COMPETITIVE EQUILIBRIUM FOR
INCOMPLETE MARKET STRUCTURES
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B - Existence and Determinacy

July 1986
Revised November 1986
N° 8701
COMPETITIVE EQUILIBRIUM FOR INCOMPLETE MARKET STRUCTURES

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ABSTRACT

Using the tools built in the first chapter, one shows that for every market structure and every vector of initial endowments, there exists a pseudo-equilibrium and that, generically, a pseudo-equilibrium is an equilibrium. Moreover, the set of pseudo-equilibria is generically finite. On the contrary, when the market structure is in some sense endogeneous, the set of equilibria is very often infinite. It is shown that the indeterminacy result gotten in the model with financial securities is a result of this general principle.

Journal of Economic Literature 020

Key words: - General Market Structures
- Demand Functions
- Pseudo-equilibrium
- Equilibrium
- Existence
- Determinacy

EQUILIBRE CONCURRENTIEL ET STRUCTURES DE MARCHE INCOMPLETES

B - Existence et nombre d'équilibres

RÉSUMÉ

Utilisant les outils construits dans le premier chapitre, on montre que pour toute structure de marché et tout vecteur de ressources initiales, il existe un pseudo-equilibre et que, génériquement, un pseudo-equilibre est un équilibre. De plus, l'ensemble des pseudo-équilibres est génériquement fini. Au contraire, quand la structure de marchés est en un certain sens endogène, l'ensemble des équilibres est très souvent infini. On montre alors que le résultat sur le nombre d'équilibres dans le cas où il existe des titres financiers constitue une application de ce principe général.

Journal of Economic Literature 020

Mots clefs: - Structures générales de marché
- Functions de demande
- Pseudo-équilibre
- Équilibre
- Existence
- Nombre d'équilibrés
Competitive Equilibrium for Incomplete Market Structures
by
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B. Existence and Determinacy*
July 1986
Revised November 1986

In the first chapter of this study entitled General Market structures and Demand Theory, I tried to build a more general definition of a market structure than the usual ones when the differential viewpoint is adopted and tried to give an account of the three basic properties of (excess) demand functions (behavior near the boundary, Slutsky matrix, and Walras law) when the typical consumer's net trade is constrained to belong to a subspace of the space of goods. The first objective of this second chapter is to show that the general differential approach (see Y. Balasko (1985) and A. MasColell (1985) can be used in a similar way as in D. Duffie and W. Shafer (1985.I) to analyze existence and determinacy. The differences with D. Duffie and W. Shafer (1985I) is that one deals with more general market structures, that our notion of pseudo equilibrium is consequently different and finally that one uses directly the demand function presented in the first chapter and particularly the behavior of the demand function near the boundary. The second objective is only to suggest that our approach covers cases for which the market structure itself varies. The formulation of the stock market with production obviously belongs to this class of examples because the market structure depends on the choice of production plans. However, we prefer to present two other examples. The first one illustrates the interest of our
more general approach. Buyers of goods cannot discriminate among "qualities" of the same good, when they buy. The bundles they buy are not fixed a priori but depends on sellers' behavior who can discriminate between different "grades". The second one is well known and bears on spot markets and financial securities. I argue that a way of looking at the important result of indeterminacy when the market structure is truly incomplete (Y. Balasko and D. Cass (1985) and J. Geanakoplos and A. MasColell (1985)) is to show that with each market structure with units of account, one can associate a continuum of market structures without units of account but which incorporate arbitrary "prices" of units of account. Obviously, when \( r - 1 = n \), all these market structures are equivalent.

Consider the set

\[
\tilde{\mathcal{F}} = \{(z, w) \in \mathcal{C}_{r, n} \times \Omega \mid \sum_{i} \xi_i(z, w) = 0\}.
\]

It is straightforward to show that \( \tilde{\mathcal{F}} \) is a manifold of dimension \((m \cdot r) + n(r-n) - n\).

Lemma 1: \( \tilde{\mathcal{F}} \) is a smooth manifold of dimension \((m \cdot r) + n(r-n) - n\).

Proof: Consider a point \((z, w) \in \tilde{\mathcal{F}}\). Choose a \( \gamma \) which induces \( z \). Using the preimage theorem, one has to show that the following equations are independent at each point at which they all vanish

\[
\xi_i(\cdot) = \sum_{i} \xi_i(h, w) = 0 \quad \text{for} \quad h \neq (h_1, \ldots, h_{r-n}).
\]
One has thus $n$ equations, and the number of variables is $n(r-n)$ (for $\gamma P$) and $m \cdot r$ (for $\omega$).

$$\exists \xi \in \xi_1(\gamma P, \omega_1)$$

From Proposition 3, $\left[ \frac{1}{\Delta \omega} \right]$ has full rank $n$. Q.E.D.

**Remark:** At a regular value $\omega$ of $\bar{P}_r : \mathbb{F}^n$ for which $\bar{P}_r^{-1}(\omega)$ is not empty, $\bar{P}_r^{-1}$ is a smooth manifold of dimension $n(r-n)-n$.

Consider now a market structure $\bar{\Lambda}$ with $n$ simple markets. One can assume that there exists $\theta \in S^{r-1}$ such that $\bar{T}_\Lambda(P(\theta))$ has rank $n$. (If not one suppress some columns of $\bar{T}_\Lambda(\Pi)$ in order to get a market structure with a smaller number of simple markets). Now there exists a $r \times \tilde{n}$ (with $0 \leq \tilde{n} \leq n$) submatrix of $\bar{T}_\Lambda(\pi)$ which has rank $\tilde{n}$, whatever is $\pi \in \mathbb{R}^{n+}$.

