STABILIZING COMPETITIVE BUSINESS CYCLES

Jean-Michel GRANDMONT

N° 8518

August 1985

A B S T R A C T

Grandmont, Jean-Michel — Stabilizing Competitive Business Cycles

This paper studies how stylized monetary (proportional money transfers), fiscal (lump sum money transfers) and public expenditure policies may eliminate endogenous deterministic cycles and stationary Markov sunspot equilibria in a simple version of the overlapping generations model. J. Econ. Theory,

(English). CEPREMAP, Paris, France.

Journal of Economic Literature Classification Numbers : 021, 022, 023, 131, 311, 321, 322

-=-=-=-=-=-=-=-=-=-=-

R E S U M E

On étudie dans cet article comment des politiques stylisées monétaires (transferts monétaires proportionnels), fiscales (transferts monétaires forfaitaires) et/ou budgétaires (dépenses publiques) peuvent éliminer les cycles déterministes endogènes et les équilibres avec taches solaires dans une version simple du modèle à générations imbriquées.

STABILIZING COMPETITIVE BUSINESS CYCLES

by

Jean-Michel GRANDMONT *

Many studies have shown that a competitive economy of which the characteristics are deterministic and stationary may display highly complex endogenous fluctuations under laisser faire, be they deterministic cycles (Benhabib and Day [3], Benhabib and Nishimura [4], Geanakoplos and Polemarchakis [7], Grandmont [9], Woodford [15]) or stochastic sunspots equilibria (Shell [14], Azariadis [1], Cass and Shell [6], Azariadis and Guesnerie [2], Woodford [15], Grandmont [12]). The present paper, which is a companion of Grandmont [9], studies how stylized monetary (proportional money transfers), fiscal (lump sum money transfers) and budgetary (public expenditures) Government policies may help reducing the cyclical properties of the economy in a simple version of the overlapping generations model.

The central features of the model are presented in Section 1. We analyse in Sections 2 and 3 how variations of predetermined rates of growth of the money supply through the three above mentioned channels affect the "cyclical" properties of the economy, i.e. how they influence the set of deterministic cycles with perfect foresight (Section 2) and the set of stationary Markov sunspot equilibria (Section 3). Predetermined proportional money transfers are found of course to be superneutral, i.e. they do not alter the set of these equilibria. By contrast, increases of the predetermined (and constant over time) money growth rate through lump sum transfers and/or through public expenditures are found to have a stabilizing impact : they eliminate altogether, if large enough, all cycles with perfect foresight that have a period k $\geq$ 2 (section 2). We extend in Section 3 an interesting result obtained recently by Azariadis and Guesnerie [2] in the case of laisser faire for two-states Markov sunspot equilibria, to the case at hand : stationary Markov sunspot equilibria (involving an arbitrary number of states) exist in the present model if and only if there is a "noncritical" period 2 cycle. Thus changes of the money growth rate through lump sum transfers and/or public expenditures that eliminate nontrivial cycles do eliminate stationary Markov sunspot equilibria as well.

Predetermined changes of the money growth rate through these channels, although they may succeed in reducing the cyclical properties of the economy, introduce inefficiencies of their own. Also, they do not guarantee the disappearance of the forecasting mistakes that the agents may make. We show in Section 4, in the same spirit as in Grandmont [9, Section 6], how preannounced and deterministic countercyclical policy rules pin down expectations by changing the informational content of prices, force accordingly the economy to follow a forward sequence of temporary equilibria with perfect foresight and to converge to the Golden Rule stationary state.

1. SHORT RUN EQUILIBRIUM

This section is devoted to the presentation of the assumptions of the model and of the relations determining a competitive equilibrium at any date.

We consider the simple specification of the overlapping generations model, without bequests, in which production is taken for simplicity as exogenous, that was used in Grandmont [9]. The model involves one nonstorable good and a single asset, "money", that is employed for transferring wealth from one period to the next. Agents live two periods and are identical (equivalently there is a single agent) in each generation. The agents' endowments of the good at each age τ = 1,2 are

$$\ell_1^* > 0,\ \ell_2^* > 0 \tag{1.a}$$

The agents' tastes among consumption streams $a_1 \geq 0$, $a_2 \geq 0$ are described by the separable utility function $V_1(a_1) + V_2(a_2)$ with

<u>For each</u> $\tau = 1,2$, $V_\tau(a_\tau)$ <u>is continuous on</u> $[0,+\infty)$ <u>and twice continuously differentiable on</u> $(0,+\infty)$. <u>Moreover</u>, $V'_\tau(a_\tau) > 0$, $V''_\tau(a_\tau) < 0$ <u>for</u> $a_\tau > 0$, <u>and</u> $\lim_{a_\tau \to 0} V'_\tau(a_\tau) = +\infty$, $\lim_{a_\tau \to +\infty} V'_\tau(a_\tau) = 0$. (1.b)

We focus on the <u>Samuelson case</u>, in which autarchy is inefficient, that is

$$\bar{\theta} = V'_1(\ell^*_1)/V'_2(\ell^*_2) < 1 \qquad (1.c)$$

We assume that there is in addition a "Government" that has three available instruments. At date t , it makes a money transfer that is proportional to the money holding of the old agent, at the rate $x_t - 1$ (<u>nominal interest payments</u>). It gives also a <u>lump sum</u> money subsidy S_t (to be interpreted as a lump sum tax if negative) to the old agent and issues the amount of money G_t when purchasing (or selling) some quantity of the good [(1)]. In the sequel it will be convenient to consider as the Government's instrumental variables the <u>rates of growth</u> of the money supply that are attributable to lump sum transfers and to the deficit $D_t = S_t + G_t$, excluding interest payments, i.e.

$$s_t = (M_{t-1}\, x_t + S_t) \,/\, M_{t-1}\, x_t \qquad (1.1)$$

$$d_t = (M_{t-1}\, x_t + S_t + G_t) \,/\, M_{t-1}\, x_t$$

where $M_{t-1} > 0$ and $M_t > 0$ designate the money stocks at the outset and the end of period t . The evolution of the money supply is then ruled by

$$M_t = M_{t-1} x_t d_t \quad , \quad M_o > 0 \quad \underline{given} \tag{1.2}$$

It will be assumed for all t

$$x_t > 0 \; , \; d_t > 0 \; , \; s_t > 0 \tag{1.d}$$

The assumption $s_t > 0$ is made for analytical convenience and means that lump sum taxes do not wipe out the old agent's money balance. Conditions (1.a) through (1.d) will be assumed to hold without any further reference.

