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THE UNIFYING FORMALISM OF DOMINATION : VALUE, PRICE, DISTRIBUTION AND GROWTH IN JOINT PRODUCTION

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Summary

The price of production framework with joint production has a number of unsolved problems, both from a mathematical and a conceptual point of view. Using the unifying construct of "domination" relations, this article sets out a number of results for a broad class of properties of the model, from the existence of a price system to the possibility of balanced growth and its efficiency. The equivalence relation between properties concerning prices and properties concerning quantities produced, as contained in Von Neumann's model, are shown to depend on very special assumptions.

The properties of the prices of production formalism in the case of simple production are now well-known but the properties of the joint production case introduced in the works of J. Von Neumann (1938) and P. Sraffa (1960), remain unfamiliar to many economists.

As a result, two attitudes have arisen. Some scholars classify the case of joint production as an obscure intellectual curiousity ; others attempt to exploit its puzzling properties in order to further the analysis of a certain number of concepts and mechanisms.

The authors of this article have opted for the latter approach and have already published three comprehensive studies on this topic encompassing the problems of prices, distribution and growth :

1. Values and Prices of Production, the Joint Production Model (G. Duménil, D. Lévy (1982 a))
2. The problems of the Factor Price Frontier (G. Duménil, D. Lévy (1982 b))
3. Prices and Quantities, the Joint Production Model (G. Duménil, D. Lévy (1983)).

In these three articles, the idea of "domination" arises as a unifying concept. The purpose of the present analysis is to demonstrate the power of this theoretical tool in dealing with a wide variety of problems.

The investigation has six parts, followed by a mathematical appendix :

1. The formal *definition of domination* will be presented.
2. It will then be applied to the problem of the so-called *positivity of values and prices*.
3. It will also be applied to the problem of *distribution*.

(i.e. under what circonstances is it impossible to give more to all agents).

4. The domination formalism will be applied to the analysis of *balanced growth.*
5. The notion of domination significantly clarifies the *relation between prices and quantities*. On this basis, the alternative theories of a simple relation of symmetry or an actual interdependence of the two will be discussed.
6. Finally, the explanatory power of this concept in studying the *choice of technology* will be considered. The term domination will thus be justified economically*.

I. THE DOMINATION FORMALISM

The object of the first section is to define the concept of "domination" in a general linear model of production and to investigate its basic properties.

1 - Notation

m	number of processes of production
n	number of products
A,B	matrices (m x n) of imputs and outputs
$\overrightarrow{V}$	column vector with m components (with $\overrightarrow{V} \geq \overrightarrow{0}$)
$\overleftarrow{W}$	row vector with n components (with $\overleftarrow{W} \geq \overleftarrow{0}$)
s	scalar (positive)

$$M(s) = B - (1 + s) A \qquad (1)$$

Each row $\overleftarrow{M}_i(0)=\overleftarrow{B}_i - \overleftarrow{A}_i$ of $M(0) = B - A$ represents the net product of each process where its level of activity is equal to one.
By extension, we refer to the row of M(s) as an "s - net product".

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* The present paper is self consistent and all the proofs are given. However, the three articles mentioned above provide more comprehensive sets of results. The firstone is related to sections 2 and 6 of the present study. The second corresponds to section 3; the third to sections 4 and 5.

For all the vectorial inequalities, the following conventions will be used :

$$\vec{V} > \vec{W} \quad \text{if} \quad \forall k \quad V_k > W_k$$

$$\vec{V} \geq \vec{W} \quad \text{if} \quad \forall k \quad V_k \geq W_k \quad \text{and} \quad V_k \neq W_k$$

$$\vec{V} \geqq \vec{W} \quad \text{if} \quad \forall k \quad V_k \geq W_k$$

2 - Definition of Domination

The matrix M(s) can exhibit either row domination $D(\vec{V}, s)$ or column domination $D(\overleftarrow{W}, s)$. According to the economic problem considered, $\vec{V}$, $\overleftarrow{W}$ and s represent different variables. For instance, $\vec{V}$ can be specified as $\vec{L}$, vector of direct labor imputs, $\overleftarrow{W}$ as $\overleftarrow{Q}$, purchasing power of the hourly wage. The scalar s can denote the rate of profit or the rate of growth.

Définition 1 : *The matrix M(s) is said to exhibit a row domination in relation to a vector $\vec{V}$ if :*

$$\exists\ \overleftarrow{\alpha} \geqq \overleftarrow{0} \text{ and } \overleftarrow{\beta} \geqq \overleftarrow{0} \text{ such that } \begin{cases} \overleftarrow{\alpha}\, M(s) \leqq \overleftarrow{\beta}\, M(s) \\ \overleftarrow{\alpha}\, \vec{V} \geq \overleftarrow{\beta}\, \vec{V} \end{cases}$$

with $\alpha_i \beta_i = 0\ \forall i$ *and at least one stict inequality among the n + 1*

In one economic application, a slightly different definition of domination is needed. This condition will be denoted $D'(\vec{V}, s)$:

$$\exists\ \overleftarrow{\alpha} \geqq \overleftarrow{0} \text{ and } \overleftarrow{\beta} \geqq \overleftarrow{0} \text{ such that } \begin{cases} \overleftarrow{\alpha}\, M(s) \leqq \overleftarrow{\beta}\, M(s) \\ \overleftarrow{\alpha}\, \vec{V} > \overleftarrow{\beta}\, \vec{V} \end{cases}$$

with $\alpha_i\ \beta_i = 0 \quad \forall\ i$

This new condition is a particular case of the previous one where the last inequality is necessarily strict.

