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REDUCTION AND IDENTIFICATION OF SIMULTANEOUS EQUATIONS MODELS WITH RATIONAL EXPECTATIONS.

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** CEPREMAP.

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S U M M A R Y

In this paper we consider simultaneous equations models with rational expectations.

Before considering estimation or testing problems, it is necessary to solve the models in order to eliminate the unobservable expectations. This is a first step for determining the observable reduced form of a R.E. model.

Then, by comparing this reduced form to the initial structural model, it is possible to address the problem of identification.

These two aspects, reduction and identification, are examined for a class of R.E. models containing both current and future expectations.

REDUCTION ET IDENTIFICATION DE MODELES A EQUATIONS SIMULTANEES COMPORTANT DES ANTICIPATIONS RATIONNELLES

L. BROZE*, C. GOURIEROUX** and A. SZAFARZ*

R E S U M E

Dans cet article, nous nous intéressons aux modèles dynamiques à équations simultanées comportant des anticipations rationnelles. Nous examinons les diverses façons d'éliminer les anticipations non observables ce qui permet de déterminer la forme réduite du modèle. Nous discutons ensuite les problèmes d'identification des paramètres structurels.

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KEY WORDS : Rational expectations, Simultanéous equations, Reduced form, Identification.

I - INTRODUCTION.

Macroeconometric models are generally written as linear systems of simultaneous equations. Introducing expectations into such models is a way to take into account dynamic aspects. Since expectations are rarely observable, forecasting schemes, such as extrapolative or rational ones, are needed. In this paper we concentrate on rational expectations (R.E.), i.e on optimal predictions given some information set (see MUTH (1961)).

Before considering estimation or testing problems, it is necessary to solve the models in order to eliminate the unobservable expectations. This is a first step for determining the observable reduced form of a R.E. model.

Then, by comparing this reduced form to the initial structural model, it is possible to address the problem of identification.

These two aspects, reduction and identification, are examined for a class of R.E. models containing both current and future expectations. In this paper, we stress the analogies and differences between R.E. models and usual simultaneous equations models.

The simple model, which only contains expectations of current variables formed in the previous period has a unique solution [see WALLIS (1980)].

However in more complex cases, for instance when future expectations are present, there may exist multiple solutions [see GOURIEROUX-LAFFONT-MONFORT (1982), BROZE-GOURIEROUX-SZAFARZ (1985)...for the one-dimensional case and WEGGE (1984) for a first approach of the multidimensional one].

In section 2, we describe a simultaneous R.E. model which extends to the multivariate case both the MUTH and CAGAN models. A first reduction method is given in section 3 and is based on some properties of matrix pencils. It is shown that the set of solutions can be described by arbitrary martingale difference sequences and that the dimension of this set depends on the number of zero roots of some determinantal equation. This exhaustive description of the set of solutions is used in section 4 to characterize the class of models which admit a unique solution and to compare the consequences of the R.E.. and the perfect foresight assumptions. Another reduction method is given in section 5 under an additional assumption ; this method seems simpler to use. This approach is applied to the two-dimensional case in section 6, where it will be seen that the dimension of the set of solutions depends on the the presence of some recursivities in the model.

Multiple solutions have obviously direct consequences on the identification problem. In section 7 we consider a model without future expectations ; this model has a unique solution and in this case, the rank conditions are similar to those derived for the usual simultaneous equations model. We also show how the identification conditions depend on the retained information set. Finally in section 8 we consider a model with multiple solutions. In such a case we have to discuss the identification not only of the structural parameters, but also of the observed solution in the set of all possible ones. In the special case of linear stationary solutions, we prove that if the structural parameters of the R.E. model are identifiable, the same is true for the observed solution.

II . THE MODEL.

We consider a simultaneous equations model, in which current and future expectations of the endogenous variables are included among the variables. The model is given by :

$$(2.1) \quad A\, y_t + B\; {}_t\hat{y}_{t+1} + C\; {}_{t-1}\hat{y}_t + D\, x_t = u_t \quad ,$$

where y_t is the G-dimensional vector of endogenous variables, x_t is the K-dimensional vector of the exogenous variables, ${}_{t-1}\hat{y}_t$ is the expectation of y_t formed at time $t-1$ and u_t is the unobservable error term. The sizes of the matrices A , B , C , D are $(G,G),(G,G),(G,G),(G,K)$. A is assumed invertible ; this ensures a well-defined expression for y_t in terms of the expectations, of the exogenous variables and of the error term.

Under the rational expectation hypothesis ${}_{t-1}\hat{y}_t$ is defined as the optimal forecast of y_t given the information I_{t-1} available at $t-1$, i.e is equal to the expectation of y_t conditional to I_{t-1} :

(2.2) $\quad {}_{t-1}\hat{y}_t = E(y_t / I_{t-1})$

In the paper we only consider two kinds of information sets. The set $I_{o,t}$, which contains the observations of past and current variables appearing in the system :

(2.3) $\quad I_{o,t} = \left\{x_t, u_t, x_{t-1}, u_{t-1}, \ldots\right\}$ (1)

or the set $I_{1,t}$, which also contains x_{t+1} :

(2.4) $\quad I_{1,t} = \left\{x_{t+1}, x_t, u_t, x_{t-1}, u_{t-1}, \ldots\right\}$

The choice between $I_{o,t}$ and $I_{1,t}$ depends on the moment at which the exogenous are known.

The error term is assumed to satisfy :

(2.5) $\quad E(u_t / I_{1,t-1}) = E(u_t / x_t, x_{t-1}, u_{t-1} \ldots) = o$.

This condition has different consequences. It implies that $E(u_t / x_t) = o$, which means that x_t can be considered as exogenous ; moreover it implies $E(u_t / u_{t-1}, u_{t-2}, \ldots) = o$, i.e the absence of correlation between u_t and its own past.

The simultaneity appears in two different ways in system (2.1), it appears as usual, through the term $A\, y_t$,but also through the expectation ${}_t\hat{y}_{t+1}$ formed at time t . As in more classical simultaneous equations models, the terms inducing simultaneity can be projected on the past. Using (2.5) and the assumption (2.4) on the information set, we see that :

$$E[A\, y_t + B\, {}_t\hat{y}_{t+1} / I_{1,t-1}] = -C\, {}_{t-1}\hat{y}_t - D\, x_t \; .$$

Therefore the conditional dynamics is only based on expectations ; in particular the model contains no dynamics such as adjustment processes, usually introduced by means of lagged endogenous variables. This allows us to concentrate on the effect of expectations.

Replacing in (2.1) expectations by their expressions (2.2), we obtain :

$$(2.6) \quad A\, y_t + B\, E(y_{t+1}/I_t) + C\, E(y_t/I_{t-1}) + D\, x_t = u_t$$

Since expectations are in general unobservable, the previous form (2.6) with conditions (2.3) or (2.4) and (2.5) will be called the <u>latent structural form</u> of the R.E. model.

III . REDUCTION OF THE R.E. MODEL [Regular case].

There exist several approaches for solving system (2.6) with respect to the endogenous y . In this section we present one of these methods and we discuss another one in section 5 . It is important to distinguish between two cases. The initial model is said to be regular iff the λ-polynomial $\det[(A+C)\lambda+B]$ is not equal to zero, singular otherwise. We essentially discuss the regular case in this paper.

3.a - Some properties of matrix pencils.

The reduction step is based on some properties of matrix pencils [see GANTMACHER (1966), a , b]. In the present case, we are only interested in the following result :

Lemma 3.1 : If $\det[(A+C)\lambda+B]$ is not equal to zero, there exist two regular matrices P and Q such that $P(A+C)Q$ and $P\ B\ Q$ have the block decompositions :

$$P(A+C)Q = \left[\begin{array}{c:c} I_{G_1} & 0 \\ \hdashline 0 & K \end{array}\right]$$

$$P\ B\ Q = \left[\begin{array}{c:c} J & 0 \\ \hdashline 0 & I_{G_2} \end{array}\right] ,$$

where I_{G_1} (resp I_{G_2}) is the identity matrix of size G_1 (resp G_2), J is a nilpotent matrix of size G_1 and K a matrix of size G_2.

Note that :

det $[\lambda P(A+C)Q + P B Q]$

$=$ det P det$(\lambda(A+C) + B)$ det Q

$=$ det $(\lambda I_{G_1} + J)$ det $(\lambda K + I_{G_2})$

Since J is nilpotent, we have :

$$\det[\lambda I_{G_1} + J] = -k\lambda^{G_1}$$, where k is constant.

Moreover for $\lambda = o$, det $(\lambda K + I_{G_2}) = 1$. We conclude that the polynomial det $(\lambda(A+C) + B)$ can be written

$$\det[\lambda(A+C) + B] = \lambda^{G_1} H(\lambda)$$ with $H(o) \neq o$. Therefore we have the following corollary.

Corollary 3.2 : G_1 is equal to the number of zero-roots of det$[\lambda(A+C) + B]$ (and G_2 to the number of non zero-roots of det $[\lambda(A+C) + B]$, iff det$(A+C) \neq o$).

If $A+C$ is invertible, it is easy to see that G_2 is equal to the number of non zero eigenvalues of the matrix $(A+C)^{-1}B$.

3.b - Constraints on the solutions.

We apply to this multivariate model the approach introduced by BROZE-GOURIEROUX-SZAFARZ [1985] for solving one-dimensional R.E. models.

i) We first define the one step ahead prediction error :

(3.3) $\varepsilon_t = y_t - E(y_t/I_{t-1})$.

