August 1982

N° 8217

ON THE EXISTENCE OF EQUILIBRIA IN ECONOMIES WITH AN INFINITE DIMENSIONAL COMMODITY SPACE

by

Monique FLORENZANO

CNRS - CEPREMAP - Paris - France

The main result of this paper was achieved when I was visiting the group for the Applications of Mathematics and Statistics to Economics at the University of California, Berkeley, Summer 1982.

The research and the production of the paper reporting it have been supported by Grant SES 79-12380-Debreu from the National Science Foundation to the University of California, Berkeley, administered through the Center for Research in Management.

I would like to thank Gerard Debreu for his hospitality and his advice. I also wish to thank Bernard Cornet who followed the successive versions of this paper and made many suggestions, and Andreu Mas Colell and Karl Vind for helpful discussions.

Present adress : CEPREMAP - 140, rue du Chevaleret - 75013 - PARIS.

ON THE EXISTENCE OF EQUILIBRIA IN ECONOMIES WITH AN INFINITE DIMENSIONAL COMMODITY SPACE

Monique FLORENZANO

CNRS - CEPREMAP - France

ABSTRACT. We prove an infinite dimensional extension of the Gale-Nikaido-Debreu lemma which includes all necessary limiting processes and allows a proof of the existence of equilibria under standard assumptions in an economy with infinitely many commodities which exactly parallels the proof of Debreu (1959) for the finite dimensional case.

1. INTRODUCTION

The assumption that the commodities are not in finite number agrees with many classical situations for economic theory: intertemporal equilibrium with an infinite horizon, uncertainty with an infinite number of states, differentiation of commodities.

Since the paper of Bewley (1972), which proved the existence of a Walrasian equilibrium for an economy with a finite number of agents and $L_\infty(M,\mathfrak{M},\mu)$ for commodity space (let us recall that M is a state space, $\mathfrak{M}$ a σ-field of subsets of M, μ a σ-finite measure on $\mathfrak{M}$ and $L_\infty(M,\mathfrak{M},\mu)$ the Banach space of all measurable functions such that $\|f\|_\infty = \sup\{r \geq 0/\mu\{m/|f(m)| \geq r\} > 0\} < \infty$), the work on the subject developed in two directions:

- While the proof of Bewley is based on a limiting process which deduces the existence of an equilibrium in the infinite dimensional case from the existence of an equilibrium in the finite dimensional case, Bojan (1974), Toussaint (1981), Magill (1981), Brown and Aliprantis (1982) look for a direct proof which is the analogue of one of the proofs in the finite dimensional case.

- At the same time, Brown and Aliprantis (1982) try to set the most general framework in which their results can be achieved. Before them, Elbarkoky (1977) extended the equilibrium existence theorem of Bewley to an outline which is essentially the same as this which will be used here.

This paper corresponds to both objectives. It states an extension of the Gale-Nakaido-Debreu lemma, in order to give a direct proof of the existence of an equilibrium which exactly parallels that of Debreu (1959) for the finite dimensional case, in a general framework which is to be defined in Section 2.

2. THE MODEL

In this paper, the space of commodities is a Banach space E, the conjugate space of another Banach space G, i.e., $E = G'$ where G' is the vector space of all continuous linear functions on G and the norm on E is the conjugate norm to the norm on G. G is ordered by a closed convex positive cone G^+ satisfying: $G = G^+ - G^+$. E itself is ordered by the positive cone:

$$E^+ = \{x \in E / \langle p,x \rangle \geq 0 \quad \forall\, p \in G^+\}.$$

where $\langle p,x \rangle$ is the value at p of the continuous linear functional $x \in G'$. We shall denote by $\geq$ the orders on G and E.

These first assumptions suffice to assume that in an exchange economy, under the usual conditions of lower boundedness of consumption sets, the attainable sets are compact for the weak topology $\sigma(E,G)$ for E associated with the duality (E,G). This will be deduced from the following elementary proposition:

<u>Proposition 1</u>. *If* G *is a real topological vector space ordered by a positive cone* G^+ *satisfying:* $G = G^+ - G^+$, *and if* E *is the conjugate space of* G, *ordered by the cone of all positive continuous linear functionals on* G^+, *let* a *and* b *be two elements of* E *and let* $A = \{x \in E / a \leq x \leq b\}$. *Then* A *is* $\sigma(E,G)$*-bounded.*

<u>Proof</u>. Let $p \in G$, $\alpha \in R$, $\alpha > 0$ and $V = \{x \in E / |\langle p,x \rangle| \leq \alpha\}$. We claim that A is absorbed by V. If $p \in G^+$ and if $\beta = \sup\{|\langle p,a \rangle|, |\langle p,b \rangle|\}$, then

$$A \subset \frac{\beta}{\alpha} V \quad \text{if} \quad \beta > 0 \qquad \text{and} \qquad A \subset V \quad \text{if} \quad \beta = 0 .$$

If $p \in -G^+$, one obtains the same result.

