May 1984

TOWARD A THEORY OF OPTIMAL PRE-COMMITMENT

I: AN ANALYSIS OF THE TIME-CONSISTENT EQUILIBRIA

Daniel Cohen
CEPREMAP, Paris

Phillippe Michel
Universite de Paris 1

N° 8412

We gratefully acknowledge the insights by D. Abreu on the topics covered in this paper and the comments by O. Blanchard, G. Oudiz and J. Sachs on an earlier version of the paper. Errors are ours.

SUMMARY

TOWARD A THEORY OF OPTIMAL PRE-COMMITMENT
I: AN ANALYSIS OF THE TIME-CONSISTENT EQUILIBRIA

It has been recognized that the optimal strategy of a government, if it is expected, is generally time-inconsistent : optimality requires to take into account expectations effects in the announcement of a policy and to ignore these effects when applying the policy.

In order to analyse the problem, we study different solutions of a simple one-dimensional linear quadratic game. The optimal time-inconsistent solution appears to be paradoxical : in the long term, the leader plays against his objective function, in order to induce the followers to take early corrective measures. The time-consistent solution, defined as a perfect equilibrium, is compared to the solution proposed by Buiter, to the Cournot-Nash equilibrium, and to an optimal policy rule. As a conclusion we try to show how this analysis helps understanding the optimality of finitely-lived precommitment, but this needs further studies we will try to do.

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RESUME

POUR UNE THEORIE DES ENGAGEMENTS POLITIQUES OPTIMAUX
I. ANALYSE DES EQUILIBRES TEMPORELLEMENT CONSEQUENTS

Une politique optimale, quand elle est anticipée, est généralement inconséquente temporellement : il est optimal de prendre en compte les effets des anticipations en annonçant une politique et d'ignorer ces effets passés quand les décisions sont prises.

On analyse ce problème en étudiant les diverses solutions d'un modèle très simple sous la forme d'un jeu linéaire-quadratique en dimension un. La solution optimal temporellement inconséquente paraît paradoxale : à long terme, le gouvernement joue contre son objectif dans le but d'entraîner les agents à compenser dès aujourd'hui ses extravagances futures. La solution temporellement conséquente —définie comme un équilibre parfait— est comparée à d'autres solutions : celle que propose Buiter, l'équilibre de Nash, l'application d'une règle optimale. Nous essayons à partir de cette analyse de comprendre ce que peut être un engagement optimal de durée finie, mais ceci devra être l'objet d'une poursuite de l'étude.

1. Introduction

Since the work by Kydland and Prescott (1977) and Calvo (1978), it has been recognized that the optimal strategy that a government would like to pursue might be time-inconsistent, by which is meant that an optimum set of policy actions announced at some initial time ceases to be optimal when time passes. Because of this property, a time-inconsistent strategy requires that a government binds its future actions so as to make it credible that they will not be revised. This feature makes a time-inconsistent equilibrium unattractive: as recently emphasized by Lucas and Stokey (1983), among others, there are few instances in our existing societies of a government which can precommit its actions from now to eternity. Governments are usually finitely-lived and must take the behavior of their successors as a given constraint on their own policy design. In the context of their model, however, Lucas and Stokey show that the first optimum of an initial government can be supported by the action of its successors if the initial government can restrict the actions of its successors by issuing a sufficiently rich menu of assets (see also Driffill (1982) on a related aspect; and Svensson and Persson (1984) and Fisher (1984) on Lucas and Stokey).

The result by Lucas and Stokey is remarkable but also specific to their model. In most circumstances (such as those we examined in this paper), there is no way by which the time-inconsistent solution can be reached by a sequence of independent governments. The purpose of our paper is then to calculate and compare, in a simple framework, the alternative solutions to the time-inconsistent one.

Our model is a one-dimensional linear quadratic game with one dominant player, a leader in the Stackleberg terminology (the government in our previous discussion) and N other players, the followers (the private sector in the above discussion). It is a one-dimensional version of the model studied by Miller and Salmon (1983) and Buiter (1983).

We first show that the time-inconsistent solution that the leader would choose as his first best policy has a peculiar feature which makes the problem very different from the Lucas-Stokey example. In our framework, we show that the time-inconsistent solution is one where the leader would like to pre-commit itself to pursue in the future a counter-productive policy: one which would make the system diverge from its stationary state. The reason of this result is that the announcement of such future extravagant policy induces the followers to take early corrective measures so that, as a net result, the dominant player can transmit the burden of the adjustment to the followers. We then calculate a time-consistent solution, that is a solution where a leader at time t only takes actions that optimize his current value function, irrespective of his previous moves. Technically, our solution is obtained by backward programming and corresponds, in Game Theory, to the Nash subgame perfect equilibrium. It differs from the solution obtained by Buiter (1983) for reasons we explain in the text. We compare our time-consistent equilibrium to the corresponding Cournot-Nash equilibrium with no-dominant player and show that a dominant player reduces the speed of adjustment of the economy toward its stationary state. The reason is as before: the leader takes little action so as to force the followers to carry the burden of the adjustment.

The time-consistent solution that we calculate assumes that the leader at time t ignores his previous actions. This assumption rules out equilibrium where he would (and could) maintain a reputation (see Barro and Gordon (1983)). We give a hint of what these equilibria would look like: we show that there exists a rule which differs from the time-consistent solution and which makes the leader better off at each period of time. This rule shows that governments may have an incentive to continue the policies of their predecessors so as to sustain their reputation. Finally, we conclude our paper by showing how our framework might help understanding the optimality of finitely-lived precommitment. We hope to show in a forthcoming paper how such equilibria can shed some light on such related aspects as political business cycle.