A newborn trader has to solve the following decision problem. Let $p > 0$ be the current money price of the good, and let $p^e > 0$, $x^e > 0$ and Δm^e stand for the price of the good, the proportional money transfer and the lump sum subsidy that he expects (with subjective certainty) for the next date, respectively. The consumer must choose then his current and future consumptions $a_1 \geq 0$, $a_2 \geq 0$, and his money demand $m \geq 0$ so as to maximize $V_1(a_1) + V_2(a_2)$ subject to

$$pa_1 + m = p\ell_1^* \qquad \text{and} \qquad p^e a_2 = p^e \ell_2^* + mx^e + \Delta m^e \tag{1.3}$$

The solution of this problem depends only on $\theta = px^e/p^e$ and on $\sigma = \Delta m^e/p^e$. The problem has a solution, which is unique, if and only if $\theta\ell_1^* + \ell_2^* + \sigma \geq 0$. In that case, we denote the optimum excess demands for the good $a_\tau - \ell_\tau^*$ as $z_\tau(\theta,\sigma)$. They satisfy

$$\theta z_1(\theta,\sigma) + z_2(\theta,\sigma) - \sigma \equiv 0 \tag{1.4}$$

On the other hand, money demand is given by

$$m^d(p, p^e/x^e, \sigma) \equiv -p\, z_1(\theta, \sigma) \tag{1.5}$$
$$\equiv (p^e/x^e)[z_2(\theta, \sigma) - \sigma]$$

An important point to note is that when $\theta \ell_1^* + \ell_2^* + \sigma > 0$, and when money demand is positive, it is the unique value of m that satisfies the first order condition

$$V_1'(\ell_1^* - \frac{m}{p}) = \theta\, V_2'(\ell_2^* + \frac{mx^e}{p^e} + \sigma) \tag{1.6}$$

In the limiting case $\theta \ell_1^* + \ell_2^* + \sigma = 0$, money demand $m = p\ell_1^*$ may be viewed also as the unique solution of (1.6) although the latter reads then $+\infty = +\infty$.

The equilibrium condition at date t is now easy to state. The excess demand for the good of the old trader and of the Government is equal to the real value M_t/p_t of the money supply, in which $p_t > 0$ is the current money price of the good. Competitive equilibrium of the good market at date t reads then, with obvious notations

$$z_1(p_t\, x^e_{t+1}/p^e_{t+1}\ ,\ S^e_{t+1}/p^e_{t+1}) + (M_t/p_t) = 0 \tag{1.7}$$

in which the superscript "e" denotes values that are anticipated by the young trader at t for the next date. This formulation takes implicitly into account the restriction

$$p_t\, x^e_{t+1}\, \ell_1^* + p^e_{t+1}\, \ell_2^* + S^e_{t+1} \geq 0 \tag{1.8}$$

which is necessary for the function z_1 to be well defined. By Walras' law, the money equilibrium equation may be obtained by multiplying both members of (1.7) by p_t — see (1.5). An equivalent formulation of (1.7) may be obtained by using (1.4) or (1.5), which yields

$$z_2(p_t\, x^e_{t+1} \,/\, p^e_{t+1} \,,\, S^e_{t+1} \,/\, p^e_{t+1}) = (M_t\, x^e_{t+1} + S^e_{t+1}) \,/\, p^e_{t+1} \tag{1.9}$$

The equilibrium condition at t can be given another equivalent form, which is often more convenient, by setting m equal to M_t in the first order condition (1.6). This yields

$$V_1'(\ell_1^* - \frac{M_t}{p_t}) = (p_t\, x^e_{t+1}/p^e_{t+1})\, V_2'(\ell_2^* + (M_t\, x^e_{t+1} + S^e_{t+1}) \,/\, p^e_{t+1}) \tag{1.10}$$

Here again, the restriction (1.8) is implicitly taken into account since the arguments of V_1' and V_2' must be nonnegative.

2. *CYCLES WITH PERFECT FORESIGHT*

We study in this section trajectories with perfect foresight when the (doubly infinite) sequences of policy parameters (x_t) , (s_t) , (d_t) are <u>predetermined</u>, in particular when $s_t = s$ and $d_t = d$ for all t . It will be shown that less and less cycles with perfect foresight do exist when s and/or d gets larger. In this sense, an increase of the size of the Government's intervention through these channels has a strong "stabilizing" impact.

<u>Trajectories with perfect foresight</u>

Given (x_t) , (s_t) , (d_t) and M_o , (1.2) determines the money supplies M_t while S_t and G_t are given by (1.1). An <u>intertemporal competitive equilibrium with perfect foresight</u> corresponding to M_o and the predetermined policy parameters is then a (again doubly infinite) sequence of prices (p_t) that verifies (1.7), or (1.9), for all t , where the superscripts "e" have been suppressed.

Such an intertemporal equilibrium is most conveniently described in terms of the variables $\theta_t = p_t x_{t+1}/p_{t+1}$, which is equal to one plus the real interest rate between dates t and t+1 , and the real balances $\mu_t = M_t/p_t$. In terms of these variables, (1.2) and (1.7) read respectively

$$\mu_{t+1} = \mu_t \theta_t d_{t+1} \tag{2.1}$$

$$z_1(\theta_t , (s_{t+1} - 1) \theta_t \mu_t) + \mu_t = 0 \tag{2.2}$$

One may employ equivalently (1.9), which yields the market equilibrium relation (stated for convenience for date t-1)

$$z_2(\theta_{t-1} , (s_t -1) d_t^{-1} \mu_t) = s_t d_t^{-1} \mu_t \tag{2.3}$$

These equations show that the <u>set</u> of trajectories with perfect foresight is independent, in real terms, of the initial money stock M_0 (money is <u>neutral</u>), and of the predetermined sequence (x_t) (proportional money transfers are <u>superneutral</u>). By contrast, the parameters s_t , d_t (fiscal and budgetary policies) <u>do influence</u> the set of real perfect foresight equilibrium magnitudes, although they are predetermined.

It can be seen further that the system <u>dichotomizes</u>, in the sense that given (s_t) , (d_t) , equilibrium of the good market at t can be expressed, under perfect foresight, in terms of real interest rates alone. This is achieved by eliminating μ_t in (2.2), (2.3) above.

We first show that (2.3) can be solved for μ_t, given θ_{t-1}, s_t and d_t, and rewritten as

$$\zeta_2(\theta_{t-1}, s_t) = d_t^{-1} \mu_t \qquad (2.4)$$

Indeed, consider the equation, for $y \geqq 0$, $\theta > 0$, $s > 0$

$$0 = sy - z_2(\theta, (s-1)y) \qquad (2.5)$$

$$= y + (s-1)y - z_2(\theta, (s-1)y)$$

The right hand member is equal to $-z(\theta,0) \leqq 0$ for $y = 0$, with strict inequality if and only if $\theta > \bar{\theta}$, and is an increasing function of y. When $s \geqq 1$, it is defined for all y and diverges to $+\infty$ when y goes to $+\infty$. When $s < 1$, the maximum admissible value of y correspond to $z_2 = -\ell_2^*$, in which case the right hand member of (2.5) is positive. Thus given $\theta > 0$, $s > 0$, there is a unique value of y, denoted $\zeta_2(\theta,s)$, that solves (2.5). With this notation, (2.4) is indeed equivalent to (2.3).