If $p = p^+ - p^-$ where p^+ and p^- belong to G^+, then $V \supset V^+ \cap V^-$ where $V^+ = \{x \in E / |\langle p^+, x\rangle| \leq \frac{\alpha}{2}\}$ and $V^- = \{x \in E / |\langle p^-, x\rangle| \leq \frac{\alpha}{2}\}$.
Let μ and ν be real numbers such that $A \subset \mu V^+$ and $A \subset \nu V^-$.
Then $A \subset \lambda V$, where $\lambda = \sup\{\mu, \nu\}$. □

Prices belong to E', the conjugate space of E. E' is ordered by the positive cone: $E'^+ = \{\pi \in E' / \langle x, \pi\rangle \geq 0 \ \forall x \in E^+\}$, where as above we denote by $\langle x, \pi\rangle$ the value at x of the continuous linear functional $\pi \in E'$. In order to define a price simplex compact for the weak topology $\sigma(E', E)$ for E' associated with the duality (E, E'), we need that E *has a non-empty interior*. For example, if, as in Aliprantis and Brown (1982), E is an A.M. space with unit (i.e., a Banach lattice, the lattice order and the norm of which are related by the two relations: $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ and there exists $e \in E^+$ such that the order interval $[-e, +e]$ coincides with the closed unit ball $\{x \in E / \|x\| \leq 1\}$) this assumption is satisfied. The following proposition is classical:

Proposition 2. *Let* E *be a real Hausdorff locally convex topological vector space,* E' *the conjugate space of* E *and* Y *be a convex cone of* E *such that* $Y^0 = \{\pi \in E' / \langle y, \pi\rangle \leq 0, \ \forall y \in Y\} \neq \{0\}$. *If* u *belongs to the interior of* Y, *the set*

$$\Delta = \{\pi \in E' / \langle u, \pi\rangle = -1\}$$

is equicontinuous and $\sigma(E', E)$*-compact and satisfies* $Y^0 = \bigcup_{\lambda \geq 0} \lambda \Delta$

Proof. Let V be a closed circled neighborhood of 0 such that:

$$\forall \pi \in Y^0, \quad \forall x \in V, \qquad \langle u + x, \pi\rangle \leq 0 .$$

From $\langle u+x,\pi\rangle \leqslant 0$ and $\langle u-x,\pi\rangle \leqslant 0$, it follows that for all $\pi \in Y^0$, $\pi \neq 0$, $\langle u,\pi\rangle < 0$ and $\frac{\pi}{|\langle u,\pi\rangle|} \in \Delta$. On the other hand, for all π in Δ an x in V: $-1+\langle x,\pi\rangle \leqslant 0$, which implies : $|\langle x,\pi\rangle| \leqslant 1$. Thus Δ is equicontinuous and it follows from the Alaoglu-Bourbaki theorem that Δ is $\sigma(E',E)$ compact. □

We consider an exchange economy $\mathfrak{E} = ((X^i,R^i,\omega^i)_{i=1,\dots,m},Y)$ with m consumers and, for each $i = 1,\dots,m$, a *consumption set* $X^i \subset E$, a *complete preordering* R^i on X^i and an *initial endowment* $\omega^i \in E$; in addition, *disposal activities* are described by a convex cone Y, contained in $(-E^+)$. To the preordering R^i, we associate the strict preference relation defined by

$$x^i P^i x'^i \iff x^i R^i x'^i \text{ and not } x'^i R^i x^i \quad .$$

Let $x^i \in X^i$; we introduce the following notations:

$$P^i(x^i) = \{x'^i \in X^i / x'^i P^i x^i\} \quad ;$$

$$R^i(x^i) = \{x'^i \in X^i / x'^i R^i x^i\} \quad ;$$

$$(P^i)^{-1}(x^i) = \{x'^i \in X^i / x^i P^i x'^i\} \quad ;$$

$$(R^i)^{-1}(x^i) = \{x'^i \in X^i / x^i R^i x'^i\} \quad .$$

For each $\pi \in E'$, we define:

$$\gamma^i(\pi) = \{x^i \in X^i / \langle x^i,\pi\rangle \leqslant \langle \omega^i,\pi\rangle\} \quad ,$$ i.e., the budget correspondence for consumer i

$$\delta^i(\pi) = \{x^i \in X^i / \langle x^i,\pi\rangle < \langle \omega^i,\pi\rangle\}$$

$$\xi^i(\pi) = \{x^i \in X^i / \ x^i \in \gamma^i(\pi) \text{ and } \gamma^i(\pi) \cap P^i(x^i) = \emptyset\} \quad ,$$

i.e., the demand correspondence for consumer i

$$\phi^i(\pi) = \{x^i \in X^i / x^i \in \gamma^i(\pi) \text{ and } \delta^i(\pi) \cap P^i(x^i) = \phi\}$$

i.e., the quasi-demand correspondence for consumer i

$$\eta(\pi) = \sum_{i=1}^{m} \phi^i(\pi) - \left\{\sum_{i=1}^{m} \omega^i\right\} ,$$

i.e., the excess quasi-demand correspondence .

