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SOLUTIONS OF DYNAMIC LINEAR RATIONAL EXPECTATIONS MODELS

L. BROZE*, C. GOURIEROUX**, A. SZAFARZ***

* IRSIA et Centre d'Economie Mathématique et d'Econométrie de l'Université Libre de Bruxelles.

** Université Paris IX-DAUPHINE et CEPREMAP

*** Centre d'Economie Mathématique et d'Econométrie de l'Université Libre de Bruxelles.

SOLUTIONS D'UN MODELE DYNAMIQUE LINEAIRE AVEC ANTICIPATIONS RATIONNELLES

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R E S U M E

Nous présentons une démarche permettant d'obtenir l'ensemble de toutes les solutions d'un modèle linéaire à anticipations rationnelles quelconque. Cette approche fait jouer un rôle fondamental aux processus de révision intervenant dans la mise à jour des prévisions. Il apparaît que l'ensemble des solutions peut être décrit au moyen d'un nombre de martingales arbitraires, nombre égal au plus grand indice des variables à prévoir. Cette approche se révèle particulièrement adaptée à la recherche de solution admettant une représentation moyenne mobile.

SOLUTIONS OF DYNAMIC LINEAR RATIONAL EXPECTATIONS MODELS

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A B S T R A C T

Linear rational expectations models generally have a large number of solutions. It is thus important to describe them exhaustively in order to study their properties and subsequently estimate which solution fits best the data.

In this paper, a global approach is suggested allowing a simultaneous treatment of all possible cases. The fundamental concepts are the revision processes appearing in the updating procedure of expectations. It comes out that the set of solutions is completely described by using a limited number of these processes.

We show how the method may be applied to determine the set of stationary solutions admitting an infinite moving average representation and we give a natural parametrization of this set and discuss the exact number of independent parameters.

1. INTRODUCTION.

Linear rational expectations (LRE) models generally have a large number of solutions. It is thus important to describe them exhaustively in order to study their properties and subsequently estimate which solution fits best the data.

In the literature various particular models have been analyzed and classes of solutions have been described. Some authors consider the solutions admitting an infinite moving average representation (MUTH (1961), SHILLER (1978), TAYLOR (1977)) or an autoregressive moving average one (EVANS-HONKAPOHJA (1984)). Some others find solutions by using a recursive procedure (BLANCHARD (1979), Mac CALLUM (1976), VISCO (1981), WEGGE(1984)).

For some models, it is also proved that the general solution may be expressed in terms of martingales (BROZE, JANSSEN and SZAFARZ (1984), BROZE and SZAFARZ (1984), GOURIEROUX, LAFFONT and MONFORT (1982), PESARAN (1981)). These techniques give in each case the solution(s), but they are not truly general.

In this paper, a global approach is suggested allowing a simultaneous treatment of all possible cases. The fundamental concepts are the revision processes appearing in the updating procedure of expectations. It comes out in sections 3 and 4 that the set of solutions is completely described by using a limited number of these processes.

In section 5, we show how the method may be applied to determine the set of stationary solutions admitting an infinite moving average representation (called linear stationary solutions). We give a natural parametrization of this set and discuss the exact number of independent parameters.

In section 6, we consider the solutions admitting an infinite moving average representation with time dependent coefficients (called linear solutions). In this case, it is impossible to parametrize the solutions.

The revision processes are martingale difference sequences It is then natural to link them to the martingales introduced by GOURIEROUX, LAFFONT and MONFORT (1982) when they establish the general solution of the homogenous version of the model with rational expectations. This problem is examined in section 7.

2. THE GENERAL FORM OF A DYNAMIC LRE MODEL.

The most general form of a unidimensional LRE model describes the evolution of the endogenous variable y_t as a function of its lagged values $y_{t-1}, y_{t-2}, \ldots$, of its expectations made in the past or the present and of the value u_t of an exogenous process

$$\text{(2.1)} \quad \boxed{y_t = \sum_{k=o}^{K} \sum_{h=o}^{H} a_{kh} \, \hat{y}_{t+h-k|t-k} + u_t}$$

where : $a_{oo} = o$

and $\hat{y}_{t+h-k|t-k} = E(y_{t+h-k} | I_{t-k})$.

The operator $E[.|.]$ refers to the conditional expectation. The information I_t is increasing with t and is composed of the current and past observations of the exogenous process u : $I_t = \{u_\tau, \tau < t\}$.
The information set is thus restricted to the exogenous variables. It is however possible to enlarge this set by introducing other variables, for instance "sunspots".

The indexes h and k are chosen in order to recognize the lead of the expectations (h) and the period in which they are made (t-k) .

The condition $k > o$ implies that one cannot make predictions using future observations of the variables. There is at most, for $k = o$, a simultaneity between the dates at which y_t and the expectations are determined . This assumption implies in particular that y_t is a function of the variables appearing in I_t .

Thus we have : $E(y_{t+h-k} | I_{t-k}) = y_{t+h-k}$ if $h < o$.

This invariance property allows to consider, without loss of generality, only non-negative values of h. In the case $h = o$, we have :

$$E(y_{t-k} | I_{t-k}) = y_{t-k} .$$

Therefore expression (2.1) covers the case of models containing lagged endogenous variables.

Finally by imposing $a_{oo} = o$, we only mean that y_t is not expressed as a function of itself.

Equation (2.1) may be reformulated as follows :

$$(2.2) \quad \boxed{y_t = \sum_{k=o}^{K} \sum_{h=1}^{H} a_{kh} \, E(y_{t+h-k} | I_{t-k}) + \sum_{k=1}^{K} a_{ko} \, y_{t-k} + u_t} \, .$$

It is also interesting to introduce another indexation of the expectations. Let $i = h - k$, we may write :

$$(2.3) \quad y_t = \sum_{(k,i) \in J} \sum a_{k,i+k} \, E(y_{t+i} | I_{t-k}) + u_t \, ,$$

where $J = \{(k,h-k)$ with $h \in \{o,1,\ldots,H\}$, $k \in \{o,1,\ldots,K\}\}$.

Generally, i takes negative as well as positive values. The model may contain expectations of future variables, but also expectations of past variables made in previous periods.

3. EXPRESSION OF THE SOLUTIONS IN TERMS OF THE SUCCESSIVE REVISIONS OF THE EXPECTATIONS.

Consider successive expectations of the variable y_{t+j} . They are generally different from each other. Let ε_t^j be the difference between the two expectations of y_{t+j} made respectively at time t and at time $t-1$. ε_t^j gives the revision of the expectation of y_{t+j} between $t-1$ and t .

$$(3.1) \quad \boxed{\varepsilon_t^j = E(y_{t+j} | I_t) - E(y_{t+j} | I_{t-1})} \quad .$$

The variable ε_t^j is I_t-measurable and its mean is zero. Moreover it is orthogonal to all past informations : $E(\varepsilon_t^j | I_{t-1}) = o$.

Therefore the stochastic process $(\varepsilon_t^j , t \in \mathbb{Z})$ is a martingale difference sequence[1] (denoted henceforth ΔM)

The prediction errors at any lead may easily be expressed as functions of the revision processes.

Let $v_{t,h}$ be the error made in predicting y_t over h periods

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(1) An integrable stochastic process $x = (x_t)$ is called a martingale difference sequence with respect to an increasing sequence of informations (I_t) iff

(i) $E(x_t | I_t) = x_t \quad \forall t$

(ii) $E(x_t | I_{t-1}) = o \quad \forall t$

$$(3.2) \quad \boxed{v_{t,h} = y_t - E(y_t | I_{t-h})} \; .$$

We have

$$v_{t,o} = o \quad \forall\, t$$

$$\begin{aligned} v_{t,1} &= y_t - E(y_t | I_{t-1}) \\ &= \varepsilon_t^o \quad \forall\, t \end{aligned}$$

In the general case, we decompose $v_{t,h}$ $(h > o)$ as follows :

$$\begin{aligned} v_{t,h} &= y_t - E(y_t | I_{t-h}) \\ &= y_t - E(y_t | I_{t-1}) \\ &\quad + E(y_t | I_{t-1}) - E(y_t | I_{t-2}) \\ &\quad + \ldots\ldots \\ &\quad + E(y_t | I_{t-h-1}) - E(y_t | I_{t-h}) \end{aligned}$$

or equivalently :

$$(3.3) \quad \boxed{\begin{aligned} v_{t,h} &= \varepsilon_t^o + \varepsilon_{t-1}^1 + \ldots + \varepsilon_{t-h+1}^{h-1} \\ &= \sum_{j=o}^{h-1} \varepsilon_{t-j}^j \quad ; \quad h > o \end{aligned}} \; .$$

Let us now consider the LRE model expressed by (2.2). If we replace each expectation in terms of the corresponding realization and of the successive revisions, we obtain :

$$y_t = \sum_{k=o}^{K} \sum_{h=1}^{H} a_{kh} \left(y_{t+h-k} - \sum_{j=o}^{h-1} \varepsilon^{j}_{t+h-k-j}\right)$$

$$+ \sum_{k=1}^{K} a_{ko}\, y_{t-k} + u_t$$

or

$$(3.4) \quad \boxed{\sum_{i=I_o}^{I_1} a_i^* \, y_{t+i} = \sum_{k=o}^{K} \sum_{h=1}^{H} a_{kh} \sum_{j=o}^{h-1} \varepsilon^{j}_{t+h-k-j} - u_t}$$

where $a_i^* = \sum_{k:(k,i)\in J} a_{k,i+k} \qquad \forall\, i \neq o$

$$a_i^* = -1 + \sum_{k:(k,o)\in J} a_{kk} \quad .$$

From now on, we assume that the coefficients a_{kh} may only be subject to nullity constraints. We denote respectively I_o and I_1 the minimal and the maximal values of index i for which a_i^* is not zero.