This random vector is I_t-measurable and orthogonal to the past : $E(\varepsilon_t/I_{t-1}) = o$. Therefore the process ε is a G-dimensional martingale difference sequence (M.D.S) with respect to the sequence of information sets $\left\{I_t\right\}$.

ii) Replacing in the latent structural form the expectations by (3.3), we see that any solution y of (2.6) satisfies a difference equation of the form :

(3.4) $B\, y_{t+1} + (A+C)\, y_t = B\, \varepsilon_{t+1} + C\, \varepsilon_t + u_t - D\, x_t$,

where ε is a G-dimensional M.D.S .

iii) Now it can be seen that, contrary to the one-dimensional case [see BROZE-GOURIEROUX-SZAFARZ (1985)], the M.D.S's cannot be chosen arbitrarily, but must generally satisfy some constraints.

Indeed let us introduce the transformed vectors : $y_t^* = Q^{-1} y_t$,

$\varepsilon_t^* = Q^{-1}\varepsilon_t = y_t^* - E(y_t^*/I_{t-1})$, $v_t^* = P(u_t - Dx_t)$, where matrices P and Q are defined as in lemma (3.1). Model (3.4) is equivalent to the system:

$$(3.5)\quad \left[\begin{array}{c|c} J & 0 \\ \hline 0 & I_{G_2} \end{array}\right] y_{t+1}^* + \left[\begin{array}{c|c} I_{G_1} & 0 \\ \hline 0 & K \end{array}\right] y_t^*$$

$$= \left[\begin{array}{c|c} J & 0 \\ \hline 0 & I_{G_2} \end{array}\right] \varepsilon_{t+1}^* + C^* \varepsilon_t^* + v_t^*$$

where $C^* = P\,C\,Q$.

From the first subsystem, we derive :

$$(3.6)\quad J\, y_{1,t+1}^* + y_{1,t}^* = J\,\varepsilon_{1,t+1}^* + C_1^*\,\varepsilon_t^* + v_{1,t}^* ,$$

where $y_{1,t}^*$, C_1^* , $v_{1,t}^*$ are the adequate blocks of y_t^* , C^* , v_t^* .

Since J is a nilpotent matrix $J^{G_1} = o$, we obtain, by successive forward replacements :

$$(3.7)\quad y_{1,t}^* = \sum_{k=o}^{G_1-1} (-J)^k \,[J\,\varepsilon_{1,t+k+1}^* + C_1^*\,\varepsilon_{t+k}^* + v_{1,t+k}^*] \;.$$

Using (3.7) and the fact that ε^* is an a M.D.S. , we have :

$$(3.8)\quad y_{1,t}^* - E(y_{1t}^*/I_{t-1}) = E(y_{1t}^*/I_t) - E(y_{1t}^*/I_{t-1})$$

$$= C_1^*\,\varepsilon_t^* + \sum_{k=o}^{G_1-1} (-J)^k \,[E(v_{1,t+k}^*/I_t) - E(v_{1,t+k}^*/I_{t-1})]$$

Therefore the prediction error satisfies the constraint :

$$(3.9) \quad \varepsilon^*_{1,t} = C^*_1 \varepsilon^*_t + \sum_{k=o}^{G_1-1} (-J)^k [E(v^*_{1,t+k}/I_t) - E(v^*_{1,t+k}/I_{t-1})]$$

3.c - Characterization of the solutions.

The interesting point is that equations (3.4) and (3.9) are sufficient to define the set of solutions of the R.E model.

THEOREM 3.10 : Using the same notation as before, the set of solutions of the R.E. model (2.6) are obtained by solving the difference equations :

$$(3.4) \quad B\, y_{t+1} + (A+C)\, y_t = B\, \varepsilon_{t+1} + C\, \varepsilon_t + u_t - D\, x_t \, ,$$

where ε describes the set of all possible G-dimensional M.D.S satisfying (3.9) .

Proof : We only have to show that (3.4), (3.9) are sufficient conditions ; withoutloss of generality it is possible to directly consider the transformed process y^* . Let us consider a solution y^* of (3.5) , where ε^* is an M.D.S. satisfying (3.9).

i) From the second subsystem of (3.5), we have :

$$y^*_{2,t+1} + K\, y^*_{2,t} = \varepsilon^*_{2,t+1} + C^*_2\, \varepsilon^*_t + v^*_{2,t} \quad ;$$

Taking the conditional expectation of both sides given I_t and substracting, we get :

$$y^*_{2,t+1} - E(y^*_{2,t+1}/I_t) = \varepsilon^*_{2,t+1} \quad ,$$

which shows that $\varepsilon^*_{2,t+1}$ is a prediction error.

ii) From the first subsystem of (3.5), it is possible to derive (3.7) and then to get :

$$y^*_{1,t} - E(y^*_{1,t}/I_{t-1}) = C^*_1 \varepsilon^*_t + \sum_{k=o}^{G_2-1} (-J)^k [E(v^*_{1,t+k}/I_t)-E(v^*_{1,t+k}/I_{t-1})]$$

Using constraints (3.9), we obtain similarly :

$$\varepsilon^*_{1,t} = y^*_{1,t} - E(y^*_{1,t}/I_{t-1}) \quad .$$

iii) Finally replacing ε^*_t by $y^*_t - E(y^*_t/I_{t-1})$ in (3.5) or equivalently ε_t by $y_t - E(y_t/I_{t-1})$ in (3.4) , we obtain the initial R.E. model (2.6).

Q.E.D.

The previous theorem shows that in general there exists a large number of solutions to R.E models. This multiplicity has two different aspects. First the existence of a difference equation allows to fix arbitrarily some initial conditions on y . The second aspect is specific to R.E. models and is due to the arbitrary choice of the M.D.S., which appears in the R.H.S. of (3.4). This second source of multiplicity has a different nature since a finite number of initial conditions is not sufficient to fix a unique path.

IV . SOME PROPERTIES OF THE SET OF SOLUTIONS [Regular case].

4.a - Dimension of the set of solutions

As seen before the M.D.S.'s can be interpreted as prediction errors of the solution. Therefore to different choices of M.D.S.'s correspond different solutions. Hence it is possible to define a notion of dimension of the set of solutions in terms of M.D.S.'s .

This dimension is equal to the number G of univariate M.D.S.'s minus the number r of independent constraints on these M.D.S.'s. Since these constraints are summarized in (3.9), we have :

$$r = \text{rank}\,[(I_{G_1}, o) - C_1^*] \quad .$$

Property 4.1 : The dimension (in terms of M.D.S.) of the set of solutions to (2.6) is equal to G_2 , i.e to the dimension of y minus the number of zero roots of $\det[(A+C)\,\lambda + B]$.

Proof : We have simply to verify that the matrix $(I_{G_1}, o) - C_1^*$ has full rank. Let us denote $A^* = P\,A\,Q$. Since A is invertible, the same is true for A^* . Moreover we have :

$$A^* + C^* = \begin{bmatrix} I_{G_1} & 0 \\ 0 & K \end{bmatrix}$$

and $A^* = \begin{bmatrix} (I_{G_1}, o) - C_1^* \\ (o, K) - C_2^* \end{bmatrix}$. Then the property is a direct consequence of

the regularity of A^* .

Q.E.D.

4.b - Models whose solutions can be fixed by a finite number of initial conditions.

The solutions can be fixed by initial conditions iff the dimension (in terms of M.D.S.'s) of the set of solutions is equal to zero or equivalently iff $\det [\lambda\ (A+C) + B] = k\ \lambda^G$, where $k \in IR - \{o\}$.

Since k is different from zero, the matrix $A+C$ is invertible and the previous conditions is equivalent to :

$$\det [\lambda\ I_G + (A+C)^{-1}\ B] = \bar{k}\ \lambda^G .$$

Therefore we have the following property :

Property 4.2 : The solutions of the R.E. model (2.6) are characterized by a finite number of initial conditions iff the matrix $A + C$ is invertible and the matrix $(A+C)^{-1}\ B$ is nilpotent.

In such a case the solutions are easily obtained in the following way.

Taking the conditional expectation of both sides of (2.6) with respect to I_{t-1} , we get :

$$(A+C)\, E(y_t/I_{t-1}) \;+\; B\, E(y_{t+1}/I_{t-1}) \;+\; D\, E(x_t/I_{t-1}) \;=\; o$$

$$\Longleftrightarrow \quad E(y_t/I_{t-1}) \;=\; -\,(A+C)^{-1}\, B\, E(y_{t+1}/I_{t-1}) - (A+C)^{-1}\, D\, E\,(x_t/I_{t-1})$$

Therefore :

$$E[y_t/I_{t-1}] \;=\; -\,(A+C)^{-1}\, B\, E[E(y_{t+1}/I_t)/I_{t-1}] \;-\; (A+C)^{-1}\, D\, E(x_t/I_{t-1})$$

$$= [\,-\,(A+C)^{-1}\, B]^2\, E(y_{t+2}/I_{t-1})$$

$$- \left\{ (\,-\,(A+C)^{-1} B)\,(A+C)^{-1} D\, E(x_{t+1}/I_{t-1}) + (A+C)^{-1} D\, E(x_t/I_{t-1}) \right\}$$

Continuing this forward replacement and using the equality $[-\,(A+C)^{-1}B]^G = o$, we derive a unique form for $E(y_t/I_{t-1})$:

$$E(y_t/I_{t-1}) \;=\; -\sum_{k=o}^{G-1} [-\,(A+C)^{-1}B]^k\, (A+C)^{-1} D\, E(x_{t+k}/I_{t-1})$$

Replacing this last expression in (2.6) we see that the model has a unique solution (the other source of multiplicity automatically disappears).

