ABSTRACT

The equilibrium with quantity rationing concept is being more and more widely used to specify macroeconomic models. The models implement at the macroeconomic level the microeconomic principles which are the justification of the equilibrium with quantity rationing. Such an approach implies in particular that, on an aggregate market, the exchanged quantity is the minimum of supply and demand. As noted for instance by Muellbauer, Kooiman, Malinvaud this assumption is difficult to maintain at the macroeconomic level.

The purpose of the present paper is to build a formal model of aggregation of commodities in a fix price equilibrium framework and to derive potentially estimable specifications. Contrary to previous works, we formalize explicitly the degree of substituability between commodities and the spillover effects accross the markets.

R E S U M E

Le concept d'équilibre avec rationnement est de plus en plus utilisé dans la spécification de modèles macroéconomiques. Ces modèles transposent au niveau macroéconomique les principes microéconomiques justifiant l'équilibre avec rationnement. Une telle pratique implique en particulier que, sur un marché agrégé, la quantité échangée est égale au minimum de l'offre et de la demande. Comme l'ont remarqué par exemple MUELLBAUER, KOOIMAN, MALINVAUD cette hypothèse est difficile à conserver au niveau macroéconomique.

Le but de ce papier est de construire un modèle formel d'agrégation de biens dans un contexte d'équilibre à prix fixe et d'en déduire des spécifications estimables. Contrairement aux travaux antérieurs, nous formalisons de manière explicite le degré de substituabilité des biens et les effets de report entre les marchés.

The equilibrium with quantity rationing concept is being more and more widely used to specify and estimate macroeconomic models. These models implement at the macroeconomic level the microeconomic principles which are the justification of the equilibrium with quantity rationing. In particular, on an aggregate market, the exchanged quantity is assumed to be the minimum of supply and demand. This assumption is difficult to maintain at the macroeconomic level. The variety of agents and the information problems associated with it on the one hand, the diversity of commodities on the other hand, both suggest that the transition from a regime where all buyers are rationed to a regime where all sellers are rationed should be smooth. This point has been made by most authors and emphasized particularly by MUELLBAUER (1978). More recently KOOIMAN (1982) and MALINVAUD (1982) have insisted on its importance when one tries to incorporate in an empirical model the information contained in opinion surveys on the prevailing regime in the economy.

The purpose of the present paper is to build a formal model of aggregation of commodities in a fix price equilibrium framework and to derive from this model potentially estimable specifications. Contrary to previous works, we formalize explicitly the degree of substituability between commodities and the spillover effects accross the markets. The paper is organized as follows. The structure of the economy is described in section 1. The basic assumptions on the degree of substitution between commodities and the associated spillover effects are presented in section 2. Section 3 defines the equilibrium and the rationing scheme which is used and states the existence and uniqueness theorem. The properties of comparative statics of the equilibrium are then described in section 4, while section 5 presents some preliminary remarks on stochastic specification and estimation. All the proofs are gathered in section 6.

1. THE MODEL

We consider an economy with a continuum of commodities and a continuum of markets. A specific commodity, which we shall call "money", serve as a medium of exchange on all the markets. Quantities of money are indexed by a subscript m . The other commodities are indexed by ω in Ω . Ω is the closed unit interval $[0,1]$ in $\mathbb{R}$. On market ω , commodity ω is exchanged against money.

In order to be able to define unambiguously an aggregate competitive demand and supply, we assume that in all situations under consideration the relative prices of commodities in Ω stay fixed. By an appropriate choice of unit, we can thus take these relative prices all equal to 1 . The aggregate competitive demand (resp. supply) is then simply the sum of all Walrasian demands (resp. supplies) of commodities ω , ω in Ω (we endow Ω with the Lebesgue measure). It depends on the common money price p of all commodities ω in Ω .

Since all relative prices are fixed, we cannot expect the money price p to clear all the micromarkets simultaneously and some quantity rationing is to appear. Typically, a part of the markets are in excess supply, another part in excess demand, and the rest clear. Thus the aggregate traded quantity, which we are looking for, is not the minimum of the aggregate competitive demand and supply. It depends on two important features of the quantity rationing mechanism. First, the *spillover effects* describe how an agent who sees his demand (or supply) constrained on some micromarkets modifies his demands (or supplies) on the other markets. They are directly linked to the degree of substituability between commodities for this agent. Second, if the spillover effects vary across the agents, the *rationing scheme*, which indicates how the available quantity is splitted among the rationed agents on each market, has to be specified in order to compute the exact extent of the spillover effects across markets.

In order to get analytical results, we keep the model very simple. Demand and supply are treated in a symmetric way. The agents in the economy are either structural demanders on *all* markets, or structural suppliers on *all* markets. There are two types of demanders (resp. suppliers). The first type of agents, indexed by s, buys (resp. sells) simultaneously on all markets. The various commodities ω, ω in Ω, are substitutes for them and they are responsible for the spillover effects. The second type of agents, indexed by a, is specialized on one commodity ω and operates on a single market. On each market, eventually, the buyers of type a and s trade with the sellers of type a and s.

In order to describe the demands and supplies on the various markets, we shall use the following notations throughout the paper. We denote by f_Ω (or for short f) a real function defined on Ω. f_ω is its value at ω and, if X is a subset of Ω, $f_X = \{f_\omega, \omega \text{ in } X\}$. All the real functions defined on Ω which appear in the paper are assumed to be square integrable with respect to the Lebesgue measure $d\omega$. We denote $\bar{f} = \int_\Omega f_\omega \, d\omega$ and $\bar{f}_X = \int_X f_\omega \, d\omega$, when X is a measurable subset of Ω.

The competitive demands on market ω of the agents of type s and a are denoted respectively by $\xi_\omega(s)$ and $\xi_\omega(a)$, two positive numbers (we omit, for the time being, the price p in the arguments, since it stays fixed in the next two sections). Similarly, the competitive supplies on market ω of the agents of type s and a are counted positively, and are denoted by $\eta_\omega(s)$ and $\eta_\omega(a)$. The competitive demand and supply on market ω are then $\xi_\omega = \xi_\omega(s) + \xi_\omega(a)$ and $\eta_\omega = \eta_\omega(s) + \eta_\omega(a)$. The aggregate competitive demand and supply are given respectively by $\bar{\xi} = \int_\Omega \xi_\omega \, d\omega$ and $\bar{\eta} = \int_\Omega \eta_\omega \, d\omega$. Finally the excess demand on market ω is $e_\omega = \xi_\omega - \eta_\omega$, and in the aggregate $\bar{e} = \bar{\xi} - \bar{\eta}$.

2. THE SPILLOVER EFFECTS.

Whenever he is rationed on some markets, an agent of type s changes his demand or supply on the other markets. This creates the spillover effects.

To fix the ideas, consider the case of a buyer (the same arguments, with the same assumptions, hold for the sellers). Let $x = \{x_\omega , \omega \text{ in } \Omega\}$ be the set of potential quantity constraints. Assume that the buyer is effectively constrained on a subset X of markets in Ω , so that $x_X = \{x_\omega , \omega \text{ in } X\}$ describe the ration allocated to him on these markets. We denote $\tilde{\xi}_\omega(x,X)$, ω in $\Omega|X$, the effective demand which is then formulated by this agent on the other markets. By definition, $\tilde{\xi}_\omega(x,X)$ depends only on x_X and is independent of the value of x_ω , ω in $\Omega|X$.

We keep the argument x for convenience of the notation.

The maximizing behavior of the agents at the microeconomic foundation of the model induces to assume a continuity property for the effective demands. Consider the limit case where the quantitative constraints become unbinding on a subset of markets X_1 of X. We expect then the effective demands on markets ω, ω in $\Omega|X$, associated with the rationing (x_X, X) to be equal to the effective demands associated with (x_{X_2}, X_2) where $X_2 = X|X_1$.