An *allocation* $x = (x^i) \in \prod_{i=1}^{m} X^i$ is *attainable* if

$$\sum_{i=1}^{m} x^i \in \left\{\sum_{i=1}^{m} \omega^i\right\} + Y .$$

An *equilibrium* of E is a point $(\bar{x},\bar{y},\bar{\pi}) \in \left(\prod_{i=1}^{m} X^i\right) \times Y \times (E'\setminus\{0\})$ satisfying the conditions:

(1) $\forall\ i = 1,\ldots,m,\ \ \bar{x}^i \in \xi^i(\bar{\pi})$;

(2) $\bar{\pi} \in Y^0 = \{\pi \in E' / \langle y,\pi\rangle \leqslant 0,\ \forall y \in Y\}$ and $\langle \bar{y},\bar{\pi}\rangle = 0$;

(3) $\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i + \bar{y}$.

A *quasi-equilibrium* of E is a point $(\bar{x},\bar{y},\bar{\pi}) \in \left(\prod_{i=1}^{m} X^i\right) \times Y \times (E'\setminus\{0\})$ satisfying the above conditions (2) and (3) and the condition (1'):

(1') $\forall\ i = 1,\ldots,m,\ \ \bar{x}^i \in \phi^i(\bar{\pi})$.

In fact, we will prove the existence of a quasi-equilibrium from an extension of the Gale-Nikaido-Debreu lemma (or market equilibrium theorem) that we state in the next section.

3. THE EXTENSION OF THE GALE-NIKAIDO-DEBREU LEMMA

In the literature, one can find two extensions of the finite dimensional Gale-Nikaido-Debreu lemma to correspondences defined on a compact subset of the space E' of all continuous linear functionals on a real Hausdorff locally convex topological vector space E, with convex values in a compact subset of E. The first is from Bojan (1974), the second from S.Toussaint (1981); the common defect of the two statements is that they cannot be applied without a limit operation to prove the existence of equilibrium under standard assumptions in a transitive economy with a finite number of agents and a commodity space as in Section 2.

Here we prove an infinite dimensional analogue of the Gale-Nikaido-Debreu lemma which is the most convenient extension for an application to a proof of existence of equilibrium which exactly parallels in the infinite dimensional case the proof of Debreu (1959) for the finite dimensional case.

First, let us introduce some definitions:

<u>Definition 1</u>. Let ϕ be a correspondence from a topological space X to a topological space Y and let x_0 be an element of X; we define the subset $\limsup_{x \to x_0} \phi(x)$ as the set of all elements y of Y such that for all neighborhoods W of y and V of x_0, there exists $x \in V$ such that $\phi(x) \cap W \neq \phi$. In other words, $\limsup_{x \to x_0} \phi(x)$ is the set of all y in X such that (x_0, y) belongs to the closure of the graph of ϕ in the product topology for $X \times Y$.

Definition 2. Given a convex subset X of a vector space E, let $\mathbb{A}$ be the collection of all finite subsets α of X. If $\alpha = \{x^1, x^2, \ldots, x^{n_\alpha}\} \in \mathbb{A}$, let S^{n_α} denote the unit simplex of the Euclidean space $\mathbb{R}^{n_\alpha}$ and $\beta_\alpha : S^{n_\alpha} \to X$ the map defined by:

$$\text{if} \quad p = \sum_{i=1}^{n_\alpha} p_i e^i, \ p_i \geq 0 \ \forall i = 1, \ldots, n_\alpha, \ \sum_{i=1}^{n_\alpha} p_i = 1 ,$$

$$\text{then} \quad \beta_\alpha(p) = \sum_{i=1}^{n_\alpha} p_i x^i .$$

The *finite topology* (see Aubin (1979), p.211) defined on X is the strongest topology for which the maps β_α are continuous for all $\alpha \in \mathbb{A}$

A correspondence ϕ from X to a topological space Y is upper semi-continuous when X is equipped with the finite topology if and only if the correspondences $\phi \circ \beta_\alpha : S^{n_\alpha} \to Y$, defined by

$$\phi \circ \beta_\alpha(p) = \phi(\beta_\alpha(p))$$

are upper semi-continuous for all $\alpha \in \mathbb{A}$.

I.e., when X is equipped with the finite topology for verifying upper semi-continuity of the correspondence ϕ, it is necessary and sufficient to verify upper semi-continuity of all the restrictions $\phi/\text{co}\,\alpha$ of the correspondence ϕ to the convex hulls co of the finite subsets of X.

The finite topology on X is stronger than any topology induced on X by a vector space topology on E.

Thus we can state:

Lemma 1. *Let* E *be a real Hausdorff locally convex topological vector space,* E' *the vector space of all continuous linear functionals on* E, Y *a convex cone of* E *such that the polar cone* $Y^0 = \{\pi \in E' / \langle y,\pi\rangle \leq 0 \ \forall y \in Y\}$ *is non-reduced to* $\{0\}$, u *a point in the interior of* Y, $\Delta = \{\pi \in E' / \langle u,\pi\rangle = -1\}$, *and* ζ *a correspondence from* Δ *to* E.