<u>Property 4.3</u> : The set of solutions of the R.E. model (2.6) is characterized by a finite number of initial conditions iff the model has a unique solution. This solution is given by :

$$y_t \;=\; +\,A^{-1} \left\{ B \sum_{k=o}^{G-1} [-\,(A+C)^{-1}B]^k\, (A+C)^{-1} D\, E(x_{t+k+1}/I_t) \right.$$

$$+\, C \sum_{k=o}^{G-1} [-\,(A+C)^{-1}B]^k\, (A+C)^{-1} D\, E(x_{t+k}/I_{t-1})$$

$$\left. -\, D\, x_t \;+\; u_t \right\} .$$

Remark 4.4 : It is known that a model without a future expectation :

$A\, y_t + C\, E(y_t/I_{t-1}) + D\, x_t = u_t$ has a unique solution (see e.g WALLIS [1980]). In this case, $B = o$ and $(A+C)^{-1} B = o$ is nilpotent. The unique solution is :

$$y_t = A^{-1} [C(A+C)^{-1} D\, E(x_t/I_{t-1}) - D\, x_t + u_t]$$

Remark 4.5 : A model without current expectation :

$A\, y_t + B\, E(y_{t+1}/I_t) + D\, x_t = u_t$ has a unique solution, iff $A^{-1}B$ is nilpotent. This solution is :

$$y_t = A^{-1} B \sum_{k=o}^{G-1} [- A^{-1}B]^k A^{-1} D\, E(x_{t+k}/I_t) - A^{-1} D\, x_t + A^{-1} u_t$$

$$\Longleftrightarrow y_t = - \sum_{k=o}^{G} [- A^{-1}B]^k A^{-1} D\, E(x_{t+k}/I_t) + A^{-1} u_t \, .$$

It coïncides with the usual forward solution.

4.c - Perfect foresight solutions (P.F).

It is of usual practice to look for perfect foresight solutions, i.e to solve the dynamic model :

$$(4.6) \quad A\, y_t + B\, y_{t+1} + C\, y_t + D\, x_t = u_t \quad ,$$

in which expectations appearing in (2.6) are replaced by realizations [see e.g BLANCHARD (1979), WEGGE (1981)].

In general the solutions of (4.6) are not related to the solutions of the initial R.E model (2.6) and, when such a relation exists, the set of perfect foresight solutions is strictly included in the set of solutions of (2.6) .

These points are now discussed.

Property 4.7 : Except for some very particular structures of the exogenous variables, the R.E. model has a solution such that $y_t = E(y_t/I_{t-1})$ iff det $[\lambda\ (A+C) + B]$ has no root equal to zero.

Proof : A perfect foresight solution exists iff it is possible to choose the M.D.S. ε or ε^* equal to zero. This is of course possible if the M.D.S. is unconstrained, i.e if $G_1 = o$. If some constraint (3.9) is effective, the condition $\varepsilon^* = o$ implies the constraint :

$$\sum_{k=o}^{G_2-1} (-J)^k \,[E(v^*_{1,t+k}/I_t) - E(v^*_{1,t+k}/I_{t-1})] = o$$

on the exogenous x and on the error u (since v^*_1 is a function of x and u). This constraint is in general not satisfied.

Q.E.D.

Property 4.8 : If det $(\lambda\ (A+C) + B)$ has no zero-root, the set of perfect foresight solutions of (2.6) coincides with the set of solutions of the perfect foresight model (4.6).

Proof : This is a direct consequence of theorem (3.10) . In effect the solutions of (2.6) are obtained by solving

$B\ y_{t+1} + (A+C)\ y_t = B\ \varepsilon_{t+1} + C\ \varepsilon_t + u_t - D\ x_t$, where ε describes the set of all possible M.D.S , and the P.F. solutions are deduced by imposing $\varepsilon = o$.

Q.E.D.

4.d - Reduction of a model with expectations of different horizons.

A direct generalization of model (2.6) consists in introducing expectations with different horizons :

$$(4.9)\quad A\,y_t + \sum_{h=1}^{H} B_h\, E(y_{t+h}/I_t) + \sum_{k=o}^{H} C_k\, E(y_{t+k}/I_{t-1}) + D\,x_t = u_t$$

Let us define the G(H+1) dimensional vector :

$$Y_t = \begin{bmatrix} Y_{ot} \\ \vdots \\ Y_{Ht} \end{bmatrix} \quad , \text{ where } \; Y_{h,t} = E(y_{t+h}/I_t) \; .$$

We have : $E\,(Y_{h,t+1}/I_t) = E\,[E(y_{t+h+1}/I_{t+1})/I_t] = Y_{h+1,t}$

and $E(Y_{h,t}/I_{t-1}) = E(y_{t+h}/I_{t-1})$.

Therefore the initial model (4.9) is equivalent to :

$$\begin{bmatrix} A & B_1 \cdots\cdots\cdots & B_H \\ & I_G \ddots & 0 \\ 0 & & \ddots\; I_G \end{bmatrix} Y_t + \begin{bmatrix} 0 & 0 \cdots\cdots & 0 \\ -I_G \ddots & 0. & 0 \\ 0 & \ddots\; -I_G & 0 \end{bmatrix} E\,[Y_{t+1}/I_t]$$

$$+ \begin{bmatrix} C_o & C_1 \cdots\cdots\cdots & C_H \\ & & \\ & 0 & \end{bmatrix} E(Y_t/I_{t-1}) + \begin{bmatrix} D\,x_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This model has the form (2.6) in the endogenous Y .

The M.D.S.'s correspond to the prediction errors of Y_t :

$$Y_t - E(Y_t/I_{t-1}) = \begin{bmatrix} y_t - E(y_t/I_{t-1}) \\ \vdots \\ E(y_{t+h}/I_t) - E(y_{t+h}/I_{t-1}) \\ \vdots \\ E(y_{t+H}/I_t) - E(y_{t+H}/I_{t-1}) \end{bmatrix}$$

and can be interpreted as revision processes associated to y [see BROZE-GOURIEROUX-SZAFARZ (1985)].

The dimension of the set of solutions depends on the number of zero roots of :

$$\det \begin{bmatrix} \lambda\,(A + C_o) & \lambda\,(B_1 + C_1) & \cdots & \lambda\,(B_H + C_H) \\ -I_G & \lambda I_G & & \\ & \ddots & \ddots & \\ 0 & & -I_G & \lambda I_G \end{bmatrix}$$

It is easily seen that this determinant is equal to :

$$\lambda^G \det(\lambda^H (A + C_o) + \lambda^{H-1}(B_1 + C_1) + \ldots + \lambda\,(B_{H-1} + C_{H-1}) + B_H + C_H) .$$

Therefore we have the following property :

<u>PROPERTY 4.10</u> : In the regular case, i.e if

$\det [\lambda^H (A+C_o) + \lambda^{H-1}(B_1+C_1) + \ldots + \lambda(B_{H-1}+C_{H-1}) + B_H + C_H] \neq o$

the dimension of the set of solutions in term of M.D.S.'s is equal to $GH - \overline{G}_1$, where $\overline{G}_1$ is the number of zero roots of the previous determinant.

It is clear that the polynomial matrix $\lambda^H (A+C_o) + \lambda^{H-1}(B_1+C_1) + \ldots + B_H + C_H$ is directly linked with the autoregressive polynomial in the P.F. model associated with (4.9) :

$$[L^H(A+C_o) + L^{H-1}(B_1+C_1) + \dots + (B_H + C_H)]\, y_{t+H} + D\, x_t = u_t \ .$$

V . ANOTHER REDUCTION TECHNIQUE.

With the solution method discussed in section 3, it is necessary to find a canonical decomposition, i.e matrices P , Q , K , J ; in general this implies finding eigenvalues and eigenvectors.

In this section, we describe another approach, which works directly on the initial endogenous variables and is often simpler to derive. However it can only be used, if the matrix $A + C$ is invertible.

PROPERTY 5.1 : If $A + C$ is invertible, the G-dimensional process y solution of the R.E. model (2.6) can be decomposed into two subprocesses y_1 and y_2 with respectively $\bar{G}_1$ and $\bar{G}_2 = G - \bar{G}_1$ elements, characterized by relations of the form :

$$(5.2) \begin{cases} y_{1t} + \bar{B}_{12}\, E(y_{2,t+1}/I_t) + \bar{C}_{12}\, E(y_{2,t}/I_{t-1}) = \bar{\omega}_{1t} \\ y_{2t} + \bar{B}_{22}\, E(y_{2,t+1}/I_t) + \bar{C}_{22}\, E(y_{2,t}/I_{t-1}) = \bar{\omega}_{2t} \end{cases}$$

where $\bar{B}_{22}$ is a regular $(\bar{G}_2 , \bar{G}_2)$ matrix and where $\bar{\omega}_{1t}$, $\bar{\omega}_{2t}$ are linear functions of present and expected values of x_t and u_t .