Choose a submatrix for some $\tilde{n}$ and assume that it contains the first $\tilde{n}$ columns of $\bar{T}_\Lambda(\pi)$. We are interested in a class $\mathcal{L}$ of $n$-market structures $\Lambda$ whose $\bar{T}_\Lambda(\pi)$ contains the first $\tilde{n}$ columns of $\bar{T}_\Lambda(\pi)$. For each $\alpha = \tilde{n} + 1, \ldots, n$, choose a partition of $(1, \ldots, r)$ into two subsets $S^\alpha_+$ and $S^\alpha_-$ such that $\lambda_\alpha^+ > 0$ only if $h \in S^\alpha_+$ and $\lambda_\alpha^- > 0$ only if $h \in S^\alpha_-$. Thus $\mathcal{L}$ is the class of all $\Lambda$'s which have the same $\tilde{n}$ first columns as $\bar{\Lambda}$ and such that $\lambda_\alpha^+ > 0$ for $h \in S^\alpha_+$ and $\lambda_\alpha^- > 0$ for $h \in S^\alpha_-$ for $\alpha = \tilde{n} + 1, \ldots, n$. For each $\Lambda \in \mathcal{L}$ one has thus $\bar{T}_\Lambda(\pi) = \left[ \bar{T}_\Lambda(\pi) \mid L(\xi) \right]$ where $\bar{T}_\Lambda(\pi)$ is the $r \times \tilde{n}$ matrix built from the first $n$ columns of $\bar{T}_\Lambda(\pi)$ and $L(\xi)$ is a $r \times (n-\tilde{n})$ matrix which respects the partitions $(S^\alpha_+, S^\alpha_-)$ with non zero entries when $\xi \in \mathbb{R}^{n-\tilde{n}}$.

Now, we have to be a little more precise for the definition of a simple market, and more particularly for the definition of a bundle of goods. In good logic, one has to define a bundle as an element of the unit simplex $S^{r-1}$, or "locally" (when one parameterizes it in a little neighborhood) by...
setting to $1/\lambda^+_1$ (resp $\lambda^+_2$) for some $h_1$ (resp $h_2$). However, one chooses to set at 1 only one element of $\lambda^+$ or $\lambda^-$ and not one element of $\lambda^+$ and one element of $\lambda^-$. As one assumes that $\lambda^a \gg 0$ for $a > n+1$, one can choose any element of $\lambda^+$ or of $\lambda^-$ for presetting it to 1. Thus one can identify $\mathcal{L}$ with $\mathbb{R}^{\rho(\tilde{n})}$ where $\rho(\tilde{n}) = (r-1)(n-\tilde{n})$. Let be $\xi$ the generic element of $\mathbb{R}^{\rho(\tilde{n})}$ and one chooses the following identification

$$\xi = \{\lambda^{(n+1)}+\lambda^{(n+1)}_-, \ldots, \lambda^n, \lambda^-\}$$

where for each $a > \tilde{n}+1$, $\lambda^a_{h(a)} = 1$ for some $h(a)$.

Let us suppress the reference to $\tilde{\lambda}$, write $\rho$ instead of $\rho(\tilde{n})$ and consider the set

$$\bar{\mathcal{I}} = \{(\pi, Z, \tilde{\lambda}) \in \mathbb{R}^n_+ \times \mathcal{G}_r \times \mathbb{R}^{\rho}_+| Z \text{ contains the subspace generated by } [\mathbb{I}(\pi), L(\rho)]\}.$$ 

Here $\pi = (\tilde{\pi}, \xi)$. Imbed $\bar{\mathcal{I}}$ and $\mathcal{I}$ in $\mathbb{R}^n_+ \times \mathcal{G}_r \times \Omega \times \mathbb{R}^{\rho}_+$. One wants to show that the "inverse intersection", $\Gamma$, of $\bar{\mathcal{I}}$ and $\mathcal{I}$ is a smooth manifold (without boundary) of dimension $(m-r) + \rho$. But before doing that, let us examine the meaning of this approach in the case of the canonical model with "real" securities.

**Example 1':** It is the continuation of example 1. Recall that there are $\sigma + 1$ states, $r'$ goods in each state $s \geq 1$, one good in state 0 and $k(k < \sigma)$ securities. Then $r = (\sigma \cdot r') + 1$ and $n = \sigma(r'-1) + k$ with $n \leq r-1$. Choose for $\tilde{n}$, the first $\sigma(r'-1)$ simple markets which formulate the $\sigma$ spot markets. This is possible because the following matrix has always rank $\sigma(r'-1)$, whatever is $[\pi^1, \ldots, \pi^\sigma]$: 
Then the parameterization will concern only the simple markets associated with the securities. In this case, it is enough to parameterize only the elements of the matrix B. It is the approach used by D. Duffie and W. Shafer (1985) with the difference that they allow for negative elements in the matrix B.

Lemma 2: \( \Gamma \) is a smooth manifold of dimension \((m \cdot r) + \rho\).

Proof: Consider a point \([(\pi, z); w, z] \in \Gamma\). Choose a normalized \( \gamma^P \) which induces \( \gamma \) and consider

\[
\begin{align*}
\ell_1(\cdot) &\equiv \gamma^P [T(\tilde{\pi}): L(\tilde{\gamma})] = 0; (r-n) \cdot n \text{ equations} \\
\ell_2(\cdot) &\equiv \sum_i \xi_i \gamma^P(\cdot, w) = 0 \text{ for } h \notin \{h_1, \ldots, h_{r-n}\}; n \text{ equations.}
\end{align*}
\]

Notice that the first set of equations does not contain \( w \) and the second one does not contain \([T(\tilde{\pi}): L(\tilde{\gamma})]\). Differentiate \( \ell_1 \) with respect to \( z \) and \( \tilde{p}_j \) \( r-n \) \( j=1 \) where \( \tilde{p}_j \) is the jth column of \( \gamma^P \) from which are deleted the elements related with goods \( h \in \{h_1, \ldots, h_{r-n}\} \). One wants to show that \( \frac{\partial \ell_1}{\partial (\tilde{p}_j, l)} \) has rank \( n \cdot (r-n) \) where \( l \) is defined by fixing \( \lambda^q = 1 \) with \( h(z) \notin \{h_1, \ldots, h_{r-n}\} \) \( \forall z = \tilde{n}+1, \ldots, n \). It can be written
where \([\bar{\mathbf{T}}(\tilde{\pi})]_\gamma\) is the \(\tilde{n} \times n\) matrix gotten from \([\bar{\mathbf{T}}(\tilde{\pi})]'\) by deleting the columns related to goods \(h \in \{h_1, \ldots, h_{\tilde{n}-n}\}\), \([L(\pi)]_\gamma\) is the \((n-\tilde{n}) \times n\) matrix gotten from \([L(\pi)]_\gamma\) by deleting the columns related with goods \(h \in \{h_1, \ldots, h_{\tilde{n}-n}\}\), \(O_n\) is the \(\tilde{n} \times (r \cdot (n-\tilde{n}))\) matrix with entries equal to 0 and

\[
\begin{pmatrix}
(p^j)'_{\tilde{n}-n+1} & 0 \\
0 & (p^j)'_{\tilde{n}}
\end{pmatrix}
\]

is a \((n-\tilde{n}) \times ((r-1) \cdot (n-\tilde{n}))\) matrix for which \((p^j)'_a\) \((a = \tilde{n}+1, \ldots, n)\) is the row-vector of derivatives of \((p^j)' \cdot [\lambda^+ - \lambda^- \cdot \alpha^-] \) with respect to \(\lambda^a\).