Let $\mathfrak{T}$ *be a Hausdorff locally convex topology for* E *weaker than* $\sigma(E,E')$, *and* $D = \{\pi \in \Delta / \pi$ is continuous if E is equipped with $\mathfrak{T}\}$. *We make the following additional assumptions:*

(1) *If* E *is equipped with* $\mathfrak{T}$ *and if* D *is equipped with the finite topology, the restriction* ζ/D *of* ζ *to* D *is upper semi-continuous with non empty closed convex values in a compact set* Z *of* E.

(2) $\forall \pi \in \Delta, \ \forall z \in \zeta(\pi), \ \langle z,\pi\rangle \leq 0$.

(3) Y *is* $\mathfrak{T}$*-closed.*

Then there exists $\bar{\pi} \in \Delta$ *such that* $Y \cap \limsup_{\pi \to \bar{\pi}} \zeta(\pi) \neq \emptyset$, *where the* lim sup *is taken for the topology on* Δ *induced by* $\sigma(E',E)$ *and the topology* $\mathfrak{T}$ *on* E.

The idea of the proof we give is borrowed partly from Bojan (1974), partly from Aliprantis and Brown (1982).

Proof. Let $\mathbb{A}$ be the collection of all finite subsets of D and, for each $\alpha \in \mathbb{A}$, $\alpha = \{\pi^1, \pi^2, \ldots, \pi^{n_\alpha}\}$, Δ_α be the convex hull of $\alpha : \Delta_\alpha = \text{co}\,\alpha$; β_α, as in Definition 2, the map from S^{n_α} onto $\Delta_\alpha : \beta_\alpha(p) = \sum_{i=1}^{n_\alpha} p_i \pi^i$; and γ_α the map from E to $\mathbb{R}^{n_\alpha}$, $\gamma_\alpha(z) = (\langle z,\pi^1\rangle, \ldots, \langle z,\pi^{n_\alpha}\rangle)$. The correspondence $\zeta \circ \beta_\alpha$ is upper semi-continuous from S^{n_α} to E equipped with $\mathfrak{T}$. The map γ_α is continuous from E equipped with $\mathfrak{T}$ to $\mathbb{R}^{n_\alpha}$. Thus the correspondence $(\gamma_\alpha \circ \zeta/D \circ \beta_\alpha)$ from S^{n_α} to R^{n_α} is upper semi-continuous with non-empty convex values in a compact subset of R^{n_α}. Furthermore, for all $p \in S^{n_\alpha}$ and for all $y \in \gamma(\zeta(\beta_\alpha(p))$, if $y = \gamma(z)$, one has:

$$p \cdot y = \sum_{i=1}^{n_\alpha} p_i \langle z, \pi^i \rangle = \langle z, \beta_\alpha(p) \rangle \leqslant 0 .$$

From the finite dimensional Gale-Nikaido-Debreu lemma (see for example Debreu (1959), p.82), there exists $\bar{p}_\alpha \in S^{n_\alpha}$ and $\bar{z}_\alpha \in \zeta(\beta_\alpha(\bar{p}_\alpha))$ such that: $\langle \bar{z}_\alpha, \pi^i \rangle \leqslant 0 \;\; \forall\, i = 1,\ldots,n_\alpha$. Thus $\bar{\pi}_\alpha = \beta_\alpha(\bar{p}_\alpha)$ and $\bar{z}_\alpha$ satisfy: $\bar{z}_\alpha \in \zeta(\bar{\pi}_\alpha) \cap Y_\alpha$, where $Y_\alpha = \{y \in E / \langle y, \pi \rangle \leqslant 0,\ \forall\, \pi \in \Delta_\alpha\}$

Now the collection $\mathbb{A}$ is directed by inclusion. Δ is $\sigma(E',E)$-compact (Proposition 2) and Z is $\mathfrak{T}$-compact. By passing to two subnets, if necessary, we can assume that $\bar{\pi}_\alpha \xrightarrow{\sigma(E',E)} \bar{\pi} \in \Delta$ and $\bar{z}_\alpha \xrightarrow{\mathfrak{T}} \bar{z} \in Z$. We have to prove that $\bar{z} \in Y$.

If $\bar{z} \notin Y$, since Y is $\mathfrak{T}$-closed, there exists $\pi \in D$ such that $\langle \bar{z}, \pi \rangle > 0$. Let $\alpha \in \mathbb{A}$ such that $\pi \in \Delta_\alpha$. Since $\langle \bar{z}, \pi \rangle > 0$, $\bar{z} \notin Y_\alpha$. Furthermore, Y_α is $\mathfrak{T}$-closed, since $\mathfrak{T}$ is compatible with the duality of E equipped with $\mathfrak{T}$ and the vector space of all $\mathfrak{T}$-continuous linear functionals on E. Thus, there exists β such that $\gamma \geqslant \beta \Rightarrow \bar{z}_\gamma \notin Y_\alpha$; and $\gamma \geqslant \sup(\beta,\alpha) \Rightarrow \bar{z}_\gamma \notin Y_\gamma$, which is a contradiction proving that $\bar{z} \in Y$. □

In particular, if E is as in Section 2 a Banach space, the conjugate space of some other Banach space G, we can take for $\mathfrak{T}$ the weak topology $\sigma(E,G)$.