From the definitions of I_o and I_1, we have

$$a^*_{I_o} \neq o \qquad \text{and} \qquad a^*_{I_1} \neq o \quad .$$

Moreover, since $i = h - k$, $o \leqslant h \leqslant H$, $o \leqslant k \leqslant K$, the maximal value I_1 is not larger than H .

Equation (3.4) shows that any solution y verifies a recursive linear equation with constant coefficients a_i^*, $i = I_o,\dots,I_1$. The right-hand side is a function of the exogenous process u and of various ΔM processes (ε_t^j). The number of these ΔM processes is equal to H, i.e. to the maximal lead of the expectations.

Formula (3.4) is a *necessary form* of the solutions y. Conversely we may wonder whether this form is sufficient, i.e. whether any process verifying (3.4) with arbitrary ΔM processes (ε_t^j) does also verify model (2.1).

As shown below, when $H = I_1$ it is possible to choose each (ε_t^j) independently from the others. On the contrary, when $I_1 < H$ it is necessary to impose constraints linking the revision processes. In such a case there is a restriction on the number of the arbitrary ΔM processes. We have the following general result.

<u>THEOREM (3.5)</u> : Let us consider model (2.1)

$$y_t = \sum_{k=o}^{K} \sum_{h=o}^{H} a_{kh} \, E(y_{t+h-k} | I_{t-k}) + u_t \, , \; a_{oo} = o \, ,$$

$$I_1 = \max \{ i : a_i^* \neq o \}$$

and let us assume that the coefficients a_{kh}

may only be subject to nullity constraints.

- If $\underline{I_1 = H}$, the reduced form of the model entails H arbitrary ΔM processes and (3.4) is a sufficient form of the solution.
- If $\underline{o < I_1 < H}$, the reduced form of the model entails I_1 arbitrary ΔM processes.
- If $\underline{I_1 = o}$, the solution is generically unique.

In order to lighten the proof of this theorem, we give it in the case H = K = 2 .

It is however easy to see that the approach is absolutely general.

4. RESOLUTION FOR H = K = 2 .

For H = K = 2 , the initial model is

$$\boxed{\begin{aligned} y_t = \; & a_{o2}\; E(y_{t+2}|I_t) \\ & + a_{o1}\; E(y_{t+1}|I_t) + a_{12}\; E(y_{t+1}|I_{t-1}) \\ & + a_{11}\; E(y_t|I_{t-1}) + a_{22}\; E(y_t|I_{t-2}) \\ & + a_{1o}\; y_{t-1} + a_{21}\; E(y_{t-1}|I_{t-2}) \\ & + a_{2o}\; y_{t-2} + u_t \; . \end{aligned}} \tag{4.1}$$

The recursive equation containing the revision processes becomes :

$$
(4.2) \quad \boxed{\begin{aligned} y_t \;=\; & a_{o2}(y_{t+2} - \varepsilon^{o}_{t+2} - \varepsilon^{1}_{t+1}) \\ & + a_{o1}(y_{t+1} - \varepsilon^{o}_{t+1}) + a_{12}(y_{t+1} - \varepsilon^{o}_{t+1} - \varepsilon^{1}_{t}) \\ & + a_{11}(y_t - \varepsilon^{o}_{t}) + a_{22}(y_t - \varepsilon^{o}_{t} - \varepsilon^{1}_{t-1}) \\ & + a_{1o}\, y_{t-1} + a_{21}(y_{t-1} - \varepsilon^{o}_{t-1}) \\ & + a_{2o}\, y_{t-2} + u_t \;. \end{aligned}}
$$

Therefore we have :

$$
\begin{aligned} a^*_2 &= a_{o2} \\ a^*_1 &= a_{o1} + a_{12} \\ a^*_o &= -1 + a_{11} + a_{22} \\ a^*_{-1} &= a_{10} + a_{21} \\ a^*_{-2} &= a_{20} \;. \end{aligned}
$$

The maximal value I_1 depends on the potential nullity of these coefficients. If $a^*_2 \neq o$, we have $I_1 = H = 2$. If $a^*_2 = o$, I_1 is smaller than H. If $a^*_2 = o$ and $a^*_1 \neq o$, we have $I_1 = H - 1 = 1$, and so on.

Equation (4.2) involves two revision processes (ε^o_t) and (ε^1_t). They are respectively defined by

$$
(4.3) \quad \left\{ \begin{aligned} \varepsilon^o_t &= y_t - E(y_t | I_{t-1}) \\ \\ \varepsilon^1_t &= E(y_{t+1} | I_t) - E(y_{t+1} | I_{t-1}) \;. \end{aligned} \right.
$$

If we apply the operator $E(.|I_{t-1})$ to both sides of model (4.1), we obtain :

$$\begin{aligned}E(y_t|I_{t-1}) = \; & a_{o2}\, E(y_{t+2}|I_{t-1}) + a_{o1}\, E(y_{t+1}|I_{t-1}) \\ & + a_{12}\, E(y_{t+1}|I_{t-1}) + a_{11}\, E(y_t|I_{t-1}) \\ & + a_{22}\, E(y_t|I_{t-2}) + a_{1o}\, y_{t-1} \\ & + a_{21}\, E(y_{t-1}|I_{t-2}) + a_{20}\, y_{t-2} + E(u_t|I_{t-1}) \; .\end{aligned}$$

Substracting this equation to (4.1), we have :

$$\begin{aligned}y_t - E(y_t|I_{t-1}) = \; & a_{o2}(E(y_{t+2}|I_t) - E(y_{t+2}|I_{t-1})) \\ & + a_{o1}\, (E(y_{t+1}|I_t) - E(y_{t+2}|I_{t-1})) \\ & + u_t - E(u_t|I_{t-1})\end{aligned}$$

or equivalently :

$$(4.4) \quad \boxed{\varepsilon_t^o = a_{o2}\, \varepsilon_t^2 + a_{o1}\, \varepsilon_t^1 + u_t - E(u_t|I_{t-1})}$$

where (ε_t^2) is given by :

$$\varepsilon_t^2 = E(y_{t+2}|I_t) - E(y_{t+2}|I_{t-1}) \; .$$

A similar algebra leads to :

$$\begin{aligned}E(y_t|I_{t-1}) - E(y_t|I_{t-2}) = \; & a_{o2}(E(y_{t+2}|I_{t-1}) - E(y_{t+2}|I_{t-2})) \\ & + (a_{o1} + a_{12})\, (E(y_{t+1}|I_{t-1}) - E(y_{t+1}|I_{t-2})) \\ & + a_{11}\, (E(y_t|I_{t-1}) - E(y_t|I_{t-2})) \\ & + a_{1o}\, (y_{t-1} - E(y_{t-1}|I_{t-2})) \\ & + E(u_t|I_{t-1}) - E(u_t|I_{t-2})\end{aligned}$$

Or equivalently :

(4.5)
$$\boxed{\begin{aligned}\varepsilon_t^1 = {} & a_{o2}\,\varepsilon_t^3 + (a_{o1}+a_{12})\,\varepsilon_t^2 + a_{11}\,\varepsilon_t^1 + a_{1o}\,\varepsilon_t^o \\ & + E(u_{t+1}|I_t) - E(u_{t+1}|I_{t-1})\end{aligned}}$$

where (ε_t^3) is given by :

$$\varepsilon_t^3 = E(y_{t+3}|I_t) - E(y_{t+3}|I_{t-1}) .$$

Equations (4.4) and (4.5) are useful to show why, in some cases, links between the revision processes have to be imposed.

<u>Case $I_1 = H = 2$</u>

When $E(y_{t+2}|I_t)$ appears in the original model the coefficient $a_2^* = a_{o2}$ is different from zero and $I_1 = H = 2$. In this case equations (4.4) and (4.5) contain "new" processes ε^2 and ε^3 . They imply no constraint on ε^o and ε^1 .

<u>Lemma (4.6)</u> : When $a_{o2} \neq o$, every process verifying the recursive equation (4.2) :

$$
\begin{aligned}
y_t = {} & a_{o2}(y_{t+2} - \varepsilon^o_{t+2} - \varepsilon^1_{t+1}) \\
& + a_{o1}(y_{t+1} - \varepsilon^o_{t+1}) + a_{12}(y_{t+1} - \varepsilon^o_{t+1} - \varepsilon^1_t) \\
& + a_{11}(y_t - \varepsilon^o_t) + a_{22}(y_t - \varepsilon^o_t - \varepsilon^1_{t-1}) \\
& + a_{1o}\, y_{t-1} + a_{21}(y_{t-1} - \varepsilon^o_{t-1}) \\
& + a_{2o}\, y_{t-2} + u_t
\end{aligned}
$$

where the revision processes ε^o and ε^1 are arbitrary ΔM processes, is a solution of model (4.1).