Let us show that \([\tilde{\mathbf{T}}(\gamma)]_\gamma\) has rank \(n\). Reindexing goods,

\[
Y^{p'}(\tilde{\mathbf{T}}(\gamma) \mid L(\pi)) = 0
\]

can be written also

\[
[I_{(r-n)}]_\gamma \tilde{\mathbf{T}}_\gamma
\]
where $[[T(\pi)],[L(\pi)]]_\gamma$ (resp. $[[T(\pi)],[L(\pi)]]_\gamma$) is the matrix gotten from $[[T(\pi)],[L(\pi)]]$ by deleting the rows associated with goods $h \in \{h_1, \ldots, h_{r-n}\}$ (resp. $h \not\in \{h_1, \ldots, h_{r-n}\}$), $\hat{P}_\gamma$ is the matrix gotten from $P^\gamma$ by deleting the rows associated with goods $h \not\in \{h_1, \ldots, h_{r-n}\}$ and $I_{(r-n)}$ is the unitary $(r-n) \times (r-n)$ matrix. It follows that

$$[T(\pi)]_\gamma + \hat{P}_\gamma[T(\pi)]_\gamma = 0$$

Ad absurdum, assume that there exists $x \neq 0$ such that $[[T(\pi)]]_\gamma x = 0$. From the above relation, it follows that

$$[[T(\pi)]]_\gamma x = 0$$

Thus

$$[[T(\pi)]]_\gamma x = 0$$

which implies that $T(\pi)$ has rank smaller than $n$, a contradiction.

In order to prove that $[\hat{T}(\pi)]_\gamma$ has rank $n \cdot (r-n)$ one has thus only to show that the rows $(p^\gamma)^\alpha, \ldots, (p^\gamma)^\rho_{\pi^\gamma}, \ldots, (p^\gamma)^{r-n}$ are linearly independent for $\alpha = r+1, \ldots, n$. But this is true by the definition of a normalized $P^\gamma$.

From the proof of Lemma 1, one concludes that $\Gamma$ is a smooth manifold of dimension $r \cdot n + \omega$.

Lemma 3: The projection $P_\gamma: \Gamma - \Omega \times \mathbb{R}^0_{++}$ is proper.

Proof: Consider a compact set $K = K_1 \times K_2$ of $\Omega \times \mathbb{R}^0_{++}$. Let
be a sequence belonging to $F^{-1}(K)$. Without loss of
generality, one can assume that $((\omega^v, \xi^v)) + (\omega_0^0, \xi_0^0) \in K$. At a pseudo-
equilibrium, no agent $i$ can be outside the set bounded from below by the
indifference curve through $w_i^1$. For each $i$, let be $\tilde{u}_i^1 = \min_{\lambda \in K_i} \tilde{u}_i^1(w_i^1)$.

Let be, for each $(i, h)$, $u_{ih}^1$ (resp $\tilde{u}_{ih}^1$) the minimum (resp the maximum) of $u_{ih}$ over the set
$R_i = \{ x_i \in IR^r_+ | \tilde{u}_i^1(x) \geq \tilde{u}_i^1 \}$. Moreover $u_{ih}^1 > 0$, $u_{i,h}(h)$. Define $K_n$ as the set of $\pi \in IR^n$ such that
it exists $i \in K_2$, $w \in K_1$, $x$ with $\bar{E}(w, \omega_i^1) = 0$ such that
$\sum_{h=1}^{n} \alpha_h\bar{E}(w, \omega_i^1) u_{ih}(x_i) = 0 \forall i$ and $\alpha = 1, \ldots, n$. From the above remarks, $K_n$
is a compact subset of $IR^n$.

Now, as $G \cap \Omega$ is compact, one can assume that the sequence $\{Z_0^v\}$
converges to $Z_0^v$. By Proposition 1, and the fact that $K_1$ is compact,
$Z_0^v \in G \cap \Omega$. One has proved that the sequence $\{((\tilde{\omega}^v, Z^v); \omega^v, \rho^v)\}$ contains a
converging subsequence. Q.E.D.

Notice that $\Gamma$ and $\Omega \times IR_+^r$ have the same dimension $(m \cdot r) + r$. By
assumption, $\exists p \in IR_+^r$ such that $\bar{F}(p)$ has rank $n$. Choose also $\bar{\omega}$ such that $(\bar{\omega}, p)$ is a Walrasian Equilibrium (without trade). Define
$\bar{F} = \bar{F}(p)$ and $\bar{\omega}$ accordingly. $F^{-1}(\bar{\omega}, \bar{F})$ contains exactly one element
$\{(\bar{\pi}, \bar{Z}), \bar{\omega}, \bar{F} \}$ where $\bar{Z}$ is the subspace of dimension $n$ generated by $\bar{F}(\bar{\pi})$
We delete now the reference to $\bar{\pi}$. If one can show that $(\bar{\omega}, \bar{F})$ is a regular
value of $P_{\Gamma}$, one gets the conclusion that there exists a pseudo-equilibrium
for every $(\omega, \xi) \in \Omega \times IR_+^r$, from the theory of Brouwer Degree.

Lemma 4: $(\bar{\omega}, \bar{F})$ is a regular value of $P_{\Gamma}: \Gamma = \Omega \times IR_+^r$. 

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Proof: Consider \( \tilde{p} \in \mathbb{R}^{r}_{++} \) which is collinear with \( \psi_1(\tilde{w}) \psi_1 \) and consider a \( \gamma \tilde{p} \) which induces \( \tilde{z} \) and such that \( \gamma \tilde{p}^1 = \tilde{p} \). Thus in the relation 
\( \psi_1(\tilde{w}_1) = \tilde{p} \beta_1 \), one shall have \( b_{i1}^1 > 0 \) and \( b_{i1}^j = 0 \) for \( j = 2, \ldots, r-n, \psi_1 \).