If we take $\mathfrak{T} = \sigma(E,E')$, we deduce the following result, proved by Bojan when E is a normed space and extended here for any real Hausdorff locally convex topological vector space:

Corollary 1. *Let* E *be a real Hausdorff locally convex topological vector space,* E' *the vector space of all continuous linear functionals on* E, Y *a closed convex cone of* E *such that the polar cone* Y^0 *is non-reduced to* $\{0\}$, u *a point in the interior of* Y, $\Delta = \{\pi \in E' / \langle u,\pi \rangle = -1\}$ *and* ζ *a correspondence from* Δ *into* E, *satisfying the following assumptions:*

(1) ζ *is upper semi-continuous with non-empty closed convex values in a compact subset* Z *of* E, *when* E' *is equipped with the topology* $\sigma(E',E)$ *and* E *with the topology* $\sigma(E,E')$

(2) $\forall \pi \in \Delta,\ \forall z \in \zeta(\pi),\ \langle z,\pi \rangle \leq 0.$

Then there exists $\bar{\pi} \in \Delta$ *such that* $Y \cap \zeta(\bar{\pi}) \neq \phi$.

Proof. Since the correspondence ζ is upper semi-continuous with closed values in a compact subset of E, one easily deduces that $\zeta(\bar{\pi}) = \limsup_{\pi \to \bar{\pi}} \zeta(\pi)$. Then, the proof follows easily from the fact that the finite topology on Δ is stronger than the topology induced on Δ by $\sigma(E',E)$. □

Corollary 2 (S.Toussaint (1981)). *The statement of Corollary 1 is true with the topology* $\sigma(E,E')$ *replaced by the initial topology of* E.

Proof. Corollary 2 is easily deduced from Corollary 1. □

Note that in the statement of Corollary 2, S.Toussaint adds an assumption of joint continuity of the restriction to $Z \times \Delta$ of the canonical bilinear functional: $(z,\pi) \to \langle z,\pi \rangle$, not needed for the result and which limits the topology to be considered for E to the Mackey topology $\tau(E,E')$ of the pairing (E,E').

4. THE EXISTENCE THEOREM

Let us come back to the exchange economy $\mathcal{E} = ((X^i, R^i, \omega^i)_{i=1,\dots,m}, Y)$ defined in Section 2. The consumption set, the preference preordering, the initial endowment of each consumer i and the cone Y are subjected to the following assumptions:

(1) $\forall\ i = 1,\dots,m,\ X^i$ is convex, bounded from below, $\sigma(E,G)$-closed.

(2) $\forall\ i = 1,\dots,m,\ \forall x^i \in X^i,\ R^i(x^i)$ is convex and $\sigma(E,G)$ closed; furthermore, $y^i \in P^i(x^i)$ and $0 \leqslant \lambda < 1$ imply: $\lambda x^i + (1-\lambda)y^i \in P^i(x^i)$.

(3) $\forall\ i = 1,\dots,m,\ \omega^i \in X^i - Y$.

(4) If $x = (x^i) \in \prod_{i=1}^{m} X^i$ is an attainable allocation, then for all $i = 1,\dots,m,\ P^i(x^i) \neq \emptyset$.

(5) Y is a $\sigma(E,G)$ closed convex cone, contained in $(-E^+)$ and has a non-empty interior for the norm topology of E.

From assumption (5) it follows that Y is not reduced to $\{0\}$. If u belongs to the interior of Y, as above we define Δ to be the $\sigma(E',E)$-compact convex set: $\Delta = \{\pi \in E' / \langle u,\pi \rangle = -1\}$.

Let $\tilde{X}^i$ denote the attainable set of consumer i:

$$\tilde{X}^i = \left\{x^i \in X^i / x^i + \sum_{j \neq i} x^j = \sum_j \omega^j + y,\ x^j \in X^j\ \forall j \neq i \text{ and } y \in Y\right\}.$$

Each $\tilde{X}^i$ is convex, bounded from below and from above (assumption (1)) and since G is a Banach space, it follows from Proposition 1 that each attainable set $\tilde{X}^i$ is relatively $\sigma(E,G)$-compact. In fact, each attainable set $\tilde{X}^i$ is $\sigma(E,G)$-compact and it follows from the assumptions (2) on preordering P^i, the assumption (3) and theorem 1 in Bergstrom (1975), that there exists $c^i \in \tilde{X}^i$ such that $P^i(c^i) \cap \tilde{X}^i \neq \emptyset$, and therefore, there exists $d^i \in P^i(c^i)$, or $d^i \in P^i(x^i)$ $\forall x^i \in \tilde{X}^i$.

