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RATIONAL EXPECTATIONS MODELS AND BOUNDED MEMORY

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S U M M A R Y

The rational expectation model :

$$y_t = a\,E\,(y_{t+1} \,/\, u_t,\, u_{t-1} \ldots) + u_t$$

has an infinity of stationary solutions. In this paper, we are concerned with the bounded memory model :

$$y_t = a\,E\,(y_{t+1} \,/\, u_t,\, u_{t-1} \ldots u_{t-K+1}) + u_t,$$

of which the previous model is a limit case. We prove that for any a, this model has a unique stationary solution and that this solution can only tend to the forward or the backward solution, when K tends to infinity. Moreover this formulation allows studying the stability properties of these solutions.

R E S U M E

Le modèle à anticipations rationnelles :

$$y_t = a\,E\,(y_{t+1} \,/\, u_t,\, u_{t-1} \ldots) + u_t$$

admet une infinité de solutions stationnaires. Dans cet article, on considère les modèles à mémoire bornée :

$$y_t = a\,E\,(y_{t+1} \,/\, u_t,\, u_{t-1} \ldots u_{t-K+1}) + u_t,$$

dont le modèle précédent est un cas limite. Nous montrons que ce modèle a une unique solution stationnaire pour tout a et que, si K tend vers l'infini, cette solution ne peut converger que vers les solutions avant ou arrière du modèle classique. De plus cette formalisation permet d'étudier la stabilité de ces solutions.

1. INTRODUCTION

The properties of the set of the solutions of R-E models have been recently studied by several authors [SHILLER (1978), GOURIEROUX - LAFFONT - MONFORT (1982), PESARAN (1981), BROZE-JANSSEN-SZAFARZ (1982)]. The main characteristics of such a set can be presented from the classical model introduced by SARGENT-WALLACE (1973). Its reduced form gives the endogenous variable at time t : y_t, as a linear function of an exogenous process u_t and of the expectation ${}_t\tilde{y}_{t+1}$ of y_{t+1} made at time t :

(1.1) $$y_t = a \, {}_t\tilde{y}_{t+1} + u_t$$

Moreover the expectation is assumed to be rational, i.e. equal to the conditional mean of y_{t+1} given the information set I_t available at time t :

(1.2) $${}_t\tilde{y}_{t+1} = E\,(y_{t+1} / I_t)$$

and I_t is assumed to be equal to :

(1.3) $$I_t = \{u_t, u_{t-1}, \ldots..\}$$

Thus the reduced form is :

(1.4) $$y_t = a\, E\,[y_{t+1} / u_t, u_{t-1} \ldots.] + u_t$$

In the sequel, model (1.4) is called "model with infinite memory".
The solutions of this model have been given in GOURIEROUX-LAFFONT-MONFORT (1982).
For convenience, we consider the case in which the process u is stationary and

has an ARMA(p,q) representation :

(1.5) $\Phi(B)\, u_t = \Theta(B)\, \varepsilon_t$

where B is the lag operator, ε is an independent white noise, $\Phi(B)$ and $\Theta(B)$ are lag polynomials with respective degrees p and q :

$$\Phi(B) = 1 - \varphi_1 B \ldots\ldots - \varphi_p B^p \quad , \quad \varphi_p \neq 0$$

$$\Theta(B) = 1 - \theta_1 B \ldots\ldots - \theta_q B^q \quad , \quad \theta_q \neq 0$$

These polynomials have their roots outside the unit circle and have no common roots.

If $|a| < 1$, the model (1.4) (1.5) has a unique stationary solution, the "forward" solution :

$$y_{ot} = \frac{1}{B - a} \left[B - a \frac{\Theta(a)}{\Phi(a)} \frac{\Phi(B)}{\Theta(B)} \right] u_t$$

If $|a| > 1$, there exists an infinity of stationary solutions : $y_t = \lambda\, y_{ot} + (1-\lambda)\, y_{1t}$ $\lambda \in \mathbb{R}$, where $y_{1t} = \frac{B}{B - a} u_t$ is the "backward" solution.

For any a , the other solutions can be obtained by adding to a stationary solution a process of the form $\frac{1}{a^t} M_t$, where M_t is a martingale (with respect to I_t) . Thus the set of all solutions, stationary or not, has a great number of elements. In particular if we consider a solution y there exists an infinity of other solutions y^* such that the variables $y^*_{t_0}$ and y_{t_0} are equal for a given date t_0 .

In this paper we examine the set of solutions for a model with "bounded memory" : $I_t = \{u_t , u_{t-1},\dots,u_{t-K+1}\}$ with given K . At each increase of t a new observation of u appears in the information set and the earliest one disappears ; there is some moving of the information set I_t . The associated model is :

$$(1.6) \qquad y_t = a\,{}_t\tilde{y}_{t+1} + u_t = a\,E\,[y_{t+1}/u_t,\dots,u_{t-K+1}] + u_t$$

What is the value of studying such a model ? In general a model with infinite memory is introduced as a limit case. It can be considered as the limit of model given by (1.6) when K tends to infinity and it is interesting to examine whether the properties of the limit model are or are not related to the properties of the bounded memory models. As it will be seen in the next section, the undesirable properties of the set of solutions of (1.5) disappear if the memory is bounded. For instance in subsection 2.1, it is shown that for any a , the model (1.6) has a unique stationary solution. If K tends to infinity (subsection 2.2), this solution can only converge to the forward solution or to the backward solution depending on the value of a . This property can be seen as a way of selecting a solution in the model with infinite memory. In subsection 2.3 it is shown that a solution is characterized when it is known at a given date. Finally we establish in 2.4 some stability results on the backward and forward solutions under some change in the exogenous process. Some proofs are gathered in section 3.

2. THE SET OF SOLUTIONS OF THE BOUNDED MEMORY MODEL.

2.1 - Stationary solutions.