$\overleftarrow{\alpha}$ M(s) represents a linear combination of the rows of the matrix M(s) :

$$\overleftarrow{\alpha}\, M(s) = \sum_{i \in I} \alpha_i \overleftarrow{M}_i(s)$$

I the set of processes for which $\alpha_i > 0$ is the *dominated* set of processes.

Similarly, $\overleftarrow{\beta}$ M(s) represents another linear combination of the rows of M(s) :

$$\overleftarrow{\beta}\, M(s) = \sum_{j \in J} \beta_j \overleftarrow{M}_j(s)$$

J the set of processes for which $\beta_j > 0$ denotes the *dominating* set of processes.

We define column domination by the same conditions, interchanging rows and columns.

Radical row domination denoted $RD_r(s)$ can be introduced as a particular case of domination where $\overleftarrow{\beta} = \overleftarrow{0}$.* The definition no longer depends on $\vec{V}$.

Définition 2 : *A situation of row radical domination is said to exist if* :

$$\exists\, RD_r(s) \iff \overleftarrow{\alpha} \geq \overleftarrow{0} \quad \textit{such that} \quad \overleftarrow{\alpha}\, M(s) \leqq \overleftarrow{0}$$

Symmetrically, radical column domination will be denoted $RD_c(s)$:

$$\exists\, RD_c(s) \iff \vec{\alpha} \geq \vec{0} \quad \text{such that} \quad M(s)\, \vec{\alpha} \leqq \vec{0}$$

3 - Graphical representation of row domination (m = n = 2)

If n = m = 2, the domination can be demonstrated graphically. We also assume $\vec{V}$ to be strictly positive ($\vec{V} > \vec{0}$) and the matrix M(s) to be regular (det M(s) $\neq$ 0).

We assume that the first process is dominated by the second in a non-radical row domination :

$$\overleftarrow{\alpha} = (\alpha_1, 0), \quad \overleftarrow{\beta} = (0, \beta_2)$$

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* $RD_r(s)$ is always a particular case of $D(\vec{V},s)$ when $\vec{V}$ is strictly positive. If $\vec{V}$ is semi-positive and if det M(s) = 0, M can exhibit a $RD_r(s)$ and no $D(\vec{V},s)$.

with $\alpha_1 \neq 0$ and $\beta_2 \neq 0$

The two conditions of definition 1 become :

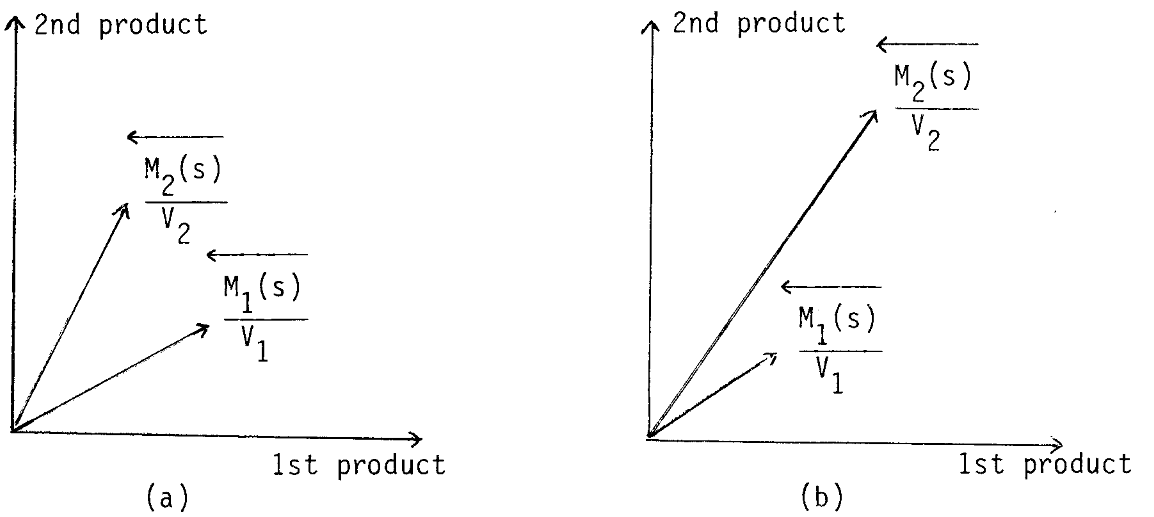
$$\alpha_1 \overleftarrow{M^1(s)} \leqq \beta_2 \overleftarrow{M^2(s)}$$

$$\alpha_1 V_1 \geq \beta_2 V_2$$

Dividing the first inequality by the second, we obtain :

$$\frac{\overleftarrow{M^1(s)}}{V_1} \leqq \frac{\overleftarrow{M^2(s)}}{V_2}$$

In order to identify such relations, we represent (fig.1) the space of row vectors of matrix M(s).



(a) no domination

(b) process 1 is dominated by process 2

Figure 1 : Graphical representation of domination

4 - Two theorems

The following two theorems state well known results of linear algebra using the concept of domination (see, for instance, Gale (1960) corollary 2 and theorem 2.10, p 49).