We choose a $\sigma(E,G)$-compact convex subset K of E containing all the attainable sets and, for each $i = 1,\ldots,m$, $a^i \in X^i$; $d^i \in X^i$ satisfying

$$a^i = \omega^i + b^i \quad , \qquad b^i \in Y \quad \text{(assumption 3)}$$

$$d^i \in P^i(x^i) \qquad \forall\, x^i \in \tilde{X}^i \quad .$$

For each $i = 1,\ldots,m$, let $\hat{X}^i = X^i \cap K$ and consider the new exchange economy

$$\hat{\mathfrak{E}} = ((\hat{X}^i, R^i, \omega^i)_{i=1,\ldots,m}, Y)$$

where each consumer retains his original endowment and preferences. We shall begin by proving the following existence theorem for the truncated economy $\hat{\mathfrak{E}}$:

<u>Proposition 3</u>. *If* $\mathfrak{E} = ((X^i, R^i, \omega^i)_{i=1,\ldots,m}, Y)$ *satisfies the assumptions (1),(2),(3),(4),(5),* $\hat{\mathfrak{E}}$ *has a quasi-equilibrium* $(\bar{x},\bar{y},\bar{\pi}) \in \prod_{i=1}^{m} \hat{X}^i \times Y \times \Delta$.

<u>Proof</u>. Let us consider, as in Section 2, the correspondence $\gamma^i, \delta^i, P^i, \xi^i, \phi^i$ and η now defined for the economy $\hat{E}$. We have to verify the assumptions of Lemma 1 in the case where $\mathfrak{T}$ is the topology $\sigma(E,G)$ and for the restriction to Δ of the correspondence η :

$$\eta(\pi) = \sum_{i=1}^{m} \phi^i(\pi) - \left\{ \sum_{i=1}^{m} \omega^i \right\}.$$

From assumption (5), Y is $\sigma(E,G)$-closed and obviously correspondence η satisfies the Walras identity:

$$\forall\ \pi \in \Delta, \quad \forall\ z \in \eta(\pi), \quad \langle z,\pi \rangle \leq 0.$$

So that only the condition (1) of Lemma 1 has to be proved true.

Let D denote the subset $D = \Delta \cap G$. On D, the correspondences ϕ^i are non-empty and convex valued. Indeed, for all $\pi \in D$, $\gamma^i(\pi)$ is non-empty (assumption 3 and definition of K) and $\sigma(E,G)$-compact (assumption 1 and definition of $\hat{X}^i$). For all $x^i \in \gamma^i(\pi)$, $\gamma^i(\pi) \cap (P^i)^{-1}(x^i)$ is $\sigma(E,G)$-open in $\gamma^i(\pi)$ (assumption 2) and it follows (Bergstrom (1975), theorem 1) from convexity and irreflexivity of P^i that $\xi^i(\pi)$ is non-empty; a fortiori $\phi^i(\pi) \neq \emptyset$. That for all π in D, $\phi^i(\pi)$ is convex follows from completeness and transitivity of the preference preordering. Thus the correspondence η is non-empty and convex valued with values in the $\sigma(E,G)$-compact subset of $E : Z = \sum_{i=1}^{m} \hat{X}^i - \left\{ \sum_{i=1}^{m} \omega^i \right\}$.

Let $\mathbb{A}$ be the collection of all finite subsets of D and, for each $\alpha \in \mathbb{A}$, $\alpha = \{\pi^1, \pi^2, \ldots, \pi^{n_\alpha}\}$, $\Delta_\alpha = co\{\pi^1, \pi^2, \ldots, \pi^{n_\alpha}\}$ be the convex hull of α. For α fixed, since Δ_α is finite dimensional, the canonical bilinear form $(x,\pi) \to \langle x,\pi \rangle$ is jointly continuous on $\hat{X}^i \times \Delta_\alpha$ equipped with the topology induced by the product of topologies $\sigma(E,G)$ and $\sigma(E',E)$; thus the restriction γ/Δ_α of the budget correspondence of any consumer i on Δ_α has a closed graph in $\Delta_\alpha \times \hat{X}^i$. It easily follows from continuity properties of P^i in assumption 2 that it is the same for the restriction ϕ^i/Δ_α of ϕ^i to Δ_α. Finally, for all $\alpha \in \mathbb{A}$, the restrictions η/Δ_α are correspondences with a closed graph in $\Delta_\alpha \times Z$ and it follows from Definition 2 that the restriction of ζ to D is upper semi-continuous and closed valued from D equipped with finite topology to E equipped with $\sigma(E,G)$.

One can apply Lemma 1 and there exist $\bar{\pi} \in \Delta$, a net $(\bar{\pi}_\alpha)$ and a net $(\bar{z}_\alpha)$ such that: $\bar{\pi}_\alpha \xrightarrow{\sigma(E',E)} \bar{\pi}$, $\bar{z}_\alpha \xrightarrow{\sigma(E,G)} \bar{z} \in Z$, $\bar{z} \in Y$ and, for all α, $\bar{z}_\alpha \in \eta(\bar{\pi}_\alpha)$.