Consider next (2.2). Multiplying by θ_t and using (1.4), we see that this equation is equivalent to

$$s_{t+1} \theta_t \mu_t = z_2(\theta_t, (s_{t+1} - 1) \theta_t \mu_t)$$

From the preceding argument, (2.2) is thus equivalent to

$$\mu_t = - \zeta_1(\theta_t, s_{t+1}) \qquad (2.6)$$

in which the function $\zeta_1(\theta,s)$ is related to $\zeta_2(\theta,s)$ by

$$\theta\zeta_1(\theta,s) + \zeta_2(\theta,s) \equiv 0 \qquad (2.7)$$

Therefore, <u>an intertemporal monetary equilibrium with perfect</u>

<u>foresight corresponding to</u> M_0 <u>and the predetermined sequences</u> (x_t), (s_t), (d_t), <u>may be defined by a sequence</u> $\theta_t > \bar{\theta}$ <u>that satisfies for all</u> t <u>the equation</u> (obtained by eliminating μ_t in (2.4), (2.6)

$$\zeta_1(\theta_t , s_{t+1}) + d_t \zeta_2(\theta_{t-1} , s_t) = 0 \tag{2.8}$$

which describes the equilibrium of the good market at date t . The system dichotomizes indeed, as claimed.

The relation (2.8) yields a difference equation on real interest rates that goes <u>backward</u> as usual. Indeed, it is easily seen from the definition of ζ_2

LEMMA 2.1. <u>One has</u> $\zeta_2(\theta,s) = 0$ <u>for</u> $\theta \leqq \bar{\theta}$ <u>and</u> $\zeta_2(\theta,s) > 0$ <u>for</u> $\theta > \bar{\theta}$. <u>Moreover</u> ζ_2 <u>is continuously differentiable, increasing in</u> θ <u>and decreasing in</u> s <u>for</u> $\theta > \bar{\theta}$ <u>and</u> $s > 0$. <u>Finally</u> ζ_2 <u>tends</u> to $+\infty$ <u>when</u> θ <u>goes to</u> $+\infty$, <u>to</u> 0 <u>when</u> s <u>tends to</u> $+\infty$ <u>and</u> $\theta > \bar{\theta}$.

Thus (2.8) can be solved for θ_{t-1} , given θ_t and the policy parameters s_t , d_t , s_{t+1}.

A trajectory with perfect foresight can be equivalently defined by the corresponding sequence of <u>real balances</u>, by making the change of variable $\mu_t = d_t \zeta_2(\theta_{t-1} , s_t)$ in (2.8) . The resulting equilibrium equation may in fact be obtained directly from (1.10). By suppressing the superscript "e" and by multiplying both members of (1.10) by M_t/p_t , one gets

$$\mu_t V_1'(\ell_1^* - \mu_t) = d_{t+1}^{-1} \mu_{t+1} V_2'(\ell_2^* + s_{t+1} d_{t+1}^{-1} \mu_{t+1}) \tag{2.9}$$

If one defines next

$$v_1(\mu) = \mu V_1'(\ell_1^* - \mu) \quad \text{for} \quad \mu \text{ in } [0,\ell_1^*) \tag{2.10}$$

$$v_2(\mu) = \mu V_2'(\ell_2^* + \mu) \quad \text{for} \quad \mu \text{ in } [0,+\infty)$$

the function v_1 is increasing and maps $[0,\ell_1^*)$ onto $[0,+\infty)$, while v_2 maps $[0,+\infty)$ into itself. Thus v_1 has an inverse and (2.9) can be solved for μ_t , given μ_{t+1} and the policy parameters s_{t+1} , d_{t+1}.

Existence of cycles

We focus now on the particular case $s_t = s$, $d_t = d$ for all t , and study how changes of s and/or d may eliminate cycles with perfect foresight.

From what precedes, such cycles may be identified with periodic solutions of the backward perfect foresight difference equation that is obtained from solving (2.9) for μ_t . Or equivalently with periodic points of the backward perfect foresight (b.p.f.) map $\chi_{s,d}$ on real balances defined by

$$\mu_t = v_1^{-1}[s^{-1} v_2(sd^{-1} \mu_{t+1})] \equiv \chi_{s,d}(\mu_{t+1}) \tag{2.11}$$

Of course, when s = d = 1 , $\chi_{s,d}$ coincides with the laisser faire b.p.f. map χ that was studied in Grandmont [9]. The following result gives useful information on $\chi_{s,d}$.

PROPOSITION 2.2. *The map* $\chi_{s,d}$ *from* $[0,+\infty)$ *into* $[0,\ell_1^*)$ *is continuously differentiable, with* $\chi_{s,d}(0) = 0$ *and* $\chi'_{s,d}(0) = (d\bar{\theta})^{-1}$.

Moreover,

1) *The intersection of the graph of* $\chi_{s,d}$ *with the line* $\mu_t = (d\theta)^{-1}\ \mu_{t+1}$ *is composed of the origin* $\mu_{t+1} = \mu_t = 0$ *and of the point* $\mu_{t+1} = d\zeta_2(\theta,s)$, $\mu_t = -\zeta_1(\theta,s)$. *The two points of intersection coincide if and only if* $\theta \leqq \bar{\theta}$.

2) *If* $d\bar{\theta} \geqq 1$, $\chi_{s,d}$ *has a unique fixed point at the origin (a nonmonetary stationary state). If* $d\bar{\theta} < 1$, $\chi_{s,d}$ *has another fixed point* $\bar{\mu}_{s,d} = d\zeta_2(d^{-1}, s) > 0$ (*the unique monetary stationary state*).

3) *Let* $R_2(a_2) = -a_2 V''_2(a_2)/V'_2(a_2)$ *measure the degree of concavity of* V_2 *for* $a_2 > 0$. *Then if* $\alpha_2 = \text{Sup } R_2(a_2) \leqq 1$, *the function* v_2 , *and thus* $\chi_{s,d}$, *is increasing everywhere. If*

$$\alpha_2 > 1 \quad \text{and} \quad R_2(a_2) \quad \text{is nondecreasing} \tag{2.12}$$

then v_2 *is unimodal with a unique critical point* $\mu^* > 0$, *i.e.* $v'_2(\mu) > 0$ *for* $0 \leqq \mu < \mu^*$, $v'_2(\mu^*) = 0$ *and* $v'_2(\mu) < 0$ *for* $\mu > \mu^*$. *In that case* $\chi_{s,d}$ *is also unimodal with a unique critical point* $\mu^*_{s,d} = s^{-1} d\mu^*$.

All these statements follow readily from the definitions. In particular, the identification of the point of intersection that differs from the origin, of the graph of $\chi_{s,d}$ and of the line $\mu_t = (d\theta)^{-1} \mu_{t+1}$, with the point $(d\zeta_2(\theta,s), -\zeta_1(\theta,s))$ results from (2.1), (2.4) and (2.6). Fig. 1 shows how the b.p.f. dynamics on real balances and on real interest rates are related.

Figure 1

We recall that this economy may have a lot of cycles under *laisser faire* $(s = d = 1)$, as shown in Grandmont [9]. So, to fix ideas, assume that the laisser faire map χ has a cycle of period $k \geqq 2$. This implies that χ, or equivalently v_2, has at least one local maximum and that the least value for which a local maximum occurs, say $\mu' > 0$, satisfies $\chi(\mu') > \mu'$. From (2.11), the least value of μ for which $\chi_{s,d}$ has a local maximum is $s^{-1} d\,\mu'$. It is then clearly possible to choose s and/or d so that $\chi_{s,d}$ has no cycle other than the stationary states : it suffices to bring the point of the graph of $\chi_{s,d}$ corresponding to $\mu = s^{-1} d\,\mu'$ *on or below the diagonal*. Formally

PROPOSITION 2.3. *Let* $\mu' > 0$ *be the least value for which* v_2 *has a local maximum. Then* $\chi_{s,d}$ *has no cycle with a period* $k \geqq 2$ *if*

$$\chi_{s,d}(s^{-1} d\,\mu') = v_1^{-1}[s^{-1} v_2(\mu')] \leqq s^{-1} d\,\mu'$$

The foregoing result encompasses three interesting particular cases.