The determination of stationary solutions is meaningful when the exogenous environment is also stationary. In this subsection, we assume that the prediction $E\,[u_{t+1}/u_t,\ldots,u_{t+1-K}]$ is a linear combination of u_{t+1-k} , $k = 1,\ldots,K$ with coefficients independent of t :

$$(1.7) \qquad E\,[u_{t+1}/u_t,\ldots,u_{t-K+1}] = \sum_{k=1}^{K} \alpha_{kK}\, u_{t+1-k} = A_K(B)\, u_t$$

with $A_K(B) = \sum_{k=1}^{K} \alpha_{kK}\, B^{k-1}$.

This assumption is for instance satisfied if u is a stationary gaussian process or if u has an ARMA(p,q) representation with an independent white noise.

PROPERTY 1 : The bounded memory model has a unique stationary solution such that the expectation $E\,[y_{t+1}/u_t,\ldots,u_{t-K+1}] = \sum_{k=1}^{K} c_{kK}\, u_{t+1-k}$ is linear with respect to u_{t+1-k} , $k = 1,\ldots,K$. This solution is given by :

$$y_t = \frac{1}{B - a} \left\{ B - \frac{a}{1-a\, A_K(a)}\, [1 - B\, A_K(B)] \right\} u_t$$

<u>Proof</u> : Let us denote by $C_K(B)$ the lag polynomial : $C_K(B) = \sum_{k=1}^{K} c_{kK} B^{k-1}$.

If we replace in the equation $y_t = a\, {}_t\tilde{y}_{t+1} + u_t$ the expectation by $C_K(B)\, u_t$, we obtain :

$$y_t = a\, C_K(B)\, u_t + u_t = (a\, C_K(B) + 1)\, u_t \ .$$

Therefore we have :

$$\begin{aligned} {}_t\tilde{y}_{t+1} &= E\,[y_{t+1}/u_t,\dots,u_{t+1-K}] \\ &= E\,[(a\, C_K(B) + 1)\, u_{t+1}/u_t,\dots,u_{t+1-K}] \\ &= (1 + a\, c_{1K})\, E[u_{t+1}/u_t,\dots,u_{t+1-K}] + \frac{a}{B}\,[C_K(B) - c_{1K}]\, u_t \\ &= (1 + a\, c_{1K})\, A_K(B)\, u_t + \frac{a}{B}\,(C_K(B) - c_{1K})\, u_t \end{aligned}$$

By comparing this expression with the initial expression of the prediction, we deduce that :

$$C_K(B) = (1 + a\, c_{1K})\, A_K(B) + \frac{a}{B}\,(C_K(B) - c_{1K})$$

By replacing B by a, we obtain :

$$(1 + a\, c_{1K})\, A_K(a) - c_{1K} = 0$$

$$\Longleftrightarrow \qquad c_{1K} = \frac{A_K(a)}{1 - a\, A_K(a)} \qquad \text{if } 1 - a\, A_K(a) \neq 0$$

Therefore :

$$\begin{aligned} C_K(B) &= \frac{1}{1 - a\, A_K(a)}\, A_K(B) + \frac{a}{B}\left[C_K(B) - \frac{A_K(a)}{1 - a\, A_K(a)}\right] \\ &= \frac{B\, A_K(B) - a\, A_K(a)}{(B - a)\,[1 - a\, A_K(a)]} \end{aligned}$$

There exists a unique stationary solution which is given by :

$$\begin{aligned} y_t &= [a\, C_K(B) + 1]\, u_t \\ &= \frac{a\, B\,[A_K(B) - A_K(a)] + B - a}{(B - a)\,[1 - a\, A_K(a)]}\, u_t \\ &= \frac{1}{B - a}\left[B - a\,\frac{1 - B\, A_K(B)}{1 - a\, A_K(a)}\right] u_t \end{aligned}$$

Q.E.D.

In general this unique stationary solution does not belong to the set of the solutions of the infinite memory model. More precisely we have the following property :

PROPERTY 2 : Let us assume that the exogenous process has an ARMA (p,q) representation. The solution of property 1 is a solution of the infinite memory model if and only if $q = o$ and $p \leqslant K$. In this case this solution coincides with the forward solution.

Proof : The stationary solutions of the infinite memory model are :

$$y_t = \lambda y_{ot} + (1 - \lambda) y_{1t}$$

$$= \frac{B \Phi(a) \Theta(B) - \lambda a \Theta(a) \Phi(B)}{(B - a) \Phi(a) \Theta(B)} u_t$$

A necessary condition for the solution of property 1 to be also a solution of the infinite memory model is that :

$$\frac{B \Phi(a) \Theta(B) - \lambda a \Theta(a) \Phi(B)}{(B - a) \Phi(a) \Theta(B)}$$

is a lag polynomial of degree smaller or equal to $K-1$.

Thus a , which is a root of the denominator, must be a root of the numerator. This implies $\lambda = 1$ and the solution can only be equal to the forward solution. Moreover, since Φ and Θ have no common roots, $B \Phi(a) \Theta(B) - a \Theta(a) \Phi(B)$ and $\Theta(B)$ have no common roots. Therefore it is necessary that $q = 0$, i.e $\Theta(B) = 1$.

The solution $y_t = \frac{1}{B - a} \left[B - a \frac{1 - B A_K(B)}{1 - a A_K(a)} \right] u_t$ is equal to the forward solution $y_t = \frac{1}{B - a} \left[B - a \frac{\Phi(B)}{\Phi(a)} \right] u_t$ if and only if : $1 - B A_K(B) = \Phi(B)$ or equivalently if and only if $K \geqslant p$.

Q.E.D.

When the exogenous process is autoregressive of order p, the solution, which depends on K, becomes fixed for $K \geqslant p$. In particular obviously this solution converges to the forward solution if K tends to infinity. Therefore for an autoregressive exogenous process, the forward solution is the only solution of the infinite memory model which may be considered as a limit case.

2.2 - Convergence of the stationary solution when K increases.

Although the solution does not belong in general to the set of solutions of the infinite memory model, it converges to this set. The limit depends on the value of a and can be the forward solution or the backward solution.

PROPERTY 3 : Let us assume that u has an ARMA (p,q) representation with $q \geqslant 1$ and let us denote by $|\underline{\xi}|$ the minimum absolute value of the roots of Θ.

i) If $|a| < |\underline{\xi}|$, the solution converges in quadratic mean to the forward solution

ii) If $|a| > |\underline{\xi}|$, the solution can only converge in quadratic mean to the backward solution.