One has to show that the \( ((r-n) \times n+n)^2 \) square matrix 
\[
\begin{bmatrix}
\frac{\partial f_1}{\partial \tilde{p}_i} & \frac{\partial f_1}{\partial \psi_1} \\
\frac{\partial f_2}{\partial \tilde{p}_i} & 0 \\
\frac{\partial f_3}{\partial \tilde{p}_i} & 0 \\
\end{bmatrix}
\]
where \( \tilde{p}_i \) is the price-vector \( \tilde{p}_i \) restrained to goods \( h \in (h_1, \ldots, h_{r-n}) \) is regular.

This matrix is equal to 
\[
\begin{bmatrix}
(\tilde{T}(\pi))_\gamma & 0 \\
0 & (\tilde{T}(\pi))_\gamma \\
\end{bmatrix}
\]

where \( (\tilde{T}(\pi))_\gamma \) is the \( n \times n \) matrix gotten from \( [\tilde{T}(\pi)]' \) by deleting the columns related to goods \( h \in (h_1, \ldots, h_{r-n}), K_\gamma \) is the \( n \times n \) matrix gotten from \( \tilde{K}_\gamma \) by deleting the rows and the columns related to goods \( h \in (h_1, \ldots, h_{r-n}), \) and \( -(\tilde{p}_i)' \tilde{\lambda} \) is the \( n \times n \) diagonal matrix of the form 
\[
\begin{bmatrix}
-(\tilde{p}_1)' \tilde{\lambda} \\
\vdots \\
-(\tilde{p}_n)' \tilde{\lambda} \\
\end{bmatrix}
\]

\[ j = 1, \ldots, r-n. \]
First \( \{\tilde{T}(\pi)\}_{i}^{j} \) has rank \( n \). For, by assumption \( \{\tilde{T}(\pi)\}_{i}^{j} \) has rank \( n \). Thus there is only one normalized \( \gamma P \) (see Part III) such that \( \{\tilde{T}(\pi)\}_{i}^{j} \gamma P = 0 \).

Thus, the same argument as that one of the proof of lemma 2 shows that \( \{\tilde{T}(\pi)\}_{i}^{j} \) has rank \( n \). Second \(- (p^{1})^T \Lambda \) has rank \( n \) because it is a diagonal \( n \times n \) matrix with non-zero elements on the diagonal \((p^{1} = \tilde{p} \in \mathbb{R}_{++}^r)\). Third \( b_{i}^{1} > 0 \) for each \( i \) and \( K_{Y_{1}} \) is a negative definite quadratic form from Proposition 2. Let us show that the rows of \( \frac{\partial f}{\partial \{\tilde{\pi},(p^{j})_{j}\}} \) are linearly independent, at \((\tilde{\omega},\tilde{\pi})\). Notice first that the collection composed of the first \( n \) rows and the last \( n \) ones is an independent one. Second consider the complementary collection of rows. It is an independent one because of the structure of \( \frac{\partial f}{\partial \{\tilde{\pi},(p^{j})_{j}\}} \). It follows from the fact that only the \( n \) first columns of \( \frac{\partial f}{\partial \{\tilde{\pi},(p^{j})_{j}\}} \) are non-zero that the collection of all rows is linearly independent.

Q.E.D.

Define \( \mathcal{L} \) as the set of all \( \Lambda \) with \( n \) simple markets such that the first \( \tilde{n} \) columns of \( \Lambda \) are the same as the first ones of \( \tilde{\alpha} \) and for \( \alpha \geq \tilde{n} + 1 \) \( \lambda_{h}^{a+} > 0 \) only if \( h \in S_{+}^{a} \) and \( \lambda_{h}^{a-} > 0 \) only if \( h \in S_{-}^{a} \).

**Proposition 4:** For every \((\omega, \Lambda)\) with \( \omega \in \Omega \) and \( \Lambda \in \mathcal{L} \) which satisfies A2 and A3, there exists a pseudo-equilibrium if A1 is made.

**Proof:** Lemmas 1-4 show that for every \( \omega \in \Omega \) and every \( \Lambda \in \mathcal{L} \), there exists a Q.W.E. (using the theory of Brouwer degree). A classical convergence argument shows that this is true for \( \Lambda \in \mathcal{L} \) and \( \omega \in \Omega \).

Fix \( \omega \) and consider a sequence \( \{\Lambda^{n}\} \) in \( \mathcal{L} \) which converges towards
Choose a pseudo-equilibrium \((\pi^v, z^v)\) for each \(\Lambda^v\). As \(G_{r,n}\) is compact, one can assume that \((z^v) + z^0 \in G_{r,n}\) and by the same argument as in Lemma 3, one can assume that \((\pi^v) + \pi^0 \in \mathbb{R}^n_+\) and that \(z^0 \in \mathcal{G}_{r,n}\). By continuity, one gets \(\xi_1(z^0, \omega_1) = 0\) and \(z^0\) contains the subspace generated by \(\Lambda_0^{(\pi^0)}\). Q.E.D.

Corollary to Proposition 4: Given \(A_1\), for every \(\Lambda\) satisfying \(A_2\) and \(A_3\), there exists a pseudo equilibrium.

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The objective of this part is to show that generically a pseudo-equilibrium is an equilibrium and that generically the set of pseudo-equilibria is finite. 4

Lemma 5: There is an open set \((\Omega_0)\) with null complement in \(\Omega \times \mathbb{R}^n_+\) such that for each \((\omega, \ell) \in \Omega_0\) there are a neighborhood \(U\) of \((\omega, \ell)\) in \(\Omega_0\) and \(q(\omega, \ell)\) functions \(\phi^v = [\phi_1^v, \phi_2^v]: U \rightarrow \mathbb{R}^n_+ \times G_{r,n}\).

\((v = 1, \ldots, q(\omega, \ell))\) such that

(i) \(P_t^{-1}(\omega, \ell) = \{(\phi_1^v(\omega, \ell), \phi_2^v(\omega, \ell)); \omega, \ell)q(\cdot)\}_{j=1}^q, \forall (\omega, \ell) \in U\)

(ii) \(\forall v\) there is \(\gamma(v)\) such that \(\phi_2^v(\omega, \ell) \in \mathcal{W}_{\gamma(v)}, \forall (\omega, \ell) \in U\).