Theorem 1 : *Either the matrix* $M(s)$ *exhibits row domination in relation to* $\vec{V} \geq \vec{0}$ $(D(\vec{V},s))$ *or the equation* $M(s)\ \vec{X} = \vec{V}$ *has a strictly positive solution* $\vec{X} = M(s)^{-1}\ \vec{V}$.

$$\nexists\, D(\vec{V},s) \iff M(s)^{-1}\ \vec{V} > \vec{0}$$

Theorem 2 : *Either the matrix* $M(s)$ *exhibits radical row domination* $(RD_r(s))$ *or the set of inequalities* $M(s)\vec{X} > \vec{0}$ *have a non-negative solution* $\vec{X} \geq 0$.*

$$\nexists\ RD_r(s) \iff \exists \vec{\lambda} \geq \vec{0} \quad \textit{such that} \quad M(s)\ \vec{\lambda} > \vec{0}$$

Symmetrical theorems hold for column domination

A slightly different version of theorem 1 exists for domination D' (see Gale (1960), theorem 2.6 p.44) :

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* Non-negativity can be replaced by strict positivity :

$$\nexists\ RD_r(s) \iff \exists\ \vec{\lambda} > 0 \ \text{ such that } \ M(s)\ \vec{\lambda} > \vec{0}$$

Theorem 1' : *Either* $M(s)$ *exhibits row domination with respect to* $\vec{V}$ $(D'(\vec{V}, s))$ *or the equation* $M(s)\vec{X} = \vec{V}$ *has a non-negative solution* $\vec{X} = M(s)^{-1}\,\vec{V}$:

$$\nexists\, D'(\vec{V},s) \iff M(s)^{-1}\,\vec{V} \geqq \vec{0}$$

II. EXISTENCE OF A PRICE SYSTEM

In this section, we use the concept of domination to study the problem of the existence of prices of production, that is price systems that imply an equal profit rate for all processes. Labor values are considered here as a particular case of prices of production with a zero rate of profit.

We will consider row dominations, both radical and non radical :

s	is the rate of profit r.
$\vec{V}$	is the vector of direct labor imputs in the production technology $\vec{L}$.

1 - Beyond Positivity

In the neoricardian perspective, the problem of the existence of a price system is directly posed in terms of prices of production : does a set of *positive* prices exist (which guarantee an evenly distributed rate of profit) ?

If a productive system leads for a given value of r to negative prices, this simply means that such a system of prices cannot exist for that rate of profit. However, other sets of prices wich do not guarantee a uniform rate of profit could be defined and allow the system to function. In what follows, this question of the existence of prices will be addressed in a perspective broader than the usual neoricardian problematic, including price systems which do not necessarily result in an evenly distributed rate of profit.

In fact, the problem of prices of production involves two concepts of equalisation of returns, an equal rate of profit on capital, and a uniform wage rate for labor. We will treat these issues separately.

2 - A Uniform Rate of Profit or a Uniform rate of Wages :

One can demonstrate the following two propositions : (see appendix).

Proposition 1 : *The simultaneous existence of a price system guaranteeing a uniform rate of profit, r, and a positive but not necessarily uniform rate of wages w_i is equivalent to the absence of radical row domination $RD_r(r)$.*

Proposition 2 : *The existence of a price system $\frac{\vec{p}}{w}$ guaranteeing a uniform rate of wages w, and a set of rates of profit r_i (possibly negative), resulting in an average rate of profit equal to a given value r is equivalent to the absence of radical row domination $RD_r(r)$.*

3 - The Existence of Prices of Production (when m = n).

The non-radical domination formalism applies to the case of the two-fold equi-remuneration of labor and capital, i.e., to the case of prices of production $\frac{\vec{p}}{w}(r)$. This price system is the solution of the set of linear equations :

$$B\,\vec{p} = (A\,\vec{p} + \vec{L}\,w)(1+r) \qquad (2)$$

or

$$M(r)\,\frac{\vec{p}}{w} = (1+r)\,\vec{L}$$

Thus, using theorem 1, one obtains :

Proposition 3 : *The existence of prices of production is equivalent to the absence of row domination for the vector $\vec{L}$ and the scalar r :*

$$\exists\ \frac{\vec{p}}{w}(r) = (1+r)\,M(r)^{-1}\,\vec{L} > \vec{0}) \Longleftrightarrow \nexists\ D(\vec{L},r)$$

Labor values $\vec{\Lambda}$ in the traditional sense of the term consti -

tute a particular case of prices of production for $r = 0$:*

Proposition 4 : *The existence of values is equivalent to the absence of row domination for the vector* $\vec{L}$ *and the scalar* 0 :

$$\exists\ \vec{\Lambda} > \vec{0} \iff \nexists\ D(\vec{L},0)$$

4 - Variations of Prices of Production with r

Wage prices $\frac{\vec{p}}{w}$ are increasing functions of r according to certain conditions which can be interpreted in terms of domination as well. It is convenient for this purpose, to consider the column vector of wage-prices of advanced capital, $\vec{K}(r)$:

$$\vec{K}(r) = \frac{1}{w}(A\vec{p} + \vec{L}w) = BM(r)^{-1}\vec{L}$$

Since :

$$\frac{d}{dr}(M(r)^{-1}) = M(r)^{-1}\,A\,M(r)^{-1} \quad ,$$

the vector of the derivatives of wage-prices in r is :

$$\frac{d}{dr}\left(\frac{\vec{p}}{w}(r)\right) = M(r)^{-1}\,\vec{K}(r)$$

One thus arrives at the following equivalence :

Proposition 5 : *The strict monotonicity of wage-prices in the profit rate is equivalent to the absence of row domination for the vector* $\vec{K}(r)$ *and the scalar* r :

$$\frac{\vec{p}}{w}(r) \ \text{strictly increasing} \iff M(r)^{-1}\vec{K}(r) > \vec{0} \iff \nexists\ D(\vec{K}(r),r).$$

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* Investigating the conditions for the positivity of values or of prices of production, several authors such as Rampa (1976), Wolfstetter (1976), Filippini (1977), Fujimori (1977) or Lippi (1978) refer to properties similar to our domination relation.
Rampa considers what he calls the "unquestionable inferior efficiency" of productive processes and Wolfstetter "absolutely inferior" processes.