Proof : Let us consider the prediction on the infinite sample :

$$E\,[u_{t+1}/u_t,u_{t-1},\ldots] = \sum_{k=1}^{\infty} \alpha_{k\infty}\, u_{t+1-k} = A_\infty(B)\, u_t \quad \text{with} \quad A_\infty(B) = \sum_{k=1}^{\infty} \alpha_{k\infty}\, B^{k-1}.$$

This asymptotic lag polynomial is equal to : $A_\infty(B) = \frac{1}{B}\left[1 - \frac{\Phi(B)}{\Theta(B)}\right]$.

Moreover since an ARMA process is a regular process (DOOB (1953)), $A_K(B)\, u_t$ converges in quadratic mean to $A_\infty(B)\, u_t$ when K tends to infinity.

Therefore to study the asymptotic behaviour of the solution $\frac{1}{B-a}\left(B - \frac{a}{1 - a A_K(a)}\,[1 - B\, A_K(B)]\right) u_t$, it is sufficient to know the asymptotic behaviour of the sequence $A_K(a)$.

It is proved in section 3 that :

i) if $|a| < |\underline{\xi}|$, $A_K(a)$ converges to $A_\infty(a)$. Thus the solution converges to :

$$\frac{1}{B-a}\left(B - \frac{a}{1 - a A_\infty(a)}\,[1 - B\, A_\infty(B)]\right) u_t$$

$$= \frac{1}{B-a}\left[B - a\,\frac{\Theta(a)}{\Phi(a)}\,\frac{\Phi(B)}{\Theta(B)}\right] u_t$$

that is to the forward solution.

ii) if $|a| > |\underline{\xi}|$, the sequence $A_K(a)$ is unbounded.

There exists a subsequence K_n such that $|A_{K_n}(a)|$ tends to infinity and for this subsequence the solution converges in quadratic mean to : $\frac{B}{B-a}\, u_t$, i.e the backward solution.

The possibility of a convergence for the whole sequence depends on the presence or absence of complex roots of Θ with modulus smaller than $|a|$.

Q.E.D.

2.3 - Determination of the solution by an initial condition.

We now consider all the solutions, stationary or not, and their characterization by their knowledge at a given date t_0 .

PROPERTY 4 : Let us assume that the exogenous process is a gaussian process, such that :

i) $E\ [u_{t+1}/u_t,\dots,u_{t+1-K}] = \sum_{k=1}^{K} \alpha_{kK}(t)\ u_{t+1-k}$

with $\alpha_{kK}(t) \neq o\ \forall\ t$ and

ii) $V\ [u_{t+1}/u_t,\dots,u_{t+1-K}] = \sigma_t^2 \neq o$

Then a solution y of the bounded memory model is determined by the knowledge of the random variable y_{t_o} .

Proof : Let us first remark that the exogenous process is in general non stationary since the coefficients α_{kK} and the conditional variances may depend on t .

i) The knowledge of y_t for $t \leqslant t_o$ is a direct consequence of the equation :

$$y_t = a\ E\ [y_{t+1}/u_t,\dots,u_{t+1-K}] + u_t$$

ii) Let us now examine the knowledge of future values : y_t , $t > t_o$.

A solution y_t is an integrable function of $u_t,\dots,u_{t+1-K}$

$y_t = h_t\ [u_t,u_{t-1},\dots,u_{t+1-K}]$.

The functions h_t satisfy :

$$h_t\ [u_t,u_{t-1},\dots,u_{t+1-K}] = a\ E\ [h_{t+1}(u_{t+1},\dots,u_{t+2-K})/u_t,\dots,u_{t+1-K}] + u_t$$

$$= a \int_{IR} h_{t+1}[u,u_t,\dots u_{t+2-K}]\ f_t(u/u_t,\dots,u_{t+1-K})\ du + u_t$$

where $f_t\ (u/u_t,\dots,u_{t+1-K})$ is the conditional density function of u_{t+1} given $u_t,\dots,u_{t+1-K}$. Since u is a gaussian process f_t is given by :

$$f_t[u/u_t,\dots,u_{t+1-K}] = \frac{1}{\sigma_t \sqrt{2\pi}} \exp - \frac{1}{2\sigma_t^2} (u - \sum_{k=1}^{K} \alpha_{kK}(t)\ u_{t+1-k})^2$$

Therefore for any values of $u_t, \ldots, u_{t+1-K}$, we have :

$$h_t(u_t, u_{t-1}, \ldots, u_{t+1-K}) = a \int_{\mathbb{R}} h_{t+1}(u, u_t, \ldots, u_{t+2-K}) \frac{1}{\sigma_t \sqrt{2\pi}}$$

$$\exp - \frac{1}{2\sigma_t^2} \left(u - \sum_{k=1}^{K} \alpha_{kK}(t)\, u_{t+1-k}\right)^2 du + u_t$$

$$\Longleftrightarrow [h_t(u_t, \ldots, u_{t+1-K}) - u_t] \exp - \frac{1}{2\sigma_t^2} \left(\sum_{k=1}^{K} \alpha_{kK}(t)\, u_{t+1-k}\right)^2$$

$$= \int_{\mathbb{R}} a\, h_{t+1}[u, u_t, \ldots, u_{t+2-K}] \frac{1}{\sigma_t \sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma_t^2}\right)$$

$$\exp\left(\frac{u}{\sigma_t^2} \sum_{k=1}^{K} \alpha_{kK}(t)\, u_{t+1-k}\right) du$$

Let us denote : $v = u_{t+1-K}$,

$$g_t(v) = [h_t(u_t, \ldots, u_{t+2-K}, v) - u_t] \exp - \frac{1}{2\sigma_t^2} \left[\sum_{k=1}^{K-1} \alpha_{kK}(t)\, u_{t+1-k} + \alpha_{KK}(t)\, v\right]^2 \text{ and}$$

$$\tilde{g}_{t+1}(u) = a\, h_{t+1}(u, u_t, \ldots, u_{t+2-K}) \frac{1}{\sigma_t \sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma_t^2}\right) \exp\left[\frac{u}{\sigma_t^2} \sum_{k=1}^{K-1} \alpha_{kK}(t)\, u_{t+1-k}\right]$$

We have for any v :

$$g_t(v) = \int_{\mathbb{R}} \tilde{g}_{t+1}(u) \exp\left[\frac{\alpha_{KK}(t)}{\sigma_t^2}\, u\, v\right] du \ .$$

Since $\alpha_{KK}(t) \neq 0$, the function g_t can be interpreted as a Laplace transform of the function $\tilde{g}_{t+1}$. Therefore by inversion of the Laplace transform, we deduce that $\tilde{g}_{t+1}$ is uniquely determined from g_t and also that h_{t+1} is uniquely determined from h_t . This gives by forward recursion the knowledge of y_t , $t > t_o$, from that of y_{t_o} .