1. Consider an increase of d from 1 to d > 1, with s being fixed and

equal to 1 (change of the money growth rate through the Government's purchases of the good, no lump sum transfers). Then the graph of $\chi_{1,d}$ is the <u>affine transformation</u> of the graph of the laisser faire map χ of axis $\overrightarrow{0\mu_t}$ and ratio d . If d is large enough, $\chi_{1,d}$ has no nontrivial cycle. One must keep $d\overline{\theta} < 1$ to ensure the existence of a monetary stationary state.

2. Consider an increase of the rate of growth of the money supply through lump sum transfers from 1 to s > 1 , the Government being inactive on the good market, i.e. d is constantly equal to s . The graph of the laisser faire map χ is <u>pushed down</u>, and the critical points of $\chi_{s,s}$ are the same as those of χ . If s = d is large enough, $\chi_{s,s}$ has no cycle of a period $k \geq 2$. Here again, one must have $d\overline{\theta} < 1$ for a monetary stationary state to exist.

3. Let the Government increase its expenditures but balance its budget, i.e. d is fixed and equal to 1 , while s decreases from 1 to s < 1 . The associated map $\chi_{s,1}$ is given by

$$\chi_{s,1}(\mu) = v_1^{-1}[\mu\, V_2'(\ell_2^* + s\mu)]$$

Thus its graph <u>goes up</u> as s decreases, the slope of the tangent at the origin being constant and equal to $\overline{\theta}^{-1}$. The least value for which $\chi_{s,1}$ has a local maximum is $s^{-1}\mu'$, which increases and gets eventually larger than ℓ_1^* when s gets lower than μ'/ℓ_1^* . Since $\chi_{s,1}(s^{-1}\mu')$ is bounded above by ℓ_1^* , the corresponding critical point of the graph $\chi_{s,1}$ is bound to cross the diagonal when s is low enough.

<u>Bifurcations</u>

We investigate now the bifurcations undergone by the difference equation

(2.11) when s and/or d are shifted gradually, by applying the methods of Grandmont [9,10] to which the reader is referred.

Assume that (2.12) is satisfied, so that $\chi_{s,d}$ is unimodal, with $s^{-1} d\mu^*$ as its critical point. Assume moreover that the laisser faire map χ satisfies $\chi^3(\mu^*) < \mu^* < \chi(\mu^*)$, where χ^3 is the third iterate of χ . This implies that χ has a cycle of period 3 and thus by Sarkovskii's theorem, that χ has an infinity of cycles : it has indeed a cycle of every period.

Consider now a gradual change of the policy parameter. Specifically assume that s and d are continuous functions $s(\lambda)$, $d(\lambda)$ of some real number λ in [0,1] , where $s(1) = d(1) = 1$ (the laisser faire case) and where $s(0)$ and $d(0)$ satisfy the condition of Proposition 2.3 with equality (there exists no cycle of period $k \geq 2$) . Assume moreover that each map $\chi_\lambda = \chi_{s(\lambda),d(\lambda)}$ of the family has its critical point above the diagonal, i.e. $\chi_\lambda(\mu_\lambda^*) > \mu_\lambda^*$ with $\mu_\lambda^* = s(\lambda)^{-1} d(\lambda) \mu^*$ for all $\lambda > 0$. The family is then full in the sense of Grandmont [9, Section 4 ; 10]. In order to apply the results pertaining to such families, however, we need to ensure that the maps have a negative Schwarzian derivative. We recall that any map f from [a,b] into itself that is thrice continuously differentiable is said to have a negative Schwarzian derivative if one has at every x such that $f'(x) \neq 0$

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0$$

The information we need is then contained in the following result, the proof of which follows the same lines as the proof of Grandmont [9, Lemma 4.6].

LEMMA 2.4. Assume that the utility functions V_τ are four times continuously differentiable. Let v_1 , v_2 be defined as in (2.10)

<u>and assume that</u> $Sv_1(\mu) \geqq 0$ <u>for all</u> μ <u>in</u> $[0,\ell_1^*)$ <u>and that</u> $Sv_2(\mu) < 0$ <u>for all</u> μ <u>such that</u> $v_2'(\mu) \neq 0$. <u>Then the map</u> $\chi_{s,d}$ <u>has a negative Schwarzian derivative</u>.

If each map χ_λ of the family satisfies the assumptions of the Lemma, one can apply the results on the bifurcations of onedimensional families of maps of the interval, as reviewed in Grandmont [10, especially Proposition 4, Theorems 7,9][3]. We do not restate them here, and content ourselves to describe their qualitative implications.

Every map χ_λ has <u>at most one</u> cycle that is weakly stable in the b.p.f. dynamics, and if such a weakly stable cycle exists, it attracts the critical point μ_λ^* and indeed almost every point — in the sense of the Lebesgue measure — of the interval $[0,\chi_\lambda(\mu_\lambda^*)]$. In fact all stable periods will occur when λ moves from $\lambda = 1$ to $\lambda = 0$. More precisely, there exists λ_∞^* in $(0,1)$ with the following properties. When λ decreases from 1 to λ_∞^* , all stable periods that differ from a power of 2 do appear. However, the map χ_λ has no weakly stable cycle for an uncountable number of values of λ in $[\lambda_\infty^*,1]$. By contrast, when $\lambda < \lambda_\infty^*$, all cycles of χ_λ have a period that is a power of 2 , and there exists <u>exactly one</u> stable cycle. Furthermore, when λ moves from λ_∞^* to 0 , stable periods are "visited" in a <u>consecutive</u> (if not monotonic) fashion according to the decreasing sequence

$$> \dots > 2^n > \dots > 8 > 4 > 2 > 1$$

leading to a sequence of "period-halving" bifurcations.

The cyclical properties of χ_λ will thus evolve according to a typical "bifurcation diagram" as in Grandmont [9, Fig.4 ; 10, Fig. 4], illustrating how the economy is progressively "stabilized" when the policy parameter moves from $\lambda = 1$ (the laisser faire case) to $\lambda = 0$ (the only remaining cycles are the stationary states).

3. STATIONARY MARKOV SUNSPOT EQUILIBRIA

Deterministic cycles with perfect foresight are but degenerate cases of stationary Markov "sunspot" equilibria, in which agents predict that prices are affected by random factors, although they do not influence the real characteristics of the economy, and in which this prophecy is self-fulfilling. We show now an extension of an interesting result obtained recently in the laisser faire case by Azariadis and Guesnerie [2] : stationary Markov sunspot equilibria involving an arbitrary number of states exist in the present model when the policy parameters s_t and d_t are constant and equal to s, d , if and only if there is a "noncritical" cycle of period 2 . In fact the set of such sunspot equilibria is infinite dimensional.