Their approaches are limited to the $n = 2$ model and cannot be regarded as general since they can only be applied to productive systems (viable economies). Filippini's and Fujimori's definitions of "dominated" processes are, however, equivalent to our own.

III. CONDITIONS FOR ANTAGONISTIC RELATIONS OF DISTRIBUTION.

In the joint production case, the wage-profit rate relation can display the most surprising behavior, in particuliar, regions in which rate of profit and wage both increase*. We model the real wage as the bundle of commodities $\overleftarrow{Q}$ corresponding to the hourly purchasing power of labor power.

To study this problem, we will employ the concept of *non-radical domination* :

s	is the rate of profit r.
$\overleftarrow{W}$	is the real hourly wage $\overleftarrow{Q}$.

1 - Antagonism between Wages and Profit Rates.

When prices of production prevail (or within the vicinity of such prices) workers are assumed to defend their purchasing power over the wage bundle **.

We now consider a situation where at the uniform rate of profit r_0 and associated wage-prices $\frac{\vec{p}}{w}(r_0)$, workers can afford the bundle $\overleftarrow{Q}$:

$$\overleftarrow{Q}\ \vec{p} = w \ .$$

It can be said that such a situation is antagonistic if no other set of prices (the rate of profit not being necessarily uniform) exists which proves preferable to every agent (wage-earners considered as a group and capitalists separately for each process of production) :

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* cf. E. Burmeister and K. Kuga (1970) or B. Schefold (1978 a).

** This bundle does not necessarily correspond to actual consumption. If a physical numeraire $\overleftarrow{N}$, existed, the workers' position could express itself through the demand for a certain nominal wage w. In this case the bundle of commodities $\overleftarrow{Q}$ would be $w\overleftarrow{N}$.

Definition 3 : *A situation of distributional antagonism is said to exist if :*

$$\nexists\ \frac{\vec{p}}{w} \text{ such that } \begin{cases} r_i(\frac{\vec{p}}{w}) \geq r_o \text{ for } i = 1,2,\dots,n \\ \overleftarrow{Q}\ \vec{p} \leq w \end{cases} \qquad (3)$$

with at least one strict inequality.

In the first condition, $r_i\ (\frac{\vec{p}}{w})$ denotes the rate of profit of process i at the price system $\frac{\vec{p}}{w}$. If there exists such a $\frac{\vec{p}}{w}$, these desiquilibrium prices benefit every capitalist.

The second condition assures that wage-earners can purchase a commodity bundle equal to or greater than $\overleftarrow{Q}$ for each of its components.

Such a definition of antagonism is stronger than the mere requirement of an increasing price of the wage bundle concomitant with an increasing rate of profit.*

2 - Conditions for the Distributional Antagonism.

In the appendix, we show that the previous definition of antagonism is equivalent to the absence of column domination according to vector $\overleftarrow{Q}$. The following logical equivalences hold :

Proposition 6 :

Distributional antagonism $\Leftrightarrow\ \nexists\ D(\overleftarrow{Q},r) \Leftrightarrow \overleftarrow{Q}\ M(r)^{-1} > \overleftarrow{0}$

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* Our definition is not the same as the one chosen by G. Abraham-Frois and E. Berrebi. For these authors, a distributional antagonism prevails if, whatever the numeraire, wages evolve inversely to the rate uniform rate of profit, (Abraham-Frois, Berrebi - 1982).

V - REPRODUCTION AND GROWTH.

We now turn to problems related to quantities of goods actually produced. In what follows, we take :

s	as the rate of growth ρ .
$\overleftarrow{W}$	as the total consumption of the period $\overleftarrow{C}$.
$\overrightarrow{V}$	as the quantities of labor employed $\overrightarrow{L}$.

1 - Productivity

The concept of the productivity of the matrix of input coefficients which governs the viability of a simple production system, i.e. its capability to reproduce itself, can be extended to the joint production case.

Definition 4 : *The system is said to be productive if one can find a semi-positive vector of levels of activity $\overleftarrow{Z}$ resulting in a net product positive in all its components :*

$$\textit{Productivity} \iff \exists\ \overleftarrow{Z} > \overleftarrow{0} \ \textit{ such that } \ \overleftarrow{Z}\, M(0) > \overleftarrow{0}$$

Theorem 2 shows that the absence of radical domination is equivalent to the productivity of the system.

Proposition 7 : *The productivity of a production system is equivalent to the absence of radical column domination for the scalar $\rho = 0$:*

$$\textit{Productivity} \iff \nexists\ RD_c(\rho = 0)$$

2 - Balanced Growth (BG)

This expression describes a growth path on which each component of the net product increases at the rate ρ which, assuming growth of labor at the same rate, satisfies the following two conditions :

a) The output of one period is sufficient to meet the input requirements of the next with some excess permitted.

b) a given consumption bundle $\overleftarrow{C}$ is also provided.

It can be shown that (see appendix) :

Proposition 8 : *The absence of radical column domination for a scalar* ρ *is a sufficient condition for the existence of a balanced growth path :*

$$\nexists\ RD_c(\rho) \Rightarrow \exists\ BG(\rho,\overleftarrow{C})$$

In some limit cases, the reciprocal can be false. However one has :

Proposition 8' : *The existence of a balanced growth path is a sufficient condition for the absence of radical column domination for the scalars* ρ' *less than* ρ :

$$\exists\ BG(\rho,\overleftarrow{C}) \Rightarrow \nexists\ RD_c(\rho') \text{ for } \rho' < \rho$$

It should be pointed out that the condition related to productivity (Proposition 7) is a particular case of this new condition. This suggests an extension of the concept of productivity to the one of ρ-productivity.

