DECEMBRE 1984

CAPITAL MARKET AND LABOUR CHOICE IN A DYNAMIC MODEL OF THE FAMILY : A SPECIAL CASE

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n° 8510

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ABSTRACT

This paper extends the analysis of a previous study of a dynamic model of the family where the number of children per family was restricted to integer values. The discontinuities which appear in the phase line destrayed the existence, uniqueness and stability of the steady state. We now try to smoothe the production function by introducing a capital market or another variable input. Some discontinuities disappear but the properties of the steady state remain.

MARCHE FINANCIER ET ARBITRAGE ENTRE TRAVAIL ET LOISIRS DANS UN MODELE DYNAMIQUE DE LA FAMILLE : UN CAS PARTICULIER

RESUME

Cet article poursuit l'analyse d'une autre étude sur un modèle dynamique démographique où le nombre d'enfants par famille était un nombre entier. Les discontinuités ainsi introduites dans la ligne de phase détruisaient l'existence, l'unicité et la stabilité de l'équilibre. Il s'agit ici d'"applanir" la courbe de production par l'introduction d'un marché financier ou d'un autre facteur de production à niveau variable. Certaines discontinuités disparaissent mais les conséquences pour les propriétés de l'équilibre demeurent.

CAPITAL MARKET AND LABOUR CHOICE IN A DYNAMIC MODEL OF THE FAMILY : A SPECIAL CASE (*)

Daniel Léonard

I - INTRODUCTION

This note is a direct extension of Léonard [1984]. In that model the technology available to each family was described by a function of capital which exhibited increasing and decreasing returns in that order. There was no explicit mention of labour input and capital markets were absent. The analysis focused on the consequences of the discontinuities introduced in the model by restricting the number of children a typical family chose to have to integer values. In this note we investigate the possibility of "smoothing" the outcome by introducing either capital markets or labour choice.

We show that although some results are drastically altered, the discontinuities remain and so do their implications for population dynamics.

In the next section we present two modifications of the technology available to the families. In Section 3 we derive the behaviour of the typical family. In Section 4 we present the dynamic path. The basic model is succintly presented ; for a more detailed discussion the reader is referred to the work cited above.

(*) The impetus for this analysis was given by comments by Gary Becker on some aspects of Léonard [1984]. The paper was written while the author was on leave at the CEPREMAP, Paris.

II - TECHNOLOGIES

There are many identical families in each generation. Each family was assumed to earn the income $f[b(t-1)]$, where $b(t-1)$ was the inheritance received from the adult's parents and f was a production function of the shape represented in Figure 1. I had been shown that a necessary and sufficient condition for ruling out undesirable outcomes was that marginal product exceed average product for small b, and the reverse for large b.

The point at which average product is maximized is b^* where $f'[b^*] = f[b^*]/b$. If each family receives $b < b^*$ it would be advantageous for some families to pool their resources to create firms with capital b^*. More formally suppose there are N families with capital b ; if N is large enough it is always possible to form I firms with capital $(bN/I) \simeq b^*$, with a suitable choice of I. The income (after trading) of each family is :

$$\tilde{f}(b) \equiv \max_{t_i} \{f(b + t_i) - R\, t_i \mid t_i + b \geq 0\}$$

where R is the competitive interest factor and t_i is the transfer of capital. When $b < b^*$ it is easy to show that two outcomes are optimal. Either $t_i = -b$ (for $N - I$ families) or $t_i + b = b^*$ (for I families). In the latter case we have $f'(b^*) = R$ which defines R and yields :

$$\tilde{f}(b) = f(b^*) - f'(b^*)(b^* - b) = f'(b^*)\, b.$$

In the former case we have $f'(0) < R$ and :

$$\tilde{f}(b) = f(0) + R\, b = f'(b^*)\, b.$$

It is therefore a matter of in difference as to which family lends or borrows. The new addition to the technology is reprented in Figure 1 by a line segment from the origin to the point of coordinates $(b^*, f(b^*))$. It is presumed that there cannot be more firms than families (perhaps because there is a fixed resource such as a plot

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of land for each family). Therefore $\tilde{f}(b)$ and $f(b)$ are identical for $b \geq b^*$. Note that $\tilde{f}$ is the concave upper contour of f. This is the result of optimizing on the capital market by individual families. We are familiar with such results from duality theory.