Proof : Taking the expectation of (4.2) given I_{t+1} yields :

$$
\begin{aligned}
y_t = {} & a_{o2}\,(E(y_{t+2}|I_{t+1}) - \varepsilon^1_{t+1}) \\
& + a_{o1}(y_{t+1} - \varepsilon^o_{t+1}) + a_{12}(y_{t+1} - \varepsilon^o_{t+1} - \varepsilon^1_t) \\
& + a_{11}(y_t - \varepsilon^o_t) + a_{22}(y_t - \varepsilon^o_t - \varepsilon^1_{t-1}) \\
& + a_{10}\, y_{t-1} + a_{21}(y_{t-1} - \varepsilon^o_{t-1}) + a_{2o}\, y_{t-2} + u_t \, .
\end{aligned}
$$

We compare this last expression with (4.2) to get (since $a_{o2} \neq o$) :

$$
(4.7) \qquad \varepsilon^o_{t+2} = y_{t+2} - E(y_{t+2}|I_{t+1})
$$

Therefore we find again the initial interpretation of ε^o .
Replacing (4.7) in (4.2), y_t may also be written as :

$$
\begin{aligned}
(4.8) \qquad y_t = {} & a_{o2}\,(E(y_{t+2}|I_{t+1}) - \varepsilon^1_{t+1}) \\
& + a_{o1}\, E(y_{t+1}|I_t) + a_{12}(E(y_{t+1}|I_t) - \varepsilon^1_t)
\end{aligned}
$$

$$+ \quad a_{11}\ E(y_t|I_{t-1}) \quad + \quad a_{22}\ (E(y_t|I_{t-1}) - \varepsilon^1_{t-1})$$
$$+ \quad a_{1o}\ y_{t-1} \quad + \quad a_{21}\ E(y_{t-1}|I_{t-2})$$
$$+ \quad a_{2o}\ y_{t-2} \quad + \quad u_t$$

Taking the expectation of (4.8) given I_t and using the condition $a_{o2} \neq o$, we get :

$$(4.9) \qquad \varepsilon^1_{t+1} = E(y_{t+2}|I_{t+1}) - E(y_{t+2}|I_t)$$

Finally, the replacement of ε^1_t by its expression (4.9) gives the original model (4.1).

Q.E.D.

<u>Case : $I_1 = H - 1 = 1$</u>

Let us now consider the case : $a_{o2} = o$. It follows that : $a^*_2 = o$. When $a^*_1 = a_{o1} + a_{12} \neq o$, we have $I_1 = H-1 = 1$ and model (4.1) reduces to :

$$(4.10) \quad \boxed{\begin{aligned} y_t = \ & a_{o1}\ E(y_{t+1}|I_t) + a_{12}\ E(y_{t+1}|I_{t-1}) \\ & + a_{11}\ E(y_t|I_{t-1}) + a_{22}\ E(y_t|I_{t-2}) \\ & + a_{1o}\ y_{t-1} + a_{21}\ E(y_{t-1}|I_{t-2}) \\ & + a_{2o}\ y_{t-2} + u_t \end{aligned}}$$

(4.4) and (4.5) become :

$$\boxed{\begin{aligned} &(4.11) \quad \varepsilon_t^o = a_{o1}\,\varepsilon_t^1 + u_t - E(u_t|I_{t-1}) \\ &(4.12) \quad \varepsilon_t^1 = (a_{o1}+a_{12})\,\varepsilon_t^2 + a_{11}\,\varepsilon_t^1 + a_{1o}\,\varepsilon_t^o \\ &\qquad\qquad + E(u_{t+1}|I_t) - E(u_{t+1}|I_{t-1}) \end{aligned}}$$

We deduce from (4.11) that ε^o and ε^1 are linked.

<u>Lemma (4.13)</u> : When $a_{o2} = o$ and $a_{o1} + a_{12} \neq o$, every process verifying the recursive equation

$$\begin{aligned} y_t &= a_{o1}(y_{t+1} - \varepsilon_{t+1}^o) + a_{12}(y_{t+1}-\varepsilon_{t+1}^o-\varepsilon_t^1) \\ (4.14) \qquad &+ a_{11}(y_t - \varepsilon_t^o) + a_{22}(y_t - \varepsilon_t^o - \varepsilon_{t-1}^1) \\ &+ a_{1o}\, y_{t-1} + a_{21}(y_{t-1}-\varepsilon_{t-1}^o) + a_{2o}y_{t-2} + u_t \end{aligned}$$

where ε^1 is an arbitrary ΔM process and where

$\varepsilon_t^o = a_{o1}\,\varepsilon_t^1 + u_t - E(u_t|I_{t-1})$

is a solution of model (4.10).

<u>Proof</u> : We take the expectation of (4.14) given I_t and we substract the result to (4.14) to yield (with $a_{o1} + a_{12} \neq o$) :

$$(4.15) \qquad \varepsilon_{t+1}^o = y_{t+1} - E(y_{t+1}|I_t)$$

Replacing (4.15) in (4.14) gives :

$$\begin{aligned} y_t &= a_{o1}\,E(y_{t+1}|I_t) - a_{12}\,(E(y_{t+1}|I_t) - \varepsilon_t^1) \\ (4.16) \qquad &+ a_{11}\,E(y_t|I_{t-1}) + a_{22}(E(y_t|I_{t-1}) - \varepsilon_{t-1}^1) \\ &+ a_{1o}\,y_{t-1} + a_{21}\,E(y_{t-1}|I_{t-2}) + a_{2o}\,y_{t-2} + u_t \end{aligned}$$

Now we take the expectation of (4.16) given I_{t-1} and we substract the result to (4.16) :

$$(4.17) \quad \begin{aligned} y_t - E(y_t|I_{t-1}) &= (a_{01}+a_{12})\,(E(y_{t+1}|I_t) - E(y_{t+1}|I_{t-1})) \\ &\quad - a_{12}\,\varepsilon_t^1 + u_t - E(u_t|I_{t-1}) \end{aligned}$$

Using (4.11), (4.15) and the assumption $a_{01} + a_{12} \neq 0$, we deduce :

$$\varepsilon_t^1 = E(y_{t+1}|I_t) - E(y_{t+1}|I_{t-1})$$

which is the initial interpretation of ε^1 .
Replacing the last expression in (4.26), we find again the original model.
In this case the revision process $\varepsilon_t^1 = E(y_{t+1}|I_t) - E(y_{t+1}|I_{t-1})$ is the only arbitrary ΔM process.
In terms of this process, the recursive equation becomes :

$$(4.18) \quad \boxed{\begin{aligned} (a_{01} + a_{12})\, y_t &= (1 - a_{11} - a_{22})y_{t-1} - (a_{10}+a_{21})y_{t-2} \\ &\quad - a_{20}\, y_{t-3} + a_{01}(a_{01}+a_{12})\,\varepsilon_t^1 \\ &\quad + (a_{12}+a_{01}(a_{11}+a_{22}))\,\varepsilon_{t-1}^1 + (a_{22}+a_{21}a_{01})\,\varepsilon_{t-2}^1 \\ &\quad + a_{01}(a_{01}+a_{12})\,(u_t - E(u_t|I_{t-1})) - u_{t-1} \\ &\quad + a_{01}(a_{11}+a_{22})\,(u_{t-1} - E(u_{t-1}|I_{t-2})) \\ &\quad + a_{21}a_{01}\,(u_{t-2} - E(u_{t-2}|I_{t-3})) \end{aligned}}$$

Q.E.D.

Case : $I_1 = H - 2 = 0$

Let us finally consider the case: $a_{o2} = a_{o1} = a_{12} = o$. It follows that $a_2^* = a_1^* = o$. Since generically $a_o^* = -1 + a_{11} + a_{22} \neq o$, we have $I_1 = H - 2 = o$ and model (4.1) reduces to :

$$
\boxed{\begin{aligned} y_t &= a_{11}\, E(y_t | I_{t-1}) + a_{22}\, E(y_t | I_{t-2}) \\ &+ a_{1o}\, y_{t-1} + a_{21}\, E(y_{t-1} | I_{t-2}) \\ &+ a_{2o}\, y_{t-2} + u_t \end{aligned}} \tag{4.19}
$$

(4.4) and (4.5) become :

$$
\varepsilon_t^o = u_t - E(u_t | I_{t-1}) \tag{4.20}
$$

$$
(1-a_{11})\, \varepsilon_t^1 = a_{1o}\, \varepsilon_t^o + E(u_{t+1} | I_t) - E(u_{t+1} | I_{t-1}) \tag{4.21}
$$

ε^o and ε^1 are uniquely determined by these relations (when $a_{11} \neq 1$).

Lemma (4.22) : When $a_{o2} = a_{o1} = a_{12} = o$, $a_{11} \neq 1$ and $a_{11} + a_{22} \neq 1$, the process verifying the recursive equation (4.23) :

$$
\begin{aligned} y_t &= a_{11}(y_t - \varepsilon_t^o) + a_{22}(y_t - \varepsilon_t^o - \varepsilon_{t-1}^1) \\ &+ a_{1o}\, y_{t-1} + a_{21}\,(y_{t-1} - \varepsilon_{t-1}^o) \\ &+ a_{2o}\, y_{t-2} + u_t \end{aligned}
$$

where ε^{o} and ε^{1} are the ΔM processes determined by (4.20) and (4.21), is the unique solution of model (4.19).