Proof : Since A is invertible, model (2.6) is equivalent to :

(5.3) $y_t + A^{-1} B\, E(y_{t+1}/I_t) + A^{-1} C\, E(y_t/I_{t-1}) = A^{-1}(u_t - D\, x_t)$.

i) If B is invertible, the property is obvious, with $G_2 = G$, $y_{2t} = y_t$, $\bar{B}_{22} = A^{-1}B$, $\bar{C}_{22} = A^{-1}C$.

ii) Let us now consider the case in which B is not of full rank and let us denote $r = \text{rank } B$. After a permutation of the components of y , $A^{-1}B$ can be written as :

$$A^{-1}B = \begin{pmatrix} \tilde{\tilde{\pi}}_{11} & \tilde{\tilde{\pi}}_{12} \\ \tilde{\tilde{\pi}}_{21} & \tilde{\tilde{\pi}}_{22} \end{pmatrix} ,$$

where $\tilde{\tilde{\pi}}_{11}$, $\tilde{\tilde{\pi}}_{12}$, $\tilde{\tilde{\pi}}_{21}$, $\tilde{\tilde{\pi}}_{22}$ are of dimension $(G-r,G-r)$, $(G-r,r)$, $(r,G-r)$, (r,r) respectively and where $r = \text{rank } (\tilde{\tilde{\pi}}_{21}, \tilde{\tilde{\pi}}_{22})$.
Therefore the $G-r$ first rows of $A^{-1}B$ are linear combinations of the r last ones and there exists a $(G-r,r)$ matrix Λ such that :

$$\tilde{\tilde{\pi}}_{11} = \Lambda\, \tilde{\tilde{\pi}}_{21} \quad , \quad \tilde{\tilde{\pi}}_{12} = \Lambda\, \tilde{\tilde{\pi}}_{22} \quad .$$

Let $\tilde{\tilde{y}}_{1t}$ be the first $G-r$ components of the image $\tilde{\tilde{y}}_t$ of y_t by the permutation and $\tilde{\tilde{y}}_{2t}$ the following ones, we get :

$$\begin{cases} \tilde{\tilde{y}}_{1t} + \Lambda\, \tilde{\tilde{\pi}}_{21}\, E(\tilde{\tilde{y}}_{1,t+1}/I_t) + \Lambda\, \tilde{\tilde{\pi}}_{22}\, E(\tilde{\tilde{y}}_{2,t+1}/I_t) + \tilde{\tilde{\phi}}_1\, E(\tilde{\tilde{y}}_t/I_{t-1}) = \tilde{\tilde{v}}_{1t} \\ \tilde{\tilde{y}}_{2t} + \tilde{\tilde{\pi}}_{21}\, E(\tilde{\tilde{y}}_{1,t+1}/I_t) + \tilde{\tilde{\pi}}_{22}\, E(\tilde{\tilde{y}}_{2,t+1}/I_t) + \tilde{\tilde{\phi}}_2\, E(\tilde{\tilde{y}}_t/I_{t-1}) = \tilde{\tilde{v}}_{2t} \end{cases}$$

where $\begin{pmatrix} \tilde{\tilde{\phi}}_1 \\ \tilde{\tilde{\phi}}_2 \end{pmatrix}$ and $\begin{pmatrix} \tilde{\tilde{v}}_{1t} \\ \tilde{\tilde{v}}_{2t} \end{pmatrix}$ are the images of $A^{-1}C$ and $A^{-1}(u_t - D\, x_t)$ by the permutation.

This system is equivalent to :

$$(5.4) \quad \begin{cases} \tilde{\tilde{y}}_{1t} + \Lambda\, \tilde{\tilde{y}}_{2t} + (\tilde{\tilde{\phi}}_1 - \Lambda\, \tilde{\tilde{\phi}}_2)\ E(\tilde{\tilde{y}}_t/I_{t-1}) = \tilde{\tilde{v}}_{1t} - \Lambda\, \tilde{\tilde{v}}_{2t} \\ \\ \tilde{\tilde{y}}_{2t} + \tilde{\tilde{\pi}}_{21}\ E(\tilde{\tilde{y}}_{1,t+1}/I_t) + \tilde{\tilde{\pi}}_{22}\ E(\tilde{\tilde{y}}_{2,t+1}/I_t) + \tilde{\tilde{\phi}}_2\ E(\tilde{\tilde{y}}_t/I_{t-1}) = \tilde{\tilde{v}}_2 \end{cases}$$

Since all previous linear transformations are invertible, the image of $A + C$, i.e

$$\begin{bmatrix} (I_{G-r}, - \Lambda) + \tilde{\tilde{\phi}}_1 - \Lambda\, \tilde{\tilde{\phi}}_2 \\ \\ (o, I_r) + \tilde{\tilde{\phi}}_2 \end{bmatrix}$$

is also invertible.

Taking the conditional expectation of both sides of the first subsystem with respect to I_{t-1} , we get :

$$[(I_{G-r}, - \Lambda) + \tilde{\tilde{\phi}}_1 - \Lambda\, \tilde{\tilde{\phi}}_2]\ E(\tilde{\tilde{y}}_t/I_{t-1}) = E(\tilde{\tilde{v}}_{1t} - \Lambda\, \tilde{\tilde{v}}_{2t}/I_{t-1})$$

Since the matrix of the L.H.S. is of full rank $G - r$, there exists a decomposition of $\tilde{\tilde{y}}_t$ into two subvectors y^*_{1t} and y^*_{2t} of dimension G^*_1 and G^*_2 such that $E(y^*_{1t}/I_{t-1})$ can be expressed as a linear combination of $E(y^*_{2t}/I_{t-1})$ plus a function of the exogenous variables and of error terms. Moreover it is easily seen that replacing $E(y^*_{1t}/I_{t-1})$ in system (5.4) [or in (5.3)] leads to a system equivalent to (5.4).

Therefore replacing in (5.3) , we obtain an equivalent system without expectations of y^*_{1t} :

$$\begin{cases} y^*_{1t} + B^*_{12}\, E(y^*_{2,t+1}/I_t) + C^*_{12}\, E(y^*_{2,t}/I_{t-1}) = \omega^*_{1t} \\ y^*_{2t} + B^*_{22}\, E(y^*_{2,t+1}/I_t) + C^*_{22}\, E(y^*_{2,t}/I_{t-1}) = \omega^*_{2t} \end{cases}$$

At this stage two cases have to be distinguished :

a) if B^*_{22} is invertible the result is obtained with $G_2 = G^*_2 = r = \text{rank } B$;

b) if B^*_{22} is singular the same approach as before can be used on the second subsystem. This induction proof stops after a finite number K of steps, since at each step the dimension G^*_2 of y^*_2 strictly decreases.

$\square$

Such a canonical form (5.2) can be directly used to obtain the set of solutions of the R.E. model (2.6).

PROPERTY 5.5 : The solutions of model (5.2) are obtained by solving the difference equations

$$(5.6) \begin{cases} y_{1t} + \bar{B}_{12}\, y_{2,t+1} + \bar{C}_{12}\, y_{2,t} = \bar{\omega}_{1t} + \bar{B}_{12}\varepsilon_{2,t+1} + \bar{C}_{12}\varepsilon_{2,t} \\ y_{2t} + \bar{B}_{22}\, y_{2,t+1} + \bar{C}_{22}\, y_{2,t} = \bar{\omega}_{2t} + \bar{B}_{22}\varepsilon_{2,t+1} + \bar{C}_{22}\varepsilon_{2,t} \end{cases}$$

where $\varepsilon_{2,t}$ describes the set of all possible $\bar{G}_2$-dimensional M.D.S.'s.

Proof : The condition is obviously necessary.

Conversely from the second subsystem of (5.6) , we deduce that :

$$(y_{2t} + \bar{B}_{22}\, y_{2,t+1} + \bar{C}_{22}\, y_{2,t}) \; - \; E[y_{2t} + \bar{B}_{22}\, y_{2,t+1} + \bar{C}_{22}\, y_{2,t}/I_t]$$

$$= \bar{B}_{22}\,[y_{2,t+1} - E(y_{2,t+1}/I_t)]$$

$$= \bar{B}_{22}\,\varepsilon_{2,t+1} \quad .$$

This gives the required form $\varepsilon_{2,t+1} = y_{2,t+1} - E(y_{2,t+1}/I_t)$, since $\bar{B}_{22}$ is invertible.

□

Since the dimension (in term of M.D.S.'s) is a characteristic of the set of solutions, we directly have the corollary :

Corollary 5.7 : $\bar{G}_2 = G_2 =$ number of non zero eigenvalues of $(A+C)^{-1}B$.

Remark 5.8 : When $C = o$, the canonical form (5.6) reduces to :

$$\begin{cases} y_{1t} + \bar{B}_{12}\, E(y_{2,t+1}/I_t) = \bar{\omega}_{1t} \\ y_{2t} + \bar{B}_{22}\, E(y_{2,t+1}/I_t) = \bar{\omega}_{2t} \end{cases}$$

or equivalently to :

$$\begin{cases} y_{1t} - \bar{B}_{12}\,\bar{B}_{22}^{-1}\, y_{2t} = \bar{\omega}_{1t} - \bar{B}_{12}\,\bar{B}_{22}^{-1}\,\bar{\omega}_{2t} \\ y_{2t} + \bar{B}_{22}\, E(y_{2,t+1}/I_t) = \bar{\omega}_{2t} \end{cases}$$

Therefore there exist some linear combinations of the endogenous variables which do not depend on endogenous expectations.

VI . AN EXAMPLE : THE TWO-DIMENSIONAL R.E MODEL WITH FUTURE EXPECTATIONS.

As an illustration we determine the second canonical forms for a two-dimensional R.E. model with future expectations.

$$(6.1) \quad \begin{cases} y_{1t} = a_{12} y_{2t} + b_{11} E(y_{1,t+1}/I_t) + b_{12} E(y_{2,t+1}/I_t) + h_1 x_t + u_{1t} \\ \\ y_{2t} = a_{21} y_{1t} + b_{21} E(y_{1,t+1}/I_t) + b_{22} E(y_{2,t+1}/I_t) + h_2 x_t + u_{2t} \end{cases}$$

We assume that the structural parameters can only be submitted to nullity constraints and we consider generic properties, i.e properties valid except on a negligeable set in the space of unconstrained parameters.

i) $\underline{G_1 = G = 2}$.

