ALLOCATION OF AGGREGATE AND INDIVIDUAL RISKS THROUGH FINANCIAL MARKETS

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ALLOCATION DES RISQUES AGREGES ET INDIVIDUELS
A TRAVERS LES MARCHES FINANCIERS

RESUME

Nous présentons un modèle canonique d'économie d'échange comportant des risques agrégés et individuels. On montre que l'économie a toujours un équilibre de marchés contingents dans lequel les prix dépendent seulement des risques agrégés (que nous appelons équilibre de base). On introduit ensuite la notion de structure d'information à laquelle on associe un nombre qui exprime la quantité maximum d'information révélée à chaque période (nombre d'embranchements). Si ce nombre correspond à la structure d'information associée aux risques agrégés est supérieur au nombre de marchés à terme, alors il est génériquement possible d'obtenir l'allocation de l'équilibre de base par un système de marchés au comptant et à termes pour les biens et marchés d'assurance pour les risques individuels.

Mots clefs : Risques, marchés financiers.

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ABSTRACT

We present a canonical pure exchange model of an economy with aggregate and individual risks. We show that the economy always has a basic contingent commodity equilibrium in which prices depend only on aggregate risks. We introduce an information structure and a number which expressed the maximum rate at which information is revealed in any time period (the branching number). We show that if the information structure associated with the aggregate risks is such that the branching number is not greater than the number of trading opportunities in futures (the number of commodities) then generically each basic contingent commodity equilibrium allocation can be achieved as an equilibrium allocation on a system of spot and futures markets for the underlying commodities and insurance markets for the individual risks.

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1. INTRODUCTION

This paper presents a canonical pure exchange model of an economy with aggregate and individual risks. Aggregate risks have the property that they directly affect the preferences and endowments of all agents simultaneously. Individual risks have the property that they affect the preferences and endowments of particular individual agents independently, and thus their effects cancel out at the aggregate level due to the operations of the law of large numbers.

We introduce two market structures. The first is the standard Arrow-Debreu system of <u>contingent commodity markets</u>. The second is a system of <u>financial markets</u>. The idea is the following: contingent commodity markets have well-known welfare properties but are essentially only a theoretical and not an actual observed type of market structure. Can we introduce instead a system of financial markets that represent an actual observed type of market structure and show that the equilibrium allocations obtainable through a system of contingent commodity markets can also be obtained as the equilibrium allocations of the system of financial markets?

In section 3 we lay out the canonical model of a pure exchange economy with aggregate and individual risks. When the first type of market structure is used we show that the economy always has an important type of equilibrium (which we call a <u>basic</u> equilibrium) in which <u>the contingent commodity prices are independent of individual risks and depend only on the aggregate risks (theorem 4)</u>. We introduce a system of financial markets in which the financial instruments consist of futures contracts on the underlying commodities and insurance contracts on the individual risks. The insurance contract offered to an individual agent depends only on his individual state and can not be used to transfer income across aggregate states. The insurance contracts are similar to those considered by Malinvaud [15] — however in our economy there is aggregate as well as individual risk and we introduce an infinite number of agents to permit an explicit application of the law of large numbers.

We want the trading on the futures markets to reflect the observed fact that agents can trade (frequently) and process information before the actual aggregate state occurs. To this end we draw on the approach of Kreps [14], recently extended by Duffie and Huang [8], which allows one to make this idea precise. We introduce an information structure and a number which expresses the maximum rate at which information is revealed in any time period (the branching number). We then show that if the information structure associated with the aggregate risks is such that the branching number is not greater than the number of trading opportunities in futures (the number of commodities), then generically each basic contingent market equilibrium allocation can be achieved as an equilibrium allocation on a system of spot markets and futures markets for the underlying commodities and insurance markets for the individual risks (theorem 7).

The analysis is completed in two steps. In the first step (section 2) we consider an economy with only aggregate risk, in the second step (section 3) we extend the analysis to an economy with individual as well as aggregate risks. In section 2 we show that in an economy with only aggregate risks if the information structure is such that the branching number never exceeds the number of commodities available for futures trading, then for all initial endowment vectors except those on a set of measure zero, each contingent commodity equilibrium allocation can be achieved as a futures market equilibrium allocation (theorem 3). Thus futures markets are an appropriate vehicle for efficiently allocating aggregate risks. The proof of this result (see section 4) is based on the _regular economy_ type arguments introduced by Debreu [3]. In Appendix A we give an example of _nonexistence_ of a futures market equilibrium which demonstrates that theorem 3 is the best result that can be obtained. The example is based on the idea in Hart's [11] famous example of nonexistence of a futures market equilibrium: however our model differs from his in that payment for futures contracts is made at the time of _delivery_ rather than at the time of _purchase_ and this necessitates a separate construction.

To keep the analysis simple the model in section 2 retains the following assumption: the terminal date is the only date at which futures contracts mature and spot markets meet. Appendix B shows that the analysis can be extended to the more general case where there are many dates at which futures contracts mature and spot markets meet. To keep the technical demands on the reader to a minimum we present the proofs of the theorems in sections 2 and 3 separately in sections 4 and 5.

2. AGGREGATE RISKS

We begin the analysis by considering a pure exchange economy with only aggregate risks. Section 3 extends the analysis to the case where there are both aggregate and individual risks. In this section we also consider the simple case where consumption takes place at a single date: appendix B extends the analysis to the case where consumption takes place at many event-dates.

m agents have random initial endowments of n goods. Let $A = \{a_1, \ldots, a_M\}$ denote the set of aggregate states of nature and let $w^i = (w^i(a))_{a \in A}$ denote the i^{th} agent's endowment vector, where $w^i(a)$ denotes his endowment if state a occurs. The i^{th} agent has a preference ordering $\underset{i}{\succsim}$ defined for consumption vectors $x^i = (x^i(a))_{a \in A}$ contained in his consumption set $X^i = R_+^{Mn}$. Since we assume each agent's preference ordering $\underset{i}{\succsim}$ is complete, transitive and continuous it can be represented by a utility function $u^i : R_+^{Mn} \rightarrow R$. In addition we assume $\underset{i}{\succsim}$ is strictly monotone and has convex preferred sets. Let $\mathscr{E} = \{\underset{i}{\succsim}, R_+^{Mn}, w^i, i = 1, \ldots, m\}$ denote the resulting pure exchange economy.

2.1 Contingent Commodity Equilibrium

The contingent commodity market model for $\mathscr{E}$ [6, ch. 7] introduces a market for each good in each state of nature. Let $P(a) \in R_+^n$ denote the vector of prices for delivery if state $a \in A$ occurs and let $P = (P(a))_{a \in A}$. A contingent commodity equilibrium is a pair $(\bar{x}^1, \ldots, \bar{x}^m; \bar{P})$ consisting of a consumption bundle for each agent and price system $\bar{P} \in R_+^{nM}$ such that each agent chooses a most preferred bundle over his budget set

$$u^i(\bar{x}^i) \geqq u^i(x^i) \quad \forall \; x^i \in B^i(\bar{P}) = \{x \in R_+^{nM} \mid \bar{P}(x - w^i) \leqq 0\} \qquad \text{and} \quad \bar{x}^i \in B^i(\bar{P}) \qquad i = 1, \ldots, m \tag{A1}$$

and each contingent market clears

$$\sum_{i=1}^{m} (\bar{x}^i(a) - w^i(a)) = 0, \qquad a \in A \tag{A2}$$

The set of feasible allocations is defined by $\{(x^1,\dots,x^m) \in R_+^{nMm} \mid \sum_{i=1}^{m} x^i - w^i \leqq 0\}$. A feasible allocation $(\bar{x}^1,\dots,\bar{x}^m)$ is a Pareto optimum if there does not exist a feasible allocation $(y^1,\dots,y^m)$ such that $u^i(y^i) \geqq u^i(\bar{x}^i)$, $i = 1,\dots,m$, and $u^k(y^k) > u^k(\bar{x}^k)$ for some k. The two fundamental welfare theorems [5 , ch. 6] assert that a contingent commodity equilibrium is a Pareto optimum and every Pareto optimum can be achieved as a contingent commodity equilibrium with transfer payments. Our object is to show that, generically for this economy with only aggregate risks, every contingent commodity equilibrium allocation can be achieved through a financial market equilibrium, where the financial instruments consist of futures contracts on the underlying commodities.

2.2 Futures Market Equilibrium

To allow trading in the financial assets to achieve additional spanning opportunities for transferring income across the states of nature, we assume that information about each state of nature $a \in A$ is revealed gradually over a sequence of time periods $t = 0,1,\dots,T$ as follows. For each t, F_t is a partition of A such that F_{t+1} is a refinement of F_t and $F_0 = \{A\}$, $F_T = \{\{a_1\},\dots,\{a_M\}\}$. Let σ_t denote a generic element of F_t and let $F = (F_t)_{t=0}^{T}$. Define

$$k(F) = \max_{\substack{\sigma_t \in F_t \\ t=0,\dots,T-1}} \#\{\sigma_{t+1} \subset \sigma_t \mid \sigma_{t+1} \in F_{t+1}\} \tag{1}$$

k(F) is the maximum number of events that can occur subsequent to any given event-date in the event tree; it is thus a measure of the maximum rate at which information is revealed by the filtration F at any event-date σ_t and is called the branching number of the filtration F.

In this section we consider a single class of futures contracts, those with maturity date at time T. In appendix B we extend the analysis to the case where there are futures contracts which have delivery dates at any of the times $t = 1,2,\ldots,T$. Consider therefore the futures contracts with maturity date at time T. At date t one of the events $\sigma_t \in F_t$ is revealed. A futures contract for good j at event-date σ_t calls for the unconditional delivery of one unit of good j at time T. Let $z^i_{jt}(\sigma_t)$ denote the number of contracts of the j^{th} good purchased by the i^{th} agent given the information σ_t and let $z^i_t(\sigma_t) = (z^i_{1t}(\sigma_t), \ldots, z^i_{nt}(\sigma_t))$. Let $q_{jt}(\sigma_t)$ denote the futures price for the j^{th} good at event-date σ_t with $q_t(\sigma_t) = (q_{1t}(\sigma_t),\ldots,q_{nt}(\sigma_t))$. If $p = (p(a))_{a\in A}$ denotes the vector of spot prices at date T, then by arbitrage $q_T(a) = p(a)$, $a\in A$. We assume without loss of generality that in period t+1 each agent closes out his futures position taken at time t. For each $a \in A$ let $\sigma_t(a)$ denote the unique $\sigma_t \in F_t$ for which $a \in \sigma_t$. The earnings obtained at date T from the trading strategy $z^i = (z^i_t(\sigma_t),\ \sigma_t \in F_t,\ t = 0,\ldots,T)$ is given by

$$R(z^i,a) = \sum_{t=0}^{T-1} z^i_t(\sigma_t(a))\,[q_{t+1}(\sigma_{t+1}(a)) - q_t(\sigma_t(a))], \qquad a \in A \tag{2}$$