Assumption 2.1 : (continuity of effective demands) : let X be a subset of Ω, and (X_1, X_2) be a partition of X.

Define :

$$x'_\omega = \begin{cases} x_\omega & \text{for } \omega \text{ in } \Omega|X_1 \\ \tilde{\xi}_\omega(x, X_2) & \text{for } \omega \text{ in } X_1 \end{cases}$$

Then :

$$\tilde{\xi}_\omega(x, X_2) = \tilde{\xi}_\omega(x', X) \qquad \text{for all } \omega \text{ in } \Omega|X$$

For the sake of simplicity, in order to be able to derive explicit formulae, we shall consider linear spillover effects. More precisely, we assume that the effective demand $\tilde{\xi}_{\omega_0}(x, X)$, ω_0 in $\Omega|X$ is linear in x_X :

$$\tilde{\xi}_{\omega_0}(x, X) = b_{\omega_0} - \int_X c_{\omega_0}(X, \omega)\, x_\omega\, d\omega \ .$$

$c_{\omega_0}(X,\omega)$ is the spillover coefficient from market ω on market ω_0 , when the agent is constrained on the markets ω in X . Again for simplicity, we assume that $c_{\omega_0}(X,\omega)$ is independent of ω_0 and ω : all commodities are treated symmetrically.

Assumption 2.2 : (linearity of spillover effects, symmetry of commodities) :

$$\tilde{\xi}_{\omega_0}(x,X) = b_{\omega_0} - c(X) \int_X x_\omega \, d\omega \quad \text{for all } \omega_0 \text{ in } \Omega|X$$

Finally, we want the effective demand to be equal to the competitive demand when the agent is unconstrained.

Assumption 2.3 :

$$\tilde{\xi}_\omega(x,\emptyset) = \xi_\omega(s) \qquad \omega \text{ in } \Omega$$

($\emptyset$ denotes the empty set)

We are now in a position to state the following property.

PROPERTY 2.4 : Under assumptions 2.1, 2.2 and 2.3, the effective demand can be written as :

$$\tilde{\xi}_{\omega_0}(x,X) = \xi_{\omega_0}(s) + c(\pi) \int_X (\xi_\omega(s) - x_\omega) \, d\omega \quad , \quad \omega_0 \text{ in } \Omega|X$$

where $\pi = \int_X d\omega$, $c(\pi) = \alpha/(1 - \alpha\pi)$ and α is a real number.

Proof : see section 6.

(Note that for the formula to be meaningful for all X in Ω , we must have $1 - \alpha\pi \neq 0$ for all π in $[0,1]$, which implies $\alpha < 1$).

In all the following, we consider an economy which satisfies assumptions 2.1, 2.2 and 2.3, so that property 2.4 holds. Furthermore, we make the additional assumption 2.5.

Assumption 2.5 :

$$c(\pi) \;=\; \alpha/(1-\alpha\pi) \quad \text{with} \quad o \leqslant \alpha < 1 \;.$$

Remark 2.6 : This type of effective demand function can of course be derived through utility maximization. Consider the case of separable utility functions, since separability seems a natural requirement to get symmetric spillover effects across commodities. The effective demands are then the solutions in ξ_ω of the following program.

$$\begin{cases} \text{Max} \quad u_m(\xi_m) + \int_\Omega u_\omega(\xi_\omega)\, d\omega \\ \xi_m + p \int_\Omega \xi_\omega \, d\omega = R \\ \xi_\omega = x_\omega \qquad\qquad \omega \text{ in } X \end{cases}$$

where m is the quantity of money held by the agent and R is his income. Since the budget constraint can be rewritten :

$$\xi_m + p \int_{\Omega|X} \xi_\omega \, d\omega = R - p \int_X x_\omega \, d\omega \;,$$

the effective demand is linear in x_X if and only if it is linear in income, i.e. the Engel curves are straight lines in some region of the commodity space. The class of utility functions yielding such demand curves is the "Bergson family", described by POLLAK (1971), which includes quasi concave functions of the form :

$$u_\omega (\xi_\omega) = a_\omega \log(\xi_\omega - b_\omega) \qquad \omega \text{ in } \Omega$$

or

$$u_\omega (\xi_\omega) = a_\omega (b_\omega + d_\omega \xi_\omega)^c \qquad \omega \text{ in } \Omega$$

or

$$u_\omega (\xi_\omega) = - a_\omega \exp (- b_\omega \xi_\omega) \qquad \omega \text{ in } \Omega \ .$$

Note that the shape of the competitive demand is restricted by this choice of utility function. Moreover, in general, the parameters of the utility function appear in the expressions both of the competitive demand and of the spillover coefficients.

Remark 2.7 : The effective demand $\tilde{\xi}_\omega(x,X)$ has been defined for ω in $\Omega|X$ only . To define an equilibrium with quantity rationing, it is useful to compute the effective demand on market ω , ω in X , which the agent would have formulated, had he not be constrained on this particular market ω . With the above notations, it is given by $\tilde{\xi}_\omega(x,X|\{\omega\})$, for ω in X . By property 2.4 :

$$\tilde{\xi}_\omega (x,X|\{\omega\}) = \xi_\omega(s) + c(\pi) \int_{X|\{\omega\}} (\xi_\omega(s) - x_\omega) \, d\omega \ .$$

Since $(\xi_\omega(s) - x_\omega)$ is integrable with respect to the Lebesgue measure, $\int_{X|\{\omega\}} (\xi_\omega(s) - x_\omega) \, d\omega = \int_X (\xi_\omega(s) - x_\omega) \, d\omega$, and we can define $\tilde{\xi}_\omega(x,X)$, for ω in X , as :

$$\tilde{\xi}_\omega(x,X) = \tilde{\xi}_\omega(x,X|\{\omega\}) \qquad \text{for} \quad \omega \text{ in } X$$

In the definition of an equilibrium with quantity rationing, the condition "no one is forced to exchange more than he wants" appears. With the above notations, this condition is expressed simply here as " x_ω is smaller than $\tilde{\xi}_\omega(x,X|\{\omega\})$, on the markets ω in X where the agent is constrained", i.e :

$$\tilde{\xi}_\omega(x,X) > x_\omega \qquad\qquad \text{for } \omega \text{ in } X .$$

Similar assumptions are made on the suppliers of type s . We denote by $y = \{y_\omega, \omega \text{ in } \Omega\}$ the set of the possible quantity constraints. When the suppliers are effectively constrained on a subset Y of markets in Ω , the effective supply which they formulate on the other markets is denoted $\tilde{\eta}_\omega(y,Y)$, ω in $\Omega|Y$. As for the consumer, we define for convenience $\tilde{\eta}_\omega(y,Y) = \tilde{\eta}_\omega(y,Y|\{\omega\})$ for ω in Y . In the rest of the paper, we assume :

$$\tilde{\eta}_{\omega_0}(y,Y) = \eta_{\omega_0}(s) + c'(\pi') \int_Y (\eta_\omega(s) - y_\omega) \, d\omega$$

where $\pi' = \int_Y d\omega$, $c'(\pi') = \alpha'/(1 - \alpha'\pi')$ and $0 \leqslant \alpha' < 1$.

3. EQUILIBRIA WITH QUANTITY RATIONING AND THE RATIONING SCHEME.

Let $x_\omega(s)$ (resp. $x_\omega(a)$) be the quantity of commodity ω, ω in Ω, given to the demanders of type s (resp. type a). Similarly, let $y_\omega(s)$ (resp. $y_\omega(a)$) be the quantity of commodity ω, ω in Ω, supplied by the agents of type s (resp. type a).