This case occurs iff $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is invertible. The solutions are obtained by considering the difference equation :

$$(6.2) \quad (I - A)\, y_t - B\, y_{t+1} = - B\, \varepsilon_{t+1} + H\, x_t + u_t \ ,$$

where $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$ is an arbitrary two-dimensional M.D.S. .

ii) $\underline{G_1 = 1}$

In this case $\det B = b_{11} b_{22} - b_{12} b_{21} = o$. Since the parameters are only submitted to nullity constraints, this condition is generally satisfied if only one equation contains expectations ($b_{11} = b_{12} = o$ or $b_{21} = b_{22} = o$) or if the model depends on the expectation of only one variable ($b_{11} = b_{21} = o$ or $b_{12} = b_{22} = o$) .

Model without expectations in the second equation.

$$\begin{cases} y_{1t} = a_{12} y_{2t} + b_{11} E(y_{1,t+1}/I_t) + b_{12} E(y_{2,t+1}/I_t) + h_1 x_t + u_{1t} \\ y_{2t} = a_{21} y_{1t} + h_2 x_t + u_{2t} \end{cases}$$

Replacing in the first equation y_{2t} by its expression we get :

$$y_{1t} = \frac{1}{1-a_{12}a_{21}} [(b_{12}a_{21}+b_{11}) E(y_{1,t+1}/I_t) + (h_1+a_{12}h_2) x_t$$
$$+ b_{12}h_2 E(x_{t+1}/I_t) + u_{1t} + b_{12} E(u_{2,t+1}/I_t)]$$

We see that the dimension G_1 is equal to 1 iff generically :

$b_{12}a_{21} + b_{11} \neq o$ and in this case the solutions satisfy :

$$(6.3) \begin{cases} y_{1t} = \frac{1}{1-a_{12}a_{21}} [(b_{12}a_{21}+b_{11}) (y_{1,t+1}-\varepsilon_{1,t+1}) + (h_1+a_{12}h_2) x_t \\ \qquad + b_{12}h_2 E(x_{t+1}/I_t) + u_{1t} + b_{12} E(u_{2,t+1}/I_t)] \\ y_{2t} = a_{21} y_{1t} + h_2 x_t + u_{2t} \end{cases}$$

where ε_1 is an arbitrary univariate M.D.S .

Model without expectations of y_2 .

The model is given by :

$$\begin{cases} y_{1t} = a_{12}y_{2t} + b_{11}\, E(y_{1,t+1}/I_t) + h_1\, x_t + u_{1t} \\ \\ y_{2t} = a_{21}y_{1t} + b_{21}\, E(y_{1,t+1}/I_t) + h_2\, x_t + u_{2t} \end{cases}$$

we assume $b_{11} \neq o$ and $b_{21} \neq o$ (if $b_{11} = o$ or $b_{21} = o$ we are back to the case of model without expectations in an equation).

We deduce that :

$$y_{1t} = a_{12}\,[a_{21}y_{1t} + b_{21}\, E(y_{1,t+1}/I_t) + h_2\, x_t + u_{2t}]$$

$$+ b_{11}\, E(y_{1,t+1}/I_t) + h_1\, x_t + u_{1t}$$

$$\Longleftrightarrow \quad y_{1t} = \frac{1}{1-a_{12}a_{21}}\,[(a_{12}b_{21}+b_{11})\, E(y_{1,t+1}/I_t) + (a_{12}h_2 + h_1)\, x_t$$

$$+ a_{12}\, u_{2t} + u_{1t}]$$

On the other hand, we have :

$$y_{1t} = a_{12}y_{2t} + \frac{b_{11}}{b_{21}}\,(y_{2t} - a_{21}y_{1t} - h_2\, x_t - u_{2t}) + h_1\, x_t + u_{1t}$$

Therefore if $a_{12}b_{21} + b_{11} \neq o$, i.e if $G_1 = 1$, we have obtained a canonical form and the solutions satisfy :

$$(6.4)\quad \begin{cases} y_{1t} = \dfrac{1 - a_{12}a_{21}}{a_{12}b_{21}+b_{11}}\, y_{1,t-1} - \dfrac{a_{12}h_2+h_1}{a_{12}b_{21}+b_{11}}\, x_{t-1} - \dfrac{a_{12}u_{2,t-1}+u_{1,t-1}}{a_{12}b_{21} + b_{11}} \\ \qquad + \varepsilon_{1,t} \\ \\ y_{2t} = \dfrac{b_{21}+b_{11}a_{21}}{a_{12}b_{21}+b_{11}}\, y_{1,t} + \dfrac{b_{11}h_2-b_{21}h_1}{a_{12}b_{21}+b_{11}}\, x_t + \dfrac{b_{11}u_{2t}-b_{21}u_{1t}}{a_{12}b_{12}+b_{11}} \end{cases},$$

where ε_1 is an arbitrary univariate M.D.S. .

iii) $\underline{G_1 = o}$

Let us first consider a model without expectations in the second equation. Since B is different from zero (the model contains expectations) and, since $G_1 = o$ iff $b_{12}a_{21} + b_{11} = o$, the condition $G_1 = o$ is generically equivalent to : $b_{11} = a_{21} = o$.

Then the model has the recursive form :

$$\begin{cases} y_{1t} = a_{12}y_{2t} + b_{12}\, E(y_{2,t+1}/I_t) + h_1\, x_t + u_{1t} \\ \\ y_{2t} = \qquad\qquad\qquad\qquad\qquad\qquad h_2\, x_t + u_{2t} \end{cases}$$

and the unique solution is given by :

$$(6.5) \quad \begin{cases} y_{1t} = (a_{12}h_2 + h_1)\, x_t \; + \; b_{12}h_2\, E(x_{t+1}/I_t) + u_{1t} \\ \\ \qquad\quad + \; a_{12}\, u_{2t} \; + \; b_{12}\, E(u_{2,t+1}/I_t) \\ \\ y_{2t} = h_2\, x_t + u_{2t} \end{cases}$$

Finally, if the expectation of y_2 does not appear in the initial system, the condition $G_1 = o$ implies the recursive form :

$$\begin{cases} y_{1t} = h_1\, x_t + u_{1t} \\ \\ y_{2t} = a_{21}y_{1t} + b_{21}\, E(y_{1,t+1}/I_t) + h_2\, x_t + u_{2t} \end{cases}$$

This is a structure similar to the one previously studied.

In summary we have the following classification of the models in the bivariate case

Class of models	Dimension of the set of solutions in terms of M.D.S.'s
B invertible	2
Model without expectations in an equation	1
Model without expectations of one of the variables	1
Model recursive both in the realizations and expectations of the endogenous variables	0

VII - IDENTIFICATION OF MODELS WITHOUT FUTURE EXPECTATIONS.

A major difficulty with the identification in R.E. models is the possibility of multiple solutions. In this section we consider a model without future expectations $B = o$, since such a model has a unique solution. This kind of models has previously been studied by WALLIS (1980) and therefore we essentially insist on some additional results discussing the importance of the information set and the links between rational expectation models and perfect foresight models.

7.a - The complete structural form.

The unique solution of model :

(7.1) $A\,y_t + C\,E(y_t/I_{t-1}) + D\,x_t = u_t$ (with $A+C$ invertible)

is given by :

(7.2) $y_t = A^{-1}C\,(A+C)^{-1}D\,E(x_t/I_{t-1}) - A^{-1}D\,x_t + u_t$.

It depends on the expectations of the exogenous variables ; hence it is now necessary to complete the structural model by defining the information set I_t and the form of the expectations of the exogenous variables.

We choose a class of information sets I_{t-1} including $I_{o,t-1}$ and $I_{1,t-1}$. More precisely we assume that the vector of exogenous variables x_t can be decomposed in $x_t = \begin{bmatrix} \tilde{x}_t \\ \approx{x}_t \end{bmatrix}$, where some of the variables, $\tilde{x}_t$, are known at t-1 and the others, $\overset{\approx}{x}_t$, are unknown. The information set is given by :

(7.3) $I_{t-1} = \left\{ \tilde{x}_t, y_{t-1}, x_{t-1}, y_{t-2}, x_{t-2} \ldots \right\} = \left\{ \tilde{x}_t, I_{o,t-1} \right\}$

Decomposing in the same way the matrix $D = [\tilde{D}, \overset{\approx}{D}]$, the solution can be written as :

$$y_t = A^{-1} C\,(A+C)^{-1}\,[\tilde{D}\,E(\tilde{x}_t/I_{t-1}) + \tilde{\tilde{D}}\,E(\tilde{\tilde{x}}_t/I_{t-1})]$$

$$- A^{-1}[\tilde{D}\,\tilde{x}_t + \tilde{\tilde{D}}\,\tilde{\tilde{x}}_t] + u_t$$

$$(7.4) \qquad y_t = -(A+C)^{-1}\tilde{D}\,\tilde{x}_t + [A^{-1}\tilde{\tilde{D}} - (A+C)^{-1}\tilde{\tilde{D}}]\,E[\tilde{\tilde{x}}_t/I_{t-1}]$$

$$- A^{-1}\tilde{\tilde{D}}\,\tilde{\tilde{x}}_t + u_t$$

To complete the R.E. model, let us assume that $E[\tilde{\tilde{x}}_t/I_{t-1}]$ is given by :

$$(7.5) \quad E[\tilde{\tilde{x}}_t/I_{t-1}] = \Gamma\,\tilde{x}_t + \Lambda\,z_{t-1}$$

where z_{t-1} are L observable variables, such that the processes $(\tilde{x}_t)$, $(\tilde{\tilde{x}}_t)$, (z_{t-1}) are linearly independent.