Q.E.D.

It is easily seen that the form of the function h_{t_o} has no importance in the previous proof. Thus a solution corresponds to any initial condition $y_{t_o} = h_{t_o}(u_{t_o}, u_{t_o-1}, \dots, u_{t_o+1-K})$.

2.4 - Effect on the solutions of a structural change in the exogenous process.

In this subsection, we consider an exogenous process such that :

$$E[u_{t+1}/u_t, \dots, u_{t+1-K}] = \sum_{k=1}^{K} \alpha_{kK}(t)\, u_{t+1-k} = A_{Kt}(B)$$ and we are interested in the solutions y such that $E[y_{t+1}/u_t, \dots, u_{t+1-K}]$

$$= \sum_{k=1}^{K} c_{kK}(t)\, u_{t+1-k} = C_{Kt}(B)\, u_t \; .$$

Such a formulation (PRIESTLEY (1981)) has the advantage to preserve the linear aspect of the model and, as it will be seen, it allows to study the influence on the (linear) solutions of some change in the exogenous process.

PROPERTY 5 : i) If for any t $\alpha_{KK}(t) \neq o$, the (linear) solutions y are such that the successive lag polynomials $C_{Kt}(B)$ satisfy :

$$C_{K,t+1}(B) = \frac{1}{a}[B\, C_{Kt}(B) - 1] + \frac{1}{a}\frac{c_{KK}(t)}{\alpha_{KK}(t)}[1 - B\, A_{Kt}(B)]$$

Each solution is characterized by the knowledge of y_{t_o} .

ii) If for a date t_o : $\alpha_{KK}(t_o) = o$, there exists an infinity of (linear) solutions compatible with a given y_{t_o} .

<u>Proof</u> : i) Since $E\,[y_{t+1}/u_t,\dots,u_{t+1-K}] = C_{Kt}(B)\,u_t$, we obtain by replacing in the model :

$$y_t = [1 + a\,C_{Kt}(B)]\,u_t$$

and

$$y_{t+1} = [1 + a\,C_{K,t+1}(B)]\,u_{t+1}$$

$$= [1 + a\,c_{1K}(t+1)]\,u_{t+1} + \frac{a\,C_{K,t+1}(B) - a\,c_{1K}(t+1)}{B}\,u_t$$

Therefore : $E[y_{t+1}/u_t,\dots,u_{t+1-K}] = \Big\{[1 + a\,c_{1K}(t+1)]\,A_{Kt}(B)$

$$+ \frac{a\,C_{K,t+1}(B) - a\,c_{1K}(t+1)}{B}\Big\}\,u_t$$

By comparing with the initial expression of the prediction, we deduce that :

$$C_{Kt}(B) = [1 + a\,c_{1K}(t+1)]\,A_{Kt}(B) + \frac{a\,C_{K,t+1}(B) - a\,c_{1K}(t+1)}{B}$$

The equality of the terms of degree K-1 implies :

$$c_{KK}(t) = [1 + a\,c_{1K}(t+1)]\,\alpha_{KK}(t)$$

and, if $\alpha_{KK}(t) \neq o$, $c_{1K}(t+1) = \frac{1}{a}\left[\frac{c_{KK}(t)}{\alpha_{KK}(t)} - 1\right]$.

By replacing in the difference equation giving $C_{K,t+1}$, we have :

$$C_{K,t+1}(B) = \frac{1}{a}\,[B\,C_{K,t}(B) - 1] + \frac{1}{a}\,\frac{c_{KK}(t)}{\alpha_{KK}(t)}\,[1 - B\,A_{Kt}(B)] \,.$$

If y_{t_o} is known, the lag polynomial $C_{Kt_o}(B)$ is also known and, by applying the previous difference equation, we see that $C_{K,t}(B)$ and y_t $t > t_o$ are perfectly determined.

ii) If $\alpha_{KK}(t_o) = o$ and if y is a solution, it is necessary that $c_{KK}(t_o) = o$. The relation :

$$C_{Kt}(B) = [1 + a\, c_{1K}(t+1)]\, A_{Kt}(B) + \frac{a\, C_{K,t+1}(B) - a\, c_{1K}(t+1)}{B}$$

leads to a system of linear equations in $c_{kK}(t+1)$ $k = 1,\ldots,K$ which is not of full rank. Therefore there exists an infinity of solutions compatible with a given possible $C_{Kt_o}(B)$.

Q.E.D.

Let us now examine the case in which the process u coïncides with a stationary process after the date $t_o - K + 1$.