Proof: Define \(\Omega_0\) to be the set of regular values of \(P_t\). From Sard's theorem \(\Omega_0^c\) is null and since \(P_t\) is proper, \(\Omega_0\) is open. Q.E.D.
Lemma 6: The set \( S = \{(w,t) \in \mathbb{R}_+^n \times \mathbb{R}_+^\rho \mid \begin{vmatrix} I(w) \\ L(t) \end{vmatrix} \) has rank \( n \) \}

is open with null complement in \( \mathbb{R}_+^n \times \mathbb{R}_+^\rho \).

Proof: The set \( S \) is obviously open. It is not empty because there is \( p \in \mathbb{R}_+^\rho \) such that \( T(p) \) has rank \( n \). For each \( a > \bar{n} \), choose \( h(a) \in S_+^a \) and set \( \lambda_{h(a)}^0 = 1 \). Given a \( r \times \bar{n} \) matrix of rank \( \bar{n} \) the set of vectors \( \lambda(n+1) \) in \( \mathbb{R}_+^{r-1} \times \{1\} \) such that \( [I; \lambda(n+1)]^{-1} \), \( \lambda(n+1) = [\lambda(n+1)^+; \lambda(n+1)^-] \) has rank \( n + 1 \) is open with null complement in \( \mathbb{R}_+^{r-1} \). In the same way, the set of vectors \( \{\lambda(n+1), \ldots, \lambda^n\} \) in \( [\mathbb{R}_+^{r-1} \times \{1\}]^{n-n} \) such that \( [I; \lambda(n+1), \ldots, \lambda^n] \) has rank \( n \) is open with null complement in \( \mathbb{R}_+^\rho \).

Now when one parameterizes \( \tilde{n} \in \tilde{\mathbb{R}}_+^n \), one has by assumption that \( T(\tilde{n}) \) has always rank \( \tilde{n} \). One can identify the set of matrices \( \tilde{T}(\tilde{n}) \), where \( \tilde{n} \in \tilde{\mathbb{R}}_+^n \), with \( \tilde{\mathbb{R}}_+^n \). Thus, by Fubini theorem, the set of \( (\tilde{n},t) \) for which \( [I; \lambda(n+1), \ldots, \lambda^n] \) has rank \( n \) is open with null complement in \( \tilde{\mathbb{R}}_+^{\rho + \tilde{n}} \). Reinstating \( \tilde{n} \in \tilde{\mathbb{R}}(n-\bar{n}) \) in the analysis, one concludes that the set \( S \) is open with null complement in \( \mathbb{R}_+^n \times \mathbb{R}_+^\rho \). Q.E.D.

Given Lemmas 5 and 6, in order to show that a pseudo equilibrium is almost always an equilibrium, one would like to show that \( \Psi^\prime: U \times \mathbb{R}_+^n \times \mathbb{R}_+^\rho \) defined by \( \Psi^\prime(w,t) = [\psi^\prime(p,t),l] \) is a submersion in order to get the result that \( [\psi^\prime]^{-1}(S) \) is open with null complement in \( U \). This would imply that \( \tilde{U} = \cap [\psi^\prime]^{-1}(S) \) is open with null complement in \( U \) and thus that for every \( (w,t) \in \tilde{U} \), every pseudo equilibrium is an equilibrium. Finally it would follow that, globally, there exist an open set \( \tilde{\Omega}_+ \) in \( \Omega \times \mathbb{R}_+^\rho \) with null complement such that for every \( (w,t) \in \tilde{\Omega}_+ \), every pseudo-equilibrium is an
equilibrium.

Now, if it is possible to choose \( \tilde{n} = n \), \( \tilde{A} \) is regular and it is possible to parameterize only \( \omega \in \Omega \), a pseudo equilibrium being always an equilibrium. If one chooses \( \tilde{n} = 0 \), which is always possible, \( \begin{bmatrix} \frac{\partial f}{\partial (\omega, \xi)} \end{bmatrix} \) has rank \((m \cdot r) + p\). On the contrary, when \( \tilde{n} \) is larger than 0 but smaller than \( n \), \( \begin{bmatrix} \frac{\partial f}{\partial (\omega, \xi)} \end{bmatrix} \) has not rank \((m \cdot r) + p\). That implies that, at a regular value of \( p_r \)

\[
\begin{bmatrix} \frac{\partial (\pi, (\tilde{\omega}, \xi))}{\partial (\omega, \xi)} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial (\pi, \tilde{\omega})} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f}{\partial (\omega, \xi)} \end{bmatrix}
\]

has not full rank. In order to show that nonetheless \( \pi = \phi^{\nu}(\omega, \xi) \) is a submersion for each \( \nu \), one shall proceed in the following way. The proof of Lemma 2 has shown that it is possible to extract a square regular \( \tilde{n} \times \tilde{n} \) matrix from \( \bar{T}(\pi) \) by suppressing from \( \bar{T}(\pi) \), \( r - \tilde{n} \) columns among which the columns related with goods \( \{h_1, \ldots, h_{r-n}\} \). Let be \( \bar{T}_n(\pi) \) the resulting submatrix and assume that it columns concern the first \( \tilde{n} \) goods. Thus let \( \tilde{p}_n = [\tilde{p}_1, \ldots, \tilde{p}_n] \) and \( \tilde{p}_{\tilde{n}} = [\tilde{p}_{\tilde{n}+1}, \ldots, \tilde{p}_{n2}] \). One gets \( \tilde{p}_n = \phi^{\nu j}(\omega, \xi) \)

and \( \tilde{p}_{\tilde{n}} = \phi^{\nu j}(\omega, \xi) \) for \( \nu = 1, \ldots, q(\omega, \xi) \) and \( j = 1, \ldots, r-n \). Thus one can write for each \( \nu \) and each \( j \):

\[
\begin{bmatrix} \tilde{p}_n \end{bmatrix} = \bar{T}_n(\pi), -1 \begin{bmatrix} \tilde{p}_{\tilde{n}} \end{bmatrix}
\]

or

\[
\phi^{\nu j}(\omega, \xi) = \bar{T}_n(\phi^{\nu j}(\omega, \xi)), -1 \phi^{\nu j}(\omega, \xi)
\]

Let be \( \bar{f} \) the system of equations gotten from \( f \) by eliminating \((r-n)\tilde{n}\) variables \( \tilde{p}_n \) and \((r-n)\tilde{n}\) equations. One gets thus
Differentiating, we verify that $\frac{\partial \mathcal{F}}{\partial (\omega, \ell)}$ has full rank (see Lemma 2) and one concludes that \( \text{rank} [d\Phi^\omega_1(\omega, \ell)] = \n(\omega, \ell) \in U \).

