^*+1},$$

implying $T^{ns}\leq T^*$, and $K^{ns}\leq K^*$. Proposition V.1 can now be proven.

(i) When $K^{ns}\leq Int[\tau]$, then $\underline{K}\leq K^{ns}\leq Int[\tau]$ by (A16) and two cases arise:

a) $\underline{K}\leq Int[\tau]-1$, implying $\underline{T}<\tau-1$, which by definition requires $\underline{T}=T^*<\tau-1$. Hence $\underline{K}=K^*=K^{ns}$, by (A16).

b) $\underline{K}=Int[\tau]<\tau$; since $K^{ns}\leq Int[\tau]$, (A16) then requires $\underline{K}=K^{ns}$, hence:

$$p_{T^{ns}+1} = (1-\delta)M^{ns}(K^{ns}) = (1-\delta)M^{ns}(\underline{K}) = (1-\delta)M(\underline{K}) = \sigma = p_{T^*+1},$$

because $\underline{K}<\underline{\tau}$ implies $q_{\underline{K}}^s=0$. Thus: $T^{ns}=T^*$, $K^*=K^{ns}=\underline{K}$. q.e.d.

(ii) When $K^{ns}>Int[\tau]$, assume that $\underline{K}<Int[\tau]$; as in (a) above, this implies $\underline{K}=K^*$; but (A16) then requires $\underline{K}\geq K^{ns}>Int[\tau]$, a contradiction. q.e.d.

(iii) Results directly from Proposition II.2, since $\overline{K}\geq K^*\geq K^{ns}$. q.e.d.

<u>Proof of Proposition IV.2</u>:

Define for all $k\in N$: $Y_k\equiv p_k+q_k^s(1-x)[p_k-\delta(q_k p_0+(1-q_k)p_{k+1})]$. Since the price is adjusted with probability q_k in state k, and $q_K=1$, (1) yields:

$$\text{(A17)}\qquad V_0-\beta = \frac{[\Sigma_{k=0}^{\overline{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})Y_k\ -\beta]}{[1-\delta\Sigma_{k=0}^{\overline{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k]}$$

where $q_{-1}\equiv 0$ by convention. Similarly, defining W_0 (resp. W_0') as the average expected present value of speculators' (resp. non-speculators') utility in state zero (with no initial stocks) and, for all $k\in \mathbb{N}$: $Z_k \equiv S-P_k+q_k'\{-\alpha-P_k+\delta[q_kP_0+(1-q_k)P_{k+1}]\}$, one can compute:

$$\text{(A18)}\qquad W_0 = \frac{[\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})Z_k]}{[1-\delta\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k]}$$

$$\text{(A19)}\qquad W_0' = \frac{[\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})(S-P_k)]}{[1-\delta\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k]}$$

Total intertemporal social welfare (net of the first adjustment cost) is $SW_0 = xW_0' + (1-x)W_0 + V_0-\beta$, or:

$$SW_0 = \frac{[\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})(x(S-P_k)+(1-x)Z_k+Y_k)) - \beta]}{[1-\delta\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k]}$$

But, for all $k\in N$: $x(S-P_k)+(1-x)Z_k+Y_k=S-c-(1-x)q_k'(\alpha+c(1-\delta))$, and:

$$(1-\delta)\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})$$

$$= 1+\Sigma_{k=1}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})-\ \Sigma_{k=0}^{\bar{K}}\ \delta^{k+1}(1-q_0)\ldots(1-q_{k-1})(1-q_k+q_k)$$

$$= 1-\delta\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k,$$ so that:

$$\text{(A20)}\qquad SW_0 = \frac{S-c}{1-\delta} - \frac{[\beta+(1-x)(\alpha+c(1-\delta))\Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k']}{[1-\delta\ \Sigma_{k=0}^{\bar{K}}\ \delta^k(1-q_0)\ldots(1-q_{k-1})q_k]}$$

which, given that $q_k=0$ for $k<\underline{K}$, completes the proof. q.e.d.