An equilibrium with quantity rationing is a collection of real numbers $(x_\omega(s)\,,\, x_\omega(a)\,,\, y_\omega(s)\,,\, y_\omega(a)\,,\ \omega \text{ in } \Omega)$ such that there exist subsets X and Y of Ω with the following properties :

(i) (equality of supply and demand)

$$x_\omega = x_\omega(s) + x_\omega(a) = y_\omega(s) + y_\omega(a) = y_\omega \qquad \omega \text{ in } \Omega$$

(ii) (the effective demands and supplies are realized on the markets where the agents are not rationed)

$$\left.\begin{array}{l} x_\omega(s) = \tilde{\xi}_\omega(x(s),X) \\ \\ x_\omega(a) = \xi_\omega(a) \end{array}\right\} \text{ for all } \omega \text{ in } \Omega|X$$

$$\left.\begin{array}{l} y_\omega(s) = \tilde{\eta}_\omega(y(s),Y) \\ \\ y_\omega(a) = \eta_\omega(a) \end{array}\right\} \text{ for all } \omega \text{ in } \Omega|Y$$

(iii) (no one is forced to exchange more than he wants)

$$\left.\begin{array}{ll} x_\omega(s) \leqslant \tilde{\xi}_\omega(x(s),X) & \text{with at least} \\ & \text{one strict} \\ x_\omega(a) \leqslant \xi_\omega(a) & \text{inequality} \end{array}\right\} \quad \text{for all } \omega \text{ in } X$$

$$\left.\begin{array}{ll} y_\omega(s) \leqslant \tilde{\eta}_\omega(y(s),Y) & \text{with at least} \\ & \text{one strict} \\ y_\omega(a) \leqslant \eta_\omega(a) & \text{inequality} \end{array}\right\} \quad \text{for all } \omega \text{ in } Y$$

(iii) (both sides of a market cannot be constrained simultaneously)

$$X \cap Y = \emptyset$$

Remarks : a) Since the demands and supplies of money are implicitly defined through the budget constraints, by Walras'law, the equalities of supply and demand (i) for commodities ω in Ω imply the equality of supply and demand for money.

b) The agents of type a operate on one market only : their effective demand on the market is equal to their competitive demand.

We have not specified in the definition how, when there is rationing on one market, the traded quantity is shared among the two types a and s of agents operating on this market. Thus we expect to obtain a large number of equilibria with quantity rationing, corresponding to the various possible rationing schemes. To be able to derive analytical results, we shall restrict ourselves to a simple form of rationing scheme, parametrized by two numbers μ and μ' , $0 \leqslant \mu \leqslant 1$, $0 \leqslant \mu' \leqslant 1$, defined by :

(r) (rationing scheme)

for all ω in X :

$$\mu\ (\xi_\omega(a) - x_\omega(a)) = (1 - \mu)\ (\tilde{\xi}_\omega(x(s),X) - x_\omega(s))$$

for all ω in Y :

$$\mu'\ (\eta_\omega(a) - y_\omega(a)) = (1 - \mu')\ (\tilde{\eta}_\omega(y(s),Y) - y_\omega(s))$$

$\xi_\omega(a) - x_\omega(a)$ is the additional quantity which the agents of type a would like to buy on market ω at the equilibrium. Similarly $\tilde{\xi}_\omega(x(s),X) - x_\omega(s)$

is the additional quantity which the agents of type s want to buy. The rationing schemes which we consider are such that the ratio between the additional quantities desired by the two types of agents is independent of the market ω, ω in X.(1) We assume the analogous property on the supply side.

The difficult part when one looks for an equilibrium with quantity rationing is to identify the sets X and Y of markets, on which respectively demand and supply are constrained. In order to proceed in this direction, we introduce four auxiliary unknowns, the proportions π and π' of markets in X and Y, and the aggregate additional trade k_X and k_Y which, respectively, the buyers and sellers of type s would undertake in a competitive environment compared with the equilibrium with quantity rationing :

$$\pi = \int_X d\omega \qquad k_X = \int_X (\xi_\omega(s) - x_\omega(s))\, d\omega$$

$$\pi' = \int_Y d\omega \qquad k_Y = \int_Y (\eta_\omega(s) - y_\omega(s))\, d\omega$$

We can then compute the total effective excess demand $\tilde{e}_\omega$ on each market :

$$\tilde{e}_\omega = \tilde{\xi}_\omega(x,X) + \xi_\omega(a) - \tilde{\eta}_\omega(y,Y) - \eta_\omega(a)$$

Given the assumptions of section 2 :

$$\tilde{e}_\omega = \xi_\omega - \eta_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y$$

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(1) Note that the traded quantities determined through the above rationing scheme may not be positive when supply is small. The rationing scheme could be modified to satisfy a requirement of positivity of the net trade. However this would make the analysis of the model much more complicated without any gain in the practice of estimation.

$$\tilde{e}_\omega = e_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y$$

Now, by (ii) and (iv) of the definition, for all ω in X :

$$y_\omega = \tilde{\eta}_\omega(y,Y) + \eta_\omega(a)$$

Thus by (i) and (iii) , for all ω in X :

$$y_\omega = x_\omega < \tilde{\xi}_\omega(x,X) + \xi_\omega(a)$$

which implies :

$$\tilde{e}_\omega > 0 \; .$$

It is easy to check that this condition characterizes X and that the same line of argument applies to Y :

$$X = \{\omega \text{ in } \Omega \;,\; e_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y > 0\}$$
$$Y = \{\omega \text{ in } \Omega \;,\; e_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y < 0\}$$

It appears thus clearly that the crucial tool for the description of the equilibrium is the empirical distribution of the competitive excess demand e_ω . Let $\bar{e}$ and σ^2 be its two first moments :

$$\bar{e} = \int_\Omega e_\omega d\omega \qquad \sigma^2 = \int_\Omega (e_\omega - \bar{e})^2\, d\omega \; .$$

We shall need the cumulative distribution F of $u_\omega = (e_\omega - \bar{e})/\sigma$:

$$F(u) = \int_\Omega \mathbb{1}_{u_\omega \leq u} \, d\omega$$

where $\mathbb{1}_{u_\omega \leq u}$ equals one if $u_\omega \leq u$ and zero otherwise.
With the above notations, we can compute π in terms of the other parameters :

$$\pi = \int_X d\omega = \int_\Omega \mathbb{1}_{e_\omega + c(\pi) k_X - c'(\pi') k_Y > 0} \, d\omega$$

$$\pi = 1 - F[(c'(\pi') k_Y - c(\pi)\, k_X - \bar{e}) / \sigma]$$

Similarly, in order to obtain an expression of k_X and k_Y , we need a function g , giving the truncated expectation of excess competitive demand on X and Y :

$$g(\pi) = \int_\Omega u_\omega \, \mathbb{1}_{u_\omega \geq F^{-1}(1-\pi)} \, d\omega$$

For simplicity, we assume in all the following that F is continuous, which implies that X and Y form a partition of Ω , together with the set of measure zero of ω's such that $e_\omega = c'(\pi')\, k_Y - c(\pi)\, k_X$.

We are now in a position to state the main result of this section.

THEOREM 3.1. : For each μ and μ' , $o \leqslant \mu \leqslant 1$, $o \leqslant \mu' \leqslant 1$, there exists a unique equilibrium with quantity rationing associated with the rationing scheme (r) .