Section 2 sets the model. Section 3 calculates the Cournot-Nash equilibrium. Section 4 calculates the time-inconsistent solution of a Stackleberg equilibrium. Section 5 analyzes the time-consistent equilibrium. Section 6 contrasts our result to those obtained by Buiter. Section 7 deals with the example of a two-country world, with one country acting as a leader. Section 8 introduces the topic of equilibrium with reputation (repeated game and finitely lived precommitments). A conclusion summarizes the paper.

2. The Model

We consider a continuous-time model of an economy inhabited by N + 1 agents and characterized at each period t by a state variable $x(t) \varepsilon R$. The law of motion of the state variable is described by the equation:

(1) $\dot{x}(t) = -ax(t) - \Sigma_{j=1}^{N+1} b_j u_j(t)$; $a > 0$; $b_j > 0$; x_0 given

where $b_j(t)$ $(j = 1,\ldots,N+1)$ is the set of control variables by which each agent j contributes to the evolution of $x(t)$ (all $u_j(t)$ are scalars). One sees that only the values at time t of the state variable and of the control variables influence the evolution $(\dot{x}(t))$ at time t of the state variable. Each agent seeks to minimize a loss function which has the form:

(2) $L_j = 1/2 \int_0^\infty [q_j x(t)^2 + r_j u_j(t)^2]dt$; $q_j > 0$; $r_j > 0$.

Here, one sees from equations (1) and (2) that the economy will converge toward a stationary state where $x = 0$; $u_j = 0$, $\forall j \varepsilon \{1,\ldots,N+1\}$. The speed of adjustment toward this stationary state is a function of the control variables used each period by the agents. As apparent from the loss functions (equation (2)), both a deviation of $x(t)$ and $u_j(t)$ from zero is costly to the agents. Each of them will then look for an optimum trade-off between speeding up the decrease of $x(t)$ and the cost to do so.

Two sorts of equilibria can be computed: the open-loop and the closed-loop solutions.

(a) The Open-Loop Equilibria

Each agent j makes a non-revisable pattern of forecast on the other agents' future actions $\{u_i(t)\}_{i \neq j; t \geq 0}$ and select accordingly his own best response $\{u_j(t)\}_{t \geq 0}$. We shall characterize perfect foresight equilibria where the agent exactly observe, as time passes, the forecasts they expected. A game representation of the open-loop equilibria will come as follows: at initial time, each agent announces a set of controls $\{u_j(t)\}_{t>0}$, and these sequences are non-revisable.

(b) The Closed-Loop Equilibria

Here, each agent forecasts a set of feedback rules from the other agents in response to the level of the state-variable. In turn, each agent responds by his own feedback rule. Through his own influence on the state variable each agent recognizes the influence of his action on the other agents' actions. In the case of closed-loop equilibria, we can then give a truly dynamic representation of the game, each agent influencing the other agent's action by his own move, at each period of time. The closed-loop equilibria and the open-loop equilibria will prove a useful benchmark. (They are those recently studied by Miller and Salmon (1983) and Buiter (1983).) We leave to a forthcoming paper the analysis of the closed-loop equilibria.

From now on we concentrate on open-loop equilibria. There are two ways by which the agents can announce their future set of controls: simultaneously or in a particular order. We shall be interested in two sorts of equilibria:

(i) Cournot-Nash Equilibria. All agents simultaneously announce their strategies.

(ii) Stackleberg Equilibria. One agent (the leader) announces his strategy first. All other agents (the followers) respond simultaneously.

In analyzing the equilibria, we shall be interested in separating time-consistent and time-inconsistent solutions, which we define as follows. To the strategy $\{u_i^*(t)\}_{t>0}$ announced by each agent corresponds an evolution of the state variable $\{x^*(t)\}_{t>0}$; x_0 given. If the game was to re-start at time t_0, with the initial value of the state variable to be $x^*_{t_0}$, we ask: Would the agents select the same solution $\{u_i^*(t)\}_{t \geq t_0}$ or would they like to change their strategy? If they stick to their original strategies we shall say that they are time-consistent; otherwise, they will be said to be time-inconsistent.

To summarize:

<u>Definition</u>

Let $\{u_i^*(t)\}$, $x^*(t)\}_{t \geq 0}$ be a solution to the game started at initial time with $x^*(0) = x_0$. If for any t_0, $\{u_i^*(s+t_0)$, $x^*(s+t_0)\}_{s \geq 0}$ is a solution to the game initiated with $x^*_{t_0}$, then the equilibrium $\{u_i^*(t), x^*(t)\}_{t \geq 0}$ is said to be time-consistent. Otherwise, it is said to be time-inconsistent.