The lag polynomial A_{Kt} is independent of t for $t \geqslant t_o$: $A_{Kt} = A_{Kt_o}$ $\forall\ t \geqslant t_o$, and the relation between C_{Kt+1} and C_{Kt} becomes :

$$C_{K,t+1}(B) = \frac{1}{a}\,[B\, C_{Kt}(B) - 1] + \frac{1}{a}\, \frac{c_{KK}(t)}{\alpha_{KK}(t_o)}\, [1 - B\, A_{Kt_o}(B)] \quad \forall\ t \geqslant t_o \ .$$

Let us introduce the vector : $C_{Kt} = [c_{1K}(t),\ldots,c_{KK}(t)]'$; we have :

$$C_{K,t+1} = [-\frac{1}{a}, o \ldots o]' + \Gamma_{Kt_o}\, C_{Kt} \qquad t \geqslant t_o$$

where : $$\Gamma_{Kt_o} = \frac{1}{a} \begin{bmatrix} 0 & \cdots & \cdots & 0 & \frac{1}{\alpha_{KK}(t_o)} \\ 1 & \ddots & & \vdots & -\frac{\alpha_{K1}(t_o)}{\alpha_{KK}(t_o)} \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{\alpha_{K-1,K}(t_o)}{\alpha_{KK}(t_o)} \end{bmatrix}$$

The solution of this linear difference equation can be written as :

$$C_{Kt} = C_K^* + \Gamma_{Kt_o}^{t-t_o} (C_{Kt_o} - C_K^*)$$

where C_K^* is a constant solution of the equation. Such a particular solution is easily obtained by taking the vector C_K^* associated with the stationary solution of the model : $y_t = a\ E[y_{t+1}/v_t,\dots,v_{t+1-K}] + v_t$ where the process v is stationary such that :

$$E[v_{t+1}/v_t, \dots, v_{t+1-K}] = A_{Kt_o}(B)\ v_t \qquad \forall\ t$$

There remains to examine the limit properties of the sequence C_{Kt} and for this purpose to study the position of the eigenvalues of Γ_{Kt_o} with respect to 1 .

<u>PROPERTY 6</u> : The eigenvalues λ_i of Γ_{Kt_o} are such that

$a\lambda_i$ is a root of $1 - z\, A_{Kt_o}(z)$.

<u>Proof</u> : We have :

$$\det [a\, \Gamma_{Kt_o} - a\lambda I] = \det \begin{bmatrix} -a\lambda & 0 & & 0 & \frac{1}{\alpha_{KK}(t_o)} \\ 1 & -a\lambda & & & -\frac{\alpha_{1K}(t_o)}{\alpha_{KK}(t_o)} \\ 0 & 1 & \ddots & 0 & \\ & & \ddots & \ddots & \\ 0 & 0 & 1 & -a\lambda & -\frac{\alpha_{K-1,K}(t_o)}{\alpha_{KK}(t_o)} \end{bmatrix}$$

By developping with respect to the elements of the last column, we obtain :

$\det\ [a\, \Gamma_{Kt_o} - a\lambda I] = \frac{(-1)^K}{\alpha_{KK}(t_o)} [1 - a\lambda A_{Kt_o}(a\lambda)]$ and the result follows.

Q.E.D.

PROPERTY 7 : C_{Kt} converges to C_K^* , when t tends to infinity if and only if : $|a| > |\bar{\eta}|$, where $|\bar{\eta}|$ is the maximum absolute value of the roots of $1 - z\, A_{Kt_o}(z)$.

Proof : This is a direct consequence of the previous property.

Q.E.D.

Let us now consider an exogenous process u obtained by disturbing an ARMA(p,q) process v such that : $E\, [v_{t+1}/v_t, \ldots, v_{t+1-K}] = A_{Kt_o}(B)\, v_t$ $\forall\ t$. For instance, we may assume that $u_t = v_t$ for $t \leqslant t_o - K_o$ and for $t \geqslant t_o - K+1$ with $K_o \geqslant K$. If before $t_o - K_o$ the evolution of the endogenous process was associated with the stationary solution C_K^* , between $t_o - K_o$ and $t_o - K+1$,the C_{Kt} satisfy the non stationary difference equation of property 5. Thus C_{Kt_o} is usually different from C_K^* .

However, since after $t_o - K+1$, the process u coïncides with v again,the solution converges to the stationary one depending on the position of $|a|$ with respect to $|\bar{\eta}|$ (property 7).

Since we know from property 3 that the stationary solution converges to the forward or to the backward solution, the previous result can be viewed asymptotically as a stability result on the backward or forward solutions. Therefore it is important to study the asymptotic behaviour of $|\bar{\eta}|$ and in particular to examine if this sequence converges to the maximum absolute value of the roots of $1 - z A_\infty(z)$. Unfortunately this result is not valid and this can easily be seen by considering the case of a MA(1) process v :

$$v_t = \varepsilon_t - \theta \varepsilon_{t-1} \qquad E \varepsilon_t = o , V \varepsilon_t = 1$$

For such a process it can be deduced from property 9 that the lag polynomial A_K is given by :

$$A_K(z) = \frac{1}{1 - \theta^{2(K+1)}} \left[- \theta \sum_{k=1}^{K} (z\theta)^{k-1} + \theta^{2K+1} \sum_{k=1}^{K} \left(\frac{z}{\theta}\right)^{k-1} \right]$$

$$= \frac{1}{1 - \theta^{2(K+1)}} \left[- \theta \frac{1 - (z\theta)^K}{1 - z\theta} + \theta^{2(K+1)} \frac{1 - \left(\frac{z}{\theta}\right)^K}{\theta - z} \right]$$

<u>PROPERTY 8</u> : If $v_t = \varepsilon_t - \theta \varepsilon_{t-1}$ is a MA(1) process :

$$\forall \alpha > o \quad \exists K_o \quad \forall K \geqslant K_o$$

$$|\bar{\eta}| \in [\frac{1}{|\theta|} - \alpha , \frac{1}{|\theta|} + \alpha] \cup [|\theta| - \alpha , |\theta| + \alpha]$$

Proof : i) Let us consider a root z of the equation : $1 - z\,A_K(z) = 0$

$$\Longleftrightarrow \quad 1 - \frac{1}{1-\theta^{2(K+1)}}\left[-\theta z\,\frac{1-(z\theta)^K}{1-z\theta} + \theta^{2(K+1)} z\,\frac{1-\left(\frac{z}{\theta}\right)^k}{\theta - z}\right] = 0$$

By reducing to the same denominator, we obtain :

$$(1-\theta^{2K+2})(1-\theta z)(\theta - z) = -\theta z(\theta - z)(1-\theta^K z^K) + (1-\theta z)(\theta^{2K+2} z - z^{K+1}\theta^{K+2})$$

or equivalently :

$$(*)\quad z^{K+2}(-\theta^{K+1} + \theta^{K+3}) + z(1-\theta^{2K+4}) + \theta^{2K+3} - \theta = 0$$