<u>Proof of Proposition IV.3</u>:

The resolution of the equation $h^*\underline{M}=h^*$ is straightforward. Normalizing the coordinates to sum to one requires: $H = \Sigma_{k=0}^{\bar{K}}\ (1-q_0)\ldots(1-q_{k-1})$.

This expression can be factored, starting from the last terms, yielding:

$H = \Sigma_{k=0}^{\bar{K}}\ (1-q_0)\ldots(1-q_{k-1})q_k(k+1) = E[\tilde{T}]$. q.e.d.

<u>Proof of Theorem IV.4</u>

<u>Claim 1</u>: All eigenvalues of $\underline{M}$ have modulus no greater than 1.

Indeed, for any complex matrix $A=(a_{ij})_{1\le i,j\le N}$ (<u>cf</u>. Varga [1965], p.17):

$$|v| \le \max\{\ \Sigma_{i=1}^{N}\ |a_{ij}|,\ 1\le j\le N\}$$

where v is any eigenvalue of A. Since the second term is equal to 1 for $A=\underline{M}$ (with $N=\bar{K}+1$), and $h^*\underline{M}=h^*$, $\underline{M}$ has radius of convergence equal to one.

<u>Claim 2</u>: $\underline{M}$ is non-cyclic, or primitive (i.e. it has only one eigenvalue equal to its radius of convergence) if and only if $q_{\underline{K}}<1$.

The proof rests on the following theorem (<u>cf</u>. Varga [1965], p.44):

<u>Theorem (Frobenius)</u>: An irreducible square matrix A with non-negative coefficients is cyclic of index n(A) (i.e. has n(A) eigenvalues with modulus equal to its radius of convergence), where n(A) is the greatest common denominator of the differences in successive degrees appearing with non-zero coefficients in its characteristic polynomial.

The matrix $\underline{M}$ has non-negative elements. By definition (Varga [1965], p.20) it is irreducible if and only if it has a strongly connected graph,i.e: for any pair $(i,j)\in\{1,..,\overline{K}+1\}^2$ there exists a sequence $(i_0=i,i_1),(i_1,i_2),...,(i_n,i_{n+1}=j)$ such that the corresponding coefficients of $\underline{M}$ are non-zero (this will be denoted as $i\sim j$). By definition of $\overline{K}$, for any $i<\overline{K}$, $m_{i\,i+1}=1-q_{i-1}>0$, therefore $i\sim j$ for any (i,j) with $j>i$. Moreover, $m_{\overline{K}+1,1}=1$, so $i\sim 1$ for all i; but $1\sim j$ for all $j>1$, hence the result. Let us now compute from (15) the characteristic polynomial of $\underline{M}$:

$$G(X) \equiv -X^{\overline{K}+1} + X^{\overline{K}}q_0 + X^{\overline{K}-1}(1-q_0)q_1 +\ldots$$
$$+\ldots+ X(1-q_0)\ldots(1-q_{\overline{K}-2})q_{\overline{K}-1} +(1-q_0)\ldots(1-q_{\overline{K}-1})$$

Moreover, since $q_k=0$ when $k<\underline{K}$, $q_k\in(0,1)$ if $\underline{K}\leq k<\overline{K}$, and $q_{\underline{K}}=1$:

$$G(X)= -X^{\overline{K}+1} + q_{\underline{K}}X^{\overline{K}-\underline{K}}$$
$$+ (1-q_{\underline{K}})[X^{\overline{K}-\underline{K}-1}q_{\underline{K}+1} + X^{\overline{K}-\underline{K}-2}(1-q_{\underline{K}+1})q_{\underline{K}+2}+\ldots$$
$$+\ldots+ (1-q_{\underline{K}+1})(1-q_{\underline{K}+2})\ldots(1-q_{\overline{K}-1})]$$

If $0<q_{\underline{K}}<1$, G(X) has non-zero coefficients on $\overline{K}-\underline{K}$ and $\overline{K}-\underline{K}-1$, so $n(\underline{M})=1$ by Frobenius' Theorem; if $q_{\underline{K}}=1$ ($\underline{K}=\overline{K}$), then $n(\underline{M})=\overline{K}+1$, hence Claim (b).

