N° 8603

A NOTE ON

THE GALE-NIKAIDO-DEBREU LEMMA AND THE EXISTENCE OF GENERAL EQUILIBRIUM

Monique FLORENZANO - Cuong LE VAN

A NOTE ON THE GALE-NIKAIDO-DEBREU LEMMA AND THE EXISTENCE OF GENERAL EQUILIBRIUM.

Monique FLORENZANO,Cuong LE VAN
(CNRS-CEPREMAP)

ABSTRACT.

In this note,we give an other version of the generalized Gale-Nikaido-Debreu lemma which has as direct corollary the usual statement without weakening any conclusion.We give also an other simple way for discarding free disposability.

Key Words:upper semi-continuous correspondence,partition of unity, retraction,transitive general quasi-equilibrium,free disposability.

NOTE SUR LE LEMME DE GALE-NIKAIDO-DEBREU ET L'EXISTENCE DE L'EQUILIBRE GENERAL

Monique FLORENZANO - Cuong LE VAN
CEPREMAP - CNRS - PARIS

RESUME.

Dans cette note,nous donnons une autre version du lemme de Gale-Nikaido-Debreu généralisé qui a comme corollaire direct le résultat de Debreu sans affaiblissement de la conclusion.Nous donnons également une autre façon simple d'écarter la libre disposition.

Mots clés:correspondance semi-continue supérieurement,partition de l'unité,rétraction,quasi-équilibre transitif,libre disposition.

Classification:JEL 021.

A NOTE ON

THE GALE-NIKAIDO-DEBREU LEMMA AND

THE EXISTENCE OF GENERAL EQUILIBRIUM

Monique FLORENZANO - Cuong LE VAN
CEPREMAP - CNRS - PARIS

In a paper published in JME, one of the two authors of this note gave a lemma presented as the most convenient version for applications in general equilibrium theory of the generalized statement of Gale-Nikaido-Debreu lemma :

Lemma 1. (Florenzano)

Let P be a closed convex cone with vertex 0 in $\mathbb{R}^{\ell}$, B the closed ball with center 0 and radius 1, S the sphere with same center and same radius and ζ an upper semi-continuous and non-empty compact convex-valued correspondence from $B \cap P$ into $\mathbb{R}^{\ell}$. If ζ satisfies the condition

$$\forall\ p \in S \cap P,\ \exists\ z \in \zeta(p),\ p.z \leq 0$$

then there exists $\bar{p} \in B \cap P$ such that $\zeta(\bar{p}) \cap P^{o} \neq \emptyset$, where

$$P^{o} = \{z \in \mathbb{R}^{\ell} / z.p \leq 0\ \forall\ p \in P\}$$ is the polar cone of P.

A very simple proof of lemma 1 was based upon the Brouwer's fixed-point theorem and the concept of partition of unity subordinated to a covering. It was evident from the proof that ζ does not need to be compact-valued but only such that $\zeta(p) - P^{o}$ is closed for each p in $B \cap P$. In particular, if P is the whole space ($P^{o} = \{0\}$), we had as corollary :

./.

Corollary 1 (Cornet)

If ζ is an upper semi-continuous and non-empty closed convex-valued correspondence from B into $\mathbb{R}^\ell$, satisfying the condition

$$\forall\, p \in S,\ \exists\, z \in \zeta(p),\ p.z \leq 0$$

then there exists $\overline{p} \in B$ such that $0 \in \zeta(\overline{p})$

Note also that the point $\overline{p}$, of which lemma 1 and corollary 1 demonstrate the existence, is not proved $\neq 0$. Actually, one can easily check that 0 is the unique point $\overline{p} \in B$ such that $0 \in \zeta(\overline{p})$, where $\zeta(p) = \{-p\}$ for every $p \in B$, and thus verifies the assumptions of corollary 1. This weakening of conclusion of lemma 1 with regard to the statement of Debreu (1956) was presented in the JME paper (1982) as the counterpart of the advantage of not leaving out the case where P is a linear subspace of $\mathbb{R}^\ell$.