**Example 1:** It is still the canonical model with "real" securities.

Recall that one chose to parameterize only the \( k \) last markets associated with the \( k(k - 1) \) securities because the matrix

$$\begin{bmatrix}
T^1(\pi^1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & T^s(\pi^s) \\
\end{bmatrix}$$

has always rank \( \sigma(r' - 1) \) because

$$T^s(\pi^s) = \begin{bmatrix}
-1 \\
\vdots \\
0 \\
-\pi^s_1 & \ldots & -\pi^s_{r'} & -\pi^s_{r' - 1} \\
\end{bmatrix}$$

Now, it is always possible, in this model, to choose \( \{h_1, \ldots, h_{r - n}\} \) as a subset of the set of numeraires: \( \{(0,r'),(1,r'),\ldots,(\sigma,r')\} \). It follows that the (square) regular \( \tilde{n} \times \tilde{n} \) matrix will be always the matrix

$$\begin{bmatrix}
\vdots \\
T^1(\pi^1) \\
\vdots \\
T^\sigma(\pi^\sigma) \\
\end{bmatrix}$$

from which the rows associated with the numeraires are eliminated. It is then possible to express, given \( \pi \), all the prices of goods \((s,h')\) for \( h' \neq r' \).
in terms of the prices of the numeraires by the relations

\[
\begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
p(s,1) \\
p(s,r'-1) \\
\vdots \\
\vdots \\
p(s,1) \\
\vdots \\
p(s,r'-1) \\
\end{bmatrix}
= \begin{bmatrix}
\pi_s^1 \\
\vdots \\
\pi_s^{r'-1} \\
\end{bmatrix}
\]

**Proposition 5:** There exists an open set \( \Omega_1 \) with null complement in \( \Omega \times \mathbb{R}^\rho_+ \) such that for each \((w,t) \in \Omega_1\), the set of pseudo equilibria is finite and each pseudo equilibrium is an equilibrium.

**Remark:** Propositions 4 and 5 imply that for every \((w,A) \in \Omega \times \mathcal{L}\), one can find \((w',A') \in \Omega \times \mathcal{L}\) arbitrary close such that all pseudo equilibria of \((w',A')\) are equilibria.

**VII**

The only objective of this part is to show that our formalism covers the case of spot markets with financial securities and is well suited to study the case for which asymmetry of information about product quality generates an incomplete market structure. It is possible to write down that for these two examples, the "real" market structure is endogenous and in some sense dependant on prices. In the case of (financial) securities involving returns in terms of units of account, it is possible to associate with this market structure a continuum of market structures for an economy without units of account in such a way that the whole set of equilibrium allocations of this continuum coincides with the set of equilibrium allocations of the original market structure with units of account. In the second case, all agents do not face the same market structure on each simple market, the structure of the
bundle of goods that "buyers" get depend on sellers' behavior which depends itself on prices.

1. There are now, \( \tilde{r} \) commodities \( h \). The first group \( (H_1) \) contains \( r \) goods (the goods of the previous parts) and the second group \( (H_2) \) contains \( \tilde{r}-r \) commodities that can be named units of account. These units of accounts do not enter in utility functions and initial endowments in these commodities are zero. Thus these commodities play only a role in the process of exchange.

A market structure \( \Lambda = (\Lambda^r, \Lambda^-) \) is defined as previously and satisfies assumption A2 - A4. Given that \( X_i = \mathbb{R}_{++}^r \), \( A_i \) is kept for the group \( H_1 \) and an equilibrium (or a pseudo-equilibrium) is defined as before. However as \( \tilde{u}_i \) is defined on \( \mathbb{R}_{++}^r \) and \( z_{ih} = 0 \) for \( h \in H_2 \), one cannot require anymore that \( \tilde{w} \) is strictly positive or bounded. Moreover, let us define \( \tilde{T}_1(\tilde{w}) \) as the matrix \( \tilde{T}(\tilde{w}) \) restrained to goods \( h \in H_1 \) and \( \tilde{T}_2(\tilde{w}) \) as the matrix \( \tilde{T}(\tilde{w}) \) restrained to goods \( h \in H_2 \).

A5: \( n = \bar{n} - (\tilde{r}-r) \) is strictly larger than 0 and strictly smaller than \( r \).

Thus \( r-n = \tilde{r}-\bar{n} > 0 \)

Define:

\[
Z(\tilde{w}) = \{ z \in \mathbb{R}^r \mid \exists \bar{a} \in \mathbb{R}^{\bar{n}} \text{ s.t. } z = \tilde{T}_1(\tilde{w}) \bar{a} \text{ and } \tilde{T}_2(\tilde{w}) \bar{a} = 0 \}
\]

\( Z(\tilde{w}) \) has at most dimension \( n \).

**Definition 3:** \( [\tilde{w}^*, Z(\tilde{w}^*), z^*] \) is an equilibrium iff:

- i) \( \sum_{i} z_i^* = 0 \)
- ii) \( \forall z_i^* \) maximizes \( \tilde{u}(\bar{w}_i + z) \) over \( Z(\tilde{w}^*) \).
Definition 4: \( \{ \tilde{\pi}^*, Z^*, z^* \} \) is a pseudo equilibrium iff:

1) \( Z^* \) is an \( n \) dimensional subspace of \( \mathbb{R}^r \) which contains \( Z(\tilde{\pi}^*) \)

2) \( \sum_{i=1}^{r} z_i^* = 0 

3) \( u_i^*: z_i^* \) maximizes \( u_i(u_i+z) \) over \( Z^* \).

As exchange ratios are not necessarily strictly positive, one can have decrease of ranks due to the fact that some exchange ratios are zero.