3. <u>Cournot-Nash Equilibrium</u>

Here, each agent j expects the other agents to play $\{u_i^*(t)\}_{i \neq j; t \geq 0}$ and minimizes his own cost function. For each agent, the problem can be written

$$\text{(3)} \qquad \text{minimize } L_j = 1/2 \int_0^\infty [q_j x(t)^2 + r_i u_j(t)^2] dt$$

$$\text{subject to } \dot{x}(t) = -ax(t) - \Sigma_{\substack{i=j \\ i \neq j}}^{N+1} b_i u_i^*(t) - b_j u_j(t)$$

A set of necessary conditions are given by minimizing with respect to u_j:

$$H_j = 1/2[q_j x(t)^2 + r_j u_j^2] + \pi_j [-ax(t) - \Sigma_{i \neq j} b_i u_i - b_j u_j]$$

with π_j a solution to

$$\dot{\pi}_j = - \frac{\partial H_j}{\partial x_j}$$

one finds

$$(4) \qquad u_j^*(t) = \frac{b_j}{r_j}\pi_j(t)$$

$$\dot{\pi}_j(t) = -q_j x(t) + a\pi_j(t)$$

Conditions (4) are sufficient provided that $\lim_{t\to\infty} \pi_j(t)x(t) = 0$ (see Arrow-Kurz (1970)). Collecting all terms for $j = 1,\dots,N+1$, one finds:

$$(5) \qquad \dot{x}(t) = -ax(t) - \pi(t)$$

$$\dot{\pi}(t) = -\beta_{N+1}^2 x(t) + a\pi(t)$$

$$\text{with } \beta_{N+1}^2 = \Sigma_{i=1}^{N+1} \frac{b_i^2 q_i}{r_i}; \ \pi(t) = \Sigma_{i=1}^{N+1} \frac{b_i^2}{r_i}\pi_i(t)$$

One sees that this formulation allows us to aggregate all agents j into a single one. It can be readily checked that all solutions to (5) satisfying $\lim_{t\to\infty} \pi(t)x(t) = 0$ determine a solution to (4) such that $\lim_{t\to\infty} \pi_j(t)x(t) = 0$; and conversely: it is enough to consider

$$\pi_j(t) = \alpha_j \pi(t), \ \alpha_j = \frac{q_j}{\beta_{N+1}^2}$$

The characteristic equation of the system (5) is

$$(6) \qquad \lambda^2 - (a^2+\beta_{N+1}^2) = 0$$

So that the solution to (5) is verifying the transversality condition is :

$$(7) \qquad x(t) = x_0 e^{-\gamma t}; \ \gamma = \sqrt{a^2+\beta_{N+1}^2};$$

$$\pi(t) = (\gamma - a)x(t)$$

The contribution of each agent to the decrease of x(t) is

$$(8) \qquad b_i u_i^*(t) = \frac{b_i^2}{r_i} \pi_i(t) = \frac{b_i^2 q_i}{r_i} \cdot \frac{1}{b_{N+1}^2} \cdot \pi(t)$$

Given equations (7) and (8), one sees that the Cournot-Nash equilibrium is time-consistent. The value of γ is independent of x_0 and the feedback $\pi(t)$ depends only on γ and on the value of the state variable.

4. Stackleberg Equilibrium: Time-Inconsistent Equilibrium

We now assume that one player (the leader) can announce his strategy in advance, and that all other N players respond simultaneously to the leader's strategy. In determining his best strategy, the leader will then take into acount what the followers' best responses will be to his own move.

Once they take as given the leader's set of future control variables, the behavior of the followers are characterized by a set of equations similar to those which we studied in the Cournot-Nash equilibrium. In the same manner, we can aggregate the N followers into a single player: With the same notations as in equation (4) and (5), we define

$$\pi(t) = \Sigma_{i=2}^{N+1} \frac{b_i^2}{r_i} \pi_i(t); \; \beta_N^2 = \Sigma_{i=2}^{N+1} \frac{b_i^2 q_i}{r_i}$$

so that the followers' behavior is characterized by $\pi(t)$ such that:

$$(8a) \qquad \dot{x}(t) = -ax(t) - \pi(t) - bu(t)$$

$$(8b) \qquad \dot{\pi}(t) = -\beta_N^2 x(t) + a\pi(t); \; \lim_{t\to\infty} \pi(t)x(t) = 0$$

where $u(t)$ now refers to $u_1(t)$, the leader's pattern of control variables. Having characterized the followers' best response by (8b), we can look for the leader's best set of actions $\{u(t)\}_{t\geq 0}$. The solution $\{u(t)\}$ will minimize

$$H = 1/2\ [gx(t)^2 + ru^2] + p_1(t)\ [-ax(t) - \pi(t) - bu]\ + p_2(t)\ [-\beta_N^2\ x(t)^2 + a\pi(t)]$$

with $p_1(t)$and $p_2(t)$solutions to

$$\dot{p}_1(t) = -\frac{\partial H}{\partial x_t};\ \dot{p}_2(t) = -\frac{\partial H}{\partial \pi_t}$$

(we dropped all subscripts from the leader's parameters).

The solution to the problem is given by the system of equations

(9a) $u^*(t) = \frac{b}{r}\, p_1(t)$

(9b) $\dot{p}_1(t) = -qx(t) + ap_1(t) + \beta_N^2 p_2(t)$

(9c) $\dot{p}_2(t) = p_1(t) - ap_2(t)$

(8a), (8b), (9b) and (9c) form a system of differential equation which can be written

$$(10)\quad \begin{bmatrix} \dot{x}(t) \\ \dot{\pi}(t) \\ \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} -a & -1 & -b^2/r & 0 \\ -\beta_N^2 & a & 0 & 0 \\ -q & 0 & a & \beta_N^2 \\ 0 & 0 & 1 & -a \end{bmatrix} \begin{bmatrix} x(t) \\ \pi(t) \\ p_1(t) \\ p_2(t) \end{bmatrix}$$

The characteristic equation of (10) is

$$(11)\quad (\lambda^2 - a^2 - \beta_N^2)^2 - (\lambda^2 - a^2)\ \frac{b^2 q}{r} = 0$$

The solutions to (11) are $\{\mu_1, -\mu_1; \mu_2, -\mu_2\}$, with $0 < \mu_2 < \mu_1$.

$$(11)\quad \mu_1^2 = a^2 + \beta_N^2 + \frac{1}{2}\beta^2 + \frac{1}{2}\sqrt{\beta^4 + 4\beta^2\beta_N^2};\quad \beta^2 \equiv \frac{b^2 q}{r}$$

$$\mu_2^2 = a^2 + \beta_N^2 + \frac{1}{2}\beta^2 - \frac{1}{2}\sqrt{\beta^4 + 4\beta^2\beta_N^2}$$

The solution x(t) to our problem will be determined by

(12a) $\quad x(t) = \alpha_1 e^{-\mu_1 t} + \alpha_2 e^{-\mu_2 t}$.