In the previous analysis no explicit mention was made of labour or other variable input. We now introduce this concept in a very restrictive fashion. Our aim is to show that optimization in regard to the choice of a variable "smoothes" the production function in the same way as did the hypothesis of a competitive capital market. The input is taken to be free ; it will shown at the end of this section that a costly input would yield similar results.

Let ℓ be the input level and with it is associated a productivity index per unit of capital $p(\ell)$, which rises from zero to a maximum at ℓ^* and then decreases. Given a capital stock b, the family must use an input level $\ell \geq b$. The linear form of this constraint is not restrictive since the scale of measurement of ℓ is arbitrary. The inequality reflects the need to man so many machines or tend to many heads of cattle for instance. The product which is to be maximized is thus $b.p(\ell)$ and the production function is defined as :

$$\phi(b) = \max_{\ell}\{b.p(\ell) \mid \ell \geq b\}$$

The outcome is obvious : if $b \leq \ell^*$ we choose $\ell = \ell^*$, hence $\phi(b) = b.p(\ell^*)$; if $b \geq \ell^*$ we choose $\ell = b$ and $\phi(b) = b.p(b)$.

We now show that the restrictions imposed on $p(.)$ are compatible with those imposed on $f(.)$ so that $\phi(b)$ has the same shape as $\tilde{f}(b)$. Suppose that the functional forms p and f are such that $p(x) = f(x)/x$ so that p is some average product. Since p reaches a maximum we have :

$$p'(x) = \frac{f'(x).x - f(x)}{x^2} < 0$$

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below the maximum and positive above it. This indicates that the average product of x exceeds its marginal product from the origin up to the maximum of p and the reverse is true afterwards. These restrictions, it will be recalled; are exactly those imposed on the function f. The ϕ function then takes the following form (note that $\ell^* = b^*$) :

$$\phi(b) = \begin{cases} b.p(b^*) = b.f(b^*)/b^* = b.f'(b^*), & \text{if } b \leq b^* \\ b.p(b) = f(b) & \text{if } b \geq b^*. \end{cases}$$

Therefore ϕ and $\tilde{f}$ are identical if we take $p(x) = f(x)/x$. Thus given a production function such as $f(\ell)/\ell$ which represents the output per unit of capital with ℓ units of labour we can define $\phi(b)$ by choosing the labour input that maximizes the productivity of labour, subjects to a constraint, and we obtain a function which is the concave upper contour of f. (If the constraint is $\ell = b$, then $\phi(b) = f(b)$.)

We now briefly consider the case where the input is costly. Suppose that the unit cost of input in terms of the capital good is w.

$$\phi(b) = \max_{\ell}\{b[\frac{f(\ell)}{\ell} - w] \mid \ell \geq b\} \quad , \quad \text{and}$$

$$\phi(b) = \begin{cases} b[f'(b^*) - w], & \text{if } b \leq b^* \\ f(b) - w\,b \quad , & \text{if } b \geq b^* \end{cases}$$

Therefore it is as if we were dealing with a net production function $f(b) - w\,b$ which exhibits a maximum, but we again obtain its concave upper contour. A similar occurence would have followed the imposition of transaction costs in the capital market version.

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III - FAMILY CHOICE

In period t the typical family chooses its consumption $c(t)$, the number of its children $n(t)$ and the bequest to each of these $b(t)$ so as to maximize its utility given by :

$$V = u[c(t)] + v[n(t).\ \tilde{f}[b(t)]]$$

subject to its budget constraint :

$$\tilde{f}[b(t-1)] = c(t) + n(t).\ b(t)$$

Let us first ignore the integer restriction on n. Denoting income by $\bar{f}$ and skipping all time arguments, the first-order conditions are :

$$\bar{f} = c + n\,b$$

$$v'[n\,\tilde{f}(b)]\tilde{f}'(b) = u'[c]$$

$$v'[n\,\tilde{f}(b)]\tilde{f}(b) = b\,u'[c]$$

The last two conditions yield $\tilde{f}'(b) = \tilde{f}(b)/b$. Whereas in Léonard [1984] where the production function had no flat segment this relation defined b^*, here we are free to choose any $b \leq b^*$. What the equation indicates is that for the relevant interval the production function has for equation $\tilde{f}(b) = f'(b^*)\,b$ and substitution of this form into the first-order conditions shows that there is an indeterminacy in the choice of b and n, only their product being well defined.