The aim of this note is to state a corollary of lemma 1 which allows to deduce the standard statement of Gale-Nikaïdo-Debreu lemma without any weakening of the conclusion.

Corollary 2 (Le Van)

Under the assumptions of lemma 1 on B, S and P, let Q be a closed subset of $B \cap P$ containing $S \cap P$ and such that there exists a retraction r from B onto Q. If ζ is an upper semi-continuous and non-empty compact convex-valued correspondence from Q into $\mathbb{R}^\ell$, satisfying the condition

$$\forall\, p \in S \cap P,\ \exists\, z \in \zeta(p),\ p.z \leq 0$$

then there exists $\overline{p} \in Q$ such that $\zeta(\overline{p}) \cap P^0 \neq \emptyset$.

Proof

Let $\zeta' : B \cap P \to \mathbb{R}^\ell$ be defined by $\zeta'(p) = \zeta(r(p))$. Since r is continuous, ζ' is upper semi-continuous. Since the restriction r/Q is the identity $Q \to Q$, ζ' satisfies the condition

$$\forall\, p \in S \cap P,\ \exists\, z \in \zeta(p) = \zeta'(p),\ p.z \leq 0$$

By lemma 1, there exists $\overline{p} \in B \cap P$ such that $\zeta'(\overline{p}) \cap P^o \neq \emptyset$; $\overline{\overline{p}} = r(\overline{p}) \in Q$ and satisfies $\zeta(\overline{\overline{p}}) \cap P^o \neq \emptyset$

$\square$

It should be emphasized that lemma 1 can be deduced from corollary 2. Indeed, if $Q = B \cap P$, Q is a closed convex subset of $\mathbb{R}^\ell$ and it follows from an extension of Tietze's theorem (see for example proposition 3.58 of Schwartz (1969)) that identity $Q \to Q$ has a continuous extension $B \to Q$; in other words, there exists a retraction $r : B \to B \cap P$. A direct proof of corollary 2 could be based on degree theory and would give an alternative proof of lemma 1.

Corollary 3 (Debreu)

Let P be a closed convex cone with vertex 0 in $\mathbb{R}^\ell$ which is not a linear subspace ; let P^o, B and S be as in lemma 1. If ζ is an upper semi-continuous and non-empty compact convex-valued correspondence from $S \cap P$ into $\mathbb{R}^\ell$ satisfying the condition :

$$\forall\, p \in S \cap P,\ \exists\, z \in \zeta(p),\ p.z \leq 0$$

then there exists $\overline{p} \in S \cap P$ such that $\zeta(\overline{p}) \cap P^o \neq \emptyset$

Proof

Corollary 3 is deduced from corollary 2 by proving the existence of a retraction from B onto $S \cap P$.

Since P is not a linear subspace of $\mathbb{R}^\ell$, there exists a non null vector $v \in P^o \cap -P$. (see K. Border (1985), Proposition 2.18). Then $w = \frac{-v}{\|v\|} \in S \cap P$ and $w.x \geq 0$ for all $x \in P$. Let $H = \{x \in \mathbb{R}^\ell / w.x=0\}$ be the hyperplane orthogonal to w and let $\pi : \mathbb{R}^\ell \to H$ be the orthogonal projection onto H, i.e. $\pi(x) = x - (x.w)w$. Firstly $\pi(B) = B \cap H$. Secondly, $\tilde{\pi}$, the restriction of π to $S \cap P$, is a homeomorphism between $S \cap P$ and $\tilde{\pi}(S \cap P)$ and $\tilde{\pi}(S \cap P)$ is convex (see Proposition 18.3 in K. Border (1985)). As above, it follows from Tietze's theorem that there exists a retraction ρ from $B \cap H$ onto $\tilde{\pi}(S \cap P)$. Define $r = \tilde{\pi}^{-1} \circ \rho \circ \pi / B$. r is a retraction from B onto $S \cap P$. Then it follows from corollary 2 that there exists $\overline{p} \in S \cap P$ such that $\zeta(\overline{p}) \cap P^o \neq \emptyset$.