Let $\bar{z}_\alpha = \sum_{i=1}^{m} \bar{x}^i_\alpha - \sum \omega^i$, with $x^i_\alpha \in \phi^i(\bar{\pi}_\alpha)$, $\forall i = 1,\dots,m$. Without a loss in generality, we can suppose that $\bar{x}^i \xrightarrow{\sigma(E,G)} \bar{x}^i \in \hat{X}^i$, $\forall i = 1,\dots,m$, and thus, since $\bar{z} = \sum_{i=1}^{m} \bar{x}^i - \sum_{i=1}^{m} \omega^i \in Y$, the allocation $\bar{x} = (\bar{x}^i)$ is attainable.

Then let $x'^i \in P^i(\bar{x}^i)$. From assumption 2, there exists $\alpha \in A$ such that $\alpha' \geq \alpha \Rightarrow x'^i \in P^i(\bar{x}^i_{\alpha'})$ and, since $\bar{x}^i_{\alpha'} \in \phi^i(\bar{\pi}_{\alpha'})$, $\langle x'^i, \bar{\pi}_{\alpha'} \rangle \geq \langle \omega^i, \bar{\pi}_{\alpha'} \rangle$. Passing to the limit, one sees that:

$$\langle x'^i, \bar{\pi} \rangle \geq \langle \omega^i, \bar{\pi} \rangle .$$

The previous relation being true for all $i = 1,\dots,m$ and for all $x'^i \in P^i(\bar{x}^i)$, one deduces from assumption 4 and assumption 2:

$$\langle \bar{x}^i, \bar{\pi} \rangle \geq \langle \omega^i, \bar{\pi} \rangle \quad \forall i = 1,\dots,m .$$

On the other hand, $\sum_{i=1}^{m} \langle \bar{x}^i, \bar{\pi} \rangle - \sum_{i=1}^{m} \langle \omega^i, \bar{\pi} \rangle = \langle \bar{z}, \bar{\pi} \rangle$ and since $\bar{z} \in Y$, one deduces: $\langle \bar{x}^i, \bar{\pi} \rangle = \langle \omega^i, \bar{\pi} \rangle \ \forall i = 1,\dots,m$

$$\langle \bar{z}, \bar{\pi} \rangle = 0.$$

The conditions (1'), (2) and (3) in Section 2 in order to $(\bar{x}, \bar{z}, \bar{\pi})$ be a quasi-equilibrium of $\hat{\mathcal{E}}$ are satisfied. □

One obtains the existence theorem for $\mathcal{E}$ by increasing the convex $\sigma(E,G)$-compact set K and by passing to limit.

Proposition 4. *If* $\mathfrak{E} = ((X^i, R^i, \omega^i)_{i=1,\dots,m}, Y)$ *satisfies the assumption (1),(2),(3),(4),(5),* $\mathfrak{E}$ *has a quasi-equilibrium* $(\bar{x},\bar{y},\bar{\pi}) \in \prod_{i=1}^{m} X^i \times Y \times \Delta$.

Proof. The collection $\mathcal{K}$ of the convex, $\sigma(E,G)$-compact subsets K of E containing all attainable consumption sets and for each $i = 1,\dots,m$, $a^i \in X^i$, $d^i \in X^i$ defined as above, is directed. For each $K \in \mathcal{K}$, the economy $\hat{E}_K = ((\hat{X}^i_k, R^i, \omega^i)_{i=1,\dots,m}, Y)$, where $\hat{X}^i_k = X^i \cap K$, has a quasi equilibrium $(\bar{x}_k, \bar{y}_k, \bar{\pi}_k) \in \prod_{i=1}^{m} X^i \times Y \times \Delta$. Since each $\tilde{X}^i$ is $\sigma(E,G)$-compact and since Δ is $\sigma(E',E)$-compact, by passing to subnets if necessary, we can assume:

$$\bar{x}^i_k \xrightarrow{\sigma(E,G)} \bar{x}^i \in \tilde{X}_i \,, \qquad \forall \; i = 1,\dots,m$$

$$\bar{\pi}_k \xrightarrow{\sigma(E',E)} \bar{\pi} \in \Delta$$

and since for all K of $\mathcal{K}$, $\sum_{i=1}^{m} \bar{x}^i_k = \sum_{i=1}^{m} \omega^i + \bar{y}_k$ and Y is $\sigma(E,G)$ closed,

$$\bar{y}_k \xrightarrow{\sigma(E,G)} \bar{y} \in Y \,.$$

By passing to limits in relations $\sum_{i=1}^{m} \bar{x}^i_k = \sum_{i=1}^{m} \omega^i + \bar{y}_k$, we see that $\bar{x} = (\bar{x}^i)$ is an attainable allocation: $\sum_{i=1}^{m} \bar{x}^i = \sum \omega^i + \bar{y}$.