Proof : We take the expectation of (4.23) given I_{t-1} and we substract the result from (4.23) :

$$y_t - E(y_t|I_{t-1}) = (a_{11}+a_{22})(y_t - E(y_t|I_{t-1})) - (a_{11}+a_{22})\ \varepsilon_t^{o} + u_t - E(u_t|I_{t-1})$$

or, using (4.20) and the assumption $1 - a_{11} - a_{22} \neq o$:

$$\varepsilon_t^{o} = y_t - E(y_t|I_{t-1}) \tag{4.24}$$

Replacing (4.24) in (4.23), we get :

$$\begin{aligned} y_t = {} & a_{11}\ E(y_t|I_{t-1}) + a_{22}(E(y_t|I_{t-1}) - \varepsilon_{t-1}^{1}) \\ & + a_{1o}\ y_{t-1} + a_{21}\ E(y_{t-1}|I_{t-2}) + a_{2o}\ y_{t-2} \\ & + u_t \end{aligned} \tag{4.25}$$

On the other hand, the difference between the expectations of (4.23) given I_{t-1} and I_{t-2} leads to :

$$\begin{aligned} E(y_t|I_{t-1}) - E(y_t|I_{t-2}) = {} & (a_{11}+a_{22})\ (E(y_t|I_{t-1}) - E(y_t|I_{t-2})) \\ & + a_{22}\ \varepsilon_{t-1}^{1} + a_{1o}(y_{t-1} - E(y_{t-1}|I_{t-2})) \\ & + E(u_t|I_{t-1}) - E(u_t|I_{t-2}) \end{aligned}$$

and using (4.21), we deduce :

$$(4.26) \qquad \varepsilon_{t-1}^{1} = E(y_t|I_{t-1}) - E(y_t|I_{t-2})$$

Finally we replace (4.26) in (4.25) to find again the original model (4.19). Therefore the solution is given by :

$$(4.27) \qquad \boxed{\begin{aligned}(1 - a_{11}-a_{22})y_t = {} & (a_{1o}+a_{21})y_{t-1} + a_{20}\, y_{t-2} \\ & + u_t - (a_{11} + a_{22})\ (u_t - E(u_t|I_{t-1})) \\ & - \frac{a_{22}}{1-a_{11}}\ (E(u_t|I_{t-1}) - E(u_t|I_{t-2})) \\ & - (\frac{a_{22}a_{1o}}{1 - a_{11}} + a_{21})\ (u_{t-1} - E(u_{t-1}|I_{t-2}))\end{aligned}}$$

Q.E.D.

The three lemmas (4.6), (4.13) and (4.22) cover all possible generic cases under nullity constraints. Theorem (3.5) is thus proved for $H = K = 2$.

Remarks

1) Theorem (3.5) shows that the LRE model (2.1) has in general a large number of solutions. This multiplicity has two causes. First it results from the natural dynamics of the model. The recursive equation allows to choose a finite number of initial conditions. The second cause is due to the arbitrary choice of the ΔM processes which determine the right-hand side of the equation.

This second source of multiplicity is of a different kind since an infinite number of initial conditions is not sufficient to fix a unique path .

2) Any second order process (y_t) , satisfying the condition $\lim_{k\to\infty} E(y_t|I_{t-k}) = o$ may be decomposed as : $y_t = \sum_{i=o}^{\infty} \varepsilon_{t-i}^{i}$ and thus may be determined by a given infinite sequence of revision processes $((\varepsilon_t^o),(\varepsilon_t^1),(\varepsilon_t^2),\dots)$. Therefore, when such a process is not constrained by a LRE model, one needs an *infinite number* of ΔM processes to fix it. Our results show that, when a LRE model is imposed, it appears a strong restriction on the number of arbitrary ΔM processes. This number is always *finite* and equal to I_1 .
In conclusion, even if the multiplicity of all solutions of a LRE model seems quite considerable, it is not comparable to the multiplicity of all second order processes.

3) When $I_1 > o$, the solutions of the model satisfy a recursive equation of the form :

$$(4.28) \qquad \boxed{\sum_{i=I_0}^{I_1} a_i^* \, y_{t+i} = \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} c_{jk} \, \varepsilon^j_{t+I_1-k} + g_t(u)}$$

where $g_t(u)$ is a known function of the exogenous process. When the model does not contain expectations of future variables, the maximal index I_1 is zero. The solution is then generically unique and satisfies the recursive equation

$$(4.29) \qquad \sum_{i=I_0}^{0} a_i^* \, y_{t+i} = f_t(u)$$

where $f_t(u)$ is a known function of the exogenous process. This last result might have been directly obtained by a projection technique (see e.g. VISCO (1981)). Here it has been established using the revision processes in order to emphasize the generality of the method.

It should also be noted that the generic unicity of the solution may be tested as soon as consistent estimators of the coefficients of the model are available, since the unicity may be expressed by means of nullity constraints on the coefficients.

4) Let us now consider the so-called "perfect foresight solution" which is the solution of the perfect foresight version of the model. The cases $I_1 = H$ and $I_1 < H$

must be distinguished. When $I_1 = H$, the "perfect foresight solution" is always a solution of the model. It corresponds to $\varepsilon_t^o = \varepsilon_t^1 = o$ for all t.
However, when $I_1 < H$ it is no more a solution of the original model since (ε_t^0) and/or (ε_t^1) are constrained and in general they cannot simultaneously vanish for all t. In such a case, the model admits no perfect foresight solution.

5) Since the revision processes have a zero mean, the recursive equation giving the evolution of the mean of the solutions does not depend on the choice of the ΔM sequences.

5. LINEAR STATIONARY SOLUTIONS.

5.1 - The general form of the solutions.

We assume that the exogenous process is stationary and has an infinite moving average representation :

$$u_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} \cdots\cdots - \theta_p \varepsilon_{t-p} \cdots\cdots = \Theta(B)\, \varepsilon_t$$

where (ε_t) is an independent white noise and $\sum_{i=o}^{\infty} |\theta_i| < +\infty$

We are interested in the stationary solutions with the same kind of representation :

$$y_t = \alpha_o \varepsilon_t - \alpha_1 \varepsilon_{t-1} \cdots - \alpha_p \varepsilon_{t-p} \cdots = A(B)\ \varepsilon_t$$

with $\sum_{i=o}^{\infty} |\alpha_i| < +\infty$.

These solutions are called *linear stationary solutions* ; they are obtained from the decomposition involving the revision processes (ε_t^j) , $j = H-I_1,\ldots,H-1$.

When $I_1 > o$, using (4.28), the general solution of model (2.1) verifies :

(5.1) $$\sum_{i=I_o}^{I_1} a_i^* \, y_{t+i} = \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} c_{jk} \, \varepsilon_{t+I_1-k}^j + g_t(u)$$

The assumption on the structure of (u_t) makes $g_t(u)$ linear in ε_t , ε_{t-1} ,... . We will denote it $\Omega(B)\ \varepsilon_t$.

<u>THEOREM (5.2)</u> : The linear stationary solutions are such that
$\varepsilon_t^j = \psi_j \ \varepsilon_t \quad \forall \ t \in \mathbb{Z}$, $\forall j = H-I_1,\ldots,H-1$
where the ψ_j are I_1 real numbers.

<u>Proof</u> : Since $y_t = \sum_{i=o}^{\infty} \alpha_i \ \varepsilon_{t-i}$, the revision processes are necessarily proportional to ε :

$$\varepsilon_t^j = E(y_{t+j}|I_t) - E(y_{t+j}|I_{t-1}) = -\alpha_j \ \varepsilon_t$$

Q.E.D.

Consequently (5.1) becomes :

$$(5.3)\qquad \boxed{\left[\sum_{i=o}^{I_1-I_o} a^*_{I_1-i}\ B^i\right] y_t = \left[\sum_{j=H-I_1}^{H-1}\ \sum_{k=k_j}^{K_j} c_{jk}\ \psi_j\ B^k + B^{I_1}\Omega(B)\right] \varepsilon_t}$$

The right-hand side of this expression is a linear function of ε_t , ε_{t-1} ,...

However arbitrary values of the coefficients ψ_j are not always associated with a stationary solution. To define a stationary process (y_t) in terms of I_1 arbitrary parameters, (5.3) must be such that all roots of the characteristic equation

$$(5.4)\qquad \sum_{i=o}^{I_1-I_o} a^*_{I_1-i}\ \lambda^i = o$$

lie outside the unit circle.

If this is not the case, one has to impose conditions on the ψ_j . Suppose first that only one root λ_1 has a modulus smaller than 1 . To obtain stationary solutions we must impose that the moving-average polynomial

$$\sum_{j=H-I_1}^{H-1}\ \sum_{k=k_j}^{K_j} c_{jk}\ \psi_j\ B^k + \Omega(B)\ B^{I_1}$$

is divisible by $(B - \lambda_1)$. This is done by fixing the value of one parameter ψ_ℓ such that

$$\sum_{j=H-I_1}^{H-1}\ \sum_{k=k_j}^{K_j} c_{jk}\ \psi_j\ \lambda_1^k + \Omega(\lambda_1)\ \lambda_1^{I_1} = o$$

Since this equation is linear in terms of the ψ_j , it admits generically a unique solution.
With the fixed ψ_ℓ , (5.3) defines a stationary process (y_t) for any choice of the $I_1 - 1$ other parameters.
The same argument is applied when N roots λ_1 , λ_2 , $\ldots, \lambda_N$ $(N < I_1)$ have a modulus not larger than 1 .
N parameters must be fixed in order to simplify the recursive equation (5.3) by $(B - \lambda_1)(B - \lambda_2)\ldots(B - \lambda_N)$.
The linear stationary solutions are then described by $I_1 - N$ arbitrary parameters.

Finally, if $N = I_1$ the model admits a generically unique linear stationary solution. If $N > I_1$ the model has generically no linear stationary solutions. These last cases may appear only when $I_o \leq o$ and $I_o < o$ respectively.

<u>THEOREM (5.5)</u> : Consider model (2.1)

$$y_t = \sum_{k=o}^{K} \sum_{h=o}^{H} a_{kh} E(y_{t+h-k} | I_{t-k}) + u_t$$

with $a_{oo} = o$

The characteristic equation associated with its reduced form is

$$\sum_{i=o}^{I_1 - I_o} a^*_{I_1 - i} \lambda^i = o \qquad (5.4)$$

Let N be the number of roots of (5.4) which are inside the unit circle ;

- If $I_1 - N > o$: the general linear stationary solution involves $I_1 - N$ arbitrary parameters ;
- if $I_1 = N$: there is a unique linear stationary solution ;
- if $I_1 - N < o$: there is no linear stationary solution.