To conclude the proof of the theorem, let $q_{\underline{K}}\neq 1$, and consider the hyperplane $L\equiv\{z\in\mathbb{R}^{\overline{K}+1} \mid \Sigma_{k=1}^{\overline{K}+1} z_k=0\}$. For any probability distribution h, $h-h^*\in L$. Moreover, $[h\dashrightarrow h\underline{M}]$ maps L into itself, therefore:

$$(\forall h\in\mathbb{R}^{\overline{K}+1}) \quad (\exists u\in L) \ (\forall n\in N) \ (|h.\underline{M}^n-h^*| = |u.\underline{M}^n| < B^n|u|)$$

where B is the radius of convergence of $\underline{M}$'s restriction to L. But, since $h^*\notin L$, Claims 1 and 2 imply: $|B| < 1$, hence the result. q.e.d.

Proof of Proposition IV.5:

With probability h^*_k, the firm is in state k; $Log(P_h)$ then increases by zero with probability $1-q_k$ and by $Log[(1+\pi)^{k+1}]$ with probability q_k; thus:

$$E^*[Log(P_h(t+1)/P_h(t))] \;/\; Log(1+\pi) = \{\Sigma_{k=0}^{\bar{K}} h^*_k q_k (k+1)\}$$

$$= \{\Sigma_{k=\underline{K}}^{\bar{K}} (1-q_0)..(1-q_{k-1})q_k(k+1)\}/H = E[\tilde{T}]/H = 1. \quad \text{q.e.d.}$$

Proof of Proposition IV.6:

By homogeneity:

$$\bar{P}(t) = P^*(t)G[\int_0^1 w(P_h(t)/P^*(t))] = P^*(t)G[\int_0^1 w(P_{it})]di,$$

and by assumption P^* grows at the rate π. Moreover: $\int_0^1 w(P_{it})]di$ is the expectation of the random variable $w(z)$ with repect to the image measure by $[i\text{-->}P_{it}]$ of the Lebesgue measure on $[0,1]$, i.e. with respect to the probability distribution h^*. This index of real prices is thus independant of t, hence the result. q.e.d.

NOTES

(1) Goods which are not storable but for which intertemporal substitution of consumption is possible will give rise to similar behaviour.

(2) Unlike durable goods, which are not themselves consumed but yield a flow of services over time, storable goods disappear after consumption.

(3) Indexed contracts may arise when the firm sells to a few large customers, but are too costly to draw and enforce with many small buyers.

(4) Buyers have instantaneous utility $U(z,y) = y + S\cdot\min(z,1)$, where z is consumption of the firm's good and y real income spent on others; their real income in each period is $I \geq S$. Their instantaneous indirect utility function (in the absence of storage) is thus $W(P,I) = I + \max(S-P, 0)$.

(5) As will be made clear below, whether inventories are consumed before or after new purchases are made is in fact irrelevant.

(6) One could also interpret x as a flow of transient customers, renewed every period, or as the fraction of customers which the firm succeds in rationing when they try to store.

(7) Because individual buyers are negligible, it makes in fact no difference at all whether the firm has observed previous storage when it sets the new price (alternating moves) or not (simultaneous moves).

(8) Formally, a state variable z is payoff relevant for player j, whose decision variable and instantaneous payoff are y_j and $G^j(y_j, y_{-j}; z)$, if for some couple of distinct values (y_{1j}, y_{2j}), the function:
$[z \dashrightarrow G^j(y_{1j}, y_{-j}; z) - G^j(y_{2j}, y_{-j}; z)]$ is not constant. In the differentiable case, this takes the form: $\partial^2 G^j/\partial y_j \partial z \neq 0$.

(9) A Markov perfect equilibrium is still perfect when arbitrary history-dependent strategies are allowed; it is simply one where all players disregard payoff irrelevant variables (<u>cf</u>. Maskin and Tirole, [1982]).

(10) In a context of perfect information, and no multiple Nash equilibria.