Consider a stationary Markov chain of random variables (sunspots) taking values in some space Ω . In order to simplify the exposition, we assume that Ω has r elements, $\Omega = \{\omega_1, \ldots, \omega_r\}$, $r \geqq 2$ and that the transition probabilities from ω_i to ω_j , say q_{ij} , satisfy $q_{ij} > 0$ for all i,j . The agents are supposed to observe the value of the sunspot at each date, and to know the transition probabilities.

Let (x_t) , $s_t = s$, $d_t = d$ be fixed and known by the agents. Given $M_o > 0$, the evolution of the money supply is then ruled by $M_t = M_{t-1} x_t d$, while S_t , G_t are given by (1.1) with $s_t = s$, $d_t = d$. The agents are supposed to know or to forecast correctly these quantities. Assume now that although sunspots do not alter the utility functions or the endowments, the agents believe nevertheless that they influence the equilibrium price at each date through the relation $p(\omega,M)$, in which ω is the sunspot observed at the date under consideration and M is the end-of-the-period money stock, with

$$p(\omega_i,M) = M/\mu_i \quad , \text{ all } i \ , \ 0 < \mu_1 \leqq \dots \leqq \mu_r \ , \ \mu_1 \neq \mu_r \ . \tag{3.1}$$

The above ordering can be obtained through a relabeling of the sunspots. The condition $\mu_1 \neq \mu_r$ means that there is genuine uncertainty about prices.

Since s and d are fixed, this will imply (see (3.6)) that agents believe that equilibrium prices are proportional to M_0 (they believe that money is neutral) and to the x_t (agents believe that proportional money transfers are superneutral). The prophecy described in (3.1) will be self-fulfilling, and will lead to a stationary Markov sunspot equilibrium if, when agents act in accordance to the belief (3.1) , $p(\omega_i,M)$ is indeed an equilibrium price when ω_i is observed, and when the money supply is M .

In order to translate this notion into a formal definition, consider a young agent at an arbitrary date t who observes ω_i and $p_t > 0$ in the current period. If he believes that prices obey (3.1) , he expects the price $p_j = p(\omega_j , M_{t+1})$ to occur with probability q_{ij} at the next date. His problem is then to choose his current consumption $a_1 \geqq 0$ and his money demand $m \geqq 0$, as well as his future consumption $a_{2j} \geqq 0$ in the event where ω_j will be observed at the next date, so as to maximize the mathematical expectation of his utility

$$V_1(a_1) + \Sigma_j \, q_{ij} \, V_2(a_{2j}) \tag{3.2}$$

subject to the current and expected budget constraints

$$p_t a_1 + m = p_t \ell_1^* \tag{3.3}$$

$$p_j a_{2j} = p_j \ell_2^* + m x_{t+1} + S_{t+1} \quad , \quad j = 1,\ldots r$$

where $S_{t+1} = (s-1) M_t x_{t+1}$. This decision problem has a solution, which is then unique, if and only if $x_{t+1} p_t \ell_1^* + p_j \ell_2^* + S_{t+1} \geqq 0$ for all j .

The most convenient way to characterize the solution is to look at the first order condition. When money demand is positive, the optimum solution satisfies

$$V_1'(a_1)/p_t = \Sigma_j q_{ij} x_{t+1} V_2'(a_{2j})/p_j$$

Therefore, money demand, when it is positive, is the only positive value of m that verifies

$$\frac{m}{p_t} V_1'(\ell_1^* - \frac{m}{p_t}) = \Sigma_j q_{ij} \frac{mx_{t+1}}{p_j} V_2'(\ell_2^* + \frac{mx_{t+1} + S_{t+1}}{p_j}) \tag{3.4}$$

Equilibrium at t is obtained by setting the young trader's demand m equal to M_t in (3.4) . By using $M_t x_{t+1} = d^{-1} M_{t+1}$ and $M_t x_{t+1} + S_{t+1} = sd^{-1} M_{t+1}$ to rearrange the right hand member, and by employing the functions v_1 , v_2 introduced in (2.10), one gets

$$v_1(\frac{M_t}{p_t}) = \Sigma_j q_{ij} s^{-1} v_2(sd^{-1} \mu_j) \tag{3.5}$$

Since v_1 is an increasing function from $[0,\ell_1^*)$ onto $[0,+\infty)$, the condition (3.5) determines uniquely the equilibrium price p_t at an arbitrary date t as a function of ω_i and of M_t. This equation leads therefore to a well defined causal (stochastic) dynamics of temporary equilibria, in which the past determines the present [(4)].

<u>Given</u> M_o, (x_t), $s_t = s$ <u>and</u> $d_t = d$ <u>for all</u> t, <u>a stationary Markov sunspot equilibrium</u> (<u>with</u> r <u>states</u>) <u>is a price function</u> $p(\omega_i, M)$ <u>satisfying</u> (3.1) <u>such that the solution in</u> p_t <u>of</u> (3.5), <u>given</u> ω_i <u>and</u> M_t, <u>is indeed</u> M_t/μ_i, <u>or equivalently, such that for all</u> i

$$v_1(\mu_i) = \Sigma_j \, q_{ij} \, s^{-1} \, v_2(sd^{-1} \, \mu_j) \tag{3.6}$$

The agents' belief is then self-fulfilling, and $p(\omega,M)/M$ follows a Markov chain on the set $\{\mu_1, \ldots, \mu_r\}$ with transition probabilities q_{ij}. In the limit case $r = 2$, $q_{12} = q_{21} = 1$, one gets of course a cycle with perfect foresight of period 2, see (2.11).

We characterize now the real balances appearing in (3.6) by using the map $\chi_{s,d}$. To simplify notations, since s,d are given, we shall pose until the end of this section

$$\bar{\chi} = \chi_{s,d} \,, \; \bar{\mu} = \bar{\mu}_{s,d} \text{ and } \bar{v}_2(\mu) = s^{-1} \, v_2(sd^{-1} \, \mu) \tag{3.7}$$

Consider a sunspot equilibrium $(\mu_1, \ldots, \mu_r)$ as in (3.6). Let now m and n be integers such that $\bar{v}_2(\mu_n) \leqq \bar{v}_2(\mu_i) \leqq \bar{v}_2(\mu_m)$ for all i. It follows from (3.6) that one has for all i, $\bar{v}_2(\mu_n) \leqq v_1(\mu_i) \leqq \bar{v}_2(\mu_m)$. Since $\mu_1 < \mu_r$, the expressions $\bar{v}_2(\mu_n)$ and $\bar{v}_2(\mu_m)$, and thus μ_m and μ_n, must differ, and one has actually $\overline{v}_2(\mu_n) < v_1(\mu_i) < \overline{v}_2(\mu_m)$ for all i , since the probabilities q_{im} and q_{in} are both positive. Applying this fact to m and n , one gets $\overline{v}_2(\mu_n) < v_1(\mu_n)$ and $v_1(\mu_m) < \overline{v}_2(\mu_m)$, or equivalently $\overline{\chi}(\mu_n) < \mu_n$ and $\mu_m < \overline{\chi}(\mu_m)$. The final step is to remark that the configuration $\mu_n < \mu_m$ is impossible, since it would imply that χ has two distinct monetary stationary states. Hence we obtain that there exist two integers m,n such that $\mu_m < \mu_n$ and