Equation 2 leads naturally to a matrix whose properties are central to an understanding of the behaviour of futures markets. Define row a and column (j,σ_t) of a matrix Q for $a\in A$, $j = 1,\ldots,n$ and $\sigma_t \in F_t$, $t = 0,\ldots,T-1$ by

$$Q_j(a,\sigma_t) = \begin{cases} 0 & \text{if } a \notin \sigma_t \\ q_{jt+1}(\sigma_{t+1}(a)) - q_{jt}(\sigma_t(a)) & \text{if } a \in \sigma_t \end{cases}$$

and let $Q_j(\sigma_t) = (Q_j(a_1,\sigma_t),\ldots,Q_j(a_M,\sigma_t)) \in R^M$ be a column vector. $Q_j(\sigma_t)$ defines the vector of earnings across the states of nature obtained at date T from the purchase of one futures contract of the j^{th} good at event-date σ_t. The earnings matrix Q is then defined as the collection of all such earnings vectors obtained from a unit trade in each of the n goods at each of the event-dates σ_t

$$Q = (Q_j(\sigma_t),\ j = 1,\ldots,n,\ \sigma_t \in F_t,\ t = 0,\ldots,T-1) \tag{4}$$

Let $R(z^i) = \left(R(z^i,a)\right)_{a\in A}$, then $R(z^i) = Qz^i$. Thus the budget set made possible by the futures trade z^i defined by

$$\mathscr{B}^i_{z^i}(p,q) = \{x \in R^{nM}_+ \mid [p(a)(x(a) - w^i(a))]_{a\in A} \leqq Qz^i\} \tag{5}$$

leads to the i^{th} agent's budget set with a system of spot and futures markets

$$\mathscr{B}^i(p,q) = \bigcup_{z^i} \mathscr{B}_{z^i}(p,q) \qquad i = 1,\ldots,m \tag{6}$$

A futures market equilibrium is a pair $[(\bar{x}^1,\bar{z}^1),\ldots,(\bar{x}^m,\bar{z}^m);\ (\bar{p},\bar{q})]$ such that each agent chooses a most preferred consumption bundle over his budget set

$$u^i(\bar{x}^i) \geqq u^i(x^i) \quad \forall \quad x^i \in \mathscr{B}^i(\bar{p},\bar{q}) \text{ and } \bar{x}^i \in \mathscr{B}^i(\bar{p},\bar{q}) \qquad i = 1,\ldots,m \tag{F1}$$

and hence selects a futures trading strategy $\bar{z}^i$ such that $\bar{x}^i \in \mathscr{B}_{\bar{z}^i}(p,q)$, $i = 1,\ldots,m$. In addition spot and futures markets clear

$$\sum_{i=1}^m (\bar{x}^i - w^i) = 0 \quad \text{(F2)}, \qquad \sum_{i=1}^m \bar{z}^i_t(\sigma_t) = 0, \quad \sigma_t \in F_t, \quad t = 0,\ldots,T-1 \quad \text{(F3)}$$

2.3 Equivalence of Equilibrium Allocations

When can a futures market equilibrium allocation be achieved as a contingent commodity equilibrium allocation?

THEOREM 1. Let $[(\bar{x}^1,\bar{z}^1),\ldots,(\bar{x}^m,\bar{z}^m);\ (\bar{p},\bar{q})]$ be a futures market equilibrium for the economy $\mathscr{E}$. If rank $(Q) = M-1$, then the allocation $(\bar{x}^1,\ldots,\bar{x}^m)$ can be achieved through a contingent commodity equilibrium $(\bar{x}^1,\ldots,\bar{x}^m;\bar{P})$.

In section 4 we show that the absence of arbitrage opportunities in a futures market equilibrium implies the existence of a vector $\beta \in R^M_{++}$ such that a candidate contingent commodity price vector must satisfy

$$\bar{P}(a) = \beta_a \bar{p}(a) \qquad a \in A \tag{7}$$

We show in addition that if rank $(Q) = M-1$, then $B^i(\bar{P}) = \mathscr{B}^i(\bar{p},\bar{q})$, $i = 1,\ldots,m$, so that the price system $\bar{P}$ defined by (7) leads to a contingent commodity equilibrium . The rank condition implies that β is unique up to multiplication by a scalar and represents the common social marginal utility of income for all agents.

Suppose we start with a contingent commodity equilibrium. If we choose any $\beta \in R^M_{++}$, $\sum_{a \in A} \beta_a = 1$ and let (7) define the spot prices, then an arbitrage argument again shows that to obtain a futures market equilibrium the futures prices must satisfy the condition

$$\beta Q = 0 \iff \sum_{a \in \sigma_t} \beta_a (\bar{q}_{t+1}(a) - \bar{q}_t(a)) = 0 \quad \text{for} \quad \sigma_t \in F_t \quad t = 0,\ldots,T-1 \tag{8}$$

where each $\bar{q}_t(\cdot)$ satisfies the measurability condition that it depends only on information available at time t. (8) is a system of first order difference equations which allows futures prices at time t to be determined recursively from those at time t+1. The whole system of futures prices can thus be determined from the spot prices defined by (7). Thus starting with the contingent commodity price system $\bar{P}$ we have been led to a well-defined system of spot and futures prices and hence to a well-defined matrix Q.

Note that if we view β_a as probabilities, then (8) is equivalent to

$$\bar{q}_t(a) = E(\bar{q}_{t+1} \mid \sigma_t(a)), \qquad a \in A, \qquad t = 0,\ldots,T-1 \tag{9}$$

which asserts that the futures price process $\{\bar{q}_t\}_{t=0}^T$ is a martingale. Since $\bar{q}_T(a) = \bar{p}(a)$, $a \in A$, (9) implies

$$\bar{q}_t(a) = E(\bar{p} \mid \sigma_t(a)), \qquad a \in A$$

so that relative to β, $\bar{q}_t$ is an unbiased predictor of the future spot price.

When can a contingent commodity equilibrium allocation be achieved as a futures market equilibrium allocation?

THEOREM 2. Let $(\bar{x}^1,\ldots,\bar{x}^m;\bar{P})$ be a contingent commodity equilibrium for the economy $\mathcal{E}$. If rank $(Q) = M-1$, then the allocation $(\bar{x}^1,\ldots,\bar{x}^m)$ can be achieved through a futures market equilibrium $[(\bar{x}^1,\bar{z}^1),\ldots,(\bar{x}^m,\bar{z}^m);\ (\bar{p},\bar{q})]$.

The intuition behind the rank condition in these two theorems is roughly as follows. With spot markets only, the dimension of an agent's budget set is $nM-M$ because of the M constraints imposed by his income in each state. With contingent commodity markets there is just one budget constraint and the dimension of each agent's budget set is $nM-1$. When futures markets are added to the system of spot markets, if rank $(Q) = M-1$, then the income transfer made possible by futures trading raises the dimension of each agent's budget set into equality with that in the case of contingent markets $(nM-M)+(M-1) = nM-1$. Thus when the rank condition is satisfied each agent has the same opportunity set on a system of contingent markets as on a system of (spot and) futures markets — hence the ability of these two market systems to achieve the same allocations.

2.4 Generic Result

How likely is it for the economy $\mathcal{E}$ that the rank condition will be satisfied? To answer this question we need to place some further restrictions on preferences and endowments. Let $\mathcal{P}_{C^1}$ denote the set of preference orderings representable by continuous utility functions $u : R_+^{nM} \to R$ with the following property. The function $f : R_{++}^{nM} \times R_{++} \to R_{++}^{nM}$ defined by $f(p,I) = \arg \max_{x \in B(p,I)} u(x)$, $B(p,I) = \{x \in R_+^{nM} \mid px \leq I\}$ is a C^1 function which satisfies the boundary condition $\lim_{(p,I)\to(\bar{p},\bar{I})} \| f(p,I)\| = \infty$ whenever $\bar{p} \neq 0$ lies in the boundary of R_{++}^{nM} and $\bar{I} > 0$. Conditions on preference orderings which generate such demand functions have been given by Debreu [4,5].

Let $w = (w^1,\ldots,w^m)$ denote the m agents' initial endowment vectors. We let the economy $\mathcal{E}$ be parameterised by $w \in R_+^{nMm}$. We can now show that the rank condition is generically satisfied (lemmas 3, 4, 5 in section 4). For a subset

$B \subset R^s$ let B^c denote the complement of B. Recalling that a subset $H \subset R^s$ is said to be null if it has s-dimensional Lebesgue measure zero, we are led to the following result. Note that in view of the nonexistence example given in appendix A this is the best result one can expect.

THEOREM 3. Consider economies $\mathscr{E}$ for which agents' preferences belong to $\mathscr{P}^1_c$. If the number of commodities is at least as great as the branching number of the information structure $(n \geq k(F))$, then there is an open subset of initial endowments $\Omega \subset R^{nMm}_{++}$, with Ω^c null, such that for each economy $\mathscr{E}$ with $w \in \Omega$, every contingent commodity equilibrium allocation is a futures market equilibrium allocation requiring nontrivial futures trading in at most k(F) commodities.

Thus if the information structure F is such that the rate at which information unfolds at any event-date is never greater than the number of trading opportunities available at each date (n), then in an economy $\mathscr{E}$ with aggregate risk only, for almost all initial endowment vectors, an allocation achieved through a system of contingent commodity markets can also be achieved through a system of spot and futures markets, and the number of futures contracts needed is at most the branching number of the information structure.

3. AGGREGATE AND INDIVIDUAL RISKS

3.1 Introducing Individual Risks

We want to enrich the structure of the economy $\mathscr{E}$ by including individual risks. Roughly speaking these are risks which while faced by individual agents, cancel out at the aggregate level. One model in which such risks are studied, and which is frequently cited, is that of Malinvaud [15]. In the simplest form of his model risks are eliminated at the aggregate level by assuming that the aggregate endowment $\sum_{i=1} w^i(a)$ is constant across the states of nature $a \in A$.

In terms of the results of the previous section, such an assumption immediately forces the set of initial endowment vectors w to lie in a set of measure zero so that theorem 3 does not apply.

More importantly, in terms of the applicability of futures markets, if we consider the case where information is revealed in one period (T=1), then under Malinvaud's assumption on preferences, the system of contingent commodity prices $(P_a)_{a \in A}$ has rank 1 so that the induced matrix Q has at most rank 1. Thus the absence of risk leads to an absence of fluctuation in the contingent commodity prices and this in turn leads to an earnings matrix Q with low rank: there is thus no way for futures trading to achieve the required transfer of income across the states of nature.

It is clear therefore that while contingent commodity markets can deal adequately with both aggregate and individual risks, a system of futures markets needs to be supplemented by another market structure to deal with the individual risks. This structure is a system of <u>insurance markets</u>. Our object is to extend the main result of the previous section by showing that, in an economy with both aggregate and individual risks, there is a precise sense in which for "most" economies, certain basic contingent commodity equilibrium allocations can be achieved through a financial market equilibrium, where the financial instruments now consist of futures contracts on the underlying commodities and insurance contracts on the individual risks.