Therefore :

$$(**)\quad |z|^{K+2}\,|\theta^{K+1} - \theta^{K+3}| - |z|\,|1-\theta^{2K+4}| - |\theta^{2K+3} - \theta| \leqslant 0$$

If asymptotically the absolute value of z were not smaller than $c > \frac{1}{|\theta|}$, we would have :

$$\limsup_K \left\{|z|^{K+2}\,|\theta^{K+1} - \theta^{K+3}| - |z|\,|1-\theta^{2K+4}| - |\theta^{2K+3} - \theta|\right\}$$

$$= \limsup_K \left\{|z|^{k+2}\,|\theta^{K+1} - \theta^{K+3}| - |z| - |\theta|\right\} = +\infty$$

and this is not compatible with the necessary inequality (**). Therefore, for any α : $|\bar{\eta}| \leqslant \frac{1}{|\theta|} + \alpha$ for K sufficiently large.

ii) On the other hand, for any given α, $1 - z\,A_K(z)$ tends uniformly to $1 - z\,A_\infty(z)$ for $z \in [0, |\theta| - \alpha] \cup [|\theta| + \alpha, \frac{1}{|\theta|} - \alpha]$. Since A_∞ has no root on this set, the same is true for A_K . Then for K sufficiently large $|\bar{\eta}|$ belongs to $[|\theta| - \alpha, |\theta| + \alpha] \cup [\frac{1}{|\theta|} - \alpha, \frac{1}{|\theta|} + \alpha]$

Q.E.D.

In this particular case of a MA(1) process, the lag polynomial $1 - zA_\infty(z)$ is equal to $\frac{1}{\Theta(z)}$ and thus it is clear that the application which associates to K the maximum absolute value of the roots of $1 - z A_K(z) = o$ is not continuous.

If we assume that $|a| < |\Theta|$, we know from property 3 that the stationary solution converges to the forward solution, when K tends to ∞ . The application of properties 7 and 8 implies the instability of this forward solution.

If $|a| > \frac{1}{|\Theta|}$, it is directly seen from the expression of $A_K(z)$ that $A_K(a)$ tends to infinity and that, from property 3, the stationary solution converges to the backward solution. We deduce from properties 7 and 8 the stability of this backward solution.

3. COEFFICIENTS OF THE PREDICTION GIVEN A FINITE SAMPLE.

To complete the proof of property 3 , it is necessary to study precisely the sequence of coefficients α_{kK} , $k = 1,\dots,K$ appearing in the conditional expectation

$$E\,[u_{t+1}/u_t,\dots,u_{t-K+1}] = \sum_{k=1}^{K} \alpha_{kK}\ u_{t+1-k} \ .$$

3.1 - Determination of the coefficients.

The problem of the determination of the regression coefficients α_{kK} , $k = 1,\dots,K$ has already been considered in the literature on time series (see for instance WHITTLE (1963), AKAIKE (1973)) ; however the resolution method we derive is slightly different from those proposed earlier.

Let us consider an ARMA (p,q) process u defined by $\Phi(B)\, u_t = \Theta(B)\, \varepsilon_t$ where B is the lag operator, Φ and Θ (of respective degrees p and q) have their zeros lying outside the unit circle and have no common zeros, and ε is an independent white noise process. The autocovariance function of u is denoted by γ . It is well known that :

i) $\gamma_{-k} = \gamma_k \quad \forall\ k$

ii) the autocovariances satisfy : $\Phi(B)\, \gamma_i = 0$ for $i \geqslant q+1$

iii) $$\Gamma(z) = \sum_{k=-\infty}^{+\infty} \gamma_k\, z^k = \frac{\sigma^2\, \Theta(z)\, \Theta(\frac{1}{z})}{\Phi(z)\, \Phi(\frac{1}{z})}$$

The regression coefficients α_{kK} , $k = 1,\dots,K$ are the solutions of the linear system :

$$\sum_{k=1}^{K} \alpha_{kK}\, \gamma_{i-k} = \gamma_i \qquad i = 1,\dots,K \tag{3.1}$$

Since this system is homogenous, it can be assumed that the process ε has a unit variance.

PROPERTY 9 : If $q \geqslant 1$ and if $K > 2\max(p,q)$, the sequence α_{kK} , $k = \max(0,p-q)+1,\dots,K-\max(0,p-q)$ satisfies a linear difference equation of order $2q$. The characteristic polynomial associated with this equation is : $z^q\, \Theta(z)\, \Theta(\frac{1}{z})$.

Proof : a) Let us apply the linear operator $\Phi(F)\ \Phi(B)$ where $F = B^{-1}$ to (3.1) for $i = \max(p,q) + 1,\ldots,K-p$. Since $\Phi(B)\ \gamma_i = 0$ for $i \geqslant q+1$, we obtain :

(3.2) $$\sum_{k=1}^{K} \alpha_{kK}\ \Phi(F)\ \Phi(B)\ \gamma_{i-k} = 0$$

i) Since $\Phi(B)\ \gamma_i = 0$ if $i \geqslant q+1$, we have :

$$\Phi(F)\ \Phi(B)\ \gamma_{i-k} = 0 \qquad \text{for } k \leqslant i - q - 1$$

ii) On the other hand, if $k \geqslant i + q + 1$, we have :

$\Phi(B)\ \gamma_{k-i} = 0$ and, since $\Phi(F)\ \Phi(B)$ is a symmetrical lag polynomial, $\Phi(F)\ \Phi(B)\ \gamma_{i-k} = \Phi(F)\ \Phi(B)\ \gamma_{k-i} = 0$.