3 - Balanced Equilibrium Growth (BEG)

A balanced equilibrium growth path is defined by the same conditions as for balanced non-equilibrium growth except that condition (a) is strengthened to rule out any excess production.

Definition 5 : *The system is said to have a Balanced Equilibrium Growth path if there exists* $\overleftarrow{Z}_h$ *such that* $\overleftarrow{Z}_h B = (1+\rho)\overleftarrow{Z}_h A + \overleftarrow{C}$.

The vector of levels of activities $\overleftarrow{Z}_h$, is a solution of the equation :

$$\overleftarrow{Z}_h M(\rho) = \overleftarrow{C},$$

and is equal to :

$$\overleftarrow{Z}_h = \overleftarrow{C} M(\rho)^{-1}$$

As a result of theorem 1' , positivity of $\overleftarrow{Z}_h$ is equivalent to the absence of domination $D'(\overleftarrow{C},\rho)$:

Proposition 9 : *The existence of BEG $(\rho, \overleftarrow{C})$ is equivalent to the absence of column domination for the vector $\overleftarrow{C}$ and the scalar ρ :*

$$\exists\ HEG(\rho, \overleftarrow{C}) \Leftrightarrow \overleftarrow{Z}_h = \overleftarrow{C}\ M(\rho)^{-1} \geq \overleftarrow{0} \Leftrightarrow \nexists\ D'(\overleftarrow{C}, \rho)$$

4 - Efficiency of BEG

Various concepts of the efficiency of growth can be defined. One is presented here as an example in the BEG case.

It might be assumed that the equilibrium characteristic of BEG provides growth with a certain efficiency since quantities are perfectly produced and utilized with no superfluous production. This is not always the case, however, as desequilibrium may result in a higher rate of growth.

In any event, the concept of efficiency has meaning only if we assume that labor is limited.

Definition 6 : *An BEG$(\rho, \overleftarrow{C})$ path is efficient if there exists no BG$(\rho', \overleftarrow{C})$ utilizing no more additional labor allowing for the same consumption but with a higher rate of growth $(\rho' \geq \rho)$:*

Efficiency of BEG$(\rho, \overleftarrow{C}) \Leftrightarrow \nexists\ \overleftarrow{Z}' \geq \overleftarrow{0}$ *and* ρ' *such that :*

- $\overleftarrow{Z}'\ B \geq (1 + \rho')(\overleftarrow{Z}'\ A + \overleftarrow{C})$

$$\left.\begin{array}{l} \bullet\ \ \rho' \geq \rho \\ \\ \bullet\ \ \overleftarrow{Z}'\ \overrightarrow{L} \leq \overleftarrow{Z}_h\ \overrightarrow{L} \end{array}\right\} \textit{with at least one strict inequality.}$$

Proposition 10 : *The absence of column domination is equivalent to the efficiency of balanced equilibrium growth paths :*

$$\text{Efficiency of BEG}(\rho, \overleftarrow{C}) \Leftrightarrow \nexists\ D(\overrightarrow{L}, \rho) \Leftrightarrow M(\rho)^{-1}\ \overrightarrow{L} > \overrightarrow{0}.$$

VI - PRICE AND QUANTITY DUALITY : FORMAL SYMMETRY OR INTERDEPENDENCE ?

The conditions thus far defined and interpreted in terms of domination address both problems of prices and distribution,and problems of quantities. All these conditions are similar; they consist of the absence of row or column dominations for given vectors and scalars. Thus, the question of the exact nature of this price-quantity relation arises : formal symmetry or interdependence ?

Already in the simple production case the existence of prices of production and values are dependent on a property related to quantities : the productivity of the system. With the famous von Neumann model, the assertion of the interdependence of price and quantity relations on balanced growth path is complete.

1 - Productivity and Positivity of Prices

In paragraph V(I), productivity is presented as a condition on the matrix M(0) ; a condition wich can be formulated in two different ways :

$$\text{Productivity} \iff \exists\ \overleftarrow{Z} > \overleftarrow{0} \text{ such that } \overleftarrow{Z}\, M(0) > \overleftarrow{0} \iff \nexists\ \overrightarrow{\lambda} \geq \overrightarrow{0} \text{ such that } M(0)\, \overrightarrow{\lambda} \leq \overrightarrow{0}$$

The positivity of prices of production is a sufficient condition for two properties :

$$\begin{array}{l} \text{Positivity of} \\ \text{prices of production} \end{array} \Longrightarrow \exists\ \overrightarrow{\lambda} \geq \overrightarrow{0} \text{ such that } M(0)\, \overrightarrow{\lambda} > \overrightarrow{0} \iff \nexists\ \overleftarrow{Z} \geq \overleftarrow{0} \text{ such that } \overleftarrow{Z}\, M(0) \leq \overleftarrow{0}$$

One is thus confronted in general with the independence of conditions for productivity and positivity of prices of production, which are related by this formal symmetry. In joint production, one can find non-productive systems where positive prices of production exist.