$\square$

./.

Remarks

In some cases, it is not necessary to invoke Tietze's theorem to exhibit a retraction. For example, if P is a closed half - space, i.e. $P = \{x \in \mathbb{R}^{\ell} / w.x \geq 0\}$ for some $w \in S$, define $\pi : B \to H$ by $\pi(x) = x - (x.w)w$ and $r : B \to S \cap P$ by $r(x) = \pi(x) + (1 - \| \pi(x) \|^2)^{1/2} w$. It is readily proved that r is a retraction from B onto $S \cap P$.

The previous example suggests a very simple trick, alternative to that one used by Bergstrom (1976), for proving the existence of a transitive general quasi-equilibrium without free disposability.

A pure exchange economy $E = ((X^i, R^i, \omega^i); i = 1,\dots, m))$ is defined by m consumption sets $X^i \subset \mathbb{R}^{\ell}$, for each i, a complete preorder R^i on X^i and an initial endowment $\omega^i \in \mathbb{R}^{\ell}$. The strict preference relation corresponding to R^i is :

$$x^i P^i x'^i \iff x^i R^i x'^i \text{ and not } x'^i R^i x^i.$$

Let us define

$$P^i(x^i) = \{x'^i \in X^i \mid x'^i P^i x^i\}$$

$$(P^i)^{-1}(x^i) = \{x'^i \in X^i \mid x^i P^i x'^i\}$$

If $p \in S^{\ell-1}$ (unit sphere of $\mathbb{R}^{\ell}$), we define

$$\gamma^i(p) = \{x^i \in X^i \mid p.x^i \leq p.\omega^i\}$$

$$\delta^i(p) = \{x^i \in X^i \mid p.x^i < p.\omega^i\}$$

$$\varphi^i(p) = \{x^i \in X^i \mid x^i \in \gamma^i(p) \text{ and } \delta^i(p) \cap P^i(x^i) = \emptyset\}$$

$$\eta(p) = \sum_{i=1}^{m} \varphi^i(p) - \left\{\sum_{i=1}^{m} \omega^i\right\}.$$

A quasi-equilibrium without free-disposability of E is $(\bar{p}, \bar{x}) \in S^{\ell-1} \times \prod_{i=1}^{m} X^i$ such that

1) $\forall i, \ \bar{x}^i \in \varphi^i(\bar{p})$

2) $\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i.$

Proposition 1.

Under the following assumptions :

H1 : $\forall i = 1,\ldots, m$, X^i is a non empty compact convex subset of $\mathbb{R}^{\ell}$.

H2 : $\forall i = 1,\ldots, m$, $\forall\, x^i \in X^i$, $(P^i)^{-1}(x^i)$ is open in X^i

H3 : $\forall i = 1,\ldots, m$, $\forall\, x^i \in X^i$, $x^i \notin \mathrm{Co}\,(P^i(x^i))$

H4 : $\forall i = 1,\ldots, m$, $\omega^i \in X^i$

H5 : If $x = (x^i) \in \prod_{i=1}^{m} X^i$, $\sum_{i=1}^{m} x^i = \sum_{i=1}^{m} \omega^i \Rightarrow \forall i,\ P^i(x^i) \neq \emptyset$

then there exists a quasi-equilibrium without free disposability.