The proportion π of markets in excess demand at the equilibrium satisfies the following equation :

$$- \mu\alpha[g(\pi)- F^{-1}(1-\pi)] + \mu'\alpha'[g(\pi)+(1-\pi)F^{-1}(1-\pi)] - F^{-1}(1-\pi) = \bar{e}/\sigma$$

The aggregate exchanged quantity, $\bar{x} = \bar{y}$, at the equilibrium, is then given by :

$$\begin{aligned} \bar{x} = \bar{y} &= \bar{\xi} - (1-\mu\alpha)\sigma[g(\pi)- F^{-1}(1-\pi)] \\ &= \bar{\eta} - (1-\mu'\alpha')\sigma[g(\pi)+(1-\pi)F^{-1}(1-\pi)] \end{aligned}$$

□

Proof : see section 6

The uniqueness result is not surprising since the assumption made on the spillover effects is similar to the sufficient conditions for uniqueness of LAROQUE (1981) and SCHULZ (1983). In fact, when the demanders of type s are forced to reduce their total purchases by 1 on the markets ω in X , a fraction $(1 - \alpha)/(1 - \alpha\pi)$ is saved in money, the rest $\alpha(1 - \pi)/(1 - \alpha\pi)$ being spent on the markets in $\Omega|X$ ($\alpha/(1 - \alpha\pi)$ on each individual market ω in $\Omega|X$) . Since $(1 - \alpha) / (1 - \alpha\pi)$ is strictly positive and smaller than one, we are in the favourable case, characterized in the standard macroeconomic model by a marginal propensity to spend income between 0 and 1 : it is most probable that the uniqueness of the equilibrium holds for a much larger class of rationing schemes than the one considered here.

4. COMPARATIVE STATICS.

The proportion π of markets constrained on the demand side and the exchanged quantity $\bar{x} = \bar{y}$ at equilibrium, obtained by solving the equation of theorem 3.1, are functions of the parameters which characterize the economy. Some results on the dependence of π and $\bar{x} = \bar{y}$ with respect to $\bar{\xi}$, $\bar{\eta}$, $\bar{e} = \bar{\xi} - \bar{\eta}$, σ, μ and μ' are given in the two following properties.

PROPERTY 4.1 : i) π increases from o to 1, when $\frac{\bar{e}}{\sigma}$ increases from $-\infty$ to $+\infty$.

ii) $\frac{\bar{\xi} - \bar{x}}{\sigma}$ increases from o to $+\infty$, when $\frac{\bar{e}}{\sigma}$ increases from $-\infty$ to $+\infty$.

Furthermore $\frac{\bar{\xi} - \bar{x}}{\sigma}$ is a convex function of $\frac{\bar{e}}{\sigma}$.

Similarly, $\frac{\bar{\eta} - \bar{y}}{\sigma}$ decreases from $+\infty$ to o, when $\frac{\bar{e}}{\sigma}$ increases from $-\infty$ to $+\infty$.

Furthermore $\frac{\bar{\eta} - \bar{y}}{\sigma}$ is a convex function of $\frac{\bar{e}}{\sigma}$.

PROPERTY 4.2 : i) π and $\bar{x}$ are increasing functions of μ and μ' ;

if $\mu = \mu' = 0$,

$\pi = 1-F(-\frac{\bar{e}}{\sigma})$ and $\bar{x} = \bar{\xi} - \sigma\{g[1-F(-\frac{\bar{e}}{\sigma})] + \frac{\bar{e}}{\sigma}[1-F(-\frac{\bar{e}}{\sigma})]\}$

ii) π is a decreasing (resp. increasing) function of σ if $\bar{e}$ is positive (resp. negative).

$\bar{x}$ is a decreasing function of σ and $\bar{x}$ tends to $\mathrm{Min}(\bar{\xi},\bar{\eta})$, if σ tends to o. A necessary condition for $\left(\frac{\partial \bar{x}}{\partial \sigma}\right)$ to be maximal is $\bar{e} = o$.

iii) The partial derivatives of $\bar{x}$ with respect to $\bar{\xi}$ and $\bar{\eta}$ are : $\frac{\partial \bar{x}}{\partial \bar{\xi}} = \frac{(1-\pi)\ (1-\mu'\alpha')}{(1-\pi)(1-\mu'\alpha')+\pi(1-\mu\alpha)}$,

$\frac{\partial \bar{x}}{\partial \bar{\eta}} = \frac{\pi(1-\mu\alpha)}{(1-\pi)(1-\mu'\alpha')+\pi(1-\mu\alpha)}$; they are such that :

$$\frac{\partial \bar{x}}{\partial \bar{\xi}} \geqslant 0 \ , \ \frac{\partial \bar{x}}{\partial \bar{\eta}} \geqslant 0 \ \text{and} \ \frac{\partial \bar{x}}{\partial \bar{\xi}} + \frac{\partial \bar{x}}{\partial \bar{\eta}} = 1 \ .$$

Proofs : see section 6

When $\mu = \mu' = o$ (or $\alpha = \alpha' = o$), there are no spillover effects and the model is similar to those considered for instance in MUELLBAUER (1978) or MALINVAUD (1981). The introduction of the spillover effects increases the exchanged quantity. Moreover when $\mu = \mu' = o$ (or $\alpha = \alpha' = o$) we obtain the formulae derived by MALINVAUD (1981) for the derivatives of the exchanged quantity with respect to $\bar{\xi}$ and $\bar{\eta}$: $\frac{\partial \bar{x}}{\partial \bar{\xi}} = 1 - \pi$, $\frac{\partial \bar{x}}{\partial \bar{\eta}} = \pi$ (*).

The second part of property 4.2 says that, if the variability of the excess demand between the markets decreases, the exchanged quantity increases, and that this increase is maximal at the competitive equilibrium price, when this price is unique. Considering the limit case $\sigma = o$,

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(*) This remark has been suggested to the authors by J.F. RICHARD.

it can be seen that the model we have introduced contains as a particular case the usual "bang-bang" model.

The foregoing properties do not directly tell how the fixed price equilibrium varies with the common price p of all the commodities. In fact, in general, we may expect that several characteristic parameters $\overline{\xi}$, $\overline{\eta}$, $\overline{e}$, σ ... change simultaneously with p .

Let us assume for simplicity that the distribution function of the excess demand e_ω depends only on p through the first and second order moments : $\overline{e} = \overline{e}(p)$, $\sigma = \sigma(p)$. Thus F is assumed to be independent of π . The derivative of the ratio $\frac{\overline{e}}{\sigma}$ with respect to p is then :

$$\frac{d}{dp}\left[\frac{\overline{e}(p)}{\sigma(p)}\right] = \frac{1}{\sigma^2(p)}\left[\sigma(p)\,\frac{d\overline{e}(p)}{dp} - \overline{e}(p)\,\frac{d\sigma(p)}{dp}\right]$$

i) Let us consider a price p_0 defined by $\overline{e}(p_0) = o$. At p_0 aggregate competitive demand is equal to the aggregate competitive supply and thus we shall call p_0 an "equilibrium price". Note however that in general the macromarkets do not clear for $p = p_0$. At an equilibrium price :

$$\frac{d}{dp}\left[\frac{\overline{e}(p)}{\sigma(p)}\right] = \frac{1}{\sigma(p_0)}\,\frac{d\overline{e}(p_0)}{dp}$$

. If this equilibrium price p_0 is locally stable, the aggregate excess demand is increasing at p_0 .

Then it follows from 4.1i) that in some neighbourhood of a stable equilibrium price, π is an increasing function of p .