2 - The Price-Quantity Symmetry

The results derived thus far concerning prices of production and BG are summarized in the following table :

- Results concerning prices of production :

Existence of prices of production for r	$\Longleftrightarrow$	No row domination in $M(r)$ with respect to $\vec{L}$
Exitence of a distributional antagonism between capita - lists and wage-earners	$\Longleftrightarrow$	No column domination in $M(r)$ with respect to $\overleftarrow{Q}$

Results concerning growth :

Existence of BEG for ρ	$\Longleftrightarrow$	No column domination in $M(\rho)$ with respect to $\overleftarrow{C}$
BEG is efficient	$\Longleftrightarrow$	No row domination in $M(\rho)$ with respect to $\vec{L}$

3 - From Symmetry to Interdependence

If one identifies ρ with r and $\overleftarrow{Q}$ with $\overleftarrow{C}$, these symmetrical relations become completely interdependent:

Existence of prices of production for r Existence of a distributional antagonism between wage-earners and capitalists whose processes are used in the BEG path *	$\Longleftrightarrow$	Existence of BEG for $\rho = r$ BEG is efficient

These remarkable results are sometimes summarized by the statement that the law the market (a price mechanism) leads to the most rapid growth (a statement about quantities). Herein lies the meaning of von Neumann's demonstration (J. von Neumann - 1938).

However, all these results evaporate as soon as one relaxes von Neumann's assumptions by permitting capitalists to consume and by allowing for a growth rate inferior to the rate of profit, (J.G. Kemeny, O. Morgenstern, G.L. Thompson 1956 or M. Morishima 1960).

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* This restriction is the consequence of the difference between the two definitions of domination (D and D').

Under our more general assumptions, the only general result which can be demonstrated and which may have genuine economic significance is the following :

Proposition 11 : *If there exists a set of prices of production and a distributional antagonism, the economy is, at a minimum, capable of BG for every $\rho \leq r$.*

VII - THE CHOICE OF TECHNIQUE

In this last section, we intend to address one aspect of a complex mechanism which cannot be thoroughly investigated within the limits of this article, namely, what occurs if the aforementioned conditions are not met? In other words, what happens if domihation situation prevail ?

We will limit ourselves to that first problem addresses in this article, the existence of a price system. Our basic claim is that the existence of relations of domination which obstruct the formation of prices of production induces a feedback mechanism upon the choice of technique. This process , therefore, opens a new chapter in the general analysis of the problem of the choice of technique.

Also, this investigation will conclude our analysis of domination by introducing an economic justification of the term itself.

1 - Incompatibility of a Set of Processes

The domination relation $D(\vec{L},r)$ expresses a type of incompatibility of production processes. No price system exists which can provide them with an equal rate of profit.

The inderlying reason for this could simply be that there are too many processes. If the economy utilizes more processes than products ($m > n$), it is impossible in general for them to yield the same rate of profit. In the simple production case, this corresponds to a situation where the same products are obtained by different processes. Except for mere coincidence or chance, there is no reason for such different processes to be equally profitable. It is therefore possible to demonstrate that if $m > n$, some domination relations will prevail.

In the case of $m \leq n$, domination can also occur. An evenly distributed rate of profit is impossible in such instances.

Domination is a sign that only investissement in the most advantageous process is appropriate as we will now illustrate in a two goods three-process model.

2 - The Choice of One Process

We consider the plane of the row vectors of $M(r)$ divided by labor inputs (cf. Section I.3). Each point (i) corresponds to one process and represents the extremity of vector $\dfrac{\overleftarrow{M_i(r)}}{L_i}$.

Figure 2 exhibits two processes (1) and (2) which have equal rates of profit at the price system corresponding to the slope of the line connecting them. In such a situation, it is possible to identify which types of processes capitalists will reject and adopt.

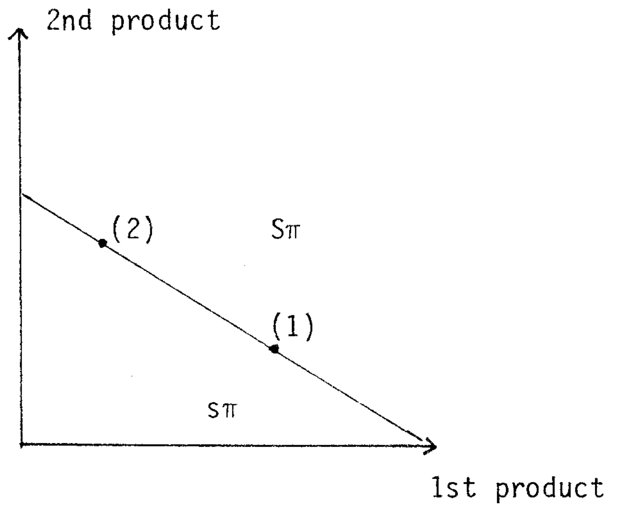

Figure 2 : Super-and sub-profitable sets

The straight line passing throught (1) and (2) is the locus of all linear combinations of processes (1) and (2) having weights whose sum is equal to one and it divides the space into two regions. Since we have normalized the processes by labor time, region Sπ consists of points representing processes which, for the ruling prices and wages, yield a higher rate of profit than r. The reverse is true for region $s\pi$.

Figure 2 therefore summarizes the domination relation and identifies what types of new processes capitalists may be induced to adopt if the original situation is defined by (1) and (2). This problem is more complex, however, as any modification of the technique defines a new situation of distribution which may undercut the original choice.