Therefore replacing (7.5) in (7.4) , we derive the expectation of y_t given all the current exogenous variables and all the past variables :

$$(7.6) \quad \begin{aligned} E[y_t/I_{t-1}] &= [-(A+C)^{-1}\tilde{D} + A^{-1}\tilde{\tilde{D}}\,\Gamma - (A+C)^{-1}\tilde{\tilde{D}}\,\Gamma]\,\tilde{x}_t \\ &\quad - A^{-1}\tilde{\tilde{D}}\,\tilde{\tilde{x}}_t + [A^{-1}\tilde{\tilde{D}} - (A+C)^{-1}\tilde{\tilde{D}}]\,\Lambda\,z_{t-1} \end{aligned}$$

7.b - <u>First order dynamical identification.</u>

We now turn to the identifiability properties of the structural parameters using as a basis the dynamic regression (7.6). The reduced parameters appearing in (7.6) : $\tilde{\pi} = -(A+C)^{-1}\tilde{D} + A^{-1}\tilde{\tilde{D}}\,\Gamma - (A+C)^{-1}\tilde{\tilde{D}}\,\Gamma$

$$\tilde{\tilde{\pi}} = -A^{-1}\tilde{\tilde{D}}$$

$$\Psi = (A^{-1}\tilde{\tilde{D}} - (A+C)^{-1}\tilde{\tilde{D}})\,\Lambda$$

are identifiable as well as the regression coefficients Γ , Λ defined in (7.5).

We are essentially interested in the structural parameters of the g^{th} equation of the R.E model. Denoting by A_g , C_g , $\tilde{D}_g$, $\tilde{\tilde{D}}_g$, the g^{th} rows of A , C , $\tilde{D}$, $\tilde{\tilde{D}}$, this equation is given by :

$$(7.7) \quad A_g y_t + C_g E(y_t/I_{t-1}) + \tilde{D}_g \tilde{x}_t + \tilde{\tilde{D}}_g \tilde{\tilde{x}}_t = u_{gt}$$

We assume that the parameters are only submitted to r_g independent linear restrictions :

$$(7.8) \quad A_g N_g + C_g M_g + \tilde{D}_g \tilde{P}_g + \tilde{\tilde{D}}_g \tilde{\tilde{P}}_g = R_g ,$$

where N_g , M_g , $\tilde{P}_g$, $\tilde{\tilde{P}}_g$, R_g are given, with dimensions $(G,r_g),(G,r_g),(\tilde{K},r_g),$ $(\tilde{\tilde{K}},r_g)$ and $(1,r_g)$. In particular there exists no constraint on the parameters of two different equations.

The structural parameters A_g , C_g , $\tilde{D}_g$, $\tilde{\tilde{D}}_g$ are identifiable iff under the constraints (7.8) , the system giving the reduced parameters has a unique solution in A_g , C_g , $\tilde{D}_g$, $\tilde{\tilde{D}}_g$. The latter system can be decomposed to extract the parameters of interest and therefore we only have to examine

$$(7.9)\quad \begin{cases} A_g N_g + C_g M_g + \tilde{D}_g \tilde{P}_g + \tilde{\tilde{D}}_g \tilde{\tilde{P}}_g = R_g \\ A_g (\tilde{\pi} + \tilde{\tilde{\pi}} \Gamma) + C_g (\tilde{\pi} + \tilde{\tilde{\pi}} \Gamma) + \tilde{D}_g + \tilde{D}_g \Gamma = o \\ A_g \tilde{\tilde{\pi}} + \tilde{\tilde{D}}_g = o \\ A_g (\Psi + \tilde{\tilde{\pi}} \Lambda) + C_g (\Psi + \tilde{\tilde{\pi}} \Lambda) + \tilde{\tilde{D}}_g \Lambda = o \quad . \end{cases}$$

$\tilde{D}_g$ and $\tilde{\tilde{D}}_g$ are uniquely defined in term of A_g , C_g and of the reduced parameters through the second and third equations. Replacing $\tilde{D}_g$ and $\tilde{\tilde{D}}_g$ by their expressions in the two other equations we get :

$$(7.10)\quad \begin{cases} A_g [N_g - \tilde{\pi} \tilde{P}_g - \tilde{\tilde{\pi}} \tilde{\tilde{P}}_g] + C_g [M_g - (\tilde{\pi} + \tilde{\tilde{\pi}} \Gamma) \tilde{P}_g] = \overline{R}_g \\ A_g \Psi + C_g (\Psi + \tilde{\tilde{\pi}} \Lambda) = \overline{\overline{R}}_g \end{cases}$$

where $\overline{R}_g$, $\overline{\overline{R}}_g$ only depend on the reduced parameters.

<u>PROPERTY 7.11</u> : The structural parameters are identifiable iff :

$$\text{Rank} \begin{bmatrix} N_g - \tilde{\pi} \tilde{P}_g - \tilde{\tilde{\pi}} \tilde{\tilde{P}}_g & \vdots & \Psi \\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ M_g - (\tilde{\pi} + \tilde{\tilde{\pi}} \Gamma) \tilde{\tilde{P}}_g & \vdots & \Psi + \tilde{\tilde{\pi}} \Lambda \end{bmatrix} = 2G \quad .$$

Another equivalent identifiability condition can be obtained in the following way. Notice first that (7.9) is equivalent to :

$$(7.12)\quad \begin{cases} A_g N_g + C_g M_g + \tilde{D}_g \tilde{P}_g + \tilde{\tilde{D}}_g \tilde{\tilde{P}}_g = R_g \\ A_g (\tilde{\pi}\, \tilde{x}_t + \tilde{\tilde{\pi}}\, \tilde{\tilde{x}}_t + \Psi\, z_{t-1}) + C_g [(\tilde{\pi} + \tilde{\tilde{\pi}}\, \Gamma)\, \hat{x}_t + (\Psi + \tilde{\tilde{\pi}}\, \Lambda)\, z_{t-1}] \\ + \tilde{D}_g \tilde{x}_t + \tilde{\tilde{D}}_g \tilde{\tilde{x}}_t = o \end{cases}$$

Moreover we have : $\tilde{\pi}\, \tilde{x}_t + \tilde{\tilde{\pi}}\, \tilde{\tilde{x}}_t + \Psi\, z_{t-1} = E(y_t / I_{1,t-1})$

$$(\hat{\pi} + \tilde{\tilde{\pi}}\, \Gamma)\, \tilde{x}_t + (\Psi + \tilde{\tilde{\pi}}\, \Lambda)\, z_{t-1} = E(y_t / I_{t-1}) \ .$$

In summary :

<u>PROPERTY 7.13</u> : The structural parameters are identifiable iff the system :

$$\begin{cases} A_g N_g + C_g M_g + \tilde{D}_g \tilde{P}_g + \tilde{\tilde{D}}_g \tilde{\tilde{P}}_g = R_g \\ A_g\, E(y_t / I_{1,t-1}) + C_g\, E(y_t / I_{t-1}) + \tilde{D}_g \tilde{x}_t + \tilde{\tilde{D}}_g \tilde{\tilde{x}}_t = o \end{cases}$$

has a unique solution in A_g , C_g , $\tilde{D}_g$, $\tilde{\tilde{D}}_g$.

7.c - <u>Case of nullity constraints.</u>

As an illustration let us examine the case of nullity constraints on the parameters, with $a_{gg} = 1$. The g^{th} equation is :

$$(7.14)\quad y_{gt} + \sum_{i \in I_g} a_{gi}\, y_{it} + \sum_{j \in J_g} c_{gj}\, E[y_{jt} / I_{t-1}] + \sum_{k \in \tilde{K}_g} \tilde{d}_{gk}\, \tilde{x}_{kt} + \sum_{k \in \tilde{\tilde{K}}_g} \tilde{\tilde{d}}_{gk}\, \tilde{\tilde{x}}_{kt} = u_{gt} \quad ,$$

where I_g , J_g , $\tilde{K}_g$, $\tilde{\tilde{K}}_g$ are the sets of unconstrained parameters with $g \notin I_g$.

Property (7.13) implies that the structural parameters are identifiable iff the variables :

$E(y_{i,t}/I_{1,t-1})$, $i \in I_g$, $E(y_{j,t}/I_{t-1})$, $j \in J_g$, $\tilde{x}_{kt}$ $k \in \tilde{K}_g$,

and $\tilde{\tilde{x}}_{k,t}$ $k \in \tilde{\tilde{K}}_g$ are linearly independent.

Remark 7.15 : Under stationarity assumptions on the processes (u_t, x_t, z_t) , it is clear that, if the previous identifiability condition is satisfied, there exists a consistent estimator of the structural parameter. Indeed let us denote by y^*_{it} the adjusted value of $y_{i,t}$ obtained from the empirical regression of y_{it} on x_t , z_{t-1} , y^{**}_{jt} the adjusted value of y_{jt} obtained from the empirical regression of y_{jt} on $\tilde{x}_t$, z_{t-1} .

$y^*_{i,t}$ (resp. $y^{**}_{j,t}$) is a consistent approximation of $E(y_{it}/I_{1,t-1})$ (resp. $E(y_{jt}/I_{t-1})$). In particular the variables $y^*_{i,t}$, $i \in I_g$, $y^{**}_{g,t}$, $j \in J_g$, $\tilde{x}_{kt}$, $k \in \tilde{K}_g$, $\tilde{\tilde{x}}_{k,t}$, $k \in \tilde{\tilde{K}}_g$ are asymptotically linearly independent and the empirical regression coefficients of y_{gt} on these variables are consistent estimators of a_{gi} , $i \in I_g$, c_{gj} , $j \in J_g$, $\tilde{d}_{gk}$, $k \in \tilde{K}_g$, $\tilde{\tilde{d}}_{g,k}$, $k \in \tilde{\tilde{K}}_g$.