Proof

For $(p, q) \in S^{\ell}$ (unit-sphère of $\mathbb{R}^{\ell+1}$), with $p \in \mathbb{R}^{\ell}$ and $q \in \mathbb{R}_+$, let us define :

$$\hat{\gamma}^i(p, q) = \{x^i \in X^i / p.x^i \leq p.\omega^i + q\}$$

$$\hat{\delta}^i(p, q) = \{x^i \in X^i / p.x^i < p.\omega^i + q\}$$

$$\hat{P}^i(x^i) = \{z^i \in X^i / z^i = \lambda y^i + (1-\lambda)x^i, \lambda \in]0,1],\ y^i \in P^i(x^i)\}$$

$$\hat{\varphi}^i(p,q) = \{x^i \in X^i / x^i \in \hat{\gamma}^i(p, q) \text{ and } \hat{\delta}^i(p, q) \cap \hat{P}^i(x^i) = \emptyset\}$$

$$\hat{\eta}(p, q) = \left(\sum_{i=1}^{m} \hat{\varphi}^i(p, q) - \{\sum_{i=1}^{m} \omega^i\}\right) \times \{-m\} .$$

Using the same techniques as in Geistdoerfer-Florenzano (1982), one can show that $\hat{\eta}$ is an upper semi-continuous correspondence from $S^{\ell} \cap (\mathbb{R}^{\ell} \times \mathbb{R}_+)$ into $\mathbb{R}^{\ell+1}$, with non-empty compact convex-values. One has :

$$\forall (p, q) \in S^{\ell} \cap (\mathbb{R}^{\ell} \times \mathbb{R}_+),\ \forall (z, -m) \in \hat{\eta}(p, q)$$

$$p.z - mq \leq 0.$$

Then by corollary 3, there exist $(\bar{p}, \bar{q}) \in S^{\ell} \cap (\mathbb{R}^{\ell} \times \mathbb{R}_+)$ and a negative number t such that :

$$(0, t) \in \hat{\eta}\,(\bar{p}, \bar{q})$$

./.

That means $\exists\, \overline{x} = (\overline{x}^i) \in \prod_{i=1}^{m} X^i$

such that $\overline{x}^i \in \hat{\varphi}^i(\overline{p}, \overline{q}) \quad , \quad \forall i = 1,\ldots, m$

$$\sum_{i=1}^{m} \overline{x}^i = \sum_{i=1}^{m} \omega^i$$

$$t = - m$$

It is straightforward to show that

$\forall i = 1,\ldots, m,\ \overline{p}.\overline{x}^i = \overline{p}.\,\omega^i + \overline{q}.$

Hence $m\,\overline{q} = 0 \Rightarrow \overline{q} = 0 \Rightarrow \overline{p} \in S^{\ell-1}$.

Finally, there exists $\overline{p} \in S^{\ell-1}$ and $\overline{x} = (\overline{x}^i) \in \prod_{i=1}^{m} X^i$ such that.

$\forall i = 1,\ldots, m \quad , \quad \overline{x}^i \in \hat{\varphi}^i(\overline{p}, 0) \subset \varphi^i(\overline{p})$

and $\displaystyle\sum_{i=1}^{m} \overline{x}^i = \sum_{i=1}^{m} \omega^i$

□

REFERENCES

BERGSTROM T.C. 1976, How to discard free disposability - At no cost, *Journal of Mathematical Economics* - 3, 131 - 134.

BORDER K.C. 1985, *Fixed point theorems with applications to economics and game theory,* Cambridge University press.

CORNET B. 1975, Fixed point and surjectivity theorems for correspondences - Applications,Working paper n° 7521 (University of Paris 9).

DEBREU G. 1956, Market equilibrium, *Proceedings of the National Academy of Sciences of the USA,* XLII, 876-878.

GEISTDOERFER-FLORENZANO M. 1982, The Gale-Nikaïdo-Debreu Lemma and the existence of transitive equilibrium with or without the free-disposal assumption, *Journal of Mathematical Economics* 9, 113-134.

SCHWARTZ J.T. 1969, *Non linear functional analysis,* Gordon and Breach Science Publishers, New York - London - Paris.