(12b) $\quad x(0) = x_0$

(12c) $\quad p_2(0) = 0$

Equation (12c) comes from the nature of the variable $\pi(t)$: since it is a non-predetermined variable, it is optimal for the leader to select a solution $\{u(t)\}$ which attributes an initial zero value to the state-variable associated with $\pi(t)$ ($p_2(t)$ is a co-costate variable; see Buiter (1983)). Equation (12c) enables us to see that the solution selected by the leader will be time-inconsistent. At initial time, the leader selects a set of actions which makes $p_2(0) = 0$; $p_2(s) \neq 0$, $s > 0$. If he was asked at time $t_0 > 0$ to select a new set of controls, then the leader would choose another sequence such that $p_2(t_0) = 0$. We give the interpretation of this choice in Section 6.

Associated with (12), we can write the solution to $\{u(t)\}$ as :

(13) $\quad u(t) = \beta_1 e^{-\mu_1 t} + \beta_2 e^{-\mu_2 t}$

We solve explicitly the system (12) and (13) in Appendix 1. The most notable feature is, considering the case $x_0 > 0$:

(14) $\quad \beta_1 > 0; \ \beta_2 < 0$

Since $0 < \mu_2 < \mu_1$, in the stationary state the system will be approximated by

(15) $\quad x(t) \sim \alpha_2 e^{-\mu_2 t}; \ \alpha_2 > 0$

$\quad\quad u(t) \sim \beta_2 e^{-\mu_2 t}; \ \beta_2 < 0$

Equation (15) then reveals that the optimum strategy of the leader in the long run is to contribute negatively to the decrease of x(t) towards its stationary state. (This point is shown through simulation exercises in Kydland and Prescott.) The interpretation of this paradoxical result comes as follows: In order to increase the contribution of the followers today, it is optimal for the leader to announce that he will seek to increase x_t in the future so as to make the followers increase their current effort to the decrease of x_t.

This result is by itself sufficient to prove the time-inconsistentcy of the optimal program of the leader: in order to obtain a decrease of x_t today (when x_0 is positive), the leader pre-commits itself to increase x_t in the long-run future. When time has passed and when the leader does come to the point when he must sustain a counter-productive policy, the outcome that is reached is worse than if the leader was simply doing nothing.

5. Time-Consistent Solutions (Stackelberg-Equilibrium)

As emphasized in Lucas and Stokey, there are few instances of a leader (a government in the context of their paper) who can pre-commit his action irrespectively of what his successors would like to do. In Game Theory, Nash subgame perfect equilibria are designed to make each move by a player optimal when it is assumed that his future moves will also obey this principle. Technically, the solutions are obtained by backward programming: today's moves minimize current losses plus future losses, when future losses solve the same problem one period ahead. In the context of our open-loop equilibrium, we can give a game representation of the time-consistent equilibrium as follows. Let us assume that the leader is formed of infinitely many players, one per period. Each leader (indexed by time $t \in R$) announces his strategy one after the other, from initial time to infinity. Each leader takes the action of his successors to only depend on the value of the state variable and, under this hypothesis, each leader of time t announces a move u(t) which minimizes the loss function at time t. When all leaders have announced their strategies, all followers simultaneously announce theirs.

An important restriction, to which we shall return in the next section, is that all leaders take the action of their successors to only depend on the value of the state variable. This assumption rules out solutions to the repeated game where the leaders could find in their interest to maintain the reputation of their predecessors. We shall return to this question in Section 8.

When the value function depends only on the state variable, the solution is simply the backward-programming equilibrium. If the model was written in discrete time, we would write:

- $L_1(\ )$ the structural form of the loss function of the leader playing a time 1
- L_0 the value of the loss function of the leader playing at time 0, associated to a given value, x_0, of the state variable. L_0 would solve

  $$L_0 = \min_{u_*}\{qx_0^2 + ru_*^2 + L_1(x_1)\}$$

With x_1 the one-period ahead value of the state variable when the leader plays u_* and when the optimal response of the followers is plugged into the law of motion of x.