Remark 7.16 : When the information set I_t is chosen as $I_{1,t}$, a necessary identifiability condition is that no endogenous variable appears simultaneously with its expectation in the g^{th} equation. This necessary condition is of course specific to the case of nullity constraints. For instance the model :

$$\begin{cases} y_{1t} = a\, y_{2t} + 2a\, E(y_{2t}/x_t \ldots) + b\, x_{1t} + u_{1t} \\ y_{2t} = c_1 x_{1t} + c_2 x_{2t} + u_{2t} \end{cases}$$

is identifiable.

On the other hand the presence of a constraint between the coefficients of y_{2t} and of $E(y_{2t}/x_t \ldots)$ in the first equation does not ensure identifiability. For instance, the model :

$$\begin{cases} y_{1t} = a\, (y_{2t} - E(y_{2t}/x_t)) + b\, x_{1t} + u_{1t} \\ y_{2t} = c_1 x_{1t} + c_2 x_{2t} + u_{2t} \end{cases}$$

is not identifiable.

7.d - Rational expectations and perfect foresight.

In this subsection we assume $I_{t-1} = I_{o,t-1} = (x_{t-1}, y_{t-1} \ldots)$.

The R.E. model (7.1) has the unique solution :

$$y_t = - A^{-1} D\, x_t + A^{-1} C\, (A+C)^{-1} D\, E(x_t/I_{t-1}) + A^{-1} u_t$$

or equivalently :

$$y_t = -(A+C)^{-1} D\, E(x_t/I_{t-1}) - A^{-1} D\, [x_t - E(x_t/I_{t-1})] + u_t$$

$$\iff y_t = -(A+C)^{-1} D\, E(x_t/I_{t-1}) + u_t - A^{-1} D\, \hat{v}_t ,$$

where $\hat{v}_t$ is the prediction error on x_t .

The P.F. model associated to (7.1) is :

$$A\, y_t + C\, y_t + D\, x_t = u_t$$

$$\iff y_t = -(A+C)^{-1} D\, x_t - (A+C)^{-1} u_t$$

$$\iff y_t = -(A+C)^{-1} D\, E(x_t/I_{t-1}) - (A+C)^{-1} u_t - (A+C)^{-1} D\, \hat{v}_t$$

The predictions $E[y_t/I_{1,t-1}]$ are different in the two models. Therefore the two models are not first order observationnally equivalent, conditional to $I_{1,t-1}$.

Moreover, since the reduced coefficients are $(A+C)^{-1}D$ in the P.F. model and $(A+C)^{-1}D$, $A^{-1}D$ in the R.E. model, it is clear that if the P.F. model can be identified, then the same follows for the R.E. model.

Another way of looking these results is to work conditionally to $I_{o,t-1}$. The models give the same prediction $E(y_t/I_{o,t-1}) = -(A+C)^{-1} D\, E(x_t/I_{o,t-1})$; however it is easily seen, using orthogonality between u_t and $\hat{v}_t$, that they do not have the same conditional variance $V\left[\begin{pmatrix} y_t \\ x_t \end{pmatrix} / I_{o,t-1}\right]$.

Conditionally to $I_{o,t-1}$, they appear to be first order observationnally equivalent (see PESARAN (1981) for such a result), but second order different.

The constraints on the reduced form coefficients implied by R.E. or P.F. scheme can be used as a basis for testing these hypotheses (See GOURIEROUX-MONFORT-RENAULT (1985)).

VIII . IDENTIFICATION OF MODELS WITH FUTURE EXPECTATIONS.

When future expectations are present in the model, there exist in general multiple solutions. It is usual in such a case to restrict the analysis to stationary solutions. This external condition can be partly justified by the possibility in a stationary environment to learn for rationality, but has essentially the advantage to lead to a tractable form.

The set of stationary solutions can be parametrized by some auxiliary parameters, i.e parameters independent from the structural ones. Thus the identification problem concerns both kinds of parameters. In the first subsection we consider as an example a unidimensional CAGAN model, we discuss some possible interpretation of the auxiliary parameters and we examine the link between the identifiability of structural parameters and the identifiability of auxiliary parameters. In the second subsection we explain how to use the general results of previous sections to determine the set of linear stationary solutions. Finally we prove in a general context that the identifiability of the structural parameters implies the identifiability of the auxiliary parameters.

8.a - An example.

Let us consider the model :

(8.1) $\quad y_t + b\,E(y_{t-1}/I_t) + dx_t = u_t \quad ,$

where (x_t) is a one dimensional first order autoregressive process :

$$x_t = \rho\, x_{t-1} + \eta_t ,$$

(u_t) is a white noise independent from the exogenous process (x_t) and the information set I_t contains the current and past values of x and u.

By definition the linear stationary solutions of (8.1) are the processes y with an infinite moving average representation in the bidimensional white noise u.

If the structural parameter b satisfies $|b| > 1$, the R.E. model (8.1) has an infinity of linear stationary solutions. These solutions are easily obtained by direct arguments. They are given by :

$$(8.2)\quad y_t = \frac{1}{b + L}\left\{\left(\lambda - \frac{dL}{1 - \rho L}\right)\eta_t + (\mu + L)\, u_t\right\}$$

where L is the lag-operator and λ, μ are two auxiliary parameters, functionally independent from the structural parameters b, d, ρ. In our example the set of linear stationary solutions has dimension two.

The innovation of the endogenous process y is equal to :

$$\varepsilon^o_t = y_t - E(y_t/I_{t-1}) = \frac{\lambda}{b}\,\eta_t + \frac{\mu}{b}\, u_t$$

with $\eta_t = x_t - E(x_t/I_{t-1})$.

Up to a multiplicative factor, λ and μ are the components of the innovation of y on the innovations of the exogenous process and of the error corresponding to the same period. Thus they can be considered as measure of the degree of instantaneous causality between y and (x,u) [see GEWEKE (1981)].

In particular the usual backward solution is obtained for $\lambda = \mu = o$ and corresponds to the absence of instantaneous causality.

Let us now consider the identification problem. We deduce from (8.2) that the reduced form is ;

$$(8.3)\quad \begin{cases} y_t = \dfrac{\lambda - (d + \rho\lambda) L}{(b+L)\,(1-\rho L)}\, \eta_t + \dfrac{\mu + L}{b + L}\, u_t \\[2ex] x_t = \dfrac{\eta_t}{1 - \rho L} \end{cases}$$

Several cases have to be distinguished.

Let us first assume that there is no common factor between $\lambda - (d + \rho\lambda)L$ and $(b+L)\,(1-\rho L)$ and between $\mu + L$ and $b + L$. By looking at the coefficients of the ARMA representation(8.3) it is easily seen that ρ , b , μ , λ , d are all identifiable.

The other cases correspond to the stationary solutions for which there is a common factor between the autoregressive and the moving average parts. We examine two such cases as illustration.

i) If the endogenous process corresponds to $\lambda = -\dfrac{db}{1 + \rho b}$ and to $\mu \neq b$, it is possible to simplify the first ratio by $b + L$ and therefore to identify ρ , μ , b and λ . Moreover since the minimal order of the autoregressive part in the regression of y on x is identifiable, it is known from the data that a simplification exists between $\lambda - (d + \rho L)$ and $b + L$, i.e it is known that the constraint $\lambda = -\dfrac{db}{1 + \rho b}$ is satisfied. Taking this constraint into account we see that all parameters can be identified.

ii) Let us now assume that the endogenous process corresponds to $\lambda = -\dfrac{db}{1 + \rho b}$ and $\mu = b$. As previously we know from the data that these simplifications exist, i.e that $\lambda = -\dfrac{db}{1+\rho b}$ and $\mu = b$ and moreover that λ , ρ can be identified. However it is not possible to identify μ , b , d , since the number of independent identifiable functions (equal to four) is smaller than the number of unknown parameters (equal to five). Let us notice that the particular solution which creates the problem of unidentifiability is given by : $y_t = -\dfrac{dx_t}{1-\rho L} + u_t$ and coincides with the usual forward solution of the R.E. model.

In summary we see in the previous example that <u>the structural parameters b,d, ρ can be identified except for some particular stationary solutions</u> (for instance the forward solution) ; moreover <u>the auxiliary parameters seem to be identifiable as soon as the structural parameters are.</u>

If we now consider the case $|b| \leqslant 1$, the R.E. model (8.1) has a unique linear stationary solution, which coincides with the forward solution. As previously seen for this solution the structural parameter can not be identified. Therefore the uniqueness of the linear stationary solution does not imply that the structural parameters can be identified.

8.b - <u>Linear stationary solutions of a R.E. model.</u>

For the general R.E. model

$$Ay_t + B\,E(y_{t+1}/I_t) + C\,E(y_t/I_{t-1}) + D\,x_t = u_t$$

we have seen in section 3 that the solutions are obtained by solving the difference equation

(3.4) $By_{t+1} + (A + C)\, y_t = B\, \varepsilon_{t+1} + C\, \varepsilon_t + u_t - D\, x_t$

where the multivariate M.D.S. ε is constrained by :

(3.9) $\varepsilon^*_{1t} = C^*_1\, \varepsilon^*_t + \sum_{k=o}^{G_1 - 1} (- J)^k\, [E(v^*_{1,t+k}/I_t) - E(v^*_{1,t+k}/I_{t-1})]$.