If we delete in (3.2) the terms corresponding to $k \leqslant i - q - 1$ and to $k \geqslant i + q + 1$, we obtain :

$$\sum_{k=\max[1,i-q]}^{\mathrm{Min}(K,i+q)} \alpha_{kK}\ \Phi(F)\ \Phi(B)\ \gamma_{i-k} = 0 \qquad \text{for } i = \max(p,q) + 1,\ldots,K-p$$

and then : $\sum_{k=i-q}^{i+q} \alpha_{kK}\ \Phi(F)\ \Phi(B)\ \gamma_{i-k} = 0$ for $i = \max(p,q) + 1,\ldots,K - \max(p,q)$

This is equivalent to :

$$\sum_{k=-q}^{q} \alpha_{i+k,K}\ \Phi(F)\ \Phi(B)\ \gamma_k = 0 \qquad \text{for } i = \max(p,q) + 1,\ldots,K - \max(p,q)$$

Therefore the sequence α_{kK} , $K = \max(0,p-q)+1,\ldots,K - \max(0,p-q)$ satifies the difference equation of order $2q$, whose coefficients are given by : $\Phi(F)\ \Phi(B)\ \gamma_k$ $k = -q,\ldots,q$.

b) The characteristic polynomial associated with this equation is :

$$z^q \sum_{k=-q}^{q} [\Phi(F)\ \Phi(B)\ \gamma_k]\ z^k \ .$$

Since $\Phi(F)\ \Phi(B)\ \gamma_k = o$ if $|k| \geqslant q+1$, this polynomial is equal to :

$$z^q \sum_{k=-\infty}^{+\infty} [\Phi(F)\ \Phi(B)\ \gamma_k]\ z^k$$

$$= z^q\ \Phi(z)\ \Phi(\tfrac{1}{z})\ \Gamma(z) \quad \text{with} \quad \Gamma(z) = \sum_{k=-\infty}^{+\infty} \gamma_k\ z^k$$

$$= z^q\ \Phi(z)\ \Phi(\tfrac{1}{z})\ \frac{\Theta(z)\ \Theta(\frac{1}{z})}{\Phi(z)\ \Phi(\frac{1}{z})}$$

$$= z^q\ \Theta(z)\ \Theta(\tfrac{1}{z})$$

Q.E.D.

Let us remark that, if $q \geqslant p$, all the coefficients satisfy this equation.

If all the roots ξ_ℓ , $\ell = 1,\dots,q$ of Θ are distinct, the α_{kK} $k = \max(o,p-q) + 1,\dots,K - \max(o,p-q)$ are given by :

$$\alpha_{kK} = \sum_{\ell=1}^{q} B_{\ell K}\ \xi_\ell^k + \sum_{\ell=1}^{q} C_{\ell K}\ \frac{1}{\xi_\ell^k}$$

The constants $B_{\ell K}$, $C_{\ell K}$ $\ell = 1,\dots,q$ together with the $\alpha_{\ell K}$ $\ell = 1,\dots,\max(o,p-q)$ and $D_{\ell K} = \alpha_{K-\ell+1}$ $\ell = 1,\dots,\max(o,p-q)$ are obtained for instance by solving the system of the following $2\max(p,q)$ linear equations :

$$(3.3) \qquad \sum_{k=1}^{K} \alpha_{kK}\ \gamma_{i-k} = \gamma_i \qquad i \in [1,\max(p,q)] \cup [K - \max(p,q)+1,K]$$

3.2 - Asymptotic behaviour of the coefficients.

The solution α_{kK} studied in the previous subsection depends on K. The regularity of the process u implies that α_{kK} converges to the component $\alpha_{k\infty}$ appearing in the development of the prediction on the infinite samples $E[u_{t+1}/u_t, u_{t-1}, \ldots] = \sum_{k=1}^{\infty} \alpha_{k\infty} u_{t+1-k} = A_\infty(B)\, u_t$.

Since $A_\infty(B) = \frac{1}{B}\left[1 - \frac{\Phi(B)}{\Theta(B)}\right]$, the coefficients $\alpha_{k\infty}$ satisfy a linear difference equation with characteristic polynomial $z^q\, \Theta(\frac{1}{z})$.

Therefore, assuming for convenience that the roots of Θ are distinct, $\alpha_{k\infty}$ can be written as $\alpha_{k\infty} = \sum_{\ell=1}^{q} C_{\ell\infty} \frac{1}{\xi_\ell^k}$.

Since the sequences ξ_ℓ^k, $\frac{1}{\xi_\ell^k}$, $\ell = 1,\ldots,q$ are linearly independent the convergence of the α_{kK} to $\alpha_{k\infty}$ implies the convergence of $B_{\ell K}$ to zero and of $C_{\ell K}$ to $C_{\ell\infty}$. We are now interested in the determination of the rates of convergence.

PROPERTY 10 : i) $B_{\ell K}$ $\ell = 1,\ldots,q$ is at most of order : $\frac{1}{|\underline{\xi}|^K} \frac{1}{|\xi_\ell|^K}$, where $|\underline{\xi}|$ is the minimum absolute value of the roots of Θ.

ii) $D_{\ell K}$, $\ell = 1,\ldots,\mathrm{Max}(o,p-q)$ is at most of order $\frac{1}{|\underline{\xi}|^K}$

Proof : The parameters $\alpha_{\ell K}$, $D_{\ell K}$ $\ell = 1,\ldots,\mathrm{Max}(o,p-q)$, $B_{\ell K}$, $C_{\ell K}$ $\ell = 1,\ldots,q$ are obtained from :

$$\sum_{k=1}^{K} \alpha_{kK} \gamma_{i-k} = \gamma_i \qquad i \in [1,\mathrm{Max}(p,q)] \cup [K-\mathrm{Max}(p,q)+1, K]$$

or equivalently from :

$$\begin{cases} \sum_{k=1}^{K} \alpha_{kK} \, \gamma_{i-k} = \gamma_i & i = 1,\dots,\mathrm{Max}(p,q) \\ \sum_{k=1}^{K} \alpha_{kK} \, \Phi(B) \, \gamma_{i-k} = 0 & i = K - \mathrm{Max}(p,q) + 1,\dots,K \,. \end{cases}$$

The second part of the system can also be written as :

$$\sum_{k=i-q}^{K} \alpha_{kK} \, \Phi(B) \, \gamma_{i-k} = 0 \qquad i = K - \mathrm{Max}(p,q) + 1,\dots,K$$

$$\Longleftrightarrow \quad \sum_{k=K-i-q+1}^{K} \alpha_{kK} \, \Phi(B) \, \gamma_{K-i-k+1} = 0 \qquad i = 1,\dots,\mathrm{Max}(p,q)$$