To establish this result we need to extend the model of the previous section. In particular to express the idea of individual risk we will need an economy with infinitely many agents. Let $A = \{a_1, \ldots, a_M\}$ denote the set of <u>aggregate states</u> with the same property stipulated in the previous section, namely that each aggregate state can affect each agent's preferences and initial endowment. Let $S = \{s_1, \ldots, s_N\}$ denote a set of <u>individual states of nature</u>. For simplicity we assume that each agent i in the economy has the same underlying set of individual states $S_i = S$. The state $s_i \in S_i$ in which the i^{th} agent finds himself has the property that it affects only his preferences and his initial endowment and not the preferences or endowments of other agents. We assume in addition that the individual state s_i is observable by outsiders (the insurance company) so that the set S_i does not include personal states such as the mood or energy of the agent. To express the idea that individual risks cancel out at the aggregate level we want to apply the law of large numbers: this requires that the economy have infinitely many agents. Thus let $I = \{1,2,\dots\}$ denote the set of agents, then the set of states of nature for the economy is given by

$$\Sigma = A \times \prod_{i \in I} S_i \quad \text{where} \quad S_i = S \text{ for all } i \in I$$

We can think of A as representing whether these are good times or bad times economy-wide, the nature of the weather, earthquakes, floods and so on — in short any state that can affect the preferences and endowments of a whole collection of agents, but not in such a way that the effect cancels out at the aggregate level. We can think of the set S_i as representing whether agent i suffers some form of personal injury (an accident, ill health, etc.) and whether his property is exposed to fire, theft, deterioration, etc.

Let $(a,s) = (a,s_1,s_2,\dots)$ denote a typical element of Σ. Let $\mathscr{S}$ denote the measurable subsets of Σ and let π denote a probability measure on $\mathscr{S}$. The property of independence of the individual states is expressed as follows: for each aggregate state $a \in A$ the conditional probability measure on $\prod_{i \in I} S_i$ is a product probability measure. This enables us to use the law of large numbers. We also need to distinguish some subσ-fields of $\mathscr{S}$ defined by aggregate and individual specific states of nature. Let

$$H_a = \{a\} \times \prod_{i \in I} S_i, \qquad H^i_{as} = \{a\} \times S_1 \times \dots \times S_{i-1} \times \{s\} \times S_{i+1} \times \dots, \qquad i \in I$$

and let $\quad \mathscr{H} = \sigma(H_a,\ a \in A), \quad \mathscr{H}^i = \sigma(H^i_{as},\ a \in A,\ s \in S), \quad i \in I$

denote the induced subσ-fields of $\mathscr{F}$.

The commodity space is taken to be the space of R^n-valued essentially bounded measurable functions defined on the probability space $(\Sigma, \mathscr{F}, \pi)$, denoted by $L_\infty(\Sigma, \pi; R^n)$. The endowment vector ω^i and consumption vector χ^i of each agent $i \in I$ is an element of the non-negative orthant $L_\infty^+(\Sigma, \pi; R^n)$.

3.2 Restrictions on Endowments and Preferences

We need to make some restrictions on agents' endowments and preferences — in particular those that formalise the concepts of aggregate and individual risks. We assume that agent i's endowment vector ω^i is $\mathscr{H}^i$-measurable. ω^i thus depends only on the aggregate state $a \in A$ and agent i's individual state $s_i \in S_i$. With this assumption ω^i can be represented by a finite-dimensional vector $w^i = (w^i_{as})_{a \in A,\, s \in S} \in R_+^{nMN}$ where $w^i_{as} = \omega^i(a, s)$, $\forall\, (a, s) \in H^i_{as}$. Agent i's preference ordering $\succsim_i$ is defined on $L_\infty^+(\Sigma, \pi; R^n)$. We express the idea that $\succsim_i$ depends only on the aggregate state $a \in A$ and his individual state $s_i \in S_i$ by assuming that

$$E(\chi \,|\mathscr{H}^i) \succ_i \chi, \qquad \forall\, \chi \in L_\infty^+(\Sigma, \pi; R^n), \qquad \chi \neq E(\chi \,|\mathscr{H}^i), \qquad i \in I \tag{10}$$

The preference ordering $\succsim_i$ on $L_\infty^+(\Sigma, \pi; R^n)$ induces a preference ordering $|\succsim_i|$ on R_+^{nMN} through the vectors $x = (x_{as})_{a \in A, s \in S}$ which represent $\mathscr{H}^i$-measurable consumption vectors $\chi \in L_\infty^+(\Sigma, \pi; R^n)$. Let $\mathscr{P}$ denote the space of continuous preferences on R_+^{nMN} with the topology of closed convergence and let $\mathscr{P}_s \subset \mathscr{P}$ denote the subspace of strictly convex preferences. The vector of probabilities $\pi^i = (\pi(H^i_{as}))_{a \in A,\, s \in S}$ must be compatible with the vector of probabilities of the aggregate events $\pi^A = (\pi(H_a))_{a \in A}$ in the sense that

$$\pi^i \in \Delta_A \quad \text{where} \quad \Delta_A = \{\rho \in R_+^{NM} \mid \sum_{\substack{a \in A \\ s \in S}} \rho_{as} = 1, \quad \sum_{s \in S} \rho_{as} = \pi(H_a),\ a \in A\}, \qquad i \in I$$

The triple $(|\succsim_i|, w^i, \pi^i)$ now defines the i^{th} agent's characteristics. We make the following boundedness assumption on the characteristics of agents. <u>There exists a compact subset $K \subset \mathscr{P}_S \times R^{nMN}_{++} \times \Delta_A$ such that $(|\underset{i}{\succsim}|, w^i, \pi^i) \in K$, $\forall\, i \in I$.</u>

One more step is required to complete our description of the economy. Consider the S-averaged economy in which agents' endowments and consumption bundles are $\mathscr{H}$-measurable and each preference ordering is restricted to such bundles. More precisely consider the map $E_i : R^{nMN}_+ \to R^{nM}_+$ and its inverse E_i^{-1} defined by

$$E_i(x) = \Big(\sum_{s \in S} \pi^i_{as} x_{as}\Big)_{a \in A} = \bar{x}, \qquad x \in R^{nMN}_+$$

$$E_i^{-1}(\xi) = \{x \in R^{nMN}_+ \mid E_i(x) = \xi\}, \qquad \xi \in R^{nM}_+$$

Define the function $\alpha_i : R^{nM}_+ \to R^{nMN}_+$ by $\alpha_i(\xi)\,|\underset{i}{\succsim}|\,z$, $\forall\, z \in E_i^{-1}(\xi)$ and $\alpha_i(\xi) \in E_i^{-1}(\xi)$ and the preference ordering $(\underset{i}{\succsim})$ over S-averaged bundles by

$$\bar{x}\,(\underset{i}{\succsim})\,\bar{y} \quad \text{if and only if} \quad \alpha_i(\bar{x})\,|\underset{i}{\succsim}|\,\alpha_i(\bar{y}), \qquad i \in I$$

Let $\mathscr{P}_{c^1}$ denote the space of preference orderings on R^{nM}_+ leading to C^1 demand functions (as defined in section 2). Endow $\mathscr{P}_{c^1}$ with the topology of closed convergence as a subset of $\mathscr{P}$ (on R^{nM}_+). We now make the assumption that the S-averaged characteristics of agents have the following property: <u>there exists a compact subset $L \subset \mathscr{P}_{c^1} \times R^{nM}_{++} \times \Delta_A$ such that $((\underset{i}{\succsim}), \bar{w}^i, \pi^i) \in L$, $\forall\, i \in I$.</u> Let $\mathscr{M}(L)$ denote the set of probability measures on L. For any measurable subset $D \subset L$, let $\mu_m(D)$ denote the proportion of the first m agents in the economy with S-averaged characteristics in D

$$\mu_m(D) = \frac{1}{m}\#\{i \mid ((\underset{i}{\succsim}), \bar{w}^i, \pi^i) \in D,\ i = 1, \ldots, m\}$$

<u>We require that there exist $\mu \in \mathscr{M}(L)$ such that $\mu_m \xrightarrow{w} \mu$ as $m \to \infty$ in the sense of weak convergence of measures.</u> We let $\mathcal{E}^*$ denote an economy with the properties outlined in sections 3.1 and 3.2.

3.3 Example of Economy $\mathscr{E}^*$

The following example will serve to illustrate an economy $\mathscr{E}^*$ with the above properties. Suppose $\underset{i}{\succsim}$ is represented by a utility function W^i of the following form

$$W^i(\chi) = \sum_{\substack{a \in A \\ s \in S}} \int_{H^i_{as}} u^i_{as}(\chi)d\pi$$

where each $u^i_{as} : R^n \rightarrow R$ is continuous, strictly concave and increasing. Clearly $\underset{i}{\succsim}$ satisfies (10). The induced preference ordering $|\underset{i}{\succsim}|$ on R^{nMN}_+ is represented by the continuous, strictly concave and increasing utility function

$$U^i(x) = \sum_{\substack{a \in A \\ s \in S}} u^i_{as}(x_{as})\pi^i_{as}$$

and thus belongs to $\mathscr{P}_s$. The induced preference ordering $(\underset{i}{\succsim})$ on R^{nM}_+ is represented by the utility function

$$\bar{U}^i(\bar{x}) = \max_x U^i(x) \quad \text{subject to} \quad \sum_{s \in S} \pi^i_{as}x_{as} = \bar{x}_a, \ a \in A$$

With appropriate smoothness and boundary assumptions on the u^i_{as}, $(\underset{i}{\succsim})$ will belong to $\mathscr{P}_{cl}$.

The i^{th} agent's characteristics are thus defined by the triple $((u^i_{as})_{a,s}, w^i, \pi^i)$. Suppose the economy consists of two "types" of agents: let agent i have the characteristics $((u'_{as})_{a,s}, w', \pi')$ if i is odd and the characteristics $((u''_{as})_{a,s}, w'', \pi'')$ if i is even and let $|\succsim'|$, $(\succsim')$ and $|\succsim''|$, $(\succsim'')$ denote the induced preferences. Then

$$K = \{(|\succsim'|, w', \pi'),\ (|\succsim''|, w'', \pi'')\}$$

$$L = \{((\succsim'), \bar{w}', \pi'),\ ((\succsim''), \bar{w}'', \pi'')\}$$

so that both are compact. Moreover μ_m on L is given by

$$\mu_m\big(((\succsim'),w',\pi')\big) = \begin{cases} \frac{1}{2} & \text{if } m \text{ even} \\ \\ \frac{m+1}{2m} & \text{if } m \text{ odd} \end{cases}$$

$$\mu_m\big(((\succsim''),\bar{w}'',\pi'')\big) = \begin{cases} \frac{1}{2} & \text{if } m \text{ even} \\ \\ \frac{m-1}{2m} & \text{if } m \text{ odd} \end{cases}$$

Clearly μ_m converges to the μ which assigns probability $\frac{1}{2}$ to each point of L. Thus the compactness assumptions and the assumption that $\{\mu_m\}$ is weakly convergent are simply generalisations of a "replica" economy with a finite number of types.

3.4 Contingent Commodity Equilibrium for $\mathscr{E}^*$

A contingent commodity price system for an economy $\mathscr{E}^*$ consists of an n-vector of measures $P = (P_1,\ldots,P_n)$ defined on $\mathscr{S}$, where $P_j(B)$ denotes the price of one unit of good j with delivery if and only if event $B \in \mathscr{S}$ occurs. We will consider only price systems which are absolutely continuous with respect to π so that $P(B) = \int_B p d\pi$ for some $p \in L_1^+(\Sigma,\pi;R^n)$, the non-negative orthant of the space of R^n-valued integrable functions defined on (Σ,π). The cost of a consumption vector $\chi \in L_\infty^+$ is thus $\int_\Sigma p\chi d\pi$. We are interested in particular in price systems p which are $\mathscr{H}$-measurable, namely those that vary only with the aggregate state $a \in A$. Such a price system can be represented by a finite-dimensional vector $\mathbf{p} = (p(H_a))_{a\in A}$.