Let us now assume that x is a stationary process with an infinite average representation associated with an innovation process η and that u is a white noise independent from x (the variance-covariance matrices of η and u are regular). The linear stationary solutions are of the form :

$y_t = \sum_{j=o}^{\infty} \Lambda_j\, \eta_{t-j} + \sum_{j=o}^{\infty} \mu_j\, u_{t-j}$, where Λ_j and μ_j have dimensions (G,K) and (G,G) respectively. In particular the innovation associated with y is necessarily of the form : $\varepsilon_t = y_t - E(y_t/I_{t-1}) = \Lambda_o\, \eta_t + \mu_o\, u_t$, i.e is a linear combination of η_t and u_t .

In general it is not possible to fix arbitrarily the coefficients Λ_o and μ_o. First they are constrained by (3.9). This implies G (K+G) linear constraints on the elements of Λ and μ , since the components of η and u are linearly independent.

Moreover in the recursion (3.4) the autoregressive part $\det[B + (A+C)\lambda]$ may contain some roots different from zero and inside the unit circle. To obtain a stationary solution it is necessary to eliminate such roots by imposing that they are also roots of the moving-average part. If G_3 denotes the number of these roots, this implies $G_3(G+K)$ linear restrictions on Λ_o and μ_o . In general these restrictions are linearly independent and we have the following result.

PROPERTY 8.4 : The dimension of the set of linear stationary solutions is equal to $G_4(G+K)$, where G_4 is the number of roots of det $[B + (A+C)\lambda]$ which are outside the unit circle.

This property is similar to Whiteman's theorem [see WHITEMAN (1983), p. 91], but take into account the possibility that the autoregressive part has some roots equal to zero.

Moreover it is seen from the previous approach that the $G(G+K)$ real auxiliary parameters are linearly related to the component of ε_t on η_t and u_t . Therefore the interpretation of these parameters in terms of instantaneous causality between y and (x,u) is always available.

8.c - Some identifiability properties.

The structural parameters A , B , C , D can be submitted to some constraints. We denote θ the set of the constrained values of A , B , C , D . Which may link parameters of two different equations of the system.

PROPERTY 8.5 : Let us assume that x has an invertible moving average representation. If the endogenous process y corresponds to a linear stationary solution for which the structural parameters $(A,B,C,D) \in \theta$ can be identified, the $G_4(G+K)$ values of the auxiliary parameters associated with y can also be identified.

This means that the identifiability of the structural parameters implies the identifiability of the solution in the set of all possible linear stationary solutions.

Proof : Since x and y are stationary processes with invertible infinite moving average representations their innovations can be writen as :

$$\eta_t = x_t - E(x_t/I_{t-1}) = x_t - \sum_{j=1}^{\infty} \phi_j x_{t-j} - \sum_{j=1}^{\infty} \tilde{\tilde{\phi}}_j y_{t-j}$$

$$\varepsilon_t = y_t - E(y_t/I_{t-1}) = y_t - \sum_{j=1}^{\infty} \Psi_j x_{t-j} - \sum_{j=1}^{\infty} \tilde{\tilde{\Psi}}_j y_{t-j}$$

Moreover since x and y are observable, the regression coefficients ϕ_j, $\tilde{\tilde{\phi}}_j$, Ψ_j, $\tilde{\tilde{\Psi}}_j$ can be identified.

The coefficient μ_o is equal to :

$$\mu_o = \text{Cov}\,[\varepsilon_t, u_t] \qquad \text{(since } \eta_t \text{ and } u_t \text{ are uncorrelated)}$$

$$= \text{Cov}\Big[y_t - \sum_{j=1}^{\infty} \Psi_j x_{t-j} - \sum_{j=1}^{\infty} \tilde{\tilde{\Psi}}_j y_{t-j} \,,\, Ay_t$$

$$+ B \left(\sum_{j=1}^{\infty} \Psi_j x_{t+1-j} - \sum_{j=1}^{\infty} \tilde{\tilde{\Psi}}_j y_{t+1-j} \right)$$

$$+ C \left(\sum_{j=1}^{\infty} \Psi_j x_{t-j} - \sum_{j=1}^{\infty} \tilde{\tilde{\Psi}}_j y_{t-j} \right) + D\, x_t \Big]$$

μ_o is a known function of the Ψ's , of the $\tilde{\tilde{\Psi}}$'s , of the autocovariance of the process (x,y) and of the structural parameters A , B , C , D .

Since all these parameters are identifiable the same is true for μ_o .

A similar approach can be used to prove that the other coefficient Λ_o is also identifiable.

Finally, when the structural parameters (which are identifiable) are given, the $G_4(G,K)$ auxiliary parameters are bijectively related to Λ_o and μ_o .
Therefore these auxiliary parameters are also identifiable.

□

Therefore it is equivalent to examine the identifiability of all the parameters (structural and auxiliary) or to examine the identifiability of the structural parameters only. We are now concentrating on this second point.

PROPERTY 8.6 : The structural parameters A , B , C , D are identifiable if and only if for the particular linear stationary solution associated with the observed endogenous process the system :

$$\begin{cases} A\,E(y_t/x_t,I_{t-1}) + B\,E(y_{t+1}/x_t,I_{t-1}) + C\,E(y_t/I_{t-1}) + D\,x_t = o \\ (A,B,C,D) \in \Theta \end{cases}$$

has a unique solution in A , B , C , D .

Proof : This is a direct consequence of the following lemma.

Lemma 8.7 : The R.E. model :

$$A\,y_t + B\,E(y_{t+1}/I_t) + C\,E(y_t/I_{t-1}) + D\,x_t = u_t$$

where u_t is a white noise, such that $E(u_t/x_t,I_{t-1}) = o$, is equivalent to the condition :

$$A\,E(y_t/x_t,I_{t-1}) + B\,E(y_{t+1}/x_t,I_{t-1}) + C\,E(y_t/I_{t-1}) + D\,x_t = o$$

The second condition is a direct consequence of the first one, taking the conditional expectation of both sides with respect to x_t , I_{t-1} and applying the law of iterated expectations.

Conversely let us assume that the second condition is satisfied and let us define the process :

$$u_t = A y_t + B E(y_{t+1}/I_t) + C E(y_t/I_{t-1}) + D x_t$$

From the second condition it is directly seen that u_t is orthogonal to x_t , I_{t-1} . Moreover u_t is a combination of y_t , $E(y_{t+1}/I_t)$, $E(y_t/I_{t-1})$, x_t; since the observed solution is stationary the same is true for process u . Finally u_t is orthogonal to $u_{t-1}, u_{t-2} \ldots$, since these lagged values are functions of I_{t-1} and therefore u is a white noise.

□

<u>Remark 8.8</u> : Property 8.6 explains the difficulty encountered with the forward solution in the example of CAGAN's model studied in subsection 8.a . If x_t has a first order autoregressive representation, we have :

$$\begin{aligned} E(y_t/x_t, I_{t-1}) &= E(y_t/x_t, x_{t-1} \ldots, u_t, u_{t-1} \ldots) \\ &= E(y_t/x_t, x_{t-1} \ldots) + E(y_t/u_t, u_{t-1} \ldots) \end{aligned}$$

(from the hypothesis $E(u_t/x_t, I_{t-1}) = o$).

Since x and u are both of order smaller than or equal to one, we have :

$$E(y_t/x_t, I_{t-1}) = E(y_t/x_t) + E(y_t/u_{t-1}) = \bar{\Psi}_o x_t + \bar{\bar{\Psi}}_o u_{t-1}$$

Similarly :

$$E(y_{t+1}/x_t, I_{t-1}) = E(y_{t+1}/x_t) + E(y_{t+1}/u_{t-1}) = \overline{\Psi}_1 x_t + \overline{\overline{\Psi}}_1 u_{t-1}$$

Therefore the three variables $E(y_t/x_t, I_{t-1})$, $E(y_{t+1}/x_t, I_{t-1})$ and x_t are not linearly independent and the condition of property 8.6 is not satisfied.

Remark 8.9 : A way of solving the identification problem in CAGAN's model discussed in 8.a consists in assuming that the exogenous process x has a second order autoregressive representation :

$$x_t = \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \eta_t .$$

The linear stationary solutions are given by :

$$y_t = \frac{1}{b + L} [(\lambda - (d + \varphi_1 \lambda)L - \varphi_2 \lambda L^2) \eta_t + (\mu + L) u_t]$$

Even in the least favorable case in which $b + L$ is simultaneously a divisor of $\lambda - (d + \varphi_1\lambda) L - \varphi_2 \lambda L^2$ and of $\mu + L$, there are four independent identifiable functions (two associated with the minimal orders and two others corresponding to the remainder of the division of $\lambda - (d + \varphi_1\lambda) L - \varphi_2\lambda L^2$ by $b + L$) for four parameters μ , b , d , λ . (Remember that φ_1 and φ_2 are identified from the distribution of the exogenous process).

Remark 8.9 : Under stationarity assumptions, there exists consistent estimates of the parameters as soon as they are identifiable. Indeed let us denote by y^*_{it} (resp $y^*_{i,t}$) the adjusted values deduced from the regression of

y_{it} (resp y_{it+1}) on x_t, x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}... and by $\bar{y}_{it}$ the adjusted values deduced from the regression of y_{it} on x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}... . Consistent estimates of the structural parameters are for instance obtained as solutions of :

$$\underset{A,B,C,D}{\text{Min}} \sum_t \| A\, y_t^* + B\, y_t^{**} + C\, \bar{y}_t + D\, x_t \|^2$$

s.t $(A,B,C,D) \in \theta$.

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