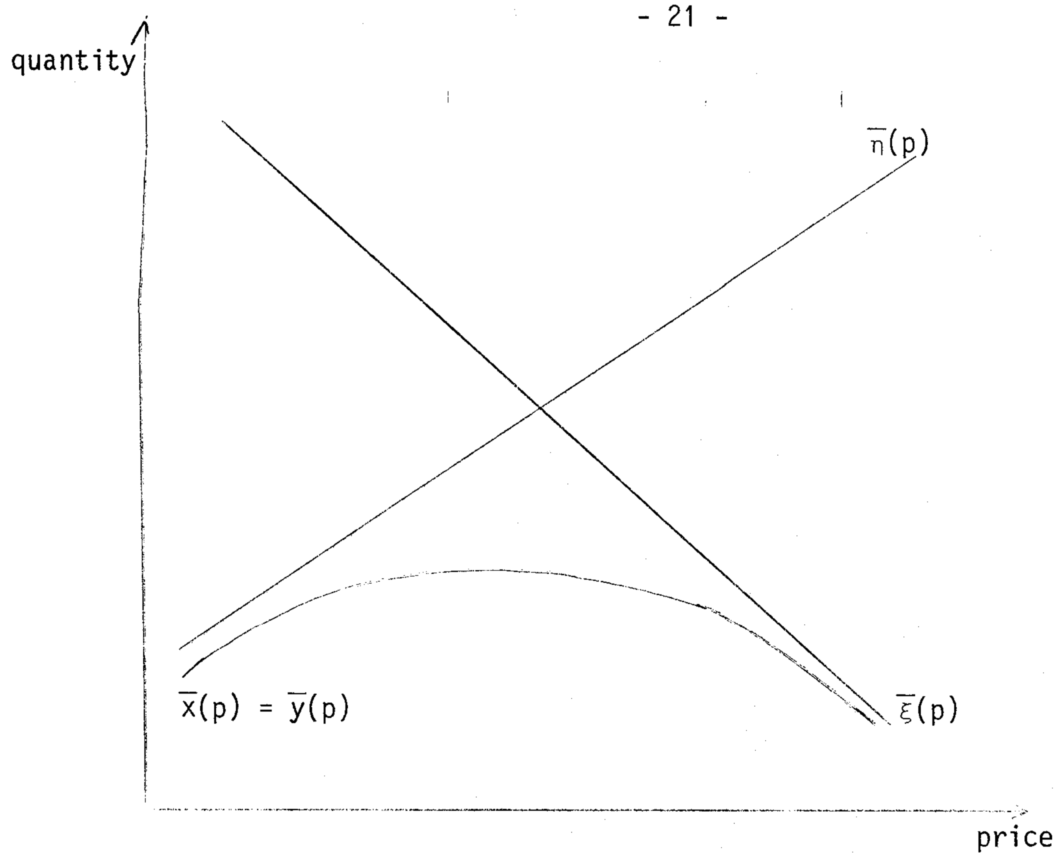
ii) To obtain more precise results, we now consider a particular form of the excess demand function. We assume that e_ω is linear in the difference between the actual price p and an equilibrium price p_o :

$$e_\omega = \beta_\omega + \gamma_\omega(p - p_o)$$

Since p_o is an equilibrium price, we have : $\bar{\beta} = \int_\Omega \beta_\omega \, d\omega = o$ and $\bar{e}(p) = \bar{\gamma}\,(p - p_o)$ where $\bar{\gamma}$ is assumed strictly positive. Moreover if we assume that β_ω and γ_ω are uncorrelated and if we denote by b^2 and c^2 their variances, we have : $\sigma^2(p) = c^2\,(p - p_o)^2 + b^2$. This variance increases with the distance between p and p_o . The derivative of the ratio $\frac{\bar{e}(p)}{\sigma(p)}$ is given by : $\frac{d}{dp}\left(\frac{\bar{e}(p)}{\sigma(p)}\right) = \frac{d}{dp}\left(\frac{\bar{\gamma}\,(p-p_o)}{\sqrt{c^2(p-p_o)^2 + b^2}}\right) = \frac{\bar{\gamma}\, b^2}{\sigma(p)^3}$

and is strictly positive for any p ; therefore π is an increasing function of p .

iii) In the econometric models, Walrasian demand and supply are often assumed to depend only on the price through their means. Furthermore these means are usually chosen as linear functions of p : $\bar{\xi} = \bar{\gamma}^{*}\, p + \bar{\beta}$, $\bar{\gamma}^{*} < o$, $\bar{\eta} = \bar{\gamma}^{**}\, p + \bar{\beta}^{**}$, $\bar{\gamma}^{**} > o$. Under these conditions, it follows directly from property 4.1 that π , $\bar{\xi} - \bar{x}$, $\bar{\eta} - \bar{y}$ are increasing functions of p , that $\bar{x}$ is concave and is asymptotically equivalent to $\bar{\eta}$ when p tends to $-\infty$, to $\bar{\xi}$ when p tends to $+\infty$. This implies an exchanged quantity whose graph has the following form :



5. STOCHASTIC SPECIFICATION AND ESTIMATION.

The equations of theorem 3.1 giving π and $\overline{x}$ may be used as a starting point to build an econometric model. For such a purpose it is necessary to introduce macroeconomic disturbances, to specify the dependence of $\overline{\xi}$, $\overline{\eta}$ with respect to the exogenous variables, to choose a tractable form for the distribution function F and to discuss the nature of the available data.

The macroeconomic disturbances may be interpreted as omitted explanatory variables or as errors of measurement on the endogenous variables.

In the first case they are additively introduced in $\bar{\xi}$ and $\bar{\eta}$. In the second, they are added for instance to the exchanged quantity (SNEESSENS (1981)). In the rest of the section, we retain the first approach and define

$$\begin{cases} \bar{\xi} = D(\gamma,z) + v \\ \bar{\eta} = S(\delta,z) + w \end{cases}$$

where v and w are zero mean random variables, γ , δ are parameters and z are the exogenous variables. v and w are assumed to be independent of the microeconomic random terms, i.e independent of e_ω . With this assumption the functions F and g do not depend on the values of v and w .

The distribution function F is usually chosen in order to allow simple computations. For instance MUELLBAUER (1978), MALINVAUD (1982) consider a uniform distribution. Other possibilities are the standard normal one, or the logistic distribution. In the latter case : $F(x) = \frac{1}{1+\exp(-x)}$. Since this distribution is centered, but not reduced, σ^2 is proportional to the variance of e_ω . The functions which appear in the equations giving π and $\bar{x}$ have the simple form :

$$F^{-1}(1-\pi) = \text{Log}(1-\pi) - \text{Log}\,\pi$$

$$g(\pi) - \pi F^{-1}(1-\pi) = -\text{Log}(1-\pi)$$

$$g(\pi) + (1-\pi)\,F^{-1}(1-\pi) = -\text{Log}\,\pi\,.$$

and π is the solution of : $\dfrac{\pi^{1-\mu'\alpha'}}{(1-\pi)^{1-\mu\alpha}} = \exp\dfrac{\bar{e}}{\sigma}$.

Once $\bar{\xi}$, $\bar{\eta}$, F and the distribution $\varphi(v,w)$ of the macroeconomic disturbances have been specified the parameters of the model can be estimated by standard estimation methods.

Let us first consider the case in which at each time $t = 1,\dots,T$, π and $\bar{x}$ are observed. We get :

$$\begin{cases} \bar{x}_t + (1-\mu\alpha)\sigma[g(\pi_t) - \pi_t F^{-1}(1-\pi_t)] = D(\gamma,z_t) + v_t & t = 1,\dots,T \\ \bar{x}_t + (1-\mu'\alpha')\sigma[g(\pi_t)+(1-\pi_t)F^{-1}(1-\pi_t)] = S(\delta,z_t) + w_t & t = 1,\dots,T \end{cases}$$

Note that these expressions show clearly that without further assumptions the macroeconomic data allow only to identify $\sigma(1-\mu\alpha)$, $\sigma(1-\mu'\alpha')$, γ , δ . This reflect the fact that it is not possible to disentangle the variability of the micromarkets represented by σ from the representative measure of the spillover effect respectively $1 - \mu\alpha$ and $1 - \mu'\alpha'$ on the demand and supply sides. Of course some further restrictions would allow to identify the parameter. For instance the specification of σ as a function of p or the relationship between α(resp. α') and the parameters which appear in the competitive demand (resp. supply) could be useful.