$$\overline{v}_2(\mu_n) < v_1(\mu_1) < v_1(\mu_r) < \overline{v}_2(\mu_m) \tag{3.8}$$

or equivalently

$$\overline{\chi}(\mu_n) < \mu_1 < \mu_r < \overline{\chi}(\mu_m) \tag{3.9}$$

Consider conversely a sequence $0 < \mu_1 \leqq \ldots \leqq \mu_r$ and assume that there exist $\mu_m < \mu_n$ such that (3.9) or (3.8) holds. Is there a Markov chain on Ω such that $(\mu_1,\ldots,\mu_r)$ defines a stationary Markov sunspot equilibrium, i.e. satisfies (3.6) ? Here the unknowns are the transition probabilities q_{ij} and it is clear that given (3.8) , one can find probabilities $q_{ij} > 0$ acting on Ω so as to satisfy (3.6). The simplest way to generate a solution is to solve (3.6) by setting first $q_{ij} = 0$ when $j \neq m,n$ (this determines uniquely $q_{im} > 0$ and $q_{in} > 0$ for all i) and then to perturbate slightly the probabilities so as to make all of them positive. Obviously, the set of solutions is nonempty, and open. We have thus obtained

PROPOSITION 3.1. _Given_ $\Omega = \langle\omega_1,\ldots,\omega_r\rangle$, _the real balances_ $0 < \mu_1 \leqq \cdots \leqq \mu_r$ _define a stationary Markov sunspot equilibrium if and only if there exist_

$\mu_m < \mu_n$ <u>that satisfy</u> (3.9).

It is now immediate that such a sunspot equilibrium exists if and only if the map $\bar{\chi}$ has a cycle of period 2 . In fact, since we have only considered "nondegenerate" sunspot equilibria ($q_{ij} > 0$ for all i,j and $\mu_1 \neq \mu_r$) , we wish to show this equivalence with cycles that are also "nondegenerate". The situation to be avoided is the limit case where $\bar{\chi}$ has a cycle of period 2 — in which case $\bar{\chi}'(0) = (d\bar{\theta})^{-1} > 1$ and the second iterate of $\bar{\chi}$, i.e. $\bar{\chi}^2$, has a fixed point that differs from 0 and from the monetary stationary state $\bar{\mu} = \bar{\mu}_{s,d}$ — but where the graph of $\bar{\chi}^2$ has no point below the diagonal when $0 < \mu < \bar{\mu}$, nor any point above it when $\mu > \bar{\mu}$. Formally, we say that $\bar{\chi}$ <u>has a noncritical cycle of period</u> 2 if there exists a period 2 cycle and if there exists either μ' in $(0,\bar{\mu})$ such that $\bar{\chi}^2(\mu') < \mu'$, or $\mu'' > \bar{\mu}$ such that $\bar{\chi}^2(\mu'') > \mu''$.

It was shown in Grandmont [12, (3.12)] that the laisser faire map $\chi = \chi_{1,1}$ had a noncritical cycle of period 2 if and only if there existed μ' and μ'' verifying $\chi(\mu'') < \mu' < \mu'' < \chi(\mu')$. The proof — which is straightforward — applies without any change to $\bar{\chi}$. Thus

PROPOSITION 3.2. <u>The map</u> $\chi = \chi_{s,d}$ <u>has a noncritical cycle of period</u> 2 <u>if and only if there exist</u> μ' and μ'' <u>such that</u> $\bar{\chi}(\mu'') < \mu' < \mu'' < \bar{\chi}(\mu')$.

Propositions 3.1 and 3.2 establish the equivalence we were looking for. In fact, Proposition 3.1 yields a constructive way to design r-states sunspots equilibria. Consider for instance the case where $\bar{\chi}'(0) > 1$ and $\bar{\chi}'(\bar{\mu}) < -1$.

It is known (Grandmont [9, 10]) that $\bar{\chi}$ has then a cycle of period 2. It has clearly a nondegenerate one since $D\bar{\chi}^2(\bar{\mu}) = [\bar{\chi}'(\bar{\mu})]^2 > 1$. The local picture around $\bar{\mu}$ is represented in Fig. 2. <u>If</u> μ_1 <u>is less than but close enough to</u> $\bar{\mu}$, <u>then</u> $\mu_1 \leqq \ldots \leqq \mu_r$ <u>satisfies the conditions of Proposition</u> 3.1 <u>and thus defines consistently a stationary Markov sunspot equilibrium if and only if</u> μ_r <u>belongs to the interval</u> $(\mu' , \bar{\chi}(\mu_1))$ (one takes m = 1 and n = r in this case). Other examples can be designed along the lines of Grandmont [12].

<u>Figure 2</u>

The above arguments show that given r , the set of stationary Markov sunspot equilibria with r states (characterized by $(\mu_1,\ldots,\mu_r)$ and the probabilities $q_{ij} > 0$ for all i,j) is finite dimensional and open. The set of all such Markov sunspot equilibria, when the number of states r is variable, is by contrast <u>infinite dimensional</u> whenever it is nonempty.

The results of this section extend easily to the case where Ω is a <u>compact metric space</u>, endowed with its Borel σ-field. Consider a Markov process on Ω given by the stationary transition probabilities $q(\omega,B)$ where the map that associates to every ω the probability $q(\omega,.)$ is <u>continuous</u> when the space of probabilities on Ω is endowed with the topology of weak convergence (Billingsley [5] or Parthasarathy [13]. We assume that $q(\omega,B) > 0$ for every ω and every nonempty <u>open</u> set B . A price function is defined, as in (3.1), as $p(\omega,M) = M/\mu(\omega)$ where $\mu(\omega)$ is <u>continuous and bounded away from</u> 0 , i.e. $\mu(\omega) \geqq a$ for all ω and some $a > 0$, and $\mu(\omega') \neq \mu(\omega'')$ for some ω' , ω'' . The counterpart of Proposition 3.1 is that $\mu(\omega)$ determines a stationary Markov sunspot equilibrium if and only if there are ω' and ω'' with

$$\mu(\omega') < \mu(\omega'') \text{ and } \bar{\chi}(\mu(\omega'')) < \mu(\omega) < \bar{\chi}(\mu(\omega')) \text{ for all } \omega .$$

A Markov sunspot equilibrium is described by the continuous map $\mu(\omega)$ and the continuous map that associates to every ω the probability $q(\omega,.)$. The set of stationary Markov sunspot equilibria is nonempty and open (in the topology of uniform convergence of maps) if and only if there is a noncritical period 2 cycle. If Ω is infinite dimensional, so is the set of stationary Markov sunspot equilibria whenever it is nonempty.