However in this situation (n real) there is no reason why a capital market should develop since b^*, which is optimal, can always be chosen whithout it. This is definitely true if there is the slightest cost. Indeed since $N\,b/I$ is only an approximation of b^*, families do better by choosing b^* on their own. It is true that if all families jointly decided the value of b, they could find N

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values which yield the optimum exactly ; they are $b_i = i\, b^*/N$, $i = 1...N$. However if some individuals depart from this strategy they bear no cost and every one else is worse off. Therefore we will assume that b^* is always chosen : parents always choose the smallest optimal number of children ; this also guarantees the highest income to each of their children.

Given this assumption the first-order conditions determine the optimal values of n and c, say $\bar{n}$ and $\bar{c}$, given income $\bar{f}$. It was shown in Léonard [1984] that if n is restricted to integer values, it is optimal to choose the integers n^+ and n^- as close as possible to $\bar{n}$ with $n^- \leq \bar{n} \leq n^+$ and to use the optimality conditions to determine the two triplets (c^+,n^+,b^+) and (c^-,n^-,b^-), one of which is the "discrete optimum". The argument hinged upon the shape of f around b^*. In the case of n^-, values of b above b^* are selected, hence the argument applies. Note that the solution (c^-,n^-,b^-) cannot best $(\bar{c},\bar{n},b^*)$.

The value of b^+ on the other hand is below b^* where f and $\tilde{f}$ differ. Consequently we must analyze this case anew. Since $b^+ \leq b^*$ we take $\tilde{f}(b) = f'(b^*)\, b$ and at the optimum we have $\bar{f} = c^+ + n^+ b^+$ and

$$v'\,[n^+ f'(b^*)\, b^+]\, f'(b^*) = u'[c^+]$$

These conditions are identical with those defining the smooth optimum. Therefore they yield the same level of utility and (c^-,n^-,b^-) is never chosen. Note also that the integer restriction causes no loss of welfare in this case.

There is of course again an indeterminacy in the choice of n^+ and b^+. We can take n as large as we wish (but not below the $\bar{n}$ value associated with b^* in the smooth optimum).

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The optimum value of consumption is unique and so is that of nb with the additional restrictions that n be an integer and $b \leq b^*$. There are therefore countably many optimal values of b if the capital market can operate firms with exactly b^* units of capital. However in this case, there is a clear incentive to choose b^+ as large as possible as we now show. The value of nb is fixed, say $n^+ b^+ = ab^*$where a is real, then after choosing n^+ the approximation to b^* is giventy N ab^*/n^+ I where I is also can integer. We can have a better approximation if n^+ is small as illustrated in the following example.

$$N = 2000 \quad a = \quad 3 = 1.732...$$

Choose $n^+ = 2$ (the smallest feasible) then I should be close to 1000 $\sqrt{3}$ and 1732 and 1733 are good approximations. If we choose $n^+ = 2000$ (the largest feasible) then I should be close to 1.732 and 1 or 2 are very poor approximations.

Furthermore since some families may be left out (it benefits the others to get to b^* exactly) there is an additional incentive to choose the largest possible b^+ value obtain a high average product.

To sum up, the optimal procedure is as follows. Find the smooth optimum values $(\bar{c}, \bar{n}, b^*)$. Select the smallest integer $n^+ \geq \bar{n}$ and $b^+ = (\bar{f} - \bar{c})/n^+$.