A contingent commodity equilibrium for an economy $\mathscr{E}^*$ is a pair $[(\tilde{\chi}^i)_{i\in I}; \tilde{p}]$ where $\tilde{\chi}^i \in L_\infty^+(\Sigma,\pi;R^n)$, $i \in I$, $p \in L_1^+(\Sigma,\pi;R^n)$ such that

$$\tilde{\chi}^i \underset{i}{\succsim} \chi^i \;\; \forall \; \chi^i \in C^i(\tilde{p}) = \{\chi \in L_\infty^+ \mid \int_\Sigma \tilde{p}(\chi - \omega^i)d\pi \leqq 0\} \text{ and } \tilde{\chi}^i \in C^i(\tilde{p}), \quad i \in I \qquad \text{(A1)*}$$

and
$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1} (\tilde{\chi}^i - \omega^i) = 0 \quad \text{a.s.} \qquad \text{(A2)*}$$

If in addition each $\tilde{\chi}^i$ is $\mathscr{H}^i$-measurable $i \in I$ and $\tilde{p}$ is $\mathscr{H}$-measurable, the pair $[(\tilde{\chi}^i)_{i\in I};\tilde{p}]$ will be called a <u>basic</u> contingent commodity equilibrium for $\mathscr{E}^*$.

THEOREM 4. <u>Every economy $\mathscr{E}^*$ has a basic contingent commodity equilibrium</u>.

<u>Remark</u>. Note that this result does not follow directly from any of the known existence results for an economy with an infinite-dimensional commodity space since there are countably many agents.

3.5 Futures-Insurance Market Equilibrium for $\mathscr{E}^*$

Futures trading takes place as in section 2. Thus we assume that information about the aggregate state $a \in A$ is revealed gradually over a sequence of time periods $t = 0,\ldots,T$, through an information partition $F = (F_t)_{t=0}^T$ where F_{t+1} is a refinement of F_t, $F_0 = \{A\}$ and $F_T = \{\{a_1\},\ldots,\{a_M\}\}$. Each agent $i \in I$ chooses a futures trading strategy $z^i = (z_t^i(\sigma_t), \sigma_t \in F_t, t = 0,\ldots,T-1)$ faced with the system of futures prices $q = (q_t(\sigma_t), \sigma_t \in F_t, t = 0,\ldots,T-1)$.

A <u>spot price system</u> $p : \Sigma \to R_+^n$ for $\mathscr{E}^*$ assigns a nonnegative vector of prices $p(a,s)$ to each state $(a,s) \in \Sigma$. We consider only spot price systems $p \in L_1^+(\Sigma,\pi;R_+^n)$. In particular we are interested in spot prices p which are $\mathscr{H}$-measurable, so that they vary only with the aggregate state of nature. Such a price system can be represented by a finite-dimensional vector $p = (p(H_a))_{a\in A}$. With such a spot price system, the induced system of futures prices would be such that a typical agent could achieve no transfer of income between two different individual states associated with the same aggregate state $a \in A$. To enable such an income transfer to take place we introduce the following insurance contracts. An <u>insurance contract for the i^{th} agent</u> is a function $\nu^i \in L_\infty(\Sigma,\pi;R)$ where $\nu^i(a,s)$ denotes the income received at date T if state $(a,s)\in\Sigma$ occurs. We consider only insurance contracts with the following three properties.

(i) <u>ν^i is $\mathscr{H}^i$-measurable</u>. Thus in particular the income transfer received by agent i does not depend on the individual state s_j of any other agent $j \neq i$.

(ii) <u>$E(\nu^i \mid H_a) = 0,\ a \in A$</u>. Thus ν^i is actuarily fair on each aggregate event H_a.

(iii) <u>$\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \nu^i = 0$</u> a.s. Thus an insurance company faces no risk.

Consider the following budget sets that are the analogues of those defined in (5) and (6) of section 2. For the i^{th} agent the trade-insurance pair (z^i,ν^i) generates the budget set

$$\mathscr{B}^i_{z^i,\nu^i}(p,q,\pi^i) = \{\chi \in L^+_\infty \mid p(a,s)(\chi(a,s) - \omega^i(a,s)) \leqq \nu^i(a,s) + R(z^i,a)\} \tag{11}$$

so that the i^{th} agent's budget set with a system of spot, futures and insurance markets becomes

$$\mathscr{B}^i(p,q,\pi^i) = \left\{ \mathscr{B}^i_{z^i,\nu^i}(p,q,\pi^i) \;\middle|\; \begin{array}{l} \text{for all } (z^i,\nu^i),\ \nu^i \text{ is } \mathscr{H}^i\text{-measurable} \\ E(\nu^i \mid H_a) = 0, \quad a \in A \end{array} \right\}$$

A <u>futures-insurance market equilibrium</u> for an economy $\mathcal{E}^*$ is a pair $[(\tilde{\chi};\tilde{z}^i,\tilde{\nu}^i)_{i\in I};(\tilde{p},\tilde{q})]$ where $\tilde{\chi}^i \in L^+_\infty(\Sigma,\pi;R^n)$, $\tilde{z}^i : F \to R^n$, $\tilde{\nu} \in L_1(\Sigma,\pi;R)$, $\tilde{p} \in L^+_1(\Sigma,\pi;R^n)$, $\tilde{q} : F \to R^n_+$ such that

$$\tilde{\chi}^i \underset{i}{\succsim} \chi^i \quad \forall\, \chi^i \in \mathscr{B}^i(\tilde{p},\tilde{q},\pi^i), \quad i \in I \tag{F1*}$$

$(\tilde{z}^i,\tilde{\nu}^i)$ is chosen so that $\tilde{\chi}^i \in \mathscr{B}^i_{\tilde{z}^i,\tilde{\nu}^i}(\tilde{p},\tilde{q},\pi^i)$ and spot, futures and insurance markets clear in the mean

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} (\tilde{\chi}^i - \omega^i) = 0 \quad \text{a.s.} \tag{F2*}$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \tilde{z}^i = 0 \qquad (F3)^*$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \tilde{\nu}^i = 0 \quad \text{a.s.} \qquad (F4)^*$$

If in addition each $\tilde{\chi}^i$ is $\mathscr{H}^i$-measurable $i \in I$ and $\tilde{p}$ is $\mathscr{H}$-measurable, then the pair $[(\tilde{\chi}^i, \tilde{z}^i, \tilde{\nu}^i)_{i \in I}; (\tilde{p}, q)]$ will be called a basic futures-insurance market equilibrium for $\mathscr{E}^*$.

3.6 Equivalence of Equilibrium Allocations for $\mathscr{E}^*$

Given a vector of futures prices q the earnings matrix Q can be defined as in section 2. We are then led to the following analogue of theorem 1 for $\mathscr{E}^*$.

THEOREM 5. *Let $[(\tilde{\chi}^i, \tilde{z}^i, \tilde{\nu}^i)_{i \in I}; (\tilde{p}, \tilde{q})]$ be a basic futures-insurance market equilibrium for the economy $\mathscr{E}^*$. If rank $(Q) = M-1$, then the allocation $(\tilde{\chi}^i)_{i \in I}$ can be achieved through a basic contingent commodity equilibrium $[(\tilde{\chi}^i)_{i \in I}; \tilde{p}]$.*

For the economy $\mathscr{E}^*$ the relation between basic contingent commodity prices and basic spot prices in futures-insurance equilibrium is similar to that defined by equation (7) for the economy $\mathscr{E}$

$$\mathbf{p}(H_a) = \beta_a p(a), \qquad a \in A \qquad (12)$$

If we start with a basic contingent commodity equilibrium, choose any $\beta \in R^M_{++}$ with $\sum_{a \in A} \beta_a = 1$ and let (12) define basic spot prices, then an arbitrage argument again shows that futures prices must satisfy $\beta Q = 0$. This leads by the same argument as in section 2 to a well-defined vector of futures prices q and hence to the matrix Q. The following result is the analogue of theorem 2 for $\mathscr{E}^*$.

THEOREM 6. *Let $[(\tilde{\chi}^i)_{i \in I}; \tilde{p}]$ be a basic contingent commodity equilibrium for the economy $\mathscr{E}^*$. If rank $(Q) = M-1$, then the allocation $(\tilde{\chi}^i)_{i \in I}$ can be achieved through a basic futures-insurance market equilibrium $[(\tilde{\chi}^i, \tilde{z}^i, \tilde{\nu}^i)_{i \in I}; (\tilde{p}, \tilde{q})]$.*

3.7 Main Result

We would like to show that for an economy $\mathscr{E}^*$ with both aggregate and individual risk, "typically" each basic contingent commodity equilibrium allocation can be achieved as an equilibrium allocation on a combined system of futures and insurance markets.

THEOREM 7. Consider economies $\mathscr{E}^*$ in which the number of commodities is at least as great as the branching number of the information structure $(n \geq k(F))$. There is an open dense set $\Gamma \subset \mathscr{M}(L)$ such that for each economy $\mathscr{E}^*$ with a limiting distribution $\mu \in \Gamma$, every basic contingent commodity equilibrium allocation can be achieved as a basic futures-insurance market equilibrium allocation, requiring non-trivial futures trading in at most $k(F)$ commodities.

4. PROOFS FOR SECTION 2

Proof of Theorem 1. The futures market equilibrium satisfies F1-F3. It suffices to show that there exists $\beta \in R^M_{++}$ such that if we define a contingent commodity price vector by (7), then the budget sets defined by (A1) and (6) coincide, $B^i(\bar{P}) = \mathscr{B}^i(\bar{p},\bar{q})$, $i = 1,\ldots,m$, since then A1, A2 are satisfied.

Let $x^i \in \mathscr{B}^i(\bar{p},\bar{q})$, then there exists z^i such that

$$[\bar{p}(a)(x^i(a) - w^i(a))]_{a\in A} \leqq Qz^i \tag{13}$$

If $Qz^i \geq 0$, $Qz^i \neq 0$ were possible then $Q\gamma z^i \geq 0$ for $\gamma > 0$ and income in some states could be increased arbitrarily with no sacrifice in income from other states. Since $u^i(\cdot)$ is strictly monotone, there could be no solution to the i^{th} agent's preference maximising problem (F1). Recall the following result.

LEMMA 1. Let Q be an $r \times n$ matrix. Only one of the following alternatives holds: (i) there exists $z \in R^n$ such that $Qz \geqq 0$, $Qz \neq 0$, (ii) there exists $\beta \in R^r_{++}$ such that $\beta Q = 0$.

Proof. (Gale [9 , Cor. 2, p. 49]).

By lemma 1 there exists $\beta \in R^M_{++}$ such that $\beta Q = 0$. Let $\bar{P}(a) = \beta_a \bar{p}(a)$, $a \in A$. Multiplying (13) by β gives $\beta[\bar{p}(a)(x^i(a) - w^i(a))]_{a\in A} \leqq \beta Q z^i = 0 \iff \sum_{a\in A} \bar{P}(a)(x^i(a) - w^i(a)) \leqq 0$ so that $x^i \in B^i(\bar{P})$.