(11) This remark draws on Gul, Sonnnenschein and Wilson [1986]. Perfect equilibrium strategies for speculators: [P --> $q_i^{**}(P)$] and the firm: [(P,q') --> $P^{**}(P,q')$] must be mutual best responses in any subgame, and in particular in those which do not result from simultaneous deviations by buyers. Conversely, consider strategies [P --> $q_i^{*}(P)$] and [P -->$P^{*}(P)$] which are mutual best responses in any such subgame; extend them to subgames which follow a simultaneous deviation, i.e. where initial aggregegate inventories q' differ from their prescribed value $q'^{*}(P) = \int_0^1 q_i^{*}(P)di$, by prescribing equilibrium behaviour in the subgame. The resulting strategies form a (Markov) perfect equilibrium with the same equilibrium path as the original strategies. In the present model, equilibrium strategies in any subgame following a simultaneous deviation (i.e. with arbitrary inherited inventories q') are easily obtained by replacing q_i by q' in equation (4) below, while keeping everything else (in particular the value function [t -->V_t]) unchanged.

(12) Attention could even be restricted to $t \in \mathbb{N}$, but it is more convenient to keep $t \in \mathbb{R}_+$ and use functions instead of sequences in the proofs and on the graphs. Also, the game is thus solved for any initial price $p \leq S$.

(13) Since the firm plays at discrete intervals of time, it will in fact adjust the price when the state $K[T^*(V)]=\min\{k \in \mathbb{N} | k \geq T^*(V)\}$ is reached.

(14) Throughout the paper, $[a,b)=(a,b)\equiv\emptyset$ for all (a,b) with a>b.

(15) While this result may not remain when customers are sufficiently heterogeneous (with for instance a continuous distribution of storage costs) it is an important warning against the fallacious intuition that the price increase should always grow more and more likely.

(16) The condition is allmost necessary as well: when $\tau \leq K[\bar{T}] < \mu$, $0 < \bar{T}-\underline{T} < 1$, but $[\underline{T},\bar{T})$ may still happen to contain an integer.

(17) For all $V \in \Gamma$; for values $V \notin \Gamma$, cf. Bénabou [1986a].

(18) With a more general (elastic) demand, the optimal policy would involve randomization over the upper bound as well (($\tilde{S},\tilde{s}$) policy).

(19) The full equilibrium of the game can indeed take any of the four basic forms identified here for continuation value equilibria, depending on parameter values: cf. Table 1 in Section IV and footnote (25) below.

(20) This condition holds generically and avoids the possible multiplicity of solutions which may arise if state τ can be reached along the equilibrium path (in this state, speculators faced with a sure price increase are indifferent between storing and not storing).

(21) In fact, generically unique: it was assumed that $\tau \notin \mathbb{N}$ and $\mu \notin \mathbb{N}$.

(22) Cf. footnote (16).

(23) The usual (S,s) model generates relative price variability but not uncertainty. Moreover, with constant returns to scale, only uncertainty induces misallocations, and empirical studies and discussions of the costs of inflation indeed interpret variability as evidence of uncertainty.

(24) Even if exogenous uncertainty is generated by parameters such as π or β, the firm must still "process" it so as to leave speculators indifferent

(25) Note also that for $\pi \leq 5\%$, the equilibrium is of type 1 (pure strategies, no storage), because $\bar{K}=K^{ns} \leq \text{Int}[\tau]$; for $\pi=7\%$, it is of type 2.1 (mixed strategies, deterministic outcome without storage), because $\underline{K}=\text{Int}[\tau]<K^{ns}=\bar{K}$; in all other cases it is of type 2.3 (stochastic outcome). Type 2.2 (deterministic outcome, full storage) occurs only for high values of x; for instance, x=.95 with $\pi=30\%$ yields $\text{Int}[\tau]=4$, $\underline{K}=K^{ns}=\bar{K}=11$.

(26) Or intertemporal substitution of consumption; cf. footnote (1).

(27) When firms are competing, the adjustment level (here S) becomes an endogenous function of the distribution of prices; cf. Bénabou [1986b].

(28) This index may include the firms' own prices, but must also include some outside good(s), if the model is to be consistent in level as well as in growth rate (cf., for instance, equation (16)).

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