There are of course some income levels for which $\bar{n}$ is an integer already. An increase in income from that point yields a slightly larger value for $\bar{n}$ hence $n^+ = \bar{n} + 1$. Consider the optimality condition corresponding to the choice of b at this switch point.

$$v'[(\bar{n} + 1)\ b^+\ f'[b^*]]f'[b^*] = u'[c^+]$$

$$v'[\bar{n}\ b^*\ f'[b^*]]\ f'[b^*] = u'[\bar{c}]$$

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Their ratio is :

$$\frac{v'[(\bar{n}+1)\ b^{+}\ f'[b^{*}]]}{v'[\bar{n}\ b^{*}\ f'[b^{*}]]} = \frac{u'[c^{+}]}{u'[\bar{c}]}$$

Suppose $(\bar{n} + 1)\ b^{+} > \bar{n}\ b^{*}$ then the left-hand side is less than unity and so is the right-hand side ; therefore $c^{+} > \bar{c}$ but both choices must be available at the same income, a contradiction. We can show that the reverse inequality also yields a contradiction and we conclude that $c^{+} = \bar{c}$ and $(\bar{n} + 1)\ b^{+} = \bar{n}\ b^{*}$. This shows that there is no drop in family consumption when family size increases and that the new bequest is exactly $b^{*}\ \bar{n}/(\bar{n} + 1)$. It therefore becomes closer to b^{*} as n increases.

With regard to the switch between $n = 0$ and $n = 1$, note that in the smooth case we would always select $n\ b^{*} > 0$ if income was positive. In the discrete case (and when the optimal value of n is less than unity) we can duplicate the level of utility by choosing $n = 1$ and b accordingly but we cannot do it with $n = 0$. Therefore the latter is never chosen and if income is positive families have at least one child.

<u>Proposition 1</u>

The emergence of capital markets leads to larger and poorer families, although childless families are now ruled out. Family consumption now rises continuously with income but discontinuities in bequests persist although the jumps are narrowed. The size of bequests never exceeds b^{*}. When a family swithches from n to $(n + 1)$ children, the new bequest is exactly $n\ b^{*}/(n + 1)$. No loss of welfare is incurred by restricting n to integer values. The specific shape of f is irrelevant execept for the values of b^{*} and $f'(b^{*})$.

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The results of Proposition 1 are illustrated in Figure 2 where the arrows indicate the rise in income.

The optimizing process which yields this income expansion path can be described in a simple way. The first step is to equate the marginal utilities of (c) and (n b) :

$$u'[c] = v'[f'(b^*)\, n\, b]\; f'(b^*)$$

This yields an increasing function $c = g[n\, b]$ where the form of g depends on u, v and $f'(b^*)$. We can then plot the graph of $c = g[b]$ for $0 \leq b \leq b^*$ and the graph of $c = g[2b]$ for $b^*/2 \leq b \leq b^*$ ... In general we plot the graph of $c = g[n\, b]$ for : $\frac{n}{n+1} b^* \leq b \leq b^*$, $n = 0, 1, 2$ .... Note that when the switch from n to $n + 1$ occurs, the slope changes from $n\, g'[n\, b^*]$ to $(n + 1)\, g'[n\, b^*]$. However we cannot claim that the slopes of the higher segments are larger because we do not know the overall shape of $g[b]$. Indeed any increasing function going through the origin is acceptable as g. We only need to choose $g[b]$ and a value for b^* to generate the whole income expansion path.

IV - DYNAMICS

From the budget constraint : $f'(b^*)\, b(t - 1) = c(t) + n(t)\, b(t)$ and the above analysis we can derive the equation of the phase line

$$f'(b^*)\, b(t - 1) = g[n(t)\, b(t)] + n(t)\, b(t)$$

Once again we draw this line, for given n, on $\frac{n}{n+1} b^* \leq b(t) \leq b^*$, $n = 0, 1, 2$ ... The relevant portion of the space is therefore a square including the origin and of side length b^*. Because the shape of g is largely arbitrary it is not possible to derive further restrictions. In particular the first nontrivial steady state (if there is one) need not be unstable because the slope of f near the origin is now irrelevant.

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Proposition 2

(i) After one period, all bequests are less than or equal to b^*.

(ii) There may be no non trivial steady state. All steady states that do exist may be stable or unstable including the trivial one.