Suppose $x^i \in B^i(\bar{P})$, let $[\bar{p}(a)(x^i(a) - w^i(a))]_{a\in A} = y^i$. We need to show that there exists z^i such that $Qz^i = y^i$. Let $H_\beta = \{y \in R^M_+ \mid \beta y = 0\}$. The condition $\beta Q = 0$ implies $Q_j(\sigma_t) \in H_\beta$, $j = 1,\ldots,n$, $\sigma_t \in F_t$, $t = 0,\ldots,T-1$. Since $\dim(H_\beta) = M-1$ and since rank $(Q) = M-1$, the columns of Q span H_β. Since $y^i \in H_\beta$ there exists z^i such that $Qz^i = y^i$. Thus $x^i \in \mathscr{B}^i_{z^i}(\bar{p},\bar{q})$ and hence $x^i \in \mathscr{B}^i(\bar{p},\bar{q})$. ∎

Proof of Theorem 2. The contingent commodity equilibrium satisfies A1, A2. It suffices to find spot and futures prices $(\bar{p},\bar{q})$ such that (i) $\mathscr{B}^i(\bar{p},\bar{q}) = B^i(\bar{P})$, $i = 1,\ldots,m$, and (ii) if $\bar{x}^i \in B^i(\bar{P})$, $i = 1,\ldots,m$, satisfy $\sum_{i=1}^{m} (\bar{x}^i - w^i) = 0$ then the $\bar{z}^i$ which are shown to solve $Q\bar{z}^i = [\bar{p}(a)(\bar{x}^i(a) - w^i(a))]_{a\in A}$, $i = 1,\ldots,m$, satisfy $\sum_{i=1}^{m} \bar{z}_i = 0$, since then $[(\bar{x}^1,\bar{z}^1),\ldots,(\bar{x}^m,\bar{z}^m);(\bar{p},\bar{q})]$ satisfies F1-F3.

Let the spot prices be defined by $\bar{P}(a) = \beta_a \bar{p}(a)$, $a \in A$ for any $\beta \in R^M_{++}$ with $\sum_{a\in A} \beta_a = 1$. The following lemma leads to the choice of futures prices.

LEMMA 2. <u>$\mathscr{B}^i(\bar{p},q) \subseteqq B^i(\bar{P})$ if and only if $\beta Q = 0$.</u>

Proof. ($\Rightarrow$) Suppose there exists z^i such that $\beta Q z^i \neq 0$ say $\beta Q z^i < 0$, then $\beta Q(-z^i) > 0$ and there exists $x^i \in \mathscr{B}_{-z^i}(\bar{p},q)$ such that $x^i \notin B^i(\bar{P})$. Thus we must have $\beta Q z^i = 0$ for all z^i. Set $z^i_j(\sigma_t) = 1$, $z^i_k(\sigma_\tau) = 0$ for $k \neq j$, $\sigma_\tau \neq \sigma_t$, then $\beta Q_j(\sigma_t) = 0$. Repeating this for $j=1,\ldots,n$, $\sigma_t \in F_t$, $t = 0,\ldots,T-1$, implies $\beta Q = 0$.
($\Leftarrow$) Immediate. ∎

As explained in section 2, $\beta Q = 0$ is equivalent to the first order system of difference equations (8) which in conjunction with the endpoint condition $\bar{q}_T(a) = \bar{p}(a)$ determines the system of futures prices, say $\bar{q}$. We need to show $\mathscr{B}^i(\bar{p},\bar{q}) \supseteqq B^i(\bar{P})$. This follows by the same argument as in the proof of Theorem 1 from the rank condition on Q.

It remains to show that futures markets clear. The contingent commodity allocation $(\bar{x}^1,\ldots,\bar{x}^m)$ satisfies $\sum_{i=1}^{m} (\bar{x}^i - \bar{w}^i) = 0$. Thus $\bar{z}^i$ which solve $Q\bar{z}^i = [\bar{p}(a)(\bar{x}^i(a) - w^i(a))]_{a\in A}$ satisfy $Q(\sum_{i=1}^{m} \bar{z}^i) = 0$. Since the same $M-1$ linearly independent columns of Q can be used to define each $\bar{z}^i$, $Q(\sum_{i=1}^{m} \bar{z}^i) = 0$ is a linear combination of linearly independent vectors equal to zero. Thus $\sum_{i=1}^{m} \bar{z}^i = 0$. ∎

Proof of Theorem 3. The proof will be broken down into a sequence of lemmas. The first step is a straightforward technical point: we want to show that we only need futures trading in the subset of the first k(F) of the n commodities. To this end, let P be a contingent commodity price vector with $P = (P(a))_{a\in A}$, $P(a) = (P_1(a),\ldots,P_n(a))$. Define the truncation $P^k = (P^k(a))_{a\in A}$ where $P^k(a) = (P_1(a), \ldots, P_k(a))$ for $k < n$. For any subset $\sigma \in A$ define $P(\sigma) = \sum_{a\in\sigma} P(a)$. Consider the the following rank condition on contingent commodity price systems.

RANK CONDITION $\mathscr{R}_k$. Let $k \leq \min(n,M)$. A contingent commodity price system P satisfies the rank condition $\mathscr{R}_k$ if for every collection of k disjoint non-empty subsets $(\sigma^1,\ldots,\sigma^k)$ of A, the vectors $(P^k(\sigma^1),\ldots,P^k(\sigma^k))$ form a linearly independent set.

Remark. For each $k \leq n$ this condition is weaker than the requirement that $(P(a_1),\ldots,P(a_m))$ have rank n and stronger than the requirement that this set have rank k.

On the futures market side consider the following reduced earnings matrix

$$\tilde{Q}(k) = (Q_j(\sigma_t),\ j = 1,\ldots,k,\quad \sigma_t \in F_t,\ t = 0,\ldots,T-1)$$

which gives the earnings vectors from unit trades in only the first k of the n commodities. In view of the following lemma the $\bar{z}^i$ obtained in the proof of theorem 2 can be chosen so that only the first k coordinates of each $\bar{z}_t(\sigma_t)$ are nonzero.

LEMMA 3. Let $k = k(F)$. If $\bar{P}$ satisfies $\mathscr{R}_k$, then rank $(\tilde{Q}(k)) = M - 1$.

Proof. Let $H_{\tilde{Q}(k)} = \{\theta \in R^M \mid \theta\tilde{Q}(k) = 0\}$. Since row rank $(\tilde{Q}(k)) = M - \dim H_{\tilde{Q}(k)}$ we need to show $\dim H_{\tilde{Q}(k)} = 1$ which is equivalent to showing that if $\theta\tilde{Q}(k) = 0$, then $\theta = \alpha\beta$ for some $\alpha \in R$. $\theta\tilde{Q}(k) = 0$ is equivalent to

$$\sum_{a \in \sigma_t} \theta(a)\left(\frac{\bar{P}^k(\sigma_{t+1}(a))}{\beta(\sigma_{t+1}(a))} - \frac{\bar{P}^k(\sigma_t)}{\beta(\sigma_t)}\right) = 0 \quad \forall\, \sigma_t \in F_t, \quad t = 0,\dots,T-1$$

which is equivalent to

$$\sum_{\sigma_{t+1} \subset \sigma_t} \sum_{a \in \sigma_{t+1}} \theta(a)\left(\frac{\bar{P}^k(\sigma_{t+1})}{\beta(\sigma_{t+1})} - \frac{\bar{P}^k(\sigma_t)}{\beta(\sigma_t)}\right) = 0 \quad \forall\, \sigma_t \in F_t, \quad t = 0,\dots,T-1$$

or

$$\sum_{\sigma_{t+1} \subset \sigma_t} \theta(\sigma_{t+1})\left(\frac{\bar{P}^k(\sigma_{t+1})}{\beta(\sigma_{t+1})} - \frac{\bar{P}^k(\sigma_t)}{\beta(\sigma_t)}\right) = 0 \quad \forall\, \sigma_t \in F_t, \quad t = 0,\dots,T-1$$

Using $\bar{P}^k(\sigma_t) = \sum_{\sigma_{t+1} \subset \sigma_t} \bar{P}^k(\sigma_{t+1})$ and $\theta(\sigma_t) = \sum_{\sigma_{t+1} \subset \sigma_t} \theta(\sigma_{t+1})$, this can be rearranged to

$$\sum_{\sigma_{t+1} \subset \sigma_t} \left(\frac{\theta(\sigma_{t+1})}{\beta(\sigma_{t+1})} - \frac{\theta(\sigma_t)}{\beta(\sigma_t)}\right)\bar{P}^k(\sigma_{t+1}) = 0$$

By the rank condition $\mathscr{R}_k$ the set of vectors $(\bar{P}^k(\sigma_{t+1}))_{\sigma_{t+1} \subset \sigma_t}$ is linearly independent so we must have

$$\frac{\theta(\sigma_{t+1})}{\beta(\sigma_{t+1})} = \frac{\theta(\sigma_t)}{\beta(\sigma_t)} \quad \forall\, \sigma_t,\ \forall\, \sigma_{t+1} \subset \sigma_t$$

Choose $a \in A$ and $(\sigma_t)_{t=0}^T$ such that $a \in \sigma_t$ for all t, where $\sigma_0 = A$ and $\sigma_T = \{a\}$. For this choice of σ_t, $t = 0,\dots,T$, solving the above difference equation yields

$$\theta(\sigma_T) = \frac{\prod_{t=0}^{T-1} \beta(\sigma_{t+1})}{\prod_{t=0}^{T-1} \beta(\sigma_t)}\,\theta(\sigma_0)$$

which is equivalent to $\theta(a) = \beta(a)\theta(A)$. This holds for every $a \in A$, so the lemma is proved. ∎

LEMMA 4. Let $\mathcal{M}_k$ denote the set of all contingent commodity price systems satisfying $\mathcal{R}_k$ for $k \leq \min(n,M)$, then $\mathcal{M}_k$ is an open subset of R^{nM}_{++} and $\mathcal{M}^c_k$ is null.

Proof. By induction on k. Let $k = 1$, then $\mathcal{R}_1$ is equivalent to $P^1(\sigma) \neq 0$ for all $\sigma \subset A$ which holds for all $P \in R^{nM}_{++}$. Suppose the lemma is true for $k-1$. Consider the set $\Sigma_k = \{(\sigma^1,\ldots,\sigma^k) \mid \sigma^i \cap \sigma^j = \emptyset,\ \sigma^i, \sigma^j \subset A,\ \sigma^j \neq \emptyset\}$. For each $B \in \Sigma_k$ define the map $\phi_B : \mathcal{M}_{k-1} \to R$ by

$$\phi_B(P) = \det\left(P^k(\sigma^1),\ldots,P^k(\sigma^k)\right)$$

then ϕ_B is a smooth map from the nM-dimensional manifold $\mathcal{M}_{k-1}$ into R and 0 is a regular value of ϕ_B, since $(P^k(\sigma^1),\ldots,P^k(\sigma^k))$ has rank at least $k-1$ for $P \in \mathcal{M}_{k-1}$. By [10, p. 21] $\phi_B^{-1}(0)$ is an $nM-1$-dimensional closed submanifold of $\mathcal{M}_{k-1}$. By definition $\mathcal{M}_k = \mathcal{M}_{k-1} \setminus \bigcup_{B \in \Sigma_k} \phi_B^{-1}(0)$. ∎

For $w = (w^1,\ldots,w^m)$ define $\tilde{w} = \sum_{i=1}^{m} w^i$. We consider normalisations of contingent commodity price vectors such that $P\tilde{w} = 1$.