In this set up, we can write L_0 as a function of the value of the state variable x_0 and of the assumed structural form of $L_1(\ )$. Formally:

$$L_0 = L_0[x_0;\ L_1(\ )]$$

The solution to our problem is then simply obtained by imposing that the structural form of L_0 coincide with $L_1(\ )$, which is the usual backward programming procedure. Formally, we impose:

$$L_0[\cdot,\ L_1(\ )] = L_1(\cdot)$$

So that one finds the Hamilton-Jacobi equation:

$$L_0(x_0) = \min_{u_*}[qx_0^2 + ru_*^2 + L_0(x_1)]$$

In a continuous-time setting, the solution to the Hamilton-Jacobi equation comes as follows. The leader solves the program:

(16) $$\underset{u(t)}{\text{Minimize}}\ 1/2[qx^2(t) + ru^2(t)] + L'(x(t))\cdot \dot{x}(t)$$

subject to (16a) $\dot{x}(t) = -ax(t) - \pi(t) - bu(t)$

(16b) $\dot{\pi}(t) = -\beta_N^2 x(t) + a\pi(t)$

$L'(\ )$ is the derivative of the loss function with respect to x(t) (its sole explanatory variable, by our assumptions). The solution to (16) is given by

(17) $$ru(t) = bL'(x(t))$$

The system is entirely determined by equations (16a), (16b) and (17) and the definition of L:

(18) $$L(x(t)) = \int_t^\infty [qx(s)^2 + ru(s)^2]\, ds \;;\; x(t) \text{ pre-determined}$$

Given the linearity of the model, we "guess" a solution

(19) $$u(t) = c \cdot x(t)$$

so that equations (16a) and (16b) can be written:

(20) $$\begin{bmatrix} \dot{x}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} -a-bc & -1 \\ -\beta_N^2 & a \end{bmatrix} \begin{bmatrix} x(t) \\ \pi(t) \end{bmatrix}$$

The negative root, $-\gamma$, to the characteristic equation associated to (20) allows us to write

(21) $$x(s) = x(t)e^{-(s-t)\gamma}$$

(22) $$L(x(t)) = \frac{q+rc^2}{4\gamma}\, x(t)^2$$

so that the system can be re-written

(22a) $$bc = \gamma - a - \frac{\beta_N^2}{\gamma+a}$$

(22b) $$rc = \frac{b}{2\gamma}(q+rc^2)$$

(22a) is simply equation (16a) when $x(t)$ is given the value (21) and when $\pi(t)$ solves (16b). (22b) is equation (17) obtained from the Hamilton-Jacobi equation.

Eliminating c in (22a), (22b), one obtains :

(23) $$(\gamma^2-a^2-\beta_N^2)\left[1 + \frac{\beta_N^2}{(\gamma+a)^2}\right] = \beta^2$$

In Appendix 2, we compare the solution to (23), which we call γ_{TC}, to the solution obtained in the Cournot-Nash equilibrium (which we call γ_{CN}). We find

(24) $\quad \gamma_{TC} < \gamma_{CN}$

Inequality (24) shows a feature of the time-consistent solution which was already apparent, in a more extreme form, in the analysis of the time-inconsistent solution. The presence of a dominant player reduces the speed of adjustment of the economy toward its stationary state. As before, the reason is that the leader finds optimal to announce a set of control variables which are below the Cournot-Nash equilibrium optimum: by doing so, the leader forces the followers to increase their share of the burden of the adjustment. As a result, the adjustment of the economy is slower, but the trade-off is beneficial to the leader.

6. The construction of the time-consistent solution: a comparison to alternative methods

Our time-consistent solution is obtained by backward-programming induction (as usual with Nash subgame perfection). Let us now see how the solution could (a posteriori) be calculated from the set of equations defining the time-inconsistent solution and how our solution compares to the one calculated by Buiter.

Let us rewrite in a compact form the set of equations defining the time-inconsistent solution.

(25a) $\quad \dot{x}(t) = -ax(t) - \pi(t) - \frac{b^2}{r} p_1(t)$

(25b) $\quad \dot{\pi}(t) = -\beta_N^2 x(t) + a\pi(t)$

(25c) $\dot{p}_1(t) = -qx(t) + ap_1(t) + \beta_N^2 p_2(t)$

(25d) $\dot{p}_2(t) = p_1(t) - ap_2(t)$

(25e) x_0 given

(25f) $p_2(0) = 0$

Equation (25f) is crucial for our purpose, since it is where time-inconsistency comes from. An interpretation comes as follows: The leader influences the action of the followers through the expected future value of x(t) (see equation (25b), which solves $\pi(t)$ as a forward-value function of x(t)). This influence is measured by $p_2(t)$. At initial time, there is no "past actions" of the followers that the leader wished to influence, so that the value of the co-state variable $p_2(t)$ associated with the co-state variable $\pi(t)$ is optimally set to be zero. When the leader can pre-commit his action to the time-inconsistent solution, each move by the leader influences previous actions of the followers through their expectation effects. The value of $p_2(t)$, $t > 0$, becomes non-zero. However, if the game was to restart at some new initial point t_0,these expectation effects would disappear: bygones are bygones, so that again it would be optimal, in this new game restarted at t_0, to set $p_2(t_0) = 0$. From these remarks, Buiter (1983) inferred that the time-consistent solution would be obtained by simply imposing $p_2(s) = 0$, $\forall s$. However, as apparent in equation (25c), this solution makes the law of motion of the co-state variable of the leader identical to the equation he would select in the Cournot-Nash equilibrium. (This point is already noted in Miller and Salmon.) The solution is time-consistent, but it one where the role of the leader is simply dismissed.

Let us now see how our solution compares to the time-inconsistent program. We show in Appendix 3 that the time-consistent solution is obtained by cancelling in the leader's optimizing program the constraint related to the law of motion of π_t (the co-state variable of the followers) and by substituting instead as a new constraint an equation of the form

$$(26) \qquad \pi_t = \theta x_t$$

where θ is taken as given by the leader.