4. POLICY RULES

The previous two sections imply that a manipulation of the predetermined money growth rates $s_t = s$, $d_t = d$ may succeed in eliminating all deterministic cycles with a period $k \geqq 2$, and thus all stationary Markov sunspot equilibria as well. Such policies, however, introduce inefficiencies of their own. Also, they do not guarantee that forecasting mistakes will disappear. We show now, in the same vein as in Grandmont [9, Section 6], how preannounced and deterministic policy rules may be designed to eliminate forecasting errors and to force the economy to converge to the Golden Rule stationary state. Monetary policy is used to control real interest rates ; fiscal and public expenditure policies are employed to ensure that forward perfect foresight dynamics do converge locally to the Golden Rule. To design such a policy rule, one needs only to know the values and the derivatives of the traders' behavioural functions at the Golden Rule.

Assume that the economy evolves under laisser faire prior to date $t = 0$. The money stock before that date is thus constant and equal to M_0 . At the outset of date 0 , i.e. before the opening of the market of that period, the Government announces that it will implement a policy — starting at the next date $t = 1$ — in which the money growth rates x , s , d will be tied to the previous rates of inflation according to the rule

$$x_t = d_t = s_t = 1 \text{ for } t \leqq 0 \text{ , and } x_t = (p_t/p_{t-1}) \, \xi(p_{t-1}/p_{t-2}), \qquad (4.a)$$

$s_t = s(p_{t-1}/p_{t-2}), d_t = d(p_{t-1}/p_{t-2})$ for all $t \geq 1$,
where the functions ξ , s , d map $(0,+\infty)$ into itself.

Assume also that the money supply M_t is public knowledge at all dates. The public adoption of the rule (4.a) changes the informational content of prices and the traders' expectations will obey for all $t \geq 0$

$$p_t x^e_{t+1}/p^e_{t+1} = \xi(p_t/p_{t-1}) \tag{4.1}$$

$$S^e_{t+1}/p^e_{t+1} = (M_t/p_t)\ \xi(p_t/p_{t-1})\ [s(p_t/p_{t-1})-1]$$

The traders may be wrong about x^e_{t+1} , p^e_{t+1} , S^e_{t+1} , but they will predict correctly $\theta_t = p_t x_{t+1}/p_{t+1}$ and S_{t+1}/p_{t+1} . The economy will accordingly follow from t = 0 on a forward sequence of temporary equilibria obtained by plugging (4.1) into the temporary equilibrium equation (1.7), along which traders have perfect foresight about real interest rates and the real values of lump sum transfers. All sunspots phenomena will thus be ruled out. This sequence of temporary equilibria may be obtained equivalently by substitution the rule (4.a) into the equations (2.6) or (2.8) that described trajectories with perfect foresight.

We focus on policy rules that satisfy

ξ continuously differentiable and decreasing, i.e. $\xi'(x) < 0$ for all x , with range $I = (1-\varepsilon\ ,\ 1+\varepsilon)$. Moreover $\xi(x^*) = 1$. (4.b)

The policy rules s(x) , d(x) are of the form $s(x) = \sigma(\xi(x))$, $d(x) = \delta(\xi(x))$, where σ and δ (4.c)

<u>are continuously differentiable functions from</u>
$(0,+\infty)$ <u>into itself, with</u> $\sigma(1) = \delta(1) = 1$. <u>One denotes</u>
$\alpha = \sigma'(1)$, $\beta = \delta'(1)$.

(4.b) means that monetary policy aims at maintaining <u>real</u> interest rates within the range I . The policy is "countercyclical" in the sense that it slows down the real rate of growth of the money supply that is engineered through proportional money transfers between t and $t+1$, when the inflation rate has been high in the immediate past, i.e. between $t-1$ and t . The inflation rate to be achieved at the Golden Rule stationary state $\theta = 1$, is $x^* - 1$. Finally, (4.c) states that s_t and d_t are in fact tied to real interest rates through σ and δ for all dates $t \geq 1$, and that the Government is inactive through these channels at the Golden Rule.

(4.c) permits a convenient formulation of the temporary equilibrium equations verified by the economy for $t \geq 0$. Let

$$Z_1(\theta) = \zeta_1(\theta,\sigma(\theta)) \quad \text{and} \quad Z_2(\theta) = \delta(\theta)\,\zeta_2(\theta,\sigma(\theta)) \tag{4.2}$$

Then substitution of (4.a) into (2.6) for $t = 0$, into (2.8) for $t \geq 1$ yields

$$Z_1(\xi(p_0/p_{-1})) + (M_0/p_0) = 0 \tag{4.3}$$

$$Z_1(\xi(p_t/p_{t-1})) + Z_2(\xi(p_{t-1}/p_{t-2})) = 0 \quad \text{for} \quad t \geq 1 \tag{4.4}$$

Real interest rates $\theta_t = \xi(p_t/p_{t-1})$ follow accordingly for $t \geq 1$

$$Z_1(\theta_t) + Z_2(\theta_{t-1}) = 0 \tag{4.5}$$

with $Z_1(1) + Z_2(1) = 0$. The respective roles of the Government's interventions is then clear. By choosing ε small enough, monetary policy forces real interest rates to stay close to the stationary state $\theta = 1$. By choosing the parameters $\alpha = \sigma'(1)$, $\beta = \delta'(1)$ in order to manipulate the derivatives $Z_1'(1)$ and $Z_2'(1)$, fiscal and public expenditure policies aim at ensuring convergence of the sequence θ_t to 1 .

Specifically, one gets by differenciation of (4.2)

$$Z_1'(1) = \frac{\partial \zeta_1}{\partial \theta}(1,1) + \alpha \frac{\partial \zeta_1}{\partial s}(1,1) \tag{4.6}$$

$$Z_2'(1) = \beta \zeta_2(1,1) + \frac{\partial \zeta_2}{\partial \theta}(1,1) + \alpha \frac{\partial \zeta_2}{\partial s}(1,1) \tag{4.7}$$

Thus adding (4.6) and (4.7) while taking into account (2.7) yields

$$Z_1'(1) + Z_2'(1) = (\beta+1)\ \zeta_2(1,1)$$

Let us fix $\beta > -1$. We have then $Z_1'(1) > -Z_2'(1)$ since $\zeta_2(1,1) > 0$. Furthermore, from Lemma 2.1 and (2.7) , $\frac{\partial \zeta_1}{\partial s}(1,1) = -\frac{\partial \zeta_2}{\partial s}(1,1) > 0$. Thus in view of (4.6), (4.7), given β , there exists $\bar{\alpha}$ large enough such that for $\alpha \geqq \bar{\alpha}$, $Z_1'(1) > Z_2'(1)$, in which case $Z_1'(1) > |\ Z_2'(1)\ |$. Note that $\bar{\alpha}$ depends only on β and on the values of ζ_2 and of its derivatives at (1.1), and is independent of ξ .

Choose from now on $\alpha \geqq \bar{\alpha}$. Pick up next $\varepsilon > 0$ low enough so that for all θ, θ' in the closure of $I = (1-\varepsilon\ ,\ 1+\varepsilon)$, and for some c such that $|Z_2'(1)| / Z_1'(1) < c < 1$, one has $|Z_2'(\theta')| \leqq c\ Z_1'(\theta)$. Since $Z_1(1) = -Z_2(1)$, the image of I by Z_1 , i.e. $Z_1(I)$, contains then the image of I by $-Z_2$, i.e.