3 - Stability of a Modification of Technique

Let us consider three processes in competition as indicated by Figure 3(a) :

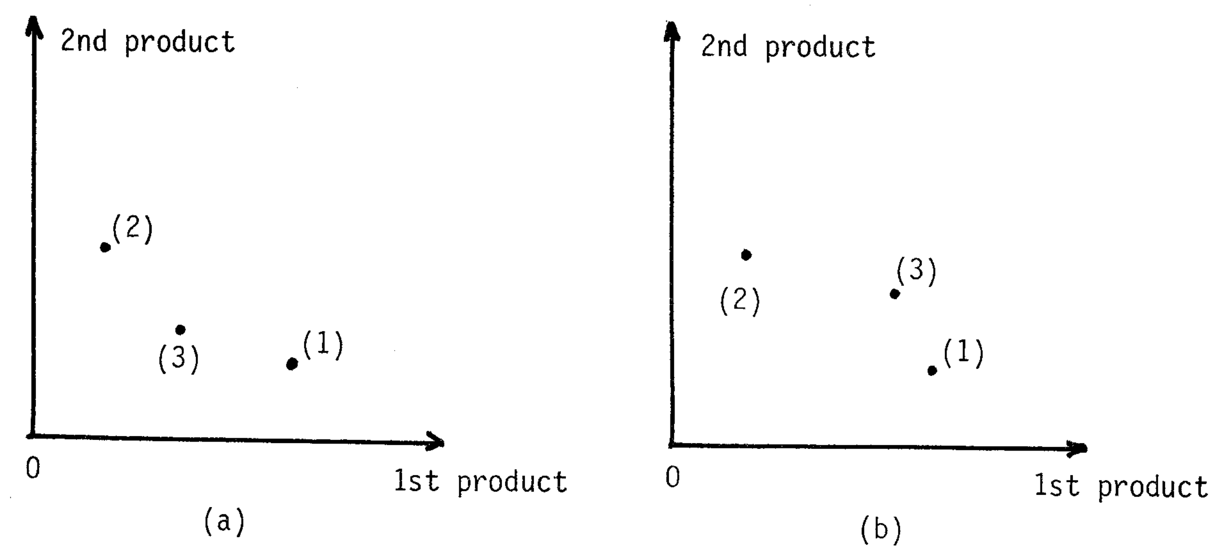

Figure 3 : How to Choose Two Processes From Among Three ?

Process (3) is dominated by a linear combination of processes (1) and (2). If (3) is suppressed, thus suppressing the domination, there exists a set of price which provide (1) and (2) with an evenly distributed rate of profit. At those prices, (3) appears sub-profitable and there is no incentive for any one to adopt it.

If process (1), which belongs to the dominating linear combination, is suppressed, the new set of price equally rewards processes (2) and (3). But in such an instance, process (1) appears super-profitable according to the new set of prices, so that capitalist entrepreneurs would be induced to adopt it.

In case 3(b), process (3) dominates the linear combination of (1) and (2). If(1) (or (2)) is discarded, this choice is confirmed by the new set of prices but there is no mechanism guaranteeing the right choice between (1) and (2).If (3) is suppressed, it necessarily appears super-profitable according to prices corresponding to an equal rate of profit for (1) and (2) ; and this could initiate its reappearance.

One thus reaches the following three conclusions :

a) If the abandoned process belongs to a dominated linear combination, in the domination relation, it is sub-profitable according to the new set of prices.

b) If the abandoned process belongs to a dominating linear combination , it proves to be super-profitable according to the new set of prices and should be reintroduced.

c) If there exist several dominated processes (as is the case in (b)), nothing can indicate which process must be abandonned. There exists a perfect symmetry between (1) and (2) in the 3(b) graph.

Two types of conclusions thus confront us. First, the proposed term "domination" appears to be economically justified. In capitalist competition, processes belonging to dominating linear combinations actually eliminate processes of the dominated linear combination. Second, one can identify new problems involving the possible indeterminancy of the choice of technology in particular situations. The answer which springs to mind in this regard is the possible impact of demand. Such considerations, however, would lead us too far astray from our present project.

CONCLUSION

A broad range of economic questions, from the existence of a system of prices to the existence of balanced growth paths can be tackled with the same basic concept of domination which has been proposed. This theoretical tool was introduced as an algebraic device to account, within a unified framework, for a numbers of properties of a productive system. In fact, the case of the existence of a price system as related to the question of the choice of technology reveals the fundamental economic significance of the concept. From various points of view, a set of production processes can have a kind of internal inconsistency, which makes it incapable of functioning as an economic system. The domination relation which refers to the domination of one set of processes by another set provides a theoretical account of these inconsistencies in various situations and from various points of view. It thus points to economically significant hierarchic relations, such as the existence of prices of production or of balanced growth paths.

MATHEMATICAL APPENDIX

Proof of proposition 1

Let be $\vec{p}$ a set of strictly positive prices and r a given rate of profit. The set of the rates of wages w_i $(i = 1, 2, \ldots, n)$ is given by the equations :

$$\overleftarrow{B}_i \, \vec{p} = (1 + r) \, (\overleftarrow{A}_i \, \vec{p} + L_i \, w_i).$$

The strict positivity of the rates of wages is equivalent to the positivity of $M(r)\vec{p}$ (see eq. (1)). Theorem 2 completes the proof.

Proof of proposition 2

If the absence of radical row domination is assumed, theorem 2 garantees the existence of a strictly positive price vector $\vec{p}$ such that $M(r) \, \vec{p} > \vec{0}$. Thus a strictly positive rate of wages $\tilde{w}$ can be defined by the relation :

$$\tilde{w} = \frac{\overleftarrow{Z} \, M(r) \, \vec{p}}{\overleftarrow{Z} \, \vec{L}(1 + r)} \tag{A.1}$$

where $\overleftarrow{Z}$ is any vector of activity levels. The price system does not guarantee an evenly distributed rate of profit. But an average rate of profit $\bar{r}$ can be defined :

$$\bar{r} = \frac{\text{Total profit}}{\text{Total advanced capital}} = \frac{\overleftarrow{Z} \, B \, \vec{p} - (\overleftarrow{Z} \, A \, \vec{p} + \overleftarrow{Z} \, \vec{L} \, \tilde{w})}{\overleftarrow{Z} \, A \, \vec{p} + \overleftarrow{Z} \, \vec{L} \, \tilde{w}}$$