(26) says that the leader recognizes that the influence of his future actions on the followers is only captured through the influence of the state variable, since it is assumed that his future behavior only relates to that. Furthermore, the requirement of time-consistency of the leader's strategy makes θ (the response by the follower) a given parameter that the leader cannot influence. This shows that the open-loop time-consistent solution that we calculated is simply an equilibrium where the leader calculates a closed-loop equilibrium while the followers select an open-loop strategy.

7. An Example

Let us consider a world economy composed of two countries H and F. The current account between the two countries is assumed to follow a law of motion:

$$(27) \qquad \dot{CA}_t = -aCA_t + bu_t - b^*u_t^*; \ a > 0$$

CA_t measures the current account of country H. CA_t^* $(= -CA_t)$ measures the current account of country F.

u_t and u_t^* are the two controls of macro-policies: they are interpreted as the unemployment rate in countries H and F respectively. Hence, equation (27) shows an autonomous tendency of the current account to return to equilibrium (say, out of a wealth effect) and assumes that unemployment improves it.

Both countries seek to minimize:

$$L_i = \int_0^{\infty}(q_i CA_{it}^2 + r_i u_{it}^2)dt; \; i = H,F$$

(All values (CA,u) must be interpreted as differences with a reference trajectory.)

Let us first write the Cournot-Nash equilibrium (one must be careful of the sign of u_t^* in equation (27)).

$$u_t = -V \cdot CA_t$$

(28)

$$u_t^* = V^* \cdot CA_t$$

When the paramters V and V* are identical in both countries, one sees that unemployment in the world economy looks like a zero-sum game, even if this property is not directly embodied in the model.

Let us now write the Stackleberg time-consistent solution when the country F acts as a leader.

(29) $$u_t = -\theta \cdot CA_t$$

$$u_t^* = \theta^* \cdot CA_t$$

By our previous results we know:

$$\theta > V$$

$\theta^* < V^*$

In order to compare this new solution to the Cournot-Nash solution, let us consider the two cases:

(a) The leader's current account is in deficit ($CA^* < 0$, $CA > 0$) (this situation can be regarded as the situation of the United States vis-à-vis the rest of the world).

In this case, equations (29) show that unemployment is reduced in both countries vis-à-vis the Cournot-Nash solution. The leader is able to reduce his deficit (vis-à-vis the Cournot-Nash trajectory) by inflating the world economy.

(b) The leader's current account is in surplus (a situation akin to Germany's within the E.M.S).

In this case, the reverse occurs: unemployment is increased in both countries. Here the leader keeps a stronger external position than in the Nash equilibrium and lets the follower reduce his deficit by more unemployment.

8. Equilibrium with Reputation

In this section we want to sketch two possible extensions of the time-consistent equilibrium. We shall deal with them in our forthcoming paper on closed-loop equilibria. The first subsection shows the existence of an optimal rule that dominates at all points of time the time-consistent solution. The second subsection deals with finitely-lived contract.

(a) The Optimal Linear Feed-back Rule

In the calculation of the time-consistent solution, we assumed that the optimal strategy selected by a leader at time t was chosen independently of his earlier moves. This constraint ruled out any solution corresponding to a repeated game where leaders at all period might find in their self-interest to sustain a reputation defended by their predecessors. We leave to a forthcoming paper dealing with closed-loop equilibria the thorough analysis of this question. At this stage we want to show in the context of the open-loop system that there exists a rule which strictly dominates the time-consistent solution at all points of time (in contrast, therefore, to the time-inconsistent solution). To calculate this optimum rule, it is enough to recognize that the solution to the time-consistent problem is obtained by a linear feed-back rule

$$u_t = C_{TC} \cdot x_t$$

This feedback rule C_{TC}, due to the way it is constructed, is not the best linear feedback rule available. To find the best of them, let us assume that all leaders at all points of time agreed to follow:

$$u_t = C^* \cdot x_t \tag{30}$$

Then, at initial time, the loss function of the first leader would be written:

$$Lx_0^2 = \frac{q+rC^{*2}}{4\gamma} x_0^2 \tag{31}$$

where γ is the equilibrium speed of adjustment of the state variable (in the open-loop system). Just like in the time-consistent case, γ relates to C by equation (22a):

$$C^* = (1/b)[\gamma^* - a - \frac{\beta_N^2}{\gamma^{*}+a}] \tag{22a}$$

The minimizing of the loss function (31) under the constraint (22a) gives the optimal linear feed-back rule. Clearly, it is independent of x_0 and then time-consistently applies at each period. We show in Appendix 4 that it relates to the time-consistent solution in the following way

$$\gamma^* < \gamma_{TC} \tag{32}$$
$$C^* < C_{TC}$$

Again, we see that the rule allows the leader to take less action ($C^* < C_{TC}$) and results in a slowing down of the adjustment of the economy toward its stationary state.

(b) Optimal Length of Pre-commitment

In this subsection, we want to sketch the implications of allowing finitely-lived pre-commitments of the leader.

Assume that a leader can pre-commit his actions for a period T, and that he expects his successors to do so in the future. If $T = \infty$, the leaders would select the time-inconsistent solution. For T finite (possibly zero), the leaders would select a sequence of finitely-lived strategies:

$$\sigma_n = \{u_{nT+s}\}_{0 \leq s < T; n \in N}$$

If one collapses periods $[nT, (n+1)T]$ into a single point, n, then the strategies σ_n would simply be the time-consistent solution of a discrete-time problem. The solutions will look like:

$$u_{nT+s} = C_s \cdot x_{nT};\ s\varepsilon[0,T];\ n\ \varepsilon\ N$$

so that one sees that the equilibrium would generate a sort of deterministic business cycle repeating itself every T periods. When T is infinite, we have seen that the time-inconsistent solution showed a bad feature: the incentive to break the original pre-commitment would become very strong as time passes, and there will be a point where the leader would prefer to simply return to the time-consistent solution rather than to continue to follow the original pattern of his pre-committed actions. Even if we assume that the renegotiation of new patterns of pre-commitment is conditional on having never broken an earlier pre-commitment, we see that the leader would like to break his original contract.