It is known from theorem 3.1 that the mapping which associates $(\pi_t,\bar{x}_t)$ to (v_t,w_t) is one to one ; therefore the density function $\ell(\pi_t,\bar{x}_t)$ of $(\pi_t,\bar{x}_t)$ can be computed from φ by the Jacobian formula.

The Jacobian is given by :

$$\frac{\partial(v_t,w_t)}{\partial(\pi_t,\bar{x}_t)} = \left| \det \begin{bmatrix} (1-\mu\alpha)\dfrac{\sigma\ \pi_t}{f[F^{-1}(1-\pi_t)]} & 1 \\ \\ -(1-\mu'\alpha')\dfrac{\sigma(1-\pi_t)}{f[F^{-1}(1-\pi_t)]} & 1 \end{bmatrix} \right|$$

$$= \frac{\sigma}{f[F^{-1}(1-\pi_t)]}\left[(1-\mu\alpha)\ \pi_t + (1-\mu'\alpha')\ (1-\pi_t)\right]$$

and the density function is :

$$\ell(\pi_t,\bar{x}_t) = \frac{\sigma}{f[F^{-1}(1-\pi_t)]}\left[(1-\mu\alpha)\ \pi_t + (1-\mu'\alpha')\ (1-\pi_t)\right]$$

$$\varphi\{\bar{x}_t + (1-\mu\alpha)\sigma[g(\pi_t)-\pi_t F^{-1}(1-\pi_t)] - D(\gamma,z_t),\bar{x}_t + (1-\mu'\alpha')\sigma[g(\pi_t)$$

$$+ (1-\pi_t)F^{-1}(1-\pi_t)] - S(\delta,z_t)\} \ .$$

The parameters $\sigma(1-\mu\alpha)$, $\sigma(1-\mu'\alpha')$, γ , δ and the variances and covariance of v and w can be estimated for instance by maximum likelihood technics, i.e by maximizing $\sum_{t=1}^{T} \log \ell(\pi_t,\bar{x}_t)$ in the case of i.i.d disturbances. These estimations may be used to test the hypothesis of symmetry of spillover effects on the demand and supply sides : $H_0 = \{1 - \mu\alpha = 1 - \mu'\alpha'\}$.

The previous approach is applicable whenever data on both π and $\bar{x}$ are available. The exchanged quantity $\bar{x}$ is in general observed. This allows in any case to implement an estimation procedure based on the marginal distribution function of $\bar{x}$ by integrating over π if π is unknown.

The proportions π and $(1-\pi)$ of markets constrained respectively on the demand and on the supply side do not appear as such in the standard statistical publications. They may sometimes be approximated from other statistics, when the rationing is clear cut, i.e. when the agents do not have any substitution possibilities through spillovers (they are all of type a). If one considers the labor markets, a possible interpretation of the model is that each worker supplies a specific quality of work, without possibility of substitution. On the other hand, firms have some substitution possibilities on the demand side. The proportion of constrained markets is then more easily computed on the supply side from the unemployment rate. The situation is not as favourable on the commodity markets. If one considers the consumption good markets, there is a lot of substitution going on through spillovers on the consumers' side. This is not true for the supply side, and observations of $(1-\pi)$ may be obtained from opinion surveys. However it is important to notice that the statistics computed from such surveys are often proportions of constrained agents weighted by the quantities which they sell. The value taken by this statistics in the formal model can be computed (see section 6) :

$$1 - \tilde{\pi} = \frac{\bar{y}_Y}{\bar{y}} = \frac{1}{\bar{y}} \{\bar{\eta}_Y - (1-\mu'\alpha'(1-\pi))\sigma[g(\pi)-(1-\pi)F^{-1}(1-\pi)]\}$$

The expression of $\tilde{\pi}$ contains $\bar{\eta}_Y = \int_Y \eta_\omega \, d\omega$ which involves the joint distribution of $(\xi_\omega, \eta_\omega)$. Thus, in order to use the published results from opinion surveys, further a priori assumptions on the shape of this distribution are needed.

6. PROOFS.

Proof of property 2.4 :

i) From assumptions 2.1 and 2.3, the effective demand coïncides with the Walrasian demand, if the quantitative constraints become unbinding for all markets : $x = \xi$.

$$\forall X \ , \ \forall \omega \ : \ \tilde{\xi}_\omega[\xi,X] = \xi_\omega$$

Under assumption 2.2 this is equivalent to :

$$\forall X \ , \ \forall \omega \ : \ b_\omega - c(X) \int_X \xi_\omega \, d\omega = \xi_\omega$$

Replacing b_ω by this expression in the effective demand, we obtain :

$$\tilde{\xi}_\omega[x,X] = \xi_\omega + c(X) \int_X (\xi_\omega - x_\omega) \, d\omega = \xi_\omega + c(X)[\bar{\xi}_X - \bar{x}_X]$$

ii) The particular form of the spillover effects also results from the continuity and the linearity of the effective demands. Let X be a subset of Ω and (X_1,X_2) a partition of X , we have :

$$\tilde{\xi}_\omega[x,X_2] = \tilde{\xi}_\omega[(\tilde{\xi}_{X_1}(x,X_2) \ , \ x_{\Omega|X_1}) \ , \ X \ ] \qquad \forall \ \omega \notin X$$

This equality can be written as :

$$\xi_\omega + c(X_2)[\bar{\xi}_{X_2} - \bar{x}_{X_2}] = \xi_\omega + c(X) \ [\int_{X_1} [\xi_\omega - \tilde{\xi}_\omega(x,X_2)] \, d\omega + \int_{X_2} (\xi_\omega - x_\omega) \, d\omega]$$

$$= \xi_\omega + c(X)\left[\int_{X_1} - c(X_2)(\bar{\xi}_{X_2} - \bar{x}_{X_2})\, d\omega + \bar{\xi}_{X_2} - \bar{x}_{X_2}\right]$$

$$= \xi_\omega + c(X) \; [1 - c(X_2)\, \pi(X_1)][\bar{\xi}_{X_2} - \bar{x}_{X_2}]$$

This implies that :

$$\forall\; X_2 \subset X \;:\; c(X) = \frac{c(X_2)}{1 - c(X_2)[\pi(X) - \pi(X_2)]}$$

$$\forall\; X_2 \subset X \;:\; \frac{c(X)}{1 + c(X)\, \pi(X)} = \frac{c(X_2)}{1 + c(X_2)\, \pi(X_2)}$$

Therefore the ratio $\frac{c(X)}{1 + c(X)\, \pi(X)}$ is independent of X .

Denoting this ratio by α , we have : $c(X) = \frac{\alpha}{1 - \alpha\, \pi(X)}$

Proof of theorem 3.1 :

Since $X = \{\omega : e_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y > o\}$, it is easily seen that the partition is characterized by the data of $\pi = 1 - \pi'$, k_X , k_Y .

i) In a first step we are looking for a system giving these three quantities.

The proportion π satisfies :

$$(*) \qquad \pi = 1 - F[(c'(\pi')\, k_Y - c(\pi)\, k_X - \bar{e}) / \sigma]$$

To derive the other equations, let us consider the markets ω in X .