LEMMA 5. Consider economies $\mathcal{E}$ for which agents' preferences belong to $\mathcal{P}_c^1$. Let $\mathcal{W} \subset R^{nM}_{++}$ be open with $\mathcal{W}^c$ null. There is an open set $\Omega \subset R^{nMm}_{++}$ with Ω^c null, such that for each economy $\mathcal{E}$ with $w \in \Omega$ every contingent commodity equilibrium price system for w with $P\tilde{w} = 1$ satisfies $P \in \mathcal{W}$.

Proof. This is basically a regular economy argument and except for the choice of normalisation of prices follows Dierker [7 , pp. 94-95] closely. Hence we can be brief. Let f^i denote the demand function of agent i. Define $Z : R^{nM}_{++} \times R^{nMm}_{++} \to R^{nM}$ by

$$Z(P,w) = \sum_{i=1}^{m} f^i(P,Pw^i) - \tilde{w}(P\tilde{w})$$

then the following properties hold.

(i) $Z(P,w)=0$ is equivalent to $\sum_{i=1}^{m} f^i(P,Pw^i)=\tilde{w}$ and $P\tilde{w}=1$. Define the equilibrium price set $\Pi(w)=\{P\in R^{nM}_{++}\mid Z(P,w)=0,\ P\tilde{w}=1\}$ then $\Pi(w)\neq 0$ since the preferences lie in $\mathscr{P}_{c^1}$.

(ii) Taking the derivative of Z with respect to the initial endowment of agent 1, $D_{w^1}Z(P,w)$ has rank nM if $P\in\Pi(w)$ so that Z is transversal to $\{0\}$. By the Transversality Theorem [13, p. 79, Thm. 2.7] 0 is a regular value of Z_w for almost every w, where $Z_w(P)\equiv Z(P,w)$.

(iii) By (ii) and the boundary condition satisfied by the f^i, there exists a family of open sets $\mathscr{U}_\alpha\subset R^{nNm}_{++}$, $\alpha\in I$ and C^1 maps $\psi^i_\alpha:\mathscr{U}_\alpha\to R^{nM}_{++}$, $i=1,\ldots,r_\alpha$ such that $r_\alpha<\infty$ and

(a) $\Pi(w)=\{\psi^1_\alpha(w),\ldots,\psi^{r_\alpha}_\alpha(w)\}\quad\forall w\in\mathscr{U}_\alpha,\quad \alpha\in I$

(b) $\left(\bigcup_{\alpha\in I}\mathscr{U}_\alpha\right)^c$ is null

(c) ψ^i_α is a submersion, $i=1,\ldots,r_a$, $a\in I$.

Only (c) needs comment, since it is usually not pointed out in regular economy type arguments. By the implicit function theorem

$$D_{w^1}\psi^i_\alpha(w) = [D_PZ(P,w)]^{-1}D_{w^1}Z(P,w),\ P=\psi^i_\alpha(w)$$

has rank nM since $D_{w^1}Z(P,w)$ has rank nM. Note also that I can be assumed countable without loss of generality.

(iv) Let $F=R^{nM}_{++}\setminus\mathscr{H}$. For each $\alpha\in I$ define $\mathscr{U}'_\alpha=\mathscr{U}_\alpha\setminus\bigcup_{i=1}^{r_\alpha}\psi^{i^{-1}}_\alpha(F)$. Since F is closed in R^{nM}_{++}, $\psi^{i^{-1}}_\alpha(F)$ is closed in $\mathscr{U}_\alpha$ and thus $\mathscr{U}'_\alpha$ is open. By lemma 6 below, each $\psi^{i^{-1}}_\alpha(F)$ is null in $\mathscr{U}_\alpha$. Take I to be countable and define $\Omega=\bigcup_{\alpha\in I}\mathscr{U}'_\alpha$. ∎

The proof of theorem 3 follows from lemmas 3-5 by letting $\mathscr{H}=\mathscr{M}_k$. It remains only to establish the following result; since it does not appear in standard tests we sketch the proof.

LEMMA 6. Let $r \geq s$, $\mathcal{U} \subset R^r$ an open set and $\Phi : \mathcal{U} \to R^s$ a submersion. If $F \subset R^s$ is null, then $\Phi^{-1}(F)$ is null in R^r.

Proof. Define the canonical submersion $\lambda : R^r \to R^s$ by $\lambda(x_1,\ldots,x_r) = (x_1,\ldots,x_s)$. Pick $x \in \mathcal{U}$. By the representation theorem for submersions [10, p. 20] there is an open set $\mathcal{U}'$, $x \in \mathcal{U}' \subset \mathcal{U}$ and a diffeomorphism Ψ of $\mathcal{U}'$ into R^r such that $\Phi(x) = \lambda \circ \Psi(x)$ $\forall$ $x \in \mathcal{U}'$. Thus

$$\Phi^{-1}\big|_{\mathcal{U}'}(F) = \Psi^{-1}(\lambda^{-1}(F)) = \Psi^{-1}(F \times R^{r-s})$$

$F \times R^{r-s}$ is null in R^r since F is null in R^s and since Ψ^{-1} is smooth, Ψ^{-1} preserves measure 0. Applying the Lindelöf principle, we get $\Phi^{-1}(F)$ is null. ∎

5. PROOFS FOR SECTION 3

Proof of Theorem 4. We want to show that a contingent commodity equilibrium exists in which p is $\mathcal{H}$-measurable and each χ^i is $\mathcal{H}^i$-measurable. In this case the finite-dimensional vectors $\mathbf{p} = (p(H_a))_{a \in A}$, $x^i = (E(\chi^i \mid H^i_{as}))_{a \in A,\ s \in S}$ serve to define p and χ^i respectively. When p is $\mathcal{H}$-measurable the budget constraint in (A1)* reduces to

$$\sum_{a \in A} \sum_{s \in S} \mathbf{p}(a)\pi^i_{as}(x_{as} - w^i_{as}) \leqq 0, \qquad i \in I$$

where $\pi^i_{as} = \pi(H^i_{as})$. Thus if χ if affordable to agent i so is $E(\chi \mid \mathcal{H}^i)$. In view of (10) a $\mathcal{H}^i$-measurable χ will always be chosen by each agent. We now proceed through a series of steps to calculate mean demand for an economy $\mathcal{E}^*$.

(i) Let π' denote a typical element of Δ_A defined in 3.2. For each $(|\succsim|, w, \pi') \in K$ let $f(|\succsim|, w, \pi'; \mathbf{p}) = (f_{as}(|\succsim|, w, \pi'; \mathbf{p}))_{a \in A,\ s \in S}$ be the $|\succsim|$ maximal element of the budget set

$$B(\mathbf{p}) = \{x \in R^{nMN}_+ \mid \sum_{a \in A} \sum_{s \in S} \mathbf{p}(a)\pi'_{as}(x_{as} - w_{as}) \leqq 0\}$$

This defines f as a continuous map $f: K \times R^{nM}_{++} \to R^{nMN}_{+}$, the proof of continuity being a variation of the argument given by Hildenbrand [12].

(ii) For each $((\succsim), \bar{w}, \pi') \in L$ and $\mathbf{p} \in R^{nM}_{+}$ let $\bar{f}((\succsim), \bar{w}, \pi'; \mathbf{p}) = \left(\bar{f}_a((\succsim), \bar{w}, \pi'; \mathbf{p})\right)_{a \in A}$ be the $(\succsim)$ maximal element in the budget set

$$\bar{B}(\mathbf{p}) = \{\bar{x} \in R^{nM} \mid \sum_{a \in A} \mathbf{p}(a)(\bar{x}_a - \bar{w}_a) \leq 0\}$$

This defines $\bar{f}$ as a continuous map $\bar{f}: L \times R^{nM}_{++} \to R^{nM}_{+}$. Since $(\succsim) \in \mathscr{P}_c 1$, $\bar{f} \in \mathscr{C}^1$ in $\mathbf{p}$ and satisfies the boundary condition stated in section 2.

(iii) Consider $(|\succsim|, w, \pi') \in K$ and the corresponding S-average characteristic $((\succsim), \bar{w}, \pi') \in L$. Then

$$\bar{f}_a((\succsim), \bar{w}, \pi'; \mathbf{p}) = \sum_{s \in S} \pi'_{as} f_{as}(|\succsim|, w, \pi'; \mathbf{p})$$

(This is just Hick's composite good theorem.)

(iv) Consider agent i's demand in state (a, δ) and with price system $\mathbf{p}$

$$\phi^i_{a\mathbf{p}}(\delta) = f_{a\delta_i}(|\underset{i}{\succsim}|, w^i, \pi^i; \mathbf{p})$$

Since the marginal probability on $\prod_{i \in I} S_i$ is a product measure, the family of random variables $\{\phi^i_{a\mathbf{p}}\}_{i \in I}$ is independent on $\prod_{i \in S} S_i$. This family is also uniformly bounded by (i)

$$\sup \|\phi^i_{a\mathbf{p}}(\delta)\| \leq \sup_{(|\succsim|, w, \pi') \in K} \max_{s \in S} \|f_{as}(|\succsim|, w, \pi'; \mathbf{p})\| < \infty$$

Thus Kolmogorov's law of large numbers applies [2],

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} [\phi^i_{a\mathbf{p}}(\delta) - E(\phi^i_{a\mathbf{p}})] = 0 \quad \text{a.s.} \qquad a \in A$$

Define $\omega^i_a(\delta) = w^i_{a\delta_i}$. Then $\{\omega^i_a\}_{i \in I}$ is an independent uniformly bounded family of random variables. Thus

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} [\omega_a^i(s) - E(\omega_a^i)] = 0 \quad \text{a.s.} \quad a\in A$$

(v) Recall that $\pi_a^A = \pi(H_a)$, $a\in A$. Since $E(\phi_{a\mathbf{p}}^i) = \frac{1}{\pi_a^A} \sum_{s\in S} \pi_{as}^i f_{as}(|\underset{i}{\succsim}|, w^i, \pi^i; \mathbf{p}) =$ $\frac{1}{\pi_a^A} \bar{f}_a((\underset{i}{\succsim}), \bar{w}^i, \pi^i; \mathbf{p})$ and $E(\omega_a^i) = \frac{1}{\pi_a^A} \sum_{s\in S} \pi_{as}^i w_a^i = \frac{1}{\pi_a^A} \bar{w}_a^i$, if we let $g^i = ((\underset{i}{\succsim}), \bar{w}^i, \pi^i)$, then

$$\frac{1}{m} \sum_{i=1}^{m} E(\phi_{a\mathbf{p}}^i) = \frac{1}{m} \frac{1}{\pi_a^A} \sum_{i=1}^{m} \bar{f}_a(g^i; \mathbf{p}) = \frac{1}{\pi_a^A} \int_L \bar{f}_a(g;\mathbf{p}) \mu_m(dg)$$

$$\frac{1}{m} \sum_{i=1}^{m} E(\omega_a^i) = \frac{1}{m} \frac{1}{\pi_a^A} \sum_{i=1}^{m} \bar{w}_a^i = \frac{1}{\pi_a^A} \int_L \bar{w}_a \, d\mu_m$$