This needs not be the same when the contracts are finitely lived. If we impose that any broken contract yields the time-consistent solution hereafter, there will be in general a finitely lived contract that the leader will never want to break. T will need to be short enough so that at all points of time, the leader is better off by allowing himself and his successors to stay within the pattern of contractual pre-commitments than by simply returning to the time-consistent solution.

We leave to our forthcoming paper the arithmetic of the equilibrium and the comparison of such patterns of pre-commitment with the solution to the repeated game. We hope that it will shed some light on the working of such contracts as the mandate of a political personnel. This would lay the basis for an analysis of the political business cycle.

Conclusion

We have calculated various solutions to a dynamic game with N + 1 players: (1) the Cournot-Nash equilibrium; (2) the time-inconsistent equilibrium when one agent acts as a leader of the Stackleberg game; (3) the time-consistent solution; (4) the optimal linear feed-back rule available to the leader.

We have shown that the speed of adjustment was reduced as we were going from equilibrium (1) to equilibrium (3) and (4). We also argued that the major problem associated with the time-inconsistent solution was that it had the extravagant feature of pre-committing the leader to pushing the economy away from the stationary state. This means that the major difficulty in dealing with the time-inconsistent solution is that the leader will always like to revise it, even if this was to restrict him from signing any other contract thereafter. To deal with this problem we introduced finitely-lived contracts of which we hope to show in a coming paper the usefulness in analyzing the rationale for having finitely-lived governments.

Appendix 1

The Time-Inconsistent Solution

The time-inconsistent equilibrium is characterized by

$$(A1) \quad \begin{bmatrix} \dot{x} \\ \dot{\pi} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -a & -1 & -b^2/r & 0 & x \\ -\beta_N^2 & a & 0 & 0 & \pi \\ -q & 0 & a & \beta_N^2 & p_1 \\ 0 & 0 & 1 & -a & p_2 \end{bmatrix}$$

$$(A2) \qquad x(0) = x_0; \ p_2(0) = 0$$

The characteristic equation has two negative solutions $(-\mu_1, -\mu_2)$, $\mu_2 < \mu_1$ which can be written:

$$(A3) \qquad \mu_1^2 = a^2 + \beta_N^2 + \frac{1}{2}\ \beta^2 + 1/2 \sqrt{\beta^4 + 4\beta^2\beta_N^2}$$

$$\mu_2^2 = a^2 + \beta_N^2 + \frac{1}{2}\ \beta^2 - 1/2 \sqrt{\beta^4 + 4\beta^2\beta_N^2}$$

The solution to (A1) and (A2) can be written

$$x(t) = \alpha_1 e^{-\mu_1 t} + \alpha_2 e^{-\mu_2 t}$$

$$\pi(t) = \delta_1 e^{-\mu_1 t} + \delta_2 e^{-\mu_2 t}$$

Since $\dot{\pi}(t) = -\beta_N^2 x(t) + a\pi(t)$ one finds

$$\delta_i = \frac{\alpha_i \beta_N^2}{a+\mu_i}; \ i = 1,2$$

This gives

$$bu(t) = \frac{b^2}{r} p_1(t) = -\dot{x}(t) - ax(t) - \pi(t)$$

$$= \alpha_1 [\mu_1 - a - \frac{\beta_N^2}{a+\mu_1}] e^{-\mu_1 t} + \alpha_2 [\mu_2 - a - \frac{\beta_N^2}{a+\mu_2}] e^{-\mu_2 t}$$

$$= b[\beta_1 e^{-\mu_1 t} + \beta_2 e^{-\mu_2 t}]$$

From the definition of μ_2 (equation (A3)), we see that if $\alpha_2 > 0$, $\beta_2 < 0$, as announced in the text. An easy calculation shows that :

$$\beta_N^2 p_2(t) = \dot{p}_1 + qx - ap_1 = - a_1 \alpha_1 e^{-\mu_1 t} + a_2 \alpha_2 e^{-\mu_2 t}$$

with

$$a_1 = \frac{r}{b^2} (\mu_1^2 - a^2 - \beta_N^2) - q = \frac{r}{2b^2} (\beta^2 + \sqrt{\beta^4 + 4\beta^2\beta_N^2}) - \frac{r\beta^2}{b^2} > 0$$

$$a_2 = -\frac{r}{b^2} (\mu_2^2 - a^2 - \beta_N^2) + q = \frac{r}{2b^2} (-\beta^2 + \sqrt{\beta^4 + 4\beta^2\beta_N^2}) + \frac{r\beta^2}{b^2} > 0$$

Thus the conditions : $w(0) = x_0$ and $p_2(0) = 0$ determine

$$\alpha_1 = \frac{a_2 x_0}{a_1 + a_2} \qquad \text{and} \qquad \alpha_2 = \frac{a_1 x_0}{a_1 + a_2}$$

which are positive if x_0 is positive.

Appendix 2

A Comparison of Time-Consistent and Cournot-Nash Solutions

Let

$$u_t = C \cdot x_t; \; C > 0$$

be a "guess" for the optimal strategy of the leader.