The form of the rationing scheme implies that the additional quantity which the agents of type s want to buy is :

$$\tilde{\xi}_\omega(x(s),X) - x_\omega(s) = \mu \tilde{e}_\omega = \mu[e_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y]$$

$$\Longleftrightarrow \quad \xi_\omega(s) - x_\omega(s) + c(\pi)\, k_X = \mu[e_\omega + c(\pi)\, k_X - c'(\pi')\, k_Y]$$

By integrating on X , we deduce that :

$$k_X + \pi\, c(\pi)\, k_X = \mu \int_X e_\omega\, d\omega + \mu\, \pi\, c(\pi)\, k_X - \mu\, \pi\, c'(\pi')\, k_Y$$

$$\Longleftrightarrow \quad k_X\, [1 + (1-\mu)\, \pi\, c(\pi) \quad] + \mu\, \pi\, c'(\pi')\, k_Y = \mu \int_X [\bar{e} + \sigma\, u_\omega]\, d\omega$$

$$(**) \quad k_X\, [1 + (1-\mu)\, \pi\, c(\pi) \quad] + \mu\, \pi\, c'(\pi')\, k_Y = \mu[\bar{e}\, \pi + \sigma g(\pi)]$$

A similar approach applied to the markets ω in Y leads to :

$$(***) \quad \mu'\, (1-\pi)\, c(\pi)\, k_X + k_Y\, [\, 1+(1-\mu')(1-\pi)c'(\pi')] = \mu'[-\, \bar{e}(1-\pi)+\sigma g(\pi)]$$

ii) From (*) and (**), we obtain k_X as a function of π :

$$k_X = \frac{\mu\sigma[g(\pi) - \pi\, F^{-1}(1-\pi)]}{1 + \pi\, c(\pi)} = \mu(1-\alpha\pi)\, \sigma[g(\pi)-\pi F^{-1}(1-\pi)]$$

Similarly, from (*) and (***), we have :

$$k_Y = \mu'\, [1 - \alpha'(1-\pi)]\, \sigma[g(\pi) + (1-\pi)\, F^{-1}(1-\pi)]$$

By replacing k_X and k_Y by their expressions in equation (*), we obtain the equation satisfied by the proportion π :

$$- \mu\alpha\ [g(\pi) - \pi\ F^{-1}(1-\pi)] + \mu'\alpha'\ [g(\pi) + (1-\pi)\ F^{-1}(1-\pi)]\ - F^{-1}(1-\pi) = \frac{\bar{e}}{\sigma}$$

iii) To prove the existence and the uniqueness of a fixed price equilibrium, we have to look for the solutions in $[0,1]$ of the above equation.

a) Let us first examine the properties of the function g .

We have : $g(o) = \int_\Omega u_\omega\ \mathbb{1}_{u_\omega \geqslant F^{-1}(1)}\ d\omega = \int_\Omega u_\omega\ \mathbb{1}_{u_\omega \geqslant +\infty}\ d\omega = o$

and $g(1) = \int_\Omega u_\omega\ \mathbb{1}_{u_\omega \geqslant F^{-1}(0)}\ d\omega = \int_\Omega u_\omega\ d\omega = o$

Moreover : $g(\pi) = \int_\Omega u_\omega\ \mathbb{1}_{u_\omega \geqslant F^{-1}(1-\pi)}\ d\omega = \int_{F^{-1}(1-\pi)}^{\infty} u\ dF(u) = \int_{1-\pi}^{1} F^{-1}(z)\ dz$

Therefore : $\frac{dg(\pi)}{d\pi} = F^{-1}(1-\pi)$

b) Let us now denote by $h(\pi)$ the left hand side of the equilibrium equation :

$$h(\pi) = - \mu\alpha\ [g(\pi) - \pi\ F^{-1}(1-\pi)] + \mu'\alpha'\ [g(\pi) + (1-\pi)\ F^{-1}(1-\pi)] - F^{-1}(1-\pi)$$

We have : $\frac{dh(\pi)}{d\pi} = \frac{1 - \pi\mu\alpha - (1-\pi)\ \mu'\alpha'}{f[F^{-1}(1-\pi)]}$ with $f(u) = \frac{dF(u)}{du}$

Since μ , μ' , α , α' are assumed to belong to $[0,1]$, $\alpha \neq 1$, $\alpha' \neq 1$ and since f is strictly positive, the derivative $\frac{dh(\pi)}{d\pi}$ is strictly positive for π in $]0,1[$. Moreover : $h(\pi)$ tends to $-\infty$ if π tends to o and tends to $+\infty$, if π tends to 1 . Since h is a strictly increasing function from $]0,1[$ onto $\mathbb{R}$, there exists a unique solution to :

$$h(\pi) = \frac{\overline{e}}{\sigma} \qquad \text{for all} \quad \frac{\overline{e}}{\sigma} \quad \text{in} \quad \mathbb{R} \quad .$$

iv) The aggregate exchanged quantity at the equilibrium can be easily expressed as a function of π :

$$\overline{x} = \int_{\Omega} [x_{\omega}(s) + x_{\omega}(a)]\, d\omega$$

$$\overline{x} = \int_{Y} [\tilde{\xi}_{\omega}[x(s),X] + \xi_{\omega}(a)]\, d\omega + \int_{X} x_{\omega}(s)\, d\omega + \int_{X} \{\xi_{\omega}(a) - \frac{1-\mu}{\mu}(\tilde{\xi}_{\omega}[x(s),X] - x_{\omega}(s))]d\omega$$

$$\overline{x} = \int_{Y} [\xi_{\omega}(s) + c(\pi)k_X + \xi_{\omega}(a)]\, d\omega + \int_{X} \xi_{\omega}(s)\, d\omega - k_X + \int_{X} \xi_{\omega}(a)\, d\omega - \frac{1-\mu}{\mu}\int_{X} (\xi_{\omega}(s) - x_{\omega}(s) + c(\pi)\, k_X)\, d\omega$$

$$\overline{x} = \overline{\xi} + k_X[(1-\pi)\, c(\pi) - 1 - \frac{1-\mu}{\mu} - \pi\, c(\pi)\, \frac{1-\mu}{\mu}]$$

$$\overline{x} = \overline{\xi} + k_X\, [c(\pi) - \frac{1+\pi\, c(\pi)}{\mu}] = \overline{\xi} + k_X\, \frac{\alpha\mu - 1}{\mu(1-\alpha\pi)}$$

Replacing k_X by its expression as a function of π , we have :

$$\overline{x} = \overline{\xi} - [1 - \alpha\mu]\, \sigma[g(\pi) - \pi\, F^{-1}(1-\pi)]$$

It is possible to introduce the walrasian supply in the previous relation.

Since : $\overline{\xi} = \overline{\eta} + \overline{e}$

$$= \overline{\eta} - \alpha\mu\sigma[g(\pi) - F^{-1}(1-\pi)] + \mu'\alpha'\sigma[g(\pi)+(1-\pi)F^{-1}(1-\pi)] - \sigma\, F^{-1}(1-\pi)$$

We have :

$$\overline{x} = \overline{\eta} - \sigma[g(\pi) - \pi F^{-1}(1-\pi)] - \sigma F^{-1}(1-\pi) + \mu'\alpha'\sigma[g(\pi)+(1-\pi)F^{-1}(1-\pi)]$$

$$\overline{x} = \overline{\eta} - (1-\mu'\alpha')\,\sigma[g(\pi) + (1-\pi)\,F^{-1}(1-\pi)]$$

<u>Proof of property 4.1</u> :

i) As seen in the proof of theorem 3.1, we have : $\frac{dh(\pi)}{d\pi} = \frac{1 - \pi\mu\alpha - (1-\pi)\,\mu'\alpha'}{f[F^{-1}(1-\pi)]}$ and the function h is strictly increasing from $]0,1[$ onto $\mathbb{R}$. Therefore h^{-1} is strictly increasing from $\mathbb{R}$ onto $]0,1[$.

ii) $\frac{d}{d\pi}\left(\frac{\overline{\xi} - \overline{x}}{\sigma}\right) = \frac{d}{d\pi}\,[(1-\mu\alpha)[g(\pi) - \pi\,F^{-1}(1-\pi)]] = (1-\mu\alpha)\,\frac{\pi}{f[F^{-1}(1-\pi)]}$

Therefore : $\frac{d}{d(\overline{e}/\sigma)}\left(\frac{\overline{\xi} - \overline{x}}{\sigma}\right) = \frac{(1-\mu\alpha)\,\pi}{1 - \pi\mu\alpha - (1-\pi)\,\mu'\alpha'}$. This derivative is strictly positive. Since this homographic function is increasing in π , we deduce fom i) that this derivative is increasing in $(\overline{e}/\sigma)$ and therefore the function $\frac{\overline{\xi} - \overline{x}}{\sigma}$ is convex in $(\overline{e}/\sigma)$.