Since by assumption there exists $\mu\in\mathcal{M}(L)$ such that $\mu_m \xrightarrow[w]{} \mu$ as $m\to\infty$ and since $\bar{f}_a$ is continuous and bounded on L

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} E(\phi_{a\mathbf{p}}^i) = \frac{1}{\pi_a^A} \int_L \bar{f}_a(g;\mathbf{p}) \mu(dg)$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} E(\omega_a^i) = \frac{1}{\pi_a^A} \int_L \bar{w}_a \, d\mu$$

Thus in view of (iv)

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \phi_{a\mathbf{p}}^i(s) = \frac{1}{\pi_a^A} \int_L \bar{f}_a(g;\mathbf{p}) \mu(dg) \quad \text{a.s.}$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \omega_a^i(s) = \frac{1}{\pi_a^A} \int_L \bar{w}_a \, d\mu \quad \text{a.s.}$$

(vi) Let $Z_a(\mathbf{p},\mu) = \int_L \bar{f}_a(g;\mathbf{p})\mu(dg) - \int_L \bar{w}_a \, d\mu$, $a\in A$, $Z(\mathbf{p},\mu) = (Z_a(\mathbf{p},\mu))_{a\in A}$ and consider the induced correspondence $\Pi : \mathcal{M}(L) \to R_{++}^{nM}$ defined by

$$\Pi(\mu) = \left\{\mathbf{p}\in R_{++}^{nM} \mid Z(\mathbf{p},\mu) = 0, \ \sum_{a\in A} \mathbf{p}(a) \int_L \bar{w}_a \, d\mu = 1\right\}$$

It follows from Hildenbrand [12] that $\Pi(\mu) \neq \emptyset$, $\forall\, \mu\in\mathcal{M}(L)$ and is upper hemicontinuous on $\mathcal{M}(L)$. Consider any $\mathbf{p} \in \Pi(\mu)$ and define χ^i by $\chi^i(a,s) = f_{as_i}(|\succsim|, w^i, \pi^i; \mathbf{p})$, $(a,s) \in \Sigma$, then $[(\chi^i)_{i \in I}; \mathbf{p}]$ is a basic contingent commodity equilibrium. ■

Proof of Theorem 5. The futures-insurance market equilibrium satisfies (F1)*-(F4)*. Since the spot price $\tilde{p}$ is $\mathcal{H}$-measurable, the constraint

$$\tilde{p}(a,s)\big(\tilde{\chi}(a,s) - \omega^i(a,s)\big) \leqq \nu^i(a,s) + R(z^i,a), \qquad (a,s) \in \Sigma$$

in equation (11) reduces to the finite-dimensional condition

$$\tilde{p}(a)(\tilde{x}_{as} - w^i_{as}) \leqq v^i_{as} + R(z^i,a), \qquad (a,s) \in A \times S \tag{14}$$

Let $R(z^i) = \big(R(z^i,a)\big)_{a \in A} = Qz^i$, then as in the proof of theorem 1 we cannot find z^i such that $Qz^i \geqq 0$, $Qz^i \neq 0$ so there must exist $\beta \in R^M_{++}$ such that $\beta Q = 0$. Multiply (14) by π^i_{as}, sum over s, use the fact that $\sum_{s \in S} \pi^i_{as} v^i_{as} = 0$; then multiply by β_a to get

$$\beta_a \tilde{p}(a) \sum_{s \in S} \pi^i_{as}(\tilde{x}_{as} - w^i_{as}) \leqq \beta_a \pi(H_a) R(z^i,a), \; a \in A \tag{15}$$

Define
$$\pi(H_a)\tilde{\mathbf{p}}(a) = \beta_a \tilde{p}(a), \qquad a \in A \tag{16}$$

then (15) reduces, by summing over a, to

$$\sum_{a \in A} \sum_{s \in S} \tilde{\mathbf{p}}(a) \pi^i_{as}(\tilde{x}_{as} - w^i_{as}) \leqq 0$$

so that $\tilde{x}$ lies in the contingent commodity budget set. Thus $\mathscr{B}^i(\tilde{p}, \tilde{q}, \pi^i) \subseteqq C^i(p)$ when contingent commodity prices are defined by (16). To show the opposite inclusion define

$$v^i_{as} = \tilde{p}(a)\Big[\Big(\tilde{x}_{as} - \frac{1}{\pi^A_a} \sum_{s \in S} \pi^i_{as} \tilde{x}^i_{as}\Big) - \Big(w^i_{as} - \frac{1}{\pi^A_a} \sum_{s \in S} \pi^i_{as} w^i_{as}\Big)\Big]$$

$$\tilde{y}^i_a = \tilde{p}(a) \frac{1}{\pi^A_a} \sum_{s \in S} \pi^i_{as}(\tilde{x}_{as} - w^i_{as}), \qquad \tilde{y}^i = (\tilde{y}^i_a)_{a \in A}$$

then $\sum_{s\in S} \pi^i_{as}\tilde{v}^i_{as} = 0$ and by the rank condition there exist $\tilde{z}^i$ such that $Q\tilde{z}^i = \tilde{y}^i$ and the proof is complete. ∎

Proof of Theorem 6. Modify the proof of theorem 2 in the natural way.

Proof of Theorem 7. By theorem 6 given a basic contingent commodity equilibrium $[(\tilde{\chi}^i)_{i\in I}; \tilde{p}]$ if the induced finite-dimensional price system $\tilde{\mathbf{p}} = (\tilde{p}(H_a))_{a\in A}$ satisfies $\mathbf{p}\in\mathcal{M}_k$, then the induced matrix Q has rank $M-1$ and $(\tilde{\chi}^i)_{i\in I}$ is achievable through a futures-insurance equilibrium. Thus to complete the proof it suffices to show that the set

$$\Gamma = \{\mu\in\mathcal{M}(L) \mid \Pi(\mu)\subset\mathcal{M}_k\}$$

is an open dense set in $\mathcal{M}(L)$. That Γ is open follows from the above mentioned upper hemicontinuity of Π. The density of Γ is a consequence of lemma 5 and the fact that the μ with finite support are dense in $\mathcal{M}(L)$. To establish the density of Γ we must show that for each $\mu\in\mathcal{M}(L)$ and each neighborhood $\mathcal{U}(\mu)$ of μ there exists μ' such that $\mu'\in\Gamma\cap\mathcal{U}(\mu)$. To this end choose μ" with finite support $(g^1,\ldots,g^\ell)$ such that $\mu''\in\mathcal{U}(\mu)$ and $\mu''(\{g^i\}) = \frac{r_i}{m}$ where r_i and m are integers. Then the economy corresponding to μ" is equivalent to an economy $\mathcal{E}$ with r_i consumers with characteristics g^i, $i = 1,\ldots,\ell$. Let $w\in R^{nMm}_{++}$ denote the initial endowment vector for this economy. By lemma 5 an arbitrarily small perturbation of w will yield an economy with endowment vector w' with associated distribution μ' such that $\mu'\in\Gamma\cap\mathcal{U}(\mu)$. ∎

APPENDIX A

EXAMPLE OF NONEXISTENCE OF FUTURES MARKET EQUILIBRIUM

Oliver Hart [11] provided an example of the nonexistence of a futures market equilibrium in a model with two commodities, two consumers and two states of nature. His model differs from ours, however, in that he assumes that agents pay for futures contracts at the time of purchase, rather than at the time of delivery. Because of this, in his example the requirement $qz^i = 0$ is imposed on agents at time 0 (in our notation). With payment at time of delivery this constraint is not required, and without this constraint the Hart example has an equilibrium; in fact it is not difficult to show that with only two states of nature in our model, a futures market equilibrium will always exist. As we show below, however, equilibria may fail to exist as soon as there are three states of nature.

The idea in Hart's example is as follows: agents have von Neumann-Morgenstern preferences and the aggregate endowment is the same in each state. This ensures that the prices in a contingent commodity equilibrium will be collinear across states. Agents' utility functions and endowments differ enough however to ensure that in a pure spot market equilibrium without futures markets, the spot prices are linearly independent. In a futures market equilibrium only two cases can arise; either spot prices are linearly dependent or they are linearly independent. In the first case, since payment is made at date zero and since there are only two states and two commodities, trading in futures achieves no additional spanning: the equilibrium must thus be a pure spot market equilibrium; but in this case spot prices are linearly independent (a contradiction). If spot prices are linearly independent then futures trading achieves complete spanning and the equilibrium is equivalent to a contingent commodity equilibrium — but in such an equilibrium prices are collinear (a contradiction).

We construct an example with three agents, three commodities and three states of nature with $T=1$ so that $k(F)=3=n$. We use the same idea as in Hart's example: constant aggregate endowment (no aggregate risk), sufficiently different individual preferences and endowments. However now three possible cases can arise: a pure spot market equilibrium, an inefficient futures market equilibrium and an equilibrium equivalent to a contingent market equilibrium. The argument needs to show that none of these cases can arise.

Assume each agent i has a log-linear von Neumann-Morgenstern utility function

$$u^i(x_1^i,x_2^i,x_3^i) = \sum_{j=1}^{3} \alpha_j^i \ln x_j^i, \qquad \sum_{j=1}^{3} \alpha_j^i = 1, \qquad \alpha_j^i > 0$$

and define $\alpha^i = (\alpha_1^i,\alpha_2^i,\alpha_3^i)$, for $i=1,2,3$. We will assume that $\{\alpha^1,\alpha^2,\alpha^3\}$ is a linearly independent set of vectors (preferences differ). Let $A=\{a_1,a_2,a_3\}$ be the set of states of nature, and suppose each a_k has a probability of one-third. $w^i(a_k)$ denotes agent i's endowment vector in state k. Assume that $\sum_{i=1}^{3} w^i(a_k) = (1,1,1)$ for each a_k, and that $w^i(a_i) \gg w^i(a_k)$ for $k \neq i$, $i=1,2,3$ (endowments differ). For example, one could take $w^i(a_i) = (1-2\varepsilon,\ 1-2\varepsilon,\ 1-2\varepsilon)$ and $w^i(a_k) = (\varepsilon,\varepsilon,\varepsilon)$ for $i \neq k$. For shorthand denote agent i's income in state a_k as $M^i(a_k,z^i) = p(a_k)w^i(a_k) + z^i(p(a_k)-q)$. We construct some equations that must be satisfied by equilibrium futures market prices $(p(a_1),p(a_2),p(a_3),q)$ and futures contracts (z^1,z^2,z^3), and then show they have no solution.