Using the expression (A.1) for $\tilde{w}$, a straightforward computation gives :

$$\bar{r} = r.$$

Proof of proposition 6

When the price system $\frac{\vec{p}}{w}$ does not lead to a uniform rate of profit, the individual rates of profit $r_i(\frac{\vec{p}}{w})$ are given by the

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equations :

$$\overleftarrow{B}_i \frac{\vec{p}}{w} = (1 + r_i(\frac{\vec{p}}{w}))(\overleftarrow{A}_i \frac{\vec{p}}{w} + L_i) \qquad i = 1, 2, \ldots, n$$

Using the prices of production equation (2) for the initial rate of profit r_o, one obtains :

$$r_i(\frac{\vec{p}}{w}) = r_o + \frac{\overleftarrow{M}_i(r_o) \; \Delta \frac{\vec{p}}{w}}{\overleftarrow{A}_i \frac{\vec{p}}{w} + L_i} \qquad (A.2)$$

with $\Delta \frac{\vec{p}}{w} = \frac{\vec{p}}{w} - \frac{\vec{p}}{w}(r_o)$.

The first condition of (3) is equivalent to the positivity of $\overleftarrow{M}_i(r_o) \; \Delta \frac{\vec{p}}{w}$, the second condition to the negativity of $\overleftarrow{Q} \; \Delta \frac{\vec{p}}{w}$, thus :

$$\text{Distribution trade-off} \iff \nexists \; \Delta \frac{\vec{p}}{w} \text{ such that } \begin{cases} M(r) \; \Delta \frac{\vec{p}}{w} \geqq \vec{0} \\ \overleftarrow{Q} \; \Delta \frac{\vec{p}}{w} \leq \vec{0} \end{cases}$$

with at least, one strict inequality.

Theorem 1 gives the end of the proof.

Proof of proposition 8

Using the theorem 2 and the absence of radical domination, one obtains :

$$\nexists \; RD_c(\rho) \iff \exists \; \overleftarrow{Z} > \overleftarrow{0} \quad \text{such that} \quad \overleftarrow{Z} \, M(\rho) > \overleftarrow{0}$$

If one chooses $\overleftarrow{Z}' = \lambda \overleftarrow{Z}$ with the scalar λ large enough, the vector $\overleftarrow{Z}' \, M(\rho)$ is larger than $\overleftarrow{C}$ and thus there exists a BG $(\rho, \overleftarrow{C})$ path.

Proof of proposition 8'

By definition, the economic system follows a BG path if :

$$\exists \; \overleftarrow{Z} \geq \overleftarrow{0} \quad \text{such that} \quad \overleftarrow{Z} \, B \geqq (1 + \rho) \, \overleftarrow{Z} \, A + \overleftarrow{C}$$

$$\text{or} \quad \overleftarrow{Z} \, M(\rho) \geqq \overleftarrow{C}$$

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If one chooses a rate of growth ρ' strictly inferior to ρ, and if the system follows a BG path, the vector

$$\overleftarrow{Z}\, M(\rho') = \overleftarrow{Z}\, M(\rho) + (\rho - \rho')\, \overleftarrow{Z}\, A$$

is bigger than $\overleftarrow{C} + (\rho - \rho')\, \overleftarrow{Z}\, A$. If all the products are useful (that is to say either they are consumption goods and/or they are production goods), this last vector is strictly positive :

$$BG(\rho, \overleftarrow{C}) \Longrightarrow \exists\, \overleftarrow{Z} \geq \overleftarrow{0} \text{ such that } \overleftarrow{Z}\, M(\rho') > \overleftarrow{0} \text{ for all } \rho' < \rho.$$

Using theorem 2, one obtains the absence of radical domination.

Proof of proposition 10

If one defines the rates of growth $\rho_i^*(\overleftarrow{Z})$ by the relation

$$\overleftarrow{Z}\, \overrightarrow{B}^i = (1 + \rho_i^*(\overleftarrow{Z}))\, (\overleftarrow{Z}\, \overrightarrow{A}^i + C_i)$$

it can be shown that definition 6 is equivalent to

Definition 6' : The BEG $(\rho, \overleftarrow{C})$ is efficient if :

$$\nexists\, \overleftarrow{Z}' \geq \overleftarrow{0} \text{ such that } \rho_i^*(\overleftarrow{Z}') \geq \rho \quad \text{for } i = 1, 2, .., n$$

$$\overleftarrow{Z}'\, \overrightarrow{L} \leq \overleftarrow{Z}_h\, \overrightarrow{L}$$

with at least one strict inequality.

The equivalence of the two definition follows from the usefulness (see proof of proposition 8') of all the products.

The rates of growth $\rho_i^*(\overleftarrow{Z})$ verify equations which are equivalent to equations (A.2) for the rates of profit $r_i(\frac{\overrightarrow{p}}{w})$:

$$\rho_i^*(\overleftarrow{Z}') = \rho + \frac{\Delta\, \overleftarrow{Z}\, \overrightarrow{M}^i(\rho)}{\overleftarrow{Z}\, \overrightarrow{A}^i + C_i} \qquad (A.3)$$

with $\Delta \overleftarrow{Z} = \overleftarrow{Z}' - \overleftarrow{Z}_h$.

So the efficiency of BEG is equivalent to the non-existence of a vector $\Delta \overleftarrow{Z}$ such that :

$$\left.\begin{array}{r} \Delta\, \overleftarrow{Z}\, M(\rho) \geqq \overleftarrow{0} \\ \Delta\, \overleftarrow{Z}\, \overrightarrow{L} \leq 0 \end{array}\right\} \quad \text{with, at least, one strict inequality.}$$

Theorem 1 completes the proof.

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