The proof is similar for $\frac{\overline{\eta} - \overline{y}}{\sigma}$.

<u>Proof of property 4.2</u> :

i) We have : $\frac{\partial\pi}{\partial\mu} = \frac{\alpha\,[g(\pi) - \pi F^{-1}(1-\pi)]}{\frac{dh(\pi)}{d\pi}} = \frac{\alpha[g(\pi) - \pi F^{-1}(1-\pi)]}{1 - \pi\mu\alpha - (1-\pi)\mu'\alpha'}\,f[F^{-1}(1-\pi)]$

Then : $\frac{\partial\overline{x}}{\partial\mu} = (1 - \mu'\alpha')\,\sigma\,\frac{(1-\pi)}{f[F^{-1}(1-\pi)]}\,\frac{\partial\pi}{\partial\mu} = \frac{(1-\mu'\alpha')\sigma\,\alpha(1-\pi)}{1 - \pi\mu\alpha - (1-\pi)\mu'\alpha'}\,[g(\pi)-\pi F^{-1}(1-\pi)]$

Since the function $g(\pi) - \pi F^{-1}(1-\pi)$ is increasing and takes the value o for $\pi = o$, we deduce that $\frac{\partial \pi}{\partial \mu} \geqslant o$, $\frac{\partial \bar{x}}{\partial \mu} \geqslant o$.
The proof is similar for the dependence in μ'.

ii) a) We have : $\frac{\partial \pi}{\partial \sigma} = -\frac{\bar{e}}{\sigma^2} \frac{1}{\frac{dh(\pi)}{d\pi}} = -\frac{\bar{e}}{\sigma^2} \frac{f[F^{-1}(1-\pi)]}{1 - \pi\mu\alpha - (1-\pi)\mu'\alpha'}$ and $\frac{\partial \pi}{\partial \sigma}$ is of the opposite sign of $\bar{e}$.

On the other hand : $$\frac{\partial \bar{x}}{\partial \sigma} = -(1-\mu\alpha)[g(\pi) - \pi F^{-1}(1-\pi)] - (1-\mu\alpha)\sigma \frac{\pi}{f[F^{-1}(1-\pi)]} \frac{\partial \pi}{\partial \sigma}$$

$$= -(1-\mu\alpha)[g(\pi) - \pi F^{-1}(1-\pi)] + \frac{(1-\mu\alpha)\pi}{1 - \pi\mu\alpha - (1-\pi)\mu\alpha'} \frac{\bar{e}}{\sigma}$$

$$= -\frac{(1-\mu\alpha)(1-\mu'\alpha')}{1 - \pi\mu\alpha - (1-\pi)\mu'\alpha'} g(\pi)$$

Let π_0 be the value such that $F^{-1}(1-\pi_0) = 0$; $\frac{dg(\pi)}{d\pi} = F^{-1}(1-\pi)$ is non negative if $\pi \leqslant \pi_0$, is non positive otherwise and since $g(0) = g(1) = o$, this implies that for all π : $g(\pi) \geqslant o$ .Therefore $\frac{\partial \bar{x}}{\partial \sigma} \leqslant o$ and $\bar{x}$ is a decreasing function of σ.

b) If σ tends to o , $\frac{\bar{e}}{\sigma}$ tends to $+\infty$ or $-\infty$ depending on the sign of $\bar{e}$.
If $\bar{e}$ is positive, π tends to 1 and $\bar{x}$ tends to $\bar{\eta} = \mathrm{Min}(\bar{\eta},\bar{\xi})$; if $\bar{e}$ is negative, π tends to o and $\bar{x}$ tends to $\bar{\xi} = \mathrm{Min}(\bar{\eta},\bar{\xi})$.

c) Let us examine the dependence of $\frac{\partial \bar{x}}{\partial \sigma}$ with respect to π. The maximum of $\left|\frac{\partial \bar{x}}{\partial \sigma}\right|$ is obtained for a value of π such that $\frac{\partial^2 \bar{x}}{\partial \sigma \partial \pi}$ vanishes. It is easily seen that : $\frac{\partial^2 \bar{x}}{\partial \sigma \partial \pi} = \frac{(1-\mu\alpha)(1-\mu'\alpha')}{(1 - \pi\mu\alpha - (1-\pi)\mu'\alpha')^2} \frac{\bar{e}}{\sigma}$ and is equal to zero for $\bar{e} = o$.

iii) $$\frac{\partial\pi}{\partial\overline{\xi}} = \frac{1}{\sigma}\;\frac{f[F^{-1}(1-\pi)]}{1-\pi\mu\alpha-(1-\pi)\,\mu'\alpha'}$$

$$\frac{\partial\overline{x}}{\partial\overline{\xi}} = 1 - (1-\mu\alpha)\,\frac{\pi\ \sigma}{f[F^{-1}(1-\pi)]}\;\frac{\partial\pi}{\partial\overline{\xi}} = \frac{(1-\pi)\ (1-\mu'\alpha')}{(1-\pi)\ (1-\mu'\alpha')+\pi(1-\mu\alpha)}$$

By symmetry, we have : $\dfrac{\partial\overline{x}}{\partial\overline{\eta}} = \dfrac{\pi(1-\mu\alpha)}{(1-\pi)\ (1-\mu'\alpha')+\pi(1-\mu\alpha)}$

and it is easily seen that these derivatives are between o and 1 .

Determination of $\tilde{\pi}$

$$\tilde{\pi} = \frac{1}{\overline{x}}\int_X [x_\omega(a)+x_\omega(s)]\,d\omega \quad \text{with}$$

$$\int_X [x_\omega(a)+x_\omega(s)]d\omega = \int_X [\xi_\omega(a)+\xi_\omega(s)]\,d\omega + \int_X [x_\omega(a)-\xi_\omega(a)]\,d\omega + \int_X [x_\omega(s)-\xi_\omega(s)]d\omega$$

$$= \int_X \xi_\omega\,d\omega - \frac{1-\mu}{\mu}\int_X \{\tilde{\xi}_\omega[x(s),X] - x_\omega(s)\}\,d\omega - k_X$$

$$= \int_X \xi_\omega\,d\omega - k_X - \frac{1-\mu}{\mu}\left[\int_X [\xi_\omega(s)-x_\omega(s)]\,d\omega + \pi\ c(\pi)\ k_X\right]$$

$$= \int_X \xi_\omega\,d\omega - \frac{k_X}{\mu(1-\alpha\pi)}\,(1-\mu\alpha\pi)$$

$$= \int_X \xi_\omega\,d\omega - (1-\mu\alpha\pi)\ \sigma\ [g(\pi) - \pi\ F^{-1}(1-\pi)]$$

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