The demand functions for agent i can be obtained by first maximising u^i subject to $p(a_k)x^i(a_k) = M^i(a_k,z^i)$ with z^i arbitrary, which yields $x_j^i(a_k) = \alpha_j^i \dfrac{M^i(a_k,z^i)}{p_j(a_k)}$, and then inserting this solution into u^i and maximising $E\,u^i$ with respect to z^i, which requires $\sum_{k=1}^{3} \dfrac{(p(a_k)-q)}{M^i(a_k,z^i)} = 0$. For supply to be equal to demand we need $\sum_{i=1}^{3} x_j^i(a_k) = \sum_{i=1}^{3} w_j^i(a_k) = 1$, or equivalently, $\sum_{i=1}^{3} \alpha_j^i \dfrac{M^i(a_k,z^i)}{p_j(a_k)} = 1 \iff$

$p_j(a_k) = \sum_{i=1}^{3} \alpha_j^i M^i(a_k,z^i)$ and $p_j(a_k) > 0$. Thus in order for $(p(a_1), p(a_2), p(a_3), q)$ to be equilibrium prices the $p(a_k)$ must be strictly positive and there must be a choice of (z^1,z^2,z^3) such that

$$p(a_k) = \sum_{i=1}^{3} \alpha^i M^i(a_k,z^i) \quad \forall\ k \tag{1}$$

$$\sum_{k=1}^{3} \frac{(p(a_k)-q)}{M^i(a_k,z^i)} = 0 \quad \forall\ i \tag{2}$$

Let $(p,1)$ denote the 4-vector obtained by appending 1 to the 3-vector p. We will consider the set of vectors $\{(p(a_1),1), (p(a_2),1), (p(a_3),1)\}$. It is easy to check that either this set is linearly independent, or there is an index $\bar{k}$ such that $p(a_{\bar{k}})$ is a convex combination of the remaining $p(a_k)$.

Suppose $(p(a_k), 1)_{k=1,2,3}$ is linearly independent. Define $\theta_i = \sum_{k=1}^{3} \frac{1}{M^i(a_k,z^i)}$. Then (2) is equivalent to $q = \sum_{k=1}^{3} \frac{p(a_k)}{\theta_i M^i(a_k,z^i)}$ for all i. Since $\sum_{k=1}^{3} \frac{1}{\theta_i M^i(a_k,z^i)} = 1$, we get

$$(q,1) - (q,1) = \sum_{k=1}^{3} \left(\frac{1}{\theta_1 M^1(a_k z^1)} - \frac{1}{\theta_i M^i(a_k,z^i)}\right)(p(a_k), 1) = 0$$

for $i = 2,3$. By linear independence we must have $\theta_i M^i(a_k,z^i) = \theta_1 M^1(a_k,z^1)\ \forall\ i$, $\forall\ k$. Then by (1), $p(a_k) = \sum_{i=1}^{3} \alpha^i \frac{\theta_1}{\theta_i} M^1(a_k,z^1) = \theta_1 M^1(a_k,z^1) \sum_{i=1}^{3} \alpha^i \frac{1}{\theta_i}$ for all k, so that the $p(a_k)$ are collinear. This would imply, however, that $(p(a_k), 1)_{k=1,2,3}$ has rank at most 2, a contradiction. (This is the part of the argument which requires more that 2 states.)

Suppose $p(a_3) = \beta p(a_1) + (1-\beta)p(a_2)$, $0 \leqq \beta \leqq 1$. By (2) one can find a τ such that $q = \tau p(a_1) + (1-\tau)p(a_2)$. By (1) $p(a_3) = \sum_{i=1}^{3} \alpha^i M^i(a_3,z^i) = \sum_{i=1}^{3} \alpha^i(\beta M^i(a_1,z^i) + (1-\beta)M^i(a_2,z^i))$.

Since $\{\alpha^i\}_{i=1,2,3}$ is linearly independent, we must have $M^i(a_3,z^i) = \beta M^i(a_1,z^i) + (1-\beta)M^i(a_2,z^i)$ for $i = 1,2,3$. Using the definition of M^i and substituting the expressions for $p(a_3)$ and q in terms of $p(a_1)$ and $p(a_2)$ yields $\beta p(a_1)(w^i(a_3) - w^i(a_1)) + (1-\beta)p(a_2)(w^i(a_3) - w^i(a_2)) = 0 \ \forall\ i$. But for $i = 3$ we have $w^3(a_3) - w^3(a_1) \gg 0$ and $w^3(a_3) - w^3(a_2) \gg 0$, a contradiction. Note that by our choice of the $w^i(a_k)$ this argument would work regardless of which $p(a_k)$ is a convex combination of the remaining spot price vectors.

APPENDIX B

MULTIPERIOD SPOT MARKETS AND MATURITY DATES

In the previous sections we have assumed that spot markets meet only at date T and that all futures contracts mature at date T. In this appendix we indicate how the results of the paper can be extended to the case where there are spot markets at a whole set of dates on the interval $[0,T]$ and where futures contracts can mature at any one of these dates. For simplicity we deal with the pure futures market case. Let A be the set of states of nature and $F = \{F_t\}_{t=0}^{T}$ a filtration, as before. Let $U \subset \{0,1,2,\ldots,T\}$ be the set of dates at which spot markets will be active. We require $\{0,T\} \subset U$, and define $U_0 = U \setminus \{0\}$. Let $D = \{(u,\sigma_u) \mid u \in U \text{ and } \sigma_u \in F_u\}$ be the set of date-event pairs at which spot trades can take place. The consumption set for each agent is $X = \{x : D \to R^n_+\}$. Each agent is characterised by a preference ordering $\underset{i}{\succsim}$ on X and an endowment vector $w^i \in X$. A contingent commodity equilibrium is a pair $(\bar{x}^1,\ldots,\bar{x}^m ; P)$ consisting of a consumption bundle $\bar{x}^i$ for each agent and a price system $P \in X$ such that $\sum_{i=1}^{m} \bar{x}^i(d) = \sum_{i=1}^{m} w^i(d) \ \forall\ d \in D$, and such that $\bar{x}^i$ is the $\underset{i}{\succsim}$ maximal element satisfying the budget equation $\sum_{d \in D} P(d)(\bar{x}^i(d) - w^i(d)) \leqq 0$.

To describe futures markets, first define $D_u = \{(t,\sigma_t) \mid t \leqq u \text{ and } \sigma_t \in F_t\}$, for $u \in U$. Define $X_u = \{x : D_u \to R^n\}$ and $X_u^+ = \{x : D_u \to R^n_+\}$. Then a futures market system can be described by a system of spot prices $p \in X$, systems $q_u \in X_u^+$ of prices for delivery at time u, and futures contracts $z_u^i \in X_u$ for delivery at date u. Let $\sigma_t(\sigma_u)$, for $t \leqq u$, denote the unique $\sigma_t \in F_t$ which contains $\sigma_u \in F_u$. As before we assume each agent at time $t+1$ closes out his position taken at time t. Then agent i's income from futures trading at date $u \in U_0$ and event $\sigma_u \in F_u$ is:

$$R_u(z_u^i,\sigma_u) = \sum_{t=0}^{u-1} z_u^i(\sigma_t(\sigma_u))[q_u(t+1, \sigma_{t+1}(\sigma_u)) - q_u(t, \sigma_t(\sigma_u))] \tag{1}$$

Since all payments are made at time of delivery, there is no way to alter income across time periods unless an additional security is used which allows such a transfer of income. To keep things simple assume agents can save and borrow freely at a zero interest rate. Let S_u^i denote agent i's savings in period u. For $u \in U_0$ define u_{-1} to be the spot market date immediately preceding u. We will constrain agents to satisfy $S_T^i = 0$. Then agent i's system of budget constraints is

$$p(0,\sigma_0)(x^i(0,\sigma_0) - w^i(0,\sigma_0)) + S_0^i \leqq 0$$

$$p(u,\sigma_u)(x^i(u,\sigma_u) - w^i(u,\sigma_u)) + S_u^i \leqq S_{u_{-1}}^i + R_u(z_u^i,\sigma_u) \quad \forall\ (u,\sigma_u) \in D, \quad u \neq 0 \quad (2)$$

$$S_T = 0$$

A futures market equilibrium is a pair $[(x^1,z_u^1,S_u^1)_{u \in U},\ldots,(x^m,z_u^m,S_u^m)_{u \in U};$ $(p,q_u)_{u \in U_0}]$ such that a) all markets clear: $\sum_{i=1}^m \bar{x}^i(d) = \sum_{i=1}^m w^i(d)\ \forall\ d \in D$, $\sum_{i=1}^m z_u^i(d) = 0\ \forall\ d \in D_u\ \forall\ u \in U_0$, $\sum_{i=1}^m S_u^i = 0\ \forall\ u \in U$; and b) $\bar{x}^i$ is the $\underset{i}{\succsim}$ maximal element for the budget system (2).

We sketch how the arguments in section 2 can be extended. Let $(\bar{x}^1,\ldots,\bar{x}^m;P)$ be a contingent commodity equilibrium. The idea is to use P to define spot and futures prices, and to show that at these prices there is a futures market equilibrium in which $(\bar{x}^1,\ldots,\bar{x}^m)$ is the commodity allocation. The crucial part of the argument is to show that there exists (z_u^i,S_u^i) which makes $\bar{x}^i$ affordable for each agent, so we will concentrate on this argument. By the same type of genericity argument as in the paper, we can assume P satisfies $\mathscr{R}_k$ for any $k \leq n$ (with D taking the place of A in the definition on page 22). If $k(F) \leq n$, then this implies that for each $u \in U_0$, $(P(u,\sigma_u))_{\sigma_u \in F_u}$ satisfies $\mathscr{R}_k$, $k = k(\{F_t\}_{t=0}^u)$, with F_u replacing A in the definition of $\mathscr{R}_k$. Now define p and q_u by first picking $\beta(a)$ such that $\sum_{a \in A} \beta(a) = 1$ and $\beta(a) > 0$, and then set

$$p(u,\sigma_u) = \frac{1}{\beta(\sigma_u)} P(u,\sigma_u)$$

$$q_u(t,\sigma_t) = \frac{1}{\beta(\sigma_t)} \sum_{\sigma_u \subset \sigma_t} P(u,\sigma_u)$$

Note that, for each u, this is exactly the same definition as in the body of the paper, if we replace A by F_u. As before we can write the vector $(R_u(z_u^i,\sigma_u))_{\sigma_u \in F_u}$ as a matrix vector product $Q_u z_u^i$, and with the rank condition $\mathcal{R}_k$ satisfied the image of the linear transformation defined by Q_u spans the space $H_u = \{y \in R^{\#F_u} \mid \sum_{\sigma_u \in F_u} \beta(\sigma_u) y_{\sigma_u} = 0\}$. We need now to show that there exists (z_u^i, S_u^i) such that the contingent commodity $\bar{x}^i$ satisfies the budget system (2). First, define S_u^i recursively as follows:

$$S_0^i = -p(0,\sigma_0)(\bar{x}^i(0,\sigma_0) - w^i(0,\sigma_0))$$

$$S_u^i = S_{u_{-1}}^i + \sum_{\sigma_u \in F_u} \beta(\sigma_u) p(u,\sigma_u)(\bar{x}^i(u,\sigma_u) - w^i(u,\sigma_u)), \quad u > 0$$

It is easy to check that $S_T^i = 0$ and that $\sum_{i=1}^m S_u^i = 0$, and that the vectors $y^i \in R^{\#F_u}$, defined by $y_{\sigma_u}^i = p(u,\sigma_u)(\bar{x}^i(u,\sigma_u) - w^i(u,\sigma_u)) + S_u - S_{u_{-1}}$, satisfy $y^i \in H_u$. Thus the equations $Q_u z_u^i = y^i$ have solutions z_u^i such that $\sum_{i=1}^m z_u^i = 0$, using the same argument as on page 22.

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