$-Z_2(I)$. It follows that for every θ_{t-1} in I , (4.5) has a unique solution θ in I

$$\theta_t = (Z_1|_I)^{-1}(-Z_2(\theta_{t-1})) \equiv f(\theta_{t-1}) \tag{4.8}$$

where $(Z_1|_I)^{-1}$ is the inverse of the restriction of Z_1 to I . The map f from I into itself so defined is continuously differenciable, with $f(1) = 1$ and $|f'(\theta)| \leqq c < 1$ on I . Finally assume that ε is small enough to ensure $Z_1(1-\varepsilon) < 0$.

Consider now the equation (4.3) . Under the above choices of α , β , ε , its left hand side is a decreasing function of p_o , which diverges to $+\infty$ when p_o goes to 0 and becomes negative when p_o is large enough. Thus (4.3) has a unique solution $\bar{p}_o$, given M_o and p_{-1} , and it is stable in any Walrasian tatonnement at $t = 0$. This determines also $\bar{\theta}_o = \xi(\bar{p}_o/p_{-1})$, an element of I .

The equations (4.4) are then solved recursively. For $t = 1$, the equation reads $Z_1(\xi(p_1/\bar{p}_o)) = -Z_2(\bar{\theta}_o)$. The left side is a decreasing function of p_1 that spans $Z_1(I)$ when p_1 runs from 0 to $+\infty$. Since $Z_1(I)$ contains $-Z_2(\bar{\theta}_o)$, the equation has a unique solution $\bar{p}_1$, and it is stable in any Walrasian tatonnement. This determines $\bar{\theta}_1 = \xi(\bar{p}_1/\bar{p}_o)$, again an element of I . Proceeding recursively, one finds that (4.4) determines a unique sequence of temporary equilibrium prices $\bar{p}_t$, and that each of them is stable in any contemporaneous Walrasian tatonnement. The sequence of equilibrium interest rates is given by $\bar{\theta}_t = \xi(\bar{p}_t/\bar{p}_{t-1})$ and it satisfies $\bar{\theta}_t = f(\bar{\theta}_{t-1})$. Thus $\bar{\theta}_t$ tends to 1 as well as $s_{t+1} = s(\bar{p}_t/\bar{p}_{t-1})$ and $d_{t+1} = d(\bar{p}_t/\bar{p}_{t-1})$, while $\bar{p}_t/\bar{p}_{t-1}$ tends to x^* as time goes to $+\infty$. To sum up, we have obtained

PROPOSITION 4.1. Assume that the policy rules verify (4.a) and that the money supplies M_t are public knowledge at all dates. Then the evolution of the economy from $t = 0$ on is given by a sequence of temporary equilibria along which traders have perfect foresight about real interest rates and the real value of lump sum transfers. If (4.c) holds, this sequence is defined by (4.3), (4.4) .

Assume that the policy rules verify in addition (4.b), (4.c) and fix $\beta = \delta'(1) > -1$. Then there exists a real number $\bar{\alpha}$, that is independent of the policy rule ξ , such that for any $\alpha = \sigma'(1) \geq \bar{\alpha}$, the following is true when ε is chosen small enough :

1) There is a unique sequence $(\bar{p}_t)$, $t \geq 0$ that satisfies (4.3), (4.4), given M_o and p_{-1} . Moreover, the temporary equilibrium price $\bar{p}_t$ is stable in any Walrasian tatonnement at date t in which the price p_t of the good responds positively to excess demand on that market, for each $t \geq 0$.

2) The sequence $\bar{\theta}_t = \xi(\bar{p}_t/\bar{p}_{t-1})$, $s_{t+1} = s(\bar{p}_t/\bar{p}_{t-1})$, $d_{t+1} = d(\bar{p}_t/\bar{p}_{t-1})$ all tend to 1, while $\bar{p}_t/\bar{p}_{t-1}$ tends to x^* when t goes to $+\infty$.

FOOTNOTES

* CEPREMAP, Paris. I wish to thank C. Bronsard and M. Rothschild for very useful conversations. Support from the Commissariat General au Plan, the University of California at San Diego and the University of Montreal is gratefully acknowledged.

1. We admit in the sequel that the Government may sell some amount of the good as well as purchase it. We maintain this fiction for analytical simplicity as a proxy for the more realistic case where public consumption would enter the trader's utility functions (production of a public good) and where the Golden Rule equilibrium would entail accordingly a positive consumption by the Government.

2. Of course when $s = 1$ (no lump sum transfer) one has $\zeta_1(\theta,1) = z_1(\theta,0)$ and $\zeta_2(\theta,1) = z_2(\theta,0)$, which is the justification for the above notation.

3. It is easy to check that $\chi''_\lambda(\mu^*_\lambda) < 0$, or equivalently $v''_2(\mu^*) < 0$. For a proof in the case of laisser faire, see Grandmont [9, Appendix, proof of Theorem 4.7].

4. This result is due to the fact that the specification of the traders' beliefs through (3.1) generates an expectation function linking a trader's forecast at date t , i.e. the probability distribution $\psi(\omega_i, M_t)$ which assigns probability q_{ij} to the price $p(\omega_j, M_{t+1})$, to his information (p_t, ω_i, M_t) at that date. Note that the expectation function is independent of the currently observed price, a condition that is familiar in temporary equilibrium analysis in order to get existence, see Grandmont [8, 11].

REFERENCES

1. C. Azariadis, Self-fulfilling prophecies, Journal of Economic Theory 25 (1981), 380-396.

2. C. Azariadis and R. Guesnerie, Sunspots and cycles, mimeo, University of Penn., Philadelphia, 1984.

3. J. Benhabib and R.H. Day, A characterization of erratic dynamics in the overlapping generations model, Journal of Economic Dynamics and Control 4 (1982), 37-55.

4. J. Benhabib and K. Nishimura, Competitive equilibrium cycles, mimeo, New York University, 1984.

5. P. Billingsley, "Convergence of Probability Measures", Wiley, New York, 1968.

6. D. Cass and K. Shell, Do sunspots matter ?, Journal of Political Economy 91 (1983), 193-227.

7. J. Geanakoplos and H. Polemarchakis, Walrasian indeterminacy and dynamic macroeconomic equilibrium, mimeo, Yale University, 1983.

8. J.M. Grandmont, Temporary general equilibrium theory, Econometrica 45 (1977), 535-572.

9. J.M. Grandmont, On endogenous competitive business cycles,Econometrica (1985), forthcoming.

10. J.M. Grandmont, Periodic and aperiodic behaviour in discrete onedimensional dynamical systems, CEPREMAP, Paris, 1983, to appear.

11. J.M. Grandmont, "Money and Value", Econometric Society Monographs Series, Cambridge University Press, New York, 1983.

12. J.M. Grandmont, Cycles concurrentiels endogènes, Cahiers du Séminaire d'Econométrie (1985), forthcoming.

13. K. Parthasarathy, "Probability Measures on Metric Spaces", Academic Press, New York, 1967.

14. K. Shell, Monnaie et allocation intertemporelle, Séminaire d'Econométrie Roy-Malinvaud, Paris, 1977.

15. M. Woodford, Indeterminacy of equilibrium in the overlapping generations model : a survey, mimeo, Columbia University, New York, 1984.