We have a natural parametrization of the set of all linear stationary solutions of a general LRE model. The dimension of this set is $I_1 - N$. The arbitrary parameters ψ_j have a simple interpretation. ψ_j is the coefficient of the regression of the revision process ε_t^j on the prediction error of the exogenous process i.e. of $E(y_{t+j}|I_t) - E(y_{t+j}|I_{t-1})$ on $u_t - E(u_t|I_{t-1})$.
This interpretation may be useful to build consistent estimators of the parameters ψ_j . So in the estimation procedure, it is possible not only to estimate the structural form of the model but also to determine the linear stationary solution corresponding to the observations (CHOW (1983 p. 361)).

5.2 - Examples.

Example (5.6) : CAGAN's model

(5.7) $$\boxed{y_t = a\, E(y_{t+1}|I_t) + u_t \quad ; \quad a \neq o}$$

The exogenous variable is assumed to have an ARMA (p,q) representation

$$\Phi(B)\, u_t = \Theta(B)\, \varepsilon_t$$

where the roots of Θ and Φ are outside the unit circle and where the two polynomials have no common root.

The general solution verifies

(5.8) $$(a - B)\, y_t = a\, \varepsilon_t^o - u_{t-1}$$

The linear stationary solutions are such that

$$\varepsilon_t^o = \psi_o\, \varepsilon_t$$

and using (5.8)

$$(a - B)\, y_t = \left(a\, \psi_o - \frac{B\ \Theta(B)}{\Phi(B)}\right) \varepsilon_t$$

or equivalently

(5.9) $$\boxed{(a - B)\, \Phi(B)\, y_t = (a\, \psi_o\, \Phi(B) - B\, \Theta(B))\, \varepsilon_t}$$

For $|a| > 1$, (5.9) defines a stationary process for any value of ψ_0 . For $|a| < 1$, the polynomial $a - B$ is not invertible and the value of ψ_0 must be fixed in order to make the MA polynomial $a\, \psi_0\, \Phi(B) - B\, \Theta(B)$ divisible by $a - B$. The condition is :

$$a\, \psi_0\, \Phi(a) \; - \; a\, \Theta(a) \; = \; o$$

or, since $\Phi(a) \neq o$:

$$\psi_0 = \frac{\Theta(a)}{\Phi(a)}$$

The linear stationary solution is unique.

Consequently, for $|a| > 1$, $\Theta(a) \neq o$ and $\Phi(a) \neq o$, the model has an infinity of linear stationary solutions with ARMA $(p+1, \max\{p,q+1\})$ representation, one with an ARMA $(p+1,q+1)$ representation (for $\psi_0 = o$) and one with an ARMA$(p,\max\{p-1,q\})$ representation (for $\psi_0 = \frac{\Theta(a)}{\Phi(a)}$).

For $|a| < 1$, the model has only one linear stationary solution and this solution has an ARMA$(p,\max\{p-1,q\})$ representation.

Example (5.10)

$$\boxed{y_t = a\, E(y_{t+2}|I_t) + b\, E(y_{t+1}|I_t) + \varepsilon_t \quad ; \quad a \neq o} \tag{5.11}$$

where (ε_t) is an independent white noise.

The general solution of (5.11) verifies

$$(5.12) \qquad (a+bB-B^2)\ y_t = (a+bB)\ \varepsilon_t^o + aB\ \varepsilon_t^1 - B^2\ \varepsilon_t \ .$$

The linear stationary solutions are such that

$$\varepsilon_t^o = \psi_o\ \varepsilon_t$$

$$\varepsilon_t^1 = \psi_1\ \varepsilon_t \ .$$

Let λ_1 and λ_2 be the roots of the characteristic equation :

$$a + b\ \lambda - \lambda^2 = o$$

(5.12) becomes

$$(5.13) \qquad \boxed{-\ (B-\lambda_1)(B-\lambda_2)\ y_t = (a\psi_o + (b\psi_o + a\psi_1)B - B^2)\ \varepsilon_t}$$

As soon as $|\lambda_1| > 1$ and $|\lambda_2| > 1$, (5.13) defines a stationary process for any value of the parameters ψ_o and ψ_1. When $|\lambda_1| < 1$ and $|\lambda_2| > 1$, ψ_1 must be chosen so that the factor $(B-\lambda_1)$ appears in the right-hand side of (5.13) i.e.

$$a\ \psi_o + (b\ \psi_o + a\ \psi_1)\ \lambda_1 - \lambda_1^2 = o \ .$$

Since λ_1 is a root of the characteristic equation, the value of ψ_1 is given by :

$$\psi_1 = \frac{1}{a}\ (\lambda_1 - \lambda_1\ \psi_o) \ .$$

By deleting the common factor, (5.13) becomes

$$(5.14) \qquad \boxed{-\ (B - \lambda_2)\ y_t \ = \ (\lambda_2\ \psi_o - B)\ \varepsilon_t\ .}$$

In this case, y_t depends on a single arbitrary parameter ψ_o.

When both roots λ_1 and λ_2 are inside the unit circle, ψ_1 and ψ_2 must be chosen so that the factors $(B-\lambda_1)$ and $(B-\lambda_2)$ appear in the right-hand side of (5.13) i.e.

$$\begin{cases} a\ \psi_o + (b\ \psi_o + a\ \psi_1)\ \lambda_1 - \lambda_1^2 = o \\ a\ \psi_o + (b\ \psi_o + a\ \psi_1)\ \lambda_2 - \lambda_2^2 = o \end{cases}$$

The corresponding values are

$$\psi_o = 1 \qquad , \qquad \psi_1 = o$$

and the unique linear stationary solution is

$$(5.15) \qquad \boxed{y_t = \varepsilon_t}$$

Note that it was assumed that λ_1 and λ_2 were real numbers. However if the characteristic equation has complex roots, i.e. if $a.b > \frac{1}{4}$, they are complex conjugate and $|\lambda_1| = |\lambda_2|$. The case $|\lambda_1| > 1$ and $|\lambda_2| < 1$ is excluded so that the polynomials appearing in (5.13), (5.14) and (5.15) have real coefficients.

Finally we may summarize the results as follows. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the model has an infinity of linear stationary solutions with ARMA(2,2) representation, an infinity of ones with ARMA(1,1) representation obtained for $\psi_1 = \frac{\lambda_1}{a}(1-\psi_o)$ or $\psi_1 = \frac{\lambda_2}{a}(1-\psi_o)$ and one white noise solution ($\psi_o = 1$, $\psi_1 = o$).

If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, the model has an infinity of linear stationary solutions with ARMA(1,1) representation and one white noise solution. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, the model has only one linear stationary solution which is a white noise.

Example (5.16) : a generalization

$$\boxed{y_t = \sum_{h=1}^{H} a_h \, E(y_{t+h}|I_t) + u_t \quad , \quad a_H \neq o} \tag{5.17}$$

where (u_t) is a stationary ARMA(p,q) process.

$$\Phi(B)\, u_t = \Theta(B)\, \varepsilon_t$$

$\Theta(B)$ and $\Phi(B)$ are invertible and have no common root.

The solutions of (5.17) satisfy :

$$\left(\sum_{i=o}^{H-1} a_{H-i}\, B^i - B^H\right) y_t = \sum_{i=o}^{H-1} a_{H-i} \sum_{j=i}^{h-1} \varepsilon_{t-j}^{j-i} - u_{t-H} \, . \tag{5.18}$$

The linear stationary solutions are such that: $\varepsilon_t^j = \psi_j \, \varepsilon_t$;

(5.18) implies :

$$(5.19) \qquad \left[\sum_{i=o}^{H-1} a_{H-i}\, B^{i} - B^{H}\right] \Phi(B) y_t = \left[\sum_{i=o}^{H-1} \sum_{j=i}^{h-1} a_{H-i}\psi_{j-i} B^{j} \Phi(B) - \Theta(B) B^{H}\right] \varepsilon_t$$

The characteristic equation is

$$\sum_{i=o}^{H-1} a_{H-i}\, \lambda^{i} - \lambda^{H} = o$$

When all roots are outside the unit circle, equation (5.19) defines the set of all stationary solutions in terms of H arbitrary parameters $\psi_o, \psi_1, \ldots, \psi_{H-1}$.

Example (5.20) : a generalized MUTH's model

Let us consider a model containing several expectations taken at the same date $t-1$:

$$(5.21) \qquad \boxed{y_t = a_o\, E(y_t | I_{t-1}) + a_1\, E(y_{t+1} | I_{t-1}) + a_2\, E(y_{t+2} | I_{t-1}) + u_t \quad ; \quad a_2 \neq o}$$

The general solution is given by :

$$y_t = a_o(y_t - \varepsilon_t^o) + a_1(y_{t+1} - \varepsilon_{t+1}^o - \varepsilon_t^1) + a_2(y_{t+2} - \varepsilon_{t+2}^o - \varepsilon_{t+1}^1 - \varepsilon_t^2) + u_t$$

where ε^1 , ε^2 are arbitrary ΔM processes and $\varepsilon_t^o = u_t - E(u_t | I_{t-1})$

In the particular case where $u_t = \varepsilon_t$ is a white noise and the roots of the characteristic equation are outside the unit circle, the linear stationary solutions satisfy :

$$a_2 y_t + a_1 y_{t-1} + (a_o - 1) y_{t-2} = a_2 \varepsilon_t + (a_1 + a_2 \psi_1) \varepsilon_{t-1} + (a_o + a_1 \psi_1 + a_2 \psi_2 - 1) \varepsilon_{t-2}$$

where ψ_1, ψ_2 are arbitrary real numbers.
Since a_2 is different from zero, these solutions have an ARMA(2,2) representation with the same autoregressive polynomial : $1 + \frac{a_1}{a_2} B + \frac{a_{o-1}}{a_2} B^2$ and with an arbitrary moving average one MA(2) with innovation ε_t (compare with EVANS-HONKAPOHJA (1984), prop. 2).