Example 2: \( r = 2 \) and \( \tilde{r} = 3 \) and \( \tilde{n} = 2 \). \( A \) is such that

\[
\tilde{\pi}(\tilde{\pi}) = \begin{bmatrix}
-\tilde{\pi}^1 \\
-\tilde{\pi}^2 \\
1 & 1
\end{bmatrix}
\]

That means that good 1 (resp. good 2) can be exchanged against the unit of account, but good 1 cannot be exchanged directly for good 2. A Walrasian equilibrium is an equilibrium for \( A \). However, there is another equilibrium for \( A: z_{i1} = z_{i2} = 0 \) for each \( i \) and \( \tilde{\pi}^1 = \tilde{\pi}^2 = 0 \). Moreover when \( \tilde{\pi} \in \mathbb{R}_{+}^2 \), only the ratio \( \tilde{\pi}^1/\tilde{\pi}^2 \) matters for getting a Walrasian Equilibrium. This is due to the fact that \( \tilde{r} - \tilde{n} = r-n = 1 \).

Consider the canonical model with spot markets and financial securities (and for simplification sake, assume there are no "real" securities). In this particular case, for each market structure of this type, it is possible to associate a continuum of market structures for an economy without units of account.

Example 3: There are two periods (0 and 1) and \( \sigma \) states of nature in period 1. In each state \( (s = 1, \ldots, \sigma) \) there are \( r' \) goods \( h = 1, \ldots, r' \) and one
unit of account (commodity \(r'+1\)). In period 0 (state 0) there is one good and one unit of account. The simplest assumption is that the unit of account of state \(s\) is exchanged for good \(r'\) of state \(s\).

There are \(k\) securities. A security is paid in period 0 in unit of account and gives only units of accounts in states \(s \geq 1\). Thus 
\[
\vec{r} = (r'+1) \sigma + 2, \quad \tau = r' \sigma + 1 \quad \text{and} \quad \vec{n} = r' \sigma + 1 + k.
\]
One gets a market structure \(\vec{\lambda}\) which gives rise to the following \(\vec{T}(\vec{\pi})\).

\[
\vec{T}(\vec{\pi}) = \begin{bmatrix}
-\vec{\pi} & 0 \\
0 & \vec{T}^s(\vec{\pi}) & \cdots \\
& \cdots \\
& & \vec{T}^{\sigma}(\vec{\pi})
\end{bmatrix}
\]

where for \(s = 1, \ldots, \sigma\)

\[
\vec{T}^s(\vec{\pi}) = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vec{\pi}^s & \vec{\pi}^s & \vec{\pi}^s & \vec{\pi}^s & 1 & 0 \\
-\vec{\pi}^s & -\vec{\pi}^s & -\vec{\pi}^s & -\vec{\pi}^s & -\vec{\pi}^s & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
\]

is a \((r'+1) \times r'\) matrix, \(\vec{\pi}\) is a \(k\) row-vector placed at the row of good 2 of period 0 and \(B\), the \((r'+1) \sigma \times k\) matrix of security returns has all its entries equal to 0 except at the rows corresponding to units of account. It follows that \(\vec{T}(\vec{\pi})\) has always full rank, if \(B\) has full rank. Thus, one assumes that the matrix \(B\) of returns of securities has full rank.

For each \(\vec{\pi}_0 = [1, \pi_0^1, \ldots, \pi_0^\sigma] \in \mathbb{R}^{\sigma+1}_+\), one can build a market structure associated with the economy with \(r = \sigma r' + 1\) goods only. For that, one eliminates the last column of \(\vec{T}^s(\vec{\pi})\) for \(s = 0, 1, \ldots, \sigma\); one substitutes
for $B'$ the matrix $B'$ with non zero entries only for good $r'$ (whatever is $s \geq 1$) and for which for each $s \geq 1$ row $r'$, of $B'(s) = \{ \text{row } r'+1 \text{ of } B(s) \} \times \pi_s^r$; and finally one assumes that securities are paid in good of period 0. Let be $\Lambda(\pi_0)$ the market structure for an economy without units of account associated with $\pi_0$. An equilibrium allocation for $\Lambda(\pi_0)$ is an equilibrium allocation for $\bar{\Lambda}$. However, when $r-n = \bar{r}-n = 1$, all the market structures $\Lambda(\pi_0)$ for $\pi_0 \in \mathbb{R}^{r+1}$ and $\pi_0 = 1$ are equivalent.

Remark. When we set $\pi_0 = 0$, one gets a market structure $\Lambda(0)$ with only spot markets. Moreover an equilibrium for $\Lambda(0)$ is an equilibrium for $\bar{\Lambda}$.

Remark: Clearly, the set $\{(\pi, Z; \omega, \pi_0) \in \mathbb{R}_+^n \times \mathbb{R}_+^r \times \Omega \times \mathbb{R}_+^q | (\pi, Z) \text{ is an equilibrium for } (\omega, \Lambda(\pi_0))\}$ is a manifold of dimension.

2. There are $m$ consumers $i$ and $r$ goods $h$. Good $r$ is "money". All is assumed. There is a partition $B = \{B^0, B^1, \ldots, B^n\}$ of $\{1, \ldots, r\}$ with $B^0 = \{r\}$. When agent $i$ buys, he cannot discriminate among goods $h \in B^v$ for each $v$. When he sells he can discriminate. There are then $n$ simple markets $\alpha$. On simple market $\alpha$, goods belonging to $B^\alpha$ are exchanged for good $r$. Let be $a^\alpha_i \geq 0$ the intensity of purchase on market $\alpha$ by agent $i$ and $a^-\alpha_i \geq 0$ the intensity of sale on market $\alpha$ by agent $i$. For each $\alpha$, let be $\lambda^\alpha \in \mathbb{R}^{r-1}$ the bundle of goods $h \in B^\alpha$: one has $\lambda_h^\alpha = 0$ for $h \notin B^\alpha$. Thus a simple market will be formulated by $\begin{pmatrix} \lambda^\alpha \\ -\pi^\alpha \end{pmatrix}$. Let be $\pi = [\pi^1, \ldots, \pi^r, \ldots, \pi^n] \in \mathbb{R}_+^n$ and $\tilde{T}_i(\pi)$ the matrix having $\begin{pmatrix} \lambda^\alpha \\ -\pi^\alpha \end{pmatrix}$ as its $a$th column.