Let i belong to $\{1,\dots,m\}$ and $x'^i \in P^i(\bar{x}^i)$. There exists $K' \in \mathcal{K}$ such that $x'^i \in K$ for all $K \geq K'$. Since $R^i(\bar{x}^i)$ is $\sigma(E,G)$ closed, there exists $K'' \in \mathcal{K}$ such that $x'^i \in P^i(\bar{x}^i_k)$ for all $K \geq K''$. Since $(\bar{x}_k, \bar{y}_k, \bar{\pi}_k)$ is a quasi-equilibrium of $\hat{E}_k$ for all K of $\mathcal{K}$, then:

$$K \geq \sup(K',K'') \Rightarrow \langle x'^i, \bar{\pi}_k \rangle \geq \langle \omega^i, \bar{\pi}_k \rangle \,.$$

And passing to limit, $\langle x'^i, \bar{\pi} \rangle \geqslant \langle \omega^i, \bar{\pi} \rangle$. Now, the previous relation is true for all $i = 1,\dots,m$ and for all $x'^i \in P^i(\bar{x}^i)$. From assumption (and assumption (4), one deduces:

$$\langle \bar{x}^i, \pi \rangle \geqslant \langle \omega^i, \pi \rangle \qquad \forall\ i = 1,\dots,m$$

On the other hand, $\sum_{i=1}^{m} \langle \bar{x}^i, \bar{\pi} \rangle - \sum_{i=1}^{m} \langle \omega^i, \bar{\pi} \rangle = \langle \bar{y}, \bar{\pi} \rangle$ and since $\bar{y} \in Y$, one deduces

$$\langle \bar{x}^i, \bar{\pi} \rangle = \langle \omega^i, \bar{\pi} \rangle \qquad \forall\ i = 1,\dots,m$$

$$\langle \bar{y}, \bar{\pi} \rangle = 0 .$$

The conditions (1'), (2) and (3) in Section 2 in order to $(\bar{x},\bar{y},\bar{\pi})$ be a quasi-equilibrium of $\mathcal{E}$ are satisfied. □

5. REMARKS

1) The assumptions (1)-(5) of Proposition 4 are standard, once the $\sigma(E,G)$ topology on E has been accepted. When the commodity space E is $L_\infty(M,\mathfrak{M},\mu)$, the continuity property of assumption (2) can be interpreted as an impatience assumption on preferences (see Brown and Lewis, 1981). Note also that if E is $L_\infty(M,\mathfrak{M},\mu)$, assumption (2) implies that, as in the finite dimensional case, each of preorderings R^i on X^i is representable by a $\sigma(E,G)$-continuous utility function, since E is $\sigma(E,G)$-separable.

2) The equilibrium price, the existence of which is proved in Propositions 3 and 4, is an element of E', the conjugate space of the commodity space E. It does not have a very natural economic interpretation. If one wants a price equilibruim in G, one needs a decomposition theorem like the Yosida-Hewitt theorem.

3) The main limitation of the result proved in this paper arises from the assumption that the disposal cone has an interior point. This prevents any extension of the result to the existence of equilibria without the free-disposal assumption. Furthermore, this stringently limits the Banach spaces to be used as commodity spaces to spaces with a positive cone with a non-empty interior.

REFERENCES

ALIPRANTIS,C.D. and D.J.BROWN, 1982, Equilibria in markets with a Riesz space of commodities. Social Science Working Paper #427, California Institute of Technology, Pasadena.

AUBIN,J.P., 1979. Mathematical Methods of Game and Economic Theory. (North-Holland, Amsterdam, New York, Oxford).

BERGSTROM,T.C., 1975. The existence of maximal elements and equilibria in the absence of transitivity. Mimeo.

BEWLEY,T.F., 1972. Existence of equilibria in economies with infinitely many commodities, Journal of Economic Theory 4, 514-540.

BOJAN,P., 1974. A generalization of theorems on the existence of competitive economic equilibrium in the case of infinitely many commodities, Mathematica Balkanica 4, 491-494.

BOURBAKI,N., 1966. Espaces Vectoriels Topologiques, 2nd edition. (Hermann, Paris).

BROWN,D.J. and L.M.LEWIS, 1981. Myopic economic agents. Econometrica 49, 359-368.

DEBREU,G., 1959. Theory of Value. (Wiley, New York).

ELBARKOKY,R.A., 1977. Existence of equilibrium in economies with Banach commodity space, Akad. Nauk. Azerdaijan SSR Dokl. 33, 8-12 [Russian].

KELLEY,J.L., J.NAMIOKA and coauthors, 1963. Linear Topological Spaces (D. Van Nostrand Co., Inc., Princeton, New Jersey).

MAGILL,M.J.P., (1981). An equilibrium existence theorem, MRG Working Paper #8118, University of Southern California, Los Angeles.

MAS-COLELL,A., 1975. A model of equilibrium with differentiated commodities, Journal of Mathematical Economics 2, 236-296.

OSTROY, J.M., 1982. On the existence of Walrasian equilibrium in large-square economies. Mimeo.

TOUSSAINT, S., 1981. On the existence of equilibria in economies with infinitely many commodities. Discussion Paper #174/81.