Example (5.22) : A model with no stationary solution.

$$(5.23) \qquad \boxed{y_t = a\, E(y_{t+1} | I_t) + b\, y_{t-1} + \varepsilon_t \quad , \quad a.b \neq o}$$

where (ε_t) is an independent white noise.
The general solution is

$$(a - B + b\, B^2)\, y_t = a\, \varepsilon_t^o - \varepsilon_{t-1}$$

The linear stationary solutions are such that

$$\varepsilon_t^o = \psi_o\, \varepsilon_t$$

Let λ_1 and λ_2 be the roots of the characteristic equation

$$a - \lambda + b\,\lambda^2 = o$$

If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the general linear stationary solution is

$$(5.24) \quad \boxed{b(B-\lambda_1)(B-\lambda_2)\; y_t = (a\,\psi_o - B)\,\varepsilon_t}$$

It depends on one arbitrary parameter ψ_o .

If $|\lambda_1| > 1$ and $|\lambda_2| < 1$, ψ_o must be chosen so that λ_2 is a root of $a\,\psi_o - B$ i.e.

$$\psi_o = \frac{\lambda_2}{a}$$

The linear stationary solution is unique and given by :

$$(5.25) \quad \boxed{b(B - \lambda_1)\; y_t = -\,\varepsilon_t}$$

Note that this case may not happen when λ_1 and λ_2 are complex roots (having the same modulus).

If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, there exist no linear stationary solution.

Finally we may summarize the results as follows. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the model has an infinity of linear stationary solutions with ARMA (2,1) representations and two AR(1) solutions corresponding to $\psi_o = \frac{\lambda_2}{a}$ and $\psi_o = \frac{\lambda_1}{a}$.

If $|\lambda_1| > 1$ and $|\lambda_2| < 1$, the model has a unique linear stationary solution with an AR(1) representation. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, the model has no linear stationary solution.

6. LINEAR SOLUTIONS.

In section 5, we considered linear stationary solutions of a LRE model. A similar procedure may be used to describe the linear solutions (stationary or not). The following results generalize those of section 5 .

We assume that the exogenous process (u_t) has the following representation :

$$u_t = \theta_{ot}\, \varepsilon_t - \theta_{1t}\, \varepsilon_{t-1} \cdots - \theta_{pt}\, \varepsilon_{t-p} \cdots$$

where (ε_t) is an independent white noise and

$$\sum_{i=o}^{\infty} |\theta_{it}| < \infty \qquad \forall \ t$$

We are interested in the linear solutions with time-dependent coefficients.

$$y_t = \alpha_{ot}\, \varepsilon_t - \alpha_{1t}\, \varepsilon_{t-1} \cdots - \alpha_{pt}\, \varepsilon_{t-p} \cdots$$

When $I_1 > o$, by (4.28), the general solution of model (2.1) verifies

$$(6.1) \qquad \sum_{i=I_0}^{I_1} a_i^* \, y_{t+i} = \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} c_{jk} \, \varepsilon_{t+I_1-k}^{j} + g_t(u)$$

The assumption on the structure of (u_t) makes $g_t(u)$ linear in ε_t , ε_{t-1} with time-dependent coefficients. We will denote it $\Omega_t(B)\ \varepsilon_t$.

The proof of the following theorem is similar to the proof of (5.2).

THEOREM (6.2) : The linear solutions with time-dependent coefficients are such that

$\varepsilon_t^j = \psi_{jt}\ \varepsilon_t$; $\forall\ t \in \mathbb{Z}$, $\forall\ j = H-I_1,\dots,H-1$

where the ψ_{jt} are I_1 functions of t .

Consequently (6.1) becomes :

$$(6.3) \qquad \boxed{\left(\sum_{i=0}^{I_1-I_0} a_{I_1-i}^* \, B^i\right) y_t = \left(\sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} c_{jk} \, \psi_{j,t-k} B^k + \Omega_{t-I_1}(B) B^{I_1}\right) \varepsilon_t}$$

Arbitrary values of the functions ψ_{jt} do not always correspond to a solution since the following condition is imposed

$$(6.4) \qquad \sum_{i=0}^{\infty} |\alpha_{it}| < \infty \quad , \quad \forall\ t\ .$$

Dropping the stationarity assumption leads to a larger set of solutions. This set cannot be described in terms of a finite number of parameters. However the extension of the set of solutions has another implication. It emphasizes the "dynamic growth" of the multiplicity of solutions. In

fact, theorem (6.2) shows that the knowledge of the ψ_{jt} for $t < T$ is not enough to determine the future values of the process y.
Therefore at time T, the value y_T of the solution may be chosen in infinite ways by means of ψ_{jT}. At time $T+1$, y_{T+1} may again be chosen in infinite ways by means of $\psi_{j,T+1}$. And so on.

Example (6.5)

$$\boxed{y_t = a\, E(y_{t+1} | I_t) + u_t , \quad a \neq o}$$

where $u_t = \theta_{ot}\, \varepsilon_t + \theta_{1t}\, \varepsilon_{t-1} + \ldots + \theta_{qt}\, \varepsilon_{t-q} = \Theta_t(B)\, \varepsilon_t$ and (ε_t) is an independent white noise.
The general solution of the model is :

$$(6.6) \qquad y_t = \frac{1}{a} y_{t-1} + \varepsilon_t^o - \frac{1}{a} \Theta_{t-1}(B)\, \varepsilon_{t-1}$$

The linear solutions are such that

$$\varepsilon_t^o = \psi_{ot}\, \varepsilon_t$$

Consequently (6.6) becomes :

$$y_t = \frac{1}{a} y_{t-1} + \left(\psi_{ot} - \frac{1}{a} \Theta_{t-1}(B) B\right) \varepsilon_t$$

and it appears clearly that, knowing y_{T-1} , $\varepsilon_{T-1}, \ldots,$ one needs a chosen ψ_{oT} to fix y_T .

7. DIFFERENCE OF TWO SOLUTIONS.

GOURIEROUX, LAFFONT and MONFORT (1979) have characterized the set of all solutions of some LRE models by using arbitrary martingales. These martingales appear in the expression of the solutions of the homogenous version of the model. Therefore it looks quite natural to find a link between these martingales and the martingale difference sequences appearing in this paper.

7.1 - The method of GOURIEROUX, LAFFONT and MONFORT (1981).

By analogy with the theory of the linear difference equations, GOURIEROUX, LAFFONT and MONFORT note that the general solution of the model is the sum of a particular solution and of the general solution of the homogenous equation obtained by excluding the exogenous process. They consider various homogenous equations.

i) The simplest one is related to the CAGAN model :

$$(7.1) \qquad y_t = a\ E(y_{t+1} | I_t)$$

Its general solution is

$$y_t = \frac{1}{a^t} M_t$$

where (M_t) is an arbitrary martingale i.e. a stochastic process for which the rational expectation coincides with the naive one :

$$E(M_t | I_{t-1}) = M_{t-1}$$

ii) A second order homogenous model, which was considered, is :

$$y_t = a_{o1}\ E(y_{t+1} | I_t) + a_{o2}\ E(y_{t+2} | I_t) \qquad .$$

Let λ_1 and λ_2 denote the roots of the characteristic equation :

$$\lambda^2 - a_{o1}\lambda - a_{o2} = o$$

Three cases were distinguished

. λ_1 and λ_2 are real and $\lambda_1 \neq \lambda_2$, the general solution is

$$y_t = \frac{1}{\lambda_1^t} M_{1t} + \frac{1}{\lambda_2^t} M_{2t}$$

. λ_1 and λ_2 are complex conjugate $\lambda_1 = \overline{\lambda}_2 = \rho\ e^{i\omega}$, the general solution is

$$y_t = \frac{1}{\rho^t} (M_{1t} \cos\omega t + M_{2t} \sin\omega t)$$

. $\lambda_1 = \lambda_2 = \lambda$, the general solution is

$$y_t = \frac{1}{\lambda^t} (M_{1t} + t\ M_{2t})$$

In each case $\begin{pmatrix} M_{1t} \\ M_{2t} \end{pmatrix}$ is an arbitrary bivariate martingale.

iii) Finally, they also considered the model

$$y_t = a_{11}E(y_t | I_{t-1}) + a_{o1}E(y_{t+1} | I_t)$$

and showed that its general solution is

$$y_t = a_{11}\left(\frac{1-a_{11}}{a_{o1}}\right)^{t-1} M_{t-1} + a_{o1}\left(\frac{1-a_{11}}{a_{o1}}\right)^{t} M_t$$

and depends on the values at t and t-1 of the same arbitrary martingale.

7.2 - Connection with the revision processes.