(iii) Possible outcomes include extinction in finite time, chaotic behaviour or cycling.

The main effect of the introduction of a capital market has been to lower the level of bequests. Discontinuities remain and with them the possibility of erratic behaviour for which the analysis of the smooth case is irrelevant.

One possible phase line is represented in Figure 3.

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REFERENCE

Léonard, D. "Solomon's Dilemma : The Consequences of Restricting Variables to Integer Values" - Paper presented at the 1984 Australasian Meeting of the Econometric Society, Sydney.

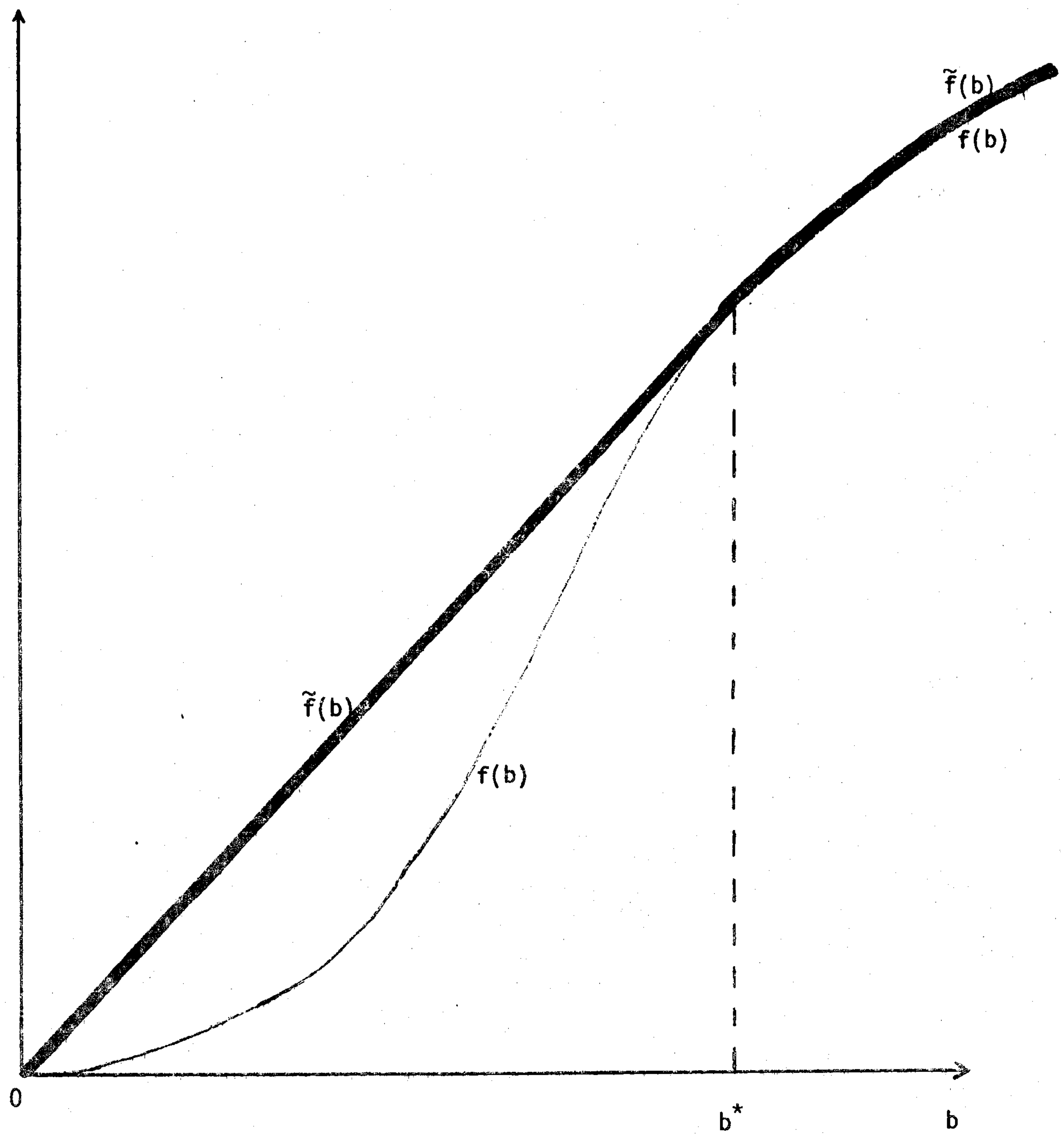

FIGURE 1

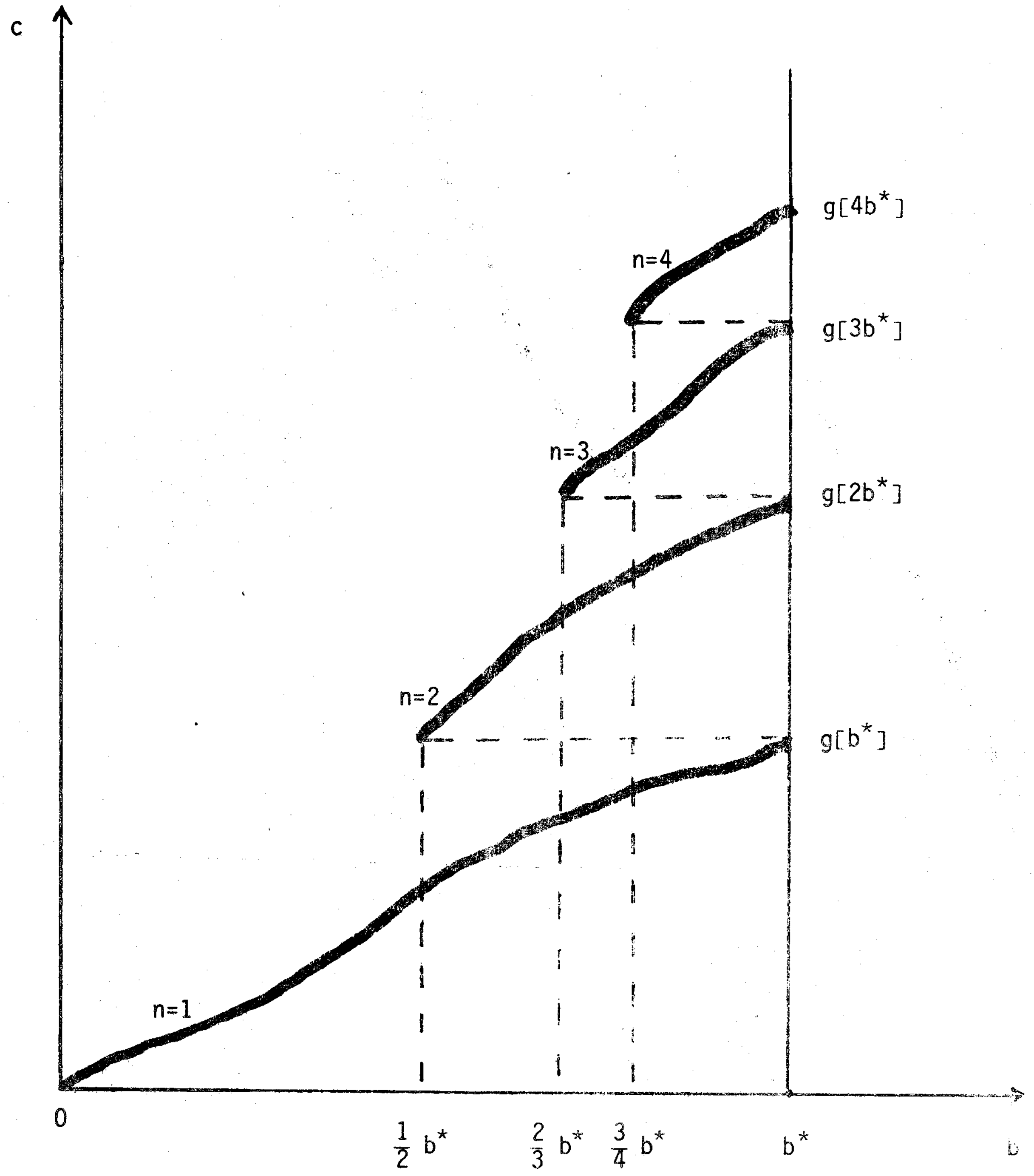

FIGURE 2

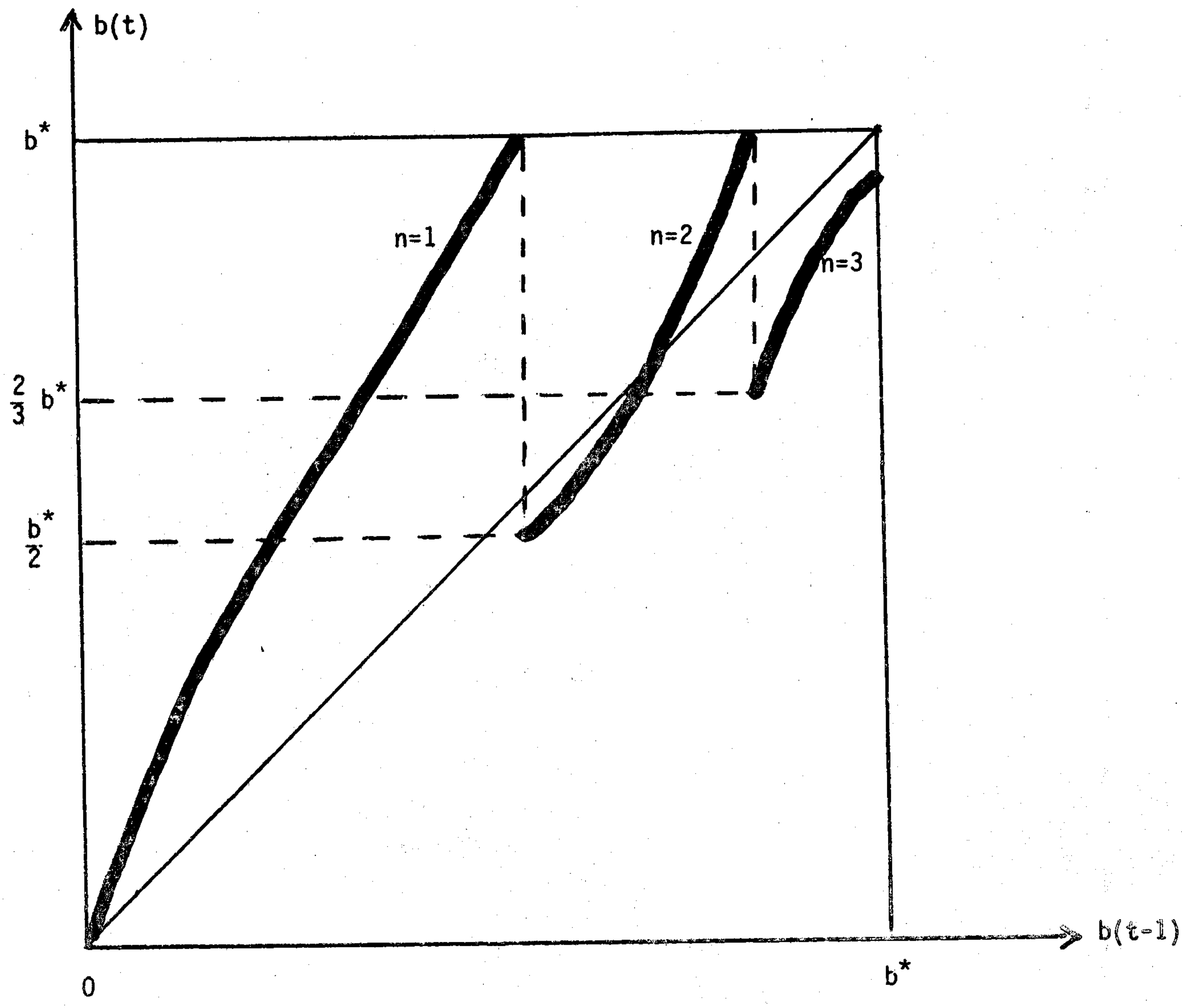

FIGURE 3