Define $z^+ = \tilde{T}_i(\pi) a^+ \alpha$ where $a^+ = [a^1, \ldots, a^\alpha, \ldots, a^n]$. When he sells,
agent \(i\) is facing a "complete" market structure \(\tilde{T}_2(\pi)\) with \(r-1\) columns:

\[
\tilde{T}_2(\pi) = \begin{bmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\ddots & \ddots & \ddots \\
1 & \cdots & 1 \\
-\pi & \cdots & -\pi \\
\end{bmatrix}
\]

Notice that all prices of goods \(h \in B^0, \forall h \geq 1\), are equal.

Define \(z^- = \tilde{T}_2(\pi) a^-\) where \(a^- = [a_1^-, \ldots, a_h^-, \ldots, a_(r-1)^-]\). One requires essentially that agent \(i\) cannot resell the goods he has bought, in order to give meaning to the asymmetric nature of relationship between buyer and sellers:

\[a_i^- = z_{ih}^-, \quad h = 1, \ldots, r-1.\]

\(\tilde{T}_1(\pi)\) is endogenous in the sense that if for some \(\alpha \neq 0\), \(\sum_{i \in B^0} z_{ih}^- > 0\), then \(\lambda^0 = \sum_{i \in B^0} z_{ih}^- \cdot [\sum_{i \in B^0} z_{ih}^-]^{-1} \forall h \in B^0\). Each consumer \(i\) is small with respect to the economy and thus does not take into account the effect on \(\lambda^0\) of a change of his sales.

Finally, balance means that

\[\sum_{i \in B^0} z_{ih}^+ = \sum_{i \in B^0} z_{ih}^- \quad h = 1, \ldots, r-1.\]

Note \(a_i = (a_i^+, a_i^-)\) and \(a = [a_1, \ldots, a_n]\). Define \(\varphi_{ih}\) as the Lagrange multiplier associated with the constraint \(a_i^- \leq \omega_{ih}\), \(\rho = [\rho_1, \ldots, \rho_{r-1}]\) and \(\sigma = [\sigma_1, \ldots, \sigma_m]\).
Definition 5: A 3-uplet \([x^*, a^*, \rho^*]\) is an equilibrium with respect to \(\mathcal{O}\) if

\[(i) \sum a_i^+ = \sum a_i^- \forall a \geq 1\]

\[(ii) \forall i: (\tilde{T}_1(x^*)) u_i a_i^+ = 0, a_i^+ \geq 0\]

\[(\tilde{T}_2(x^*) \cup_{i+\rho^*}) a_i^- = 0, \forall a \geq 1, \forall \rho \geq 1\]

\[(iii) \forall a \text{ such that } \sum a_i^+ > 0, a_h^+ = \sum a_i^- / \sum a_i^+ \forall a \geq 1\]

Notice that this definition includes weak inequalities. It follows that the use of manifold with corners is needed to study this case from the differential viewpoint. The need for this tool arises also when one wants to consider equilibria implying rationing (with flexible \(\pi\)) in this case (see Y. Younes (1984)).

Conclusion

When \(A2'\) is substituted for \(A2\), the same kind of analysis can be made.

Recall that \(A2'\) means that one requires only that for each \(a: \lambda^a - \in \mathbb{R}_+ \setminus \{0\}\). It follows that \(\lambda^a \in \mathbb{R}\) for each \(a\) such that \(\lambda^a \not= \mathbb{R}_+ \setminus \{0\}\). One starts from some \(\tilde{a} = (\tilde{a}^+, \tilde{a}^-)\) which satisfies \(A2'\) and \(A3\). If parameterization of a part of \(\tilde{a}\) is necessary or useful, one defines still two subsets \(S^+\) and \(S^-\) of \(\{1, \ldots, r\}\), for each \(a\) for which one seeks parameterization. One has \(S^+ \cup S^- = (\{1, \ldots, r\})\) but no more necessarily \(S^+ \cap S^- = \emptyset\). Let be \(r^+(\alpha)\) the cardinality of \(S^+\) and \(r^-(\alpha)\) the cardinality \(S^-\). Let be

\[\rho = \frac{1}{n!} \left[ r^+(\alpha) + r^-(\alpha) \right] - (n-\tilde{n}).\]

Lemma 2 carries over for this case i.e., \(F\) is still a smooth manifold.
of dimension \( (m-r)+\rho \), when \( \lambda^{a+} \) is parameterized in \( \mathbb{R}^{r+(a)} \) and \( \lambda^{a-} \) is parameterized in \( \mathbb{R}^{r-(a)} \). One can still write:

\[
\bar{f}_1(\cdot) \equiv \gamma \Gamma'[\bar{T}(\pi) \mid \Pi(\pi)] = 0; (r-n) \text{ n equations}
\]

\[
\bar{f}_2(\cdot) \equiv \sum_{i=1}^{n} \sum_{j=1}^{r-n} \chi_{h_j}(\pi, \lambda_j) = 0 \text{ for } h \in \{h_1, \ldots, h_{r-n}\}; \text{ n equations.}
\]

In order to check that 0 is a regular value, one differentiate \( \bar{f}_1 \) with respect to \( (p^j)_{j=1}^{r-n} \) and \( \pi \) (because it may happen that \( \pi^a = 0 \) for some \( a \), but \( \lambda^{a-} > 0 \) for some) and \( \bar{f}_2 \) with respect to \( \lambda \).

The argument at the end of the Annex is used to show that \( P_{\Gamma} : \Gamma \times \Omega \times \bar{\mathbb{R}}^{\delta} \) is proper. The counterpart of lemma 4 is true because \( \chi^{a-} > 0 \). Thus a pseudoequilibrium always exists if \( A_{2'} \) and \( A_{3} \) are made, given \( A_{1'} \).

However, we think that the economic meaning of the exchange of "bundles" with goods in "negative proportions" and which can overlap is not very clear.
Footnotes

* I benefited from D. Cass' advice and suggestions from C. Le Van.

(3) See e.g., Y. Younes (1984).

(4) Notice that there can be equilibria of rank smaller than n.

(5) Instead of unit of account, one can use the term inside money. D. Cass keeps the name inside money for a security which is paid in units of account of period 0 and gives one unit of account in every state of period 1.