We consider the homogenous version of model (2.1)

$$(7.1) \qquad y_t = \sum_{k=o}^{K}\sum_{h=o}^{H} a_{kh}\, E(y_{t+h-k} | I_{t-k})$$

The solutions of (7.1) are equal to the differences of two solutions of the original model (2.1). If $\tilde{y}$ and $\tilde{\tilde{y}}$ are two solutions of (2.1), we have, from (4.28) used for $I_1 > o$ (I_1 is assumed to be strictly positive to have a multiple solutions model) :

$$\sum_{i=I_o}^{I_1} a_i^* \tilde{y}_{t+i} = \sum_{j=H-I_1}^{H-1}\sum_{k=k_j}^{K_j} c_{jk}\, \tilde{\varepsilon}^j_{t+I_1-k} + g_t(u)$$

$$\sum_{i=I_o}^{I_1} a_i^* \tilde{\tilde{y}}_{t+1} = \sum_{j=H-I_1}^{H-1}\sum_{k=k_j}^{K_j} c_{jk}\, \tilde{\tilde{\varepsilon}}^j_{t+I_1-k} + g_t(u)$$

The difference $y_t = \tilde{y}_t - \tilde{\tilde{y}}_t$ verifies the equation

$$(7.2) \qquad \boxed{\sum_{i=I_o}^{I_1} a_i^* y_{t+i} = \sum_{j=H-I_1}^{H-1}\sum_{k=k_j}^{K_j} c_{jk}\, \varepsilon^j_{t+I_1-k}}$$

where $\varepsilon^j_t = \tilde{\varepsilon}^j_t - \tilde{\tilde{\varepsilon}}^j_t$ is also a ΔM process. Since $\tilde{\varepsilon}^j_t$ and $\tilde{\tilde{\varepsilon}}^j_t$

are arbitrary, the same is true for ε_t^j and (7.2) gives the general form of the solutions of the homogenous equation. It is equivalent to

$$(7.3) \qquad y_{t+I_1} = - \sum_{i=I_0}^{I_1-1} \frac{a_i^*}{a_{I_1}^*} y_{t+i} + \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} \frac{c_{jk}}{a_{I_1}^*} \varepsilon_{t+I_1-k}^j$$

This relation can be expressed with matrix notations :

$$Y_t = A\, Y_{t-1} + \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} d_{jk}\, V_{t-k}^j$$

where

$$Y_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t+I_0-I_1+1} \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{a_{I_1-1}^*}{a_{I_1}^*} & \cdots\cdots & -\frac{a_{I_0}^*}{a_{I_1}^*} \\ 1 \quad 0 & \cdots & 0 \\ & & \vdots \\ 0 \cdots & 1 & 0 \end{pmatrix}$$

$$V_{t-k}^j = \begin{pmatrix} \varepsilon_{t-k}^j \\ o \\ \vdots \\ 0 \end{pmatrix}, \quad \frac{c_{jk}}{a_{I_1}^*} = d_{jk}$$

If initial values Y_0 are given and if the values of the revision processes before $t = o$ are fixed to zero: $V_t^j = o \quad \forall\, t < o$, we obtain :

$$(7.4) \qquad Y_t = A^t\, Y_0 + \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} d_{jk}\, (V_{t-k}^j + A V_{t-k-1}^j + \ldots + A^{t-1} V_{1-}^j$$

Since $a^*_{I_o} \neq o$, the matrix A is invertible.
Let us consider the process (M^j_t) defined by :

$$(7.5) \qquad M^j_t = A^{-t} \; (V^j_t + A \; V^j_{t-1} + \ldots + A^{t-1} \; V^j_1)$$

Taking the expectation of (7.5) given I_{t-1} yields

$$\begin{aligned} E(M^j_t | I_{t-1}) &= A^{-t}(A \; V^j_{t-1} + \ldots + A^{t-1} V^j_1) \\ &= A^{-(t-1)} \; (V^j_{t-1} + \ldots + A^{(t-1)-1} V^j_1) \\ &= M^j_{t-1} \end{aligned}$$

Consequently, (M^j_t) is a multidimensional martingale with zero mean. Its components are all generated by the same ΔM process (ε^j_t) .
Substituting (7.5) in (7.4) gives

$$(7.6) \qquad \boxed{Y_t = A^t \; Y_o + \sum_{j=H-I_1}^{H-1} \; \sum_{k=k_j}^{K_j} \; d_{jk} \; A^t \; M^j_{t-k}}$$

The eigenvalues of A are denoted by λ_i ; they are equal to the roots of the characteristic equation :

$$(7.7) \qquad \sum_{i=I_o}^{I_1} \; a^*_i \; x^{i-I_o} = o$$

In the simplest case, matrix A has a diagonal representation. Let Q refer to the matrix of eigenvectors and Λ refer to the diagonal matrix :

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{I_1 - I_o} \end{pmatrix}$$

we have : $A = Q^{-1} \Lambda Q$ and equation (7.6) becomes :

$$Y_t = A^t Y_o + \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} d_{jk} Q^{-1} \Lambda^t Q M_{t-k}^j .$$

The processes $(\tilde{M}_t^j) = Q (M_t^j)$ are also martingales and we have :

$$(7.8) \quad \boxed{Y_t = A^t Y_o + \sum_{j=H-I_1}^{H-1} \sum_{k=k_j}^{K_j} d_{jk} Q^{-1} \Lambda^t \tilde{M}_{t-k}^j}$$

It is seen from equation (7.8) that the first component y_t of the vector Y_t is the sum of a deterministic function corresponding to the first component of $A^t Y_o$ and a combination of terms of type

$$\sum_{\ell} \alpha_{jk\ell} \lambda_{\ell}^{t} \tilde{M}^{j}_{\ell,t-k}$$

where $\tilde{M}^{j}_{\ell,\cdot}$ is the ℓ^{th} coordinate of the martingale $\tilde{M}^{j}$. We have exponentially weighted combinations of martingales.

There are I_1 vector martingales $(\tilde{M}^{j}_{t})$. Each one is generated by a scalar ΔM process and may appear with lagged terms.

On the other hand, when matrix A cannot be diagonalized, it is however possible to replace Λ by the JORDAN's reduced form. The exponential factors λ_{ℓ}^{t} become products of exponential functions by polynomials.

We will now consider some examples to illustrate the previous procedure.

Example (7.9)

We consider CAGAN's model

$$\boxed{y_t = a\, E(y_{t+1} | I_t) + u_t} \tag{7.10}$$

with $a \neq o$. Its solutions verify the recursive equation

$$y_t = a\, y_{t+1} - a\, \varepsilon^{o}_{t+1} + u_t .$$

The solution of the homogenous version of the model are the processes satisfying :

$$y_t = a\, y_{t+1} - a\, \varepsilon^o_{t-1}$$

or equivalently for $t > o$:

$$y_t = \frac{1}{a}\, y_{t-1} + \varepsilon^o_t$$

$$= \frac{1}{a^t}\, y_o + (\varepsilon^o_t + \frac{1}{a}\, \varepsilon^o_{t-1} + \ldots + \frac{1}{a^{t-1}}\, \varepsilon^o_1)$$

$$= \frac{1}{a^t}\, y_o + \frac{1}{a^t}\, \tilde{M}_t$$

where $\tilde{M}_t = a^t\, (\varepsilon^o_t + \frac{1}{a}\, \varepsilon^o_{t-1} + \ldots + \frac{1}{a^{t-1}}\, \varepsilon^o_1)$ is a martingale with zero mean.

The difference sequence associated with $\tilde{M}_t$ is :

$$\tilde{M}_t - \tilde{M}_{t-1} = a^t\, \varepsilon^o_t$$

It is linked to the ΔM process (ε^o_t) by means of a non stationary function of t .

Example (7.11)

We consider the following homogenous model :

$$(7.12) \qquad \boxed{y_t = a_{o1}\, E(y_{t+1}|I_t) + a_{12}\, E(y_{t+1}|I_{t-1}) + a_{11}\, E(y_t|I_{t-1})}$$

Inserting the revision processes, we get

$$y_t = (a_{o1}+a_{12})y_{t+1} + a_{11}y_t - (a_{o1}+a_{12})\varepsilon^o_{t+1} - a_{12}\,\varepsilon^1_t - a_{11}\varepsilon^o_t \quad .$$

The comparison of this relation with its projection on I_{t-1} yields :

$$\varepsilon^o_t = a_{o1}\,\varepsilon^1_t$$

and :

$$y_t = (a_{o1}+a_{12})y_{t+1} + a_{11}y_t - (a_{o1}+a_{12})a_{o1}\varepsilon^1_{t+1} - (a_{12}+a_{11}a_{o1})\,\varepsilon^1_t \quad ,$$

or equivalently for $a_{o1} + a_{12} \neq o$:

$$(7.13) \quad \boxed{y_t = \frac{1-a_{11}}{a_{o1}+a_{12}}\,y_{t-1} + a_{o1}\varepsilon^1_t + \frac{a_{12}+a_{11}a_{o1}}{a_{o1}+a_{12}}\,\varepsilon^1_{t-1}}$$

where (ε^1_t) in an arbitrary ΔM process.

Let us now introduce the zero mean martingale :

$$\tilde{M}_t = (\varepsilon^1_t + \frac{1-a_{11}}{a_{o1}+a_{12}}\,\varepsilon^1_{t-1} + \ldots + \left(\frac{1-a_{11}}{a_{o1}+a_{12}}\right)^{t-1} \varepsilon^1_1) \left(\frac{a_{o1}+a_{12}}{1-a_{11}}\right)^t$$

We have for $t > o$

$$(7.14) \quad \begin{aligned} y_t &= \left(\frac{1-a_{11}}{a_{o1}+a_{12}}\right)^t y_o + \left(\frac{1-a_{11}}{a_{o1}+a_{12}}\right)^t a_{o1} \tilde{M}_t \\ &+ \left(\frac{1-a_{11}}{a_{o1}+a_{12}}\right)^{t-1} \frac{a_{12}+a_{o1}a_{11}}{a_{o1}+a_{12}} \tilde{M}_{t-1} \end{aligned}$$

It should be noted that in both previous examples, the martingale is scalar. $\tilde{M}_t - \tilde{M}_{t-1}$ is proportional to ε_t^1 and may be chosen arbitrarily. Therefore the zero mean martingale $\tilde{M}_t$ is also arbitrary.