The time consistent solution will be characterized by

$$\dot{x}_t = -[a + bC]\,x(t) - \pi(t)$$

$$\dot{\pi}(t) = -\beta_N^2 x(t) + a\pi(t)$$

The system has one negative eigenvalue, $-\gamma$, such that

$$\gamma = bC/2 + \sqrt{[a+(bC/2)]^2 + \beta_N^2}$$

C relates to γ by:

(B1) $$C = (1/b)[\gamma - a - \frac{\beta_N^2}{r+a}]$$

Following the method in the text, the time-consistent solution imposes

(B2) $$rC = (b/2\gamma)[q + rC^2]$$

(B1) and (B2) yield the following equation for γ

(B3) $$Q(\gamma) \equiv (\gamma^2 - a^2 - \beta_N^2)[1 + \frac{\beta_N^2}{(\gamma+a)^2}] = \beta^2; \; \beta^2 = \frac{b^2 q}{r}$$

which has a unique positive solution. ($Q(\gamma)$ is an increasing function of $\gamma > 0, Q(0) <$ and $Q(+\infty) = +\infty$).

To see that the solution to (B3) (which we call γ_{TC}) is below the solution to the Cournot-Nash equilibrium, γ_{CN}, it is enough to check:

$$Q(\gamma_{CN}) > \beta^2$$

Since $\gamma_{CN}^2 = a^2 + \beta^2 + \beta_N^2$, this is clearly true. Hence

$$\gamma_{TC} < \gamma_{CN}$$

Appendix 3

Derivation of the Time-Consistent Solution

Let us show that the time-consistent solution can be obtained by postulating, in the leader's optimizing program, a rule

(C1) $\pi_t = \theta \cdot x_t$

The leader minimizes

$$L = \frac{1}{2}\int_0^\infty (qx_t^2 + ru_t^2)dt$$

subject to $\dot{x}_t = -ax_t - bu_t - \pi_t$

and (C1): $\pi_t = \theta x_t$

The solution comes from minimizing:

$$H = \frac{1}{2}[qx_t^2 + ru_t^2] + p_1(t)[-ax(t) - \theta x(t) - bu(t)]$$

with $p_1(t)$ a solution to

$$\dot{p}_1(t) = -qx(t) + ap_1(t) + \theta p_1(t)$$

The solution is

$$u(t) = (b/r)p_1(t); \; x(t) = x_0 e^{-\gamma t}$$

Since

$$\dot{\pi}(t) = -\beta_N^2 x(t) + a\pi(t)$$

The equilibrium value of θ is

$$\theta = \frac{\beta_N^2}{\gamma+a}$$

so that the law of motion of $p_1(t)$ is

$$\dot{p}_1(t) = -qx(t) + [a + \frac{\beta_N^2}{\gamma+a}]p_1(t)$$

This leads to

$$p_1(t) = \frac{q}{\gamma+a+\frac{\beta_N^2}{\gamma+a}} \qquad x(t)$$

Since $u(t) = (b/r)p_1(t) = C \cdot x(t)$,

one finds

(C4) $C = \frac{qb}{r} \cdot \frac{1}{\gamma+a+\frac{\beta_N^2}{\gamma+a}}$

Given the law of motion of $x(t)$, we also know

(C5) $\qquad C = 1/b[\gamma - a + \frac{\beta_N^2}{\gamma+a}]$

(C4) and (C5) lead to

(C6) $\qquad \beta^2 = [\gamma^2 - a^2 - \beta_N^2][1 + \frac{\beta_N^2}{(\gamma+a)^2}]$

Comparison to the calculation of the time-consistent equilibrium, equation (B3), shows that it coincides with the solution to equation (C6)

Appendix 4

The Optimal Linear Feed-back Rule

From equations (30) and (22a) in the text, the optimal linear feed-back rule is obtained by the following program:

$$\underset{C^*}{\text{Minimize}}\ L = \frac{q + rC^{*2}}{4\gamma}$$

$$\text{Subject to } C^* = 1/b\left[\gamma^* - a - \frac{\beta_N^2}{\gamma^* + a}\right]$$

Plugging the value of C into the definition of L and writing the first order condition leads to the following equation

$$\text{(D1)} \qquad \Delta(\gamma) \equiv \left\{ [\gamma^2 - a^2 - \beta_N^2]\left[1 + \frac{\beta_N^2}{(\gamma + a)^2}\right] - \beta^2 \right\}$$

$$+ \left\{ [\gamma^2 - a^2 - \beta_N^2] \frac{2\gamma}{(\gamma + a)^3} \beta_N^2 \right\} = 0$$

Let $Q(\gamma)$ be the first term in brackets in the definition of $\Delta(\gamma)$. We saw in Appendix 3 that $Q(\gamma) = 0$ gave the solution to the time-consistent equilibrium. This proves that the optimal linear feed-back rule differs from the time-consistent equilibrium. Furthermore we can see that $(\gamma + a)^3 D(\gamma)$ is an increasing function when $\gamma^2 > a^2 + \beta_N^2$ (strictly negative when $\gamma^2 < a^2 + \beta_N^2$) so that

$$\sqrt{a^2 + \beta_N^2} < \gamma^* < \gamma_{TC}$$

Since $C = 1/b[\gamma - a - \frac{\beta_N}{\gamma + a}]$ and since the LHS is an increasing function of γ, one also sees that

$$0 < C^* < C_{TC} < C_{Nash}$$

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