If K is sufficiently large the α_{kK} appearing in the previous equations depend on the parameters through $B_{\ell K}$, $C_{\ell K}$, $D_{\ell K}$. By replacing the α_{kK} in terms of $B_{\ell K}$, $C_{\ell K}$, $D_{\ell K}$ we obtain :

$$\sum_{k=K-i-q+1}^{K-\mathrm{Max}(o,p-q)} \left(\sum_{\ell=1}^{q} B_{\ell K} \, \xi_\ell^k + \sum_{\ell=1}^{q} C_{\ell K} \, \frac{1}{\xi_\ell^k} \right) \Phi(B) \, \gamma_{K-i-k+1}$$

$$+ \sum_{k=K-\mathrm{Max}(o,p-q)+1}^{K} D_{K-k+1,K} \, \Phi(B) \, \gamma_{K-i-k+1} = 0 \quad i = 1,\dots,\mathrm{Max}(p,q)$$

$$\Longleftrightarrow \quad \sum_{\ell=1}^{q} B_{\ell K} \, \xi_\ell^K \sum_{k=K-i-q+1}^{K-\mathrm{Max}(o,p-q)} \xi_\ell^{k-K} \, \Phi(B) \, \gamma_{K-i-k+1}$$

$$+ \sum_{\ell=1}^{q} C_{\ell K} \frac{1}{\xi_\ell^K} \sum_{k=K-i-q+1}^{K-\mathrm{Max}(o,p-q)} \frac{1}{\xi_\ell^{k-K}} \, \Phi(B) \, \gamma_{K-i-k+1}$$

$$+ \sum_{\ell=1}^{\mathrm{Max}(o,p-q)} D_{\ell K} \, \Phi(B) \, \gamma_{\ell-i} = 0 \qquad i = 1,\dots,\mathrm{Max}(p,q)$$

This system is of the form :

$$\sum_{\ell=1}^{q} B_{\ell K}\, \xi_\ell^K\, \lambda_{i\ell} + \sum_{\ell=1}^{q} C_{\ell K} \frac{1}{\xi_\ell^K}\, \mu_{i\ell} + \sum_{\ell=1}^{Max(o,p-q)} D_{\ell K}\, \nu_{i\ell} = 0 \quad , i=1,\dots,Max(p-q$$

where $\lambda_{i\ell}$, $\mu_{i\ell}$, $\nu_{i\ell}$ are independent of K .

It gives $B_{\ell K}$, $D_{\ell K}$ as function of $C_{\ell K}$:

$$\sum_{\ell=1}^{q} B_{\ell K}\, \xi_\ell^K\, \lambda_{i\ell} + \sum_{\ell=1}^{Max(o,p-q)} D_{\ell K}\, \nu_{i\ell} = - \sum_{\ell=1}^{q} C_{\ell K} \frac{1}{\xi_\ell^K}\, \mu_{i\ell} \quad i=1,\dots,Max(p,q)$$

Since : $- \sum_{\ell=1}^{q} C_{\ell K} \frac{1}{\xi_\ell^K}\, \mu_{i\ell}$ is at most of order $\frac{1}{|\underline{\xi}|^K}$, the solutions

$B_{\ell K}\, \xi_\ell^K$ and $D_{\ell K}$ are also at most of order $\frac{1}{|\underline{\xi}|^K}$.

<u>PROPERTY 11</u> : i) If $|a| < |\underline{\xi}|$, $A_K(a)$ tends to $A_\infty(a) = \frac{1}{a}\,[1 - \frac{\Phi(a)}{\Theta(a)}]$

ii) If $|a| > |\underline{\xi}|$, $A_K(a)$ is unbounded .

<u>Proof</u> : Let us denote $r = Max(o,p-q)$. We have :

$$\begin{aligned} A_K(a) &= \sum_{k=1}^{K} \alpha_{kK}\, a^{k-1} \\ &= \sum_{k=1}^{r} \alpha_{kK}\, a^{k-1} + \sum_{k=r+1}^{k-r} \alpha_{kK}\, a^{k-1} + \sum_{k=K-r+1}^{K} \alpha_{kK}\, a^{k-1} \\ &= \sum_{k=1}^{r} \alpha_{kK}\, a^{k-1} + \sum_{k=r+1}^{K-r} \left(\sum_{\ell=1}^{q} B_{\ell K}\, \xi_\ell^k + \sum_{\ell=1}^{q} C_{\ell K} \frac{1}{\xi_\ell^k} \right) a^{k-1} \\ &\quad + \sum_{\ell=1}^{r} D_{\ell K}\, a^{K-\ell} \end{aligned}$$

$$= \sum_{k=1}^{r} \alpha_{kK} a^{k-1} + \sum_{\ell=1}^{q} B_{\ell K} \frac{1}{a} (\xi_\ell a)^{r+1} \frac{(a\xi_\ell)^{K-2r} - 1}{a\xi_\ell - 1}$$

$$+ \sum_{\ell=1}^{q} C_{\ell K} \frac{1}{a} \left(\frac{a}{\xi_\ell}\right)^{r+1} \frac{\left(\frac{a}{\xi_\ell}\right)^{K-2r} + 1}{\frac{a}{\xi_\ell} - 1} + \sum_{\ell=1}^{r} D_{\ell K} a^{K-\ell}$$

i) Let us first consider the case $|a| < |\underline{\xi}|$. It directly follows from property 10 that $A_k(a)$ converges to

$$\sum_{k=1}^{r} \alpha_{k\infty} a^{k-1} + \sum_{\ell=1}^{q} C_{\ell\infty} \frac{1}{a} \left(\frac{a}{\xi_\ell}\right)^{r+1} \frac{1}{\frac{a}{\xi_\ell} - 1}$$ and this quantity is equal

to : $A_\infty(a) = \sum_{k=1}^{\infty} \alpha_{k\infty} a^{k-1}$.

ii) If $|a| > |\underline{\xi}|$, the third term contains geometric series with a rate of modulus greater than one and therefore this term is unbounded.

Q.E.D.

Let us finally remark that the approach developped in section 3 may be applied to other problems of time series, in particular to the problem of inversion of an autocovariance matrix (see for instance AKAIKE (1973)).

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