Example (7.15)

Consider the model

$$(7.16) \quad \boxed{y_t = a\ E(y_{t+1} | I_t) + b\ y_{t-1} + \varepsilon_t} \quad , \quad a.b \neq o$$

The solutions of the homogenous version verify the relation

$$y_t = \frac{1}{a} y_{t-1} - \frac{b}{a} y_{t-2} + \varepsilon_t^o$$

or using the lag operator B :

$$(a - B + b\ B^2)\ y_t = a\ \varepsilon_t^o$$

Let λ_1 and λ_2 be the roots of the equation $a - \lambda + b\ \lambda^2 = o$. If λ_1 and λ_2 are real and distinct, we may write, for $t > o$,

$y_t = \frac{a}{b} (B - \lambda_1)^{-1} (B - \lambda_2)^{-1} \tilde{\varepsilon}_t^o + f(t, y_o, y_{-1})$, where f is a deterministic function and where $\tilde{\varepsilon}_t^o$ is taken equal to zero for $t < o$ and equal to ε_t^o for $t > o$. (This latter condition gives a sense to $(B - \lambda_1)^{-1} (B - \lambda_2)^{-1} \tilde{\varepsilon}_t^o$ whatever are the modulis of λ_1 and λ_2)

$$y_t = \left[\frac{a}{b(\lambda_1 - \lambda_2)} (B-\lambda_1)^{-1} + \frac{a}{b(\lambda_2 - \lambda_1)} (B-\lambda_2)^{-1}\right] \tilde{\varepsilon}_t^o + f(t, y_o, y_{-1})$$

$$= \frac{a}{b(\lambda_1 - \lambda_2)\lambda_1} \sum_{i=o}^{t-1} \frac{1}{\lambda_1^i} \varepsilon_{t-i}^o + \frac{a}{b(\lambda_2 - \lambda_1)\lambda_2} \sum_{i=o}^{t-1} \frac{1}{\lambda_2^i} \varepsilon_{t-i}^o + f(t, y_o, y_{-1})$$

or equivalently

$$(7.17) \quad \boxed{y_t = \frac{1}{\lambda_1^t} M_t^1 + \frac{1}{\lambda_2^t} M_t^2 + f(t, y_o, y_{-1}), t > o}$$

where $M_t^i = \frac{(-1)^{i+1} a \lambda_i^t}{b \lambda_i (\lambda_1 - \lambda_2)} (\varepsilon_t^o + \frac{1}{\lambda_i} \varepsilon_{t-1}^o + \frac{1}{\lambda_i^2} \varepsilon_{t-2}^o + \ldots + \frac{1}{\lambda_i^{t-1}} \varepsilon_1^o)$

is a zero mean martingale.

(7.17) expresses the solution of the homogenous model in terms of two martingales (M_t^1) and (M_t^2) functions of the same ΔM processes (ε_t^o) .

Example (7.18)

We consider the homogenous model

$$\text{(7.19)} \quad \boxed{y_t = a\, E(y_{t+2} | I_t)} \quad , \quad a >$$

The solutions of this model, for $t > o$, verify

$$\text{(7.20)} \quad y_t - \frac{1}{a} y_{t-2} = \varepsilon_t^o + \varepsilon_{t-1}^1$$

where (ε_t^o) and (ε_t^1) are arbitrary ΔM processes with $\varepsilon_t^o = o$ for $t \leq o$ and $\varepsilon_t^1 = o$ for $t < o$.

We also have

$$y_t = \frac{1}{2}\left[\frac{1}{1 - \frac{B}{\sqrt{a}}} + \frac{1}{1 + \frac{B}{\sqrt{a}}}\right] (\varepsilon_t^o + \varepsilon_{t-1}^1) + f(t\ y_o$$

$$= \frac{1}{2} \sum_{i=o}^{\infty} \frac{1}{\sqrt{a}^i} (\varepsilon_{t-i}^o + \varepsilon_{t-i-1}^1) + \frac{1}{2} \sum_{i=o}^{\infty} \frac{1}{(-\sqrt{a})^i} (\varepsilon_{t-i}^o + \varepsilon_{t-i-1}^1)$$

$$+ f(t, y_o, y_{-1})$$

Adding and substracting $\frac{\sqrt{a}}{2} \varepsilon_t^1$, we get :

$$y_t = \frac{1}{2} \sum_{i=o}^{\infty} \frac{B^i}{\sqrt{a}^i} (\varepsilon_t^o + \sqrt{a}\, \varepsilon_t^1) + \frac{1}{2} \sum_{i=o}^{\infty} \frac{B^i}{(-\sqrt{a})^i} (\varepsilon_t^o - \sqrt{a}\, \varepsilon_t^1) + f(t, y_o, y_{-1}$$

$$\text{(7.21)} \quad \boxed{y_t = \frac{1}{\sqrt{a}^t} M_t^1 + \frac{1}{(-\sqrt{a})^t} M_t^2 + f(t, y_o, y_{-1}), t > o}$$

where $M_t^1 = \frac{\sqrt{a}^t}{2} \sum_{i=o}^{\infty} \frac{B^i}{\sqrt{a}^i} (\varepsilon_t^o + \sqrt{a}\, \varepsilon_t^1)$

and $M_t^2 = \frac{(-\sqrt{a})^t}{2} \sum_{i=o}^{\infty} \frac{B^i}{(-\sqrt{a})^i} (\varepsilon_t^o - \sqrt{a}\, \varepsilon_t^1)$

are two zero mean martingales.

Another form of the general solution might have been obtained by introducing the process :

$$v_t = \varepsilon_t^o + \varepsilon_{t-1}^1$$

and by substituting in (7.20)

$$\boxed{y_t = \frac{1}{a} y_{t-2} + v_t}$$

(v_t) is an arbitrary process such that
$E(v_t | I_t) = v_t$ and $E(v_t | I_{t-2}) = o$

8. <u>CONCLUDING REMARKS</u>.

1) Revision processes are particularly adapted to the description of all solutions of the LRE models. In the previous sections, we have shown that a finite number of initial conditions is not sufficient to single out one solution. It should however be noted that final conditions are more restrictive. A martingale process M is completely determined by a final condition since $M_t = E(M_T/I_t) \;\forall\; t < T$. Consequently, there is an asymmetry between the impacts of initial and final conditions. In the

case of linear solutions with time-dependent coefficients, the imposition of final conditions cuts off the "dynamic growth" of the multiplicity obtained with initial conditions.

2) In section 5, we restricted the study to the linear stationary solutions in order to emphasize the existence of a parametrization. However a large number of other stationary solutions do also exist. To develop this point, we consider CAGAN's model

$$y_t = a\, E(y_{t+1} | I_t) + u_t \quad , \quad a > 1 \quad .$$

The general solution is

$$y_t = \frac{1}{a}\, y_{t-1} + \varepsilon_t^o - \frac{1}{a}\, u_{t-1}$$

where (ε_t^o) is an arbitrary ΔM process. We assume that the exogenous variable is an independent white noise.

$$u_t = \varepsilon_t \qquad \forall\ t \quad .$$

It is then clear that the stochastic process defined by

$$\varepsilon_t^o = (\varepsilon_t)^2 - E((\varepsilon_t^2))$$

is a ΔM process. It leads to the stationary solution

$$y_t = \frac{1}{a} y_{t-1} - \frac{1}{a} \varepsilon_{t-1} + (\varepsilon_t)^2 - E((\varepsilon_t)^2).$$

Actually there exist a large number of non-linear stationary solutions.

They constitute a set which cannot be described in terms of a finite number of parameters.

3) Finally it can be noted that the general procedure based on revision processes, which has been used for determining the solutions of a unidimensional dynamic LRE model, can be extended to the case of multi-dimensional dynamic LRE model and also to some non-linear R.E models, such as mean-variance models.

R E F E R E N C E S

BLANCHARD, O.J. (1979) : "Backward and Forward Solutions for Economies with Rational Expectations", American Economic Review, Papers and Proceedings, 69, 114-118.

BROZE, L., J. JANSSEN and A. SZAFARZ (1984) : "On Solutions of Linear Models with Rational Expectations", in Alternative Approaches to Time Series Analysis (Proceedings of the 3rd Franco-Belgian Meeting of Statisticians, November 1982), edited by J.P. FLORENS, M. MOUCHART, J.P. RAOULT and L. SIMAR, Publications des Facultés Universitaires Saint-Louis, Bruxelles, 210-217.

BROZE, L. and A. SZAFARZ (1984) : "On linear Models with Rational Expectations which Admit a Unique Solution", European Economic Review, 24, 103-111.

CHOW, G.C. (1983) : "Econometrics", Mc Graw Hill.

EVANS, G. and S. HONKAPOHJA (1984) : "A complete characterization of ARMA solutions to linear rational expectations models", University of Helsinki, W.P. 209.

GOURIEROUX, C., LAFFONT J.J. and A. MONFORT (1981) : Modèles linéaires avec anticipations rationnelles : solutions et critères de sélection, Cahiers du CNRS 23, 15-46.

GOURIEROUX, C., LAFFONT J.J. and A. MONFORT (1982) : "Rational Expectations in Dynamic Linear Models : Analysis of the Solutions", Econometrica, 50, 409-425.

Mac CALLUM, B.T. (1976) : "Rational Expectations and the Estimation of Econometric Models : An Alternative Procedure", International Economic Review, 17, 484-490.

MUTH, J.F. (1961) : "Rational Expectations and the Theory of Price Movements", Econometrica, 29, 315-335.

PESARAN, M.H. (1981) : "Identification of Rational Expectations Models", Journal of Econometrics, 16, 375-398.

SHILLER, R. (1978) : "Rational Expectations and the Dynamic Structure of Macroeconomic Models", Journal of Monetary Economics, 4, 1-44.

TAYLOR, J. (1977) : "Conditions for Unique Solution in Stochastic Macroeconomic Models with Rational Expectations", Econometrica, 45, 1377-1387.

VISCO, I. (1981) : "On the Derivation of Reduced Forms of Rational Expectations Models", European Economic Review, 16, 355-365.

WEGGE, L. (1984) : "Identifiability of Structural Models Containing MUTH-rational Current and Future Expectations", Univ. of California, W.P. 233.