SECOND BEST PRICING RULES
IN THE BOITEUX TRADITION:
DERIVATION, REVIEW AND DISCUSSION.

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INTRODUCTION.

In a first best world, pricing policies obey a simple principle: the price of the marginal unit sold to each consumer should equate its marginal cost. Even if the implementation of such a rule may raise further problems (cf. the implementation of peak load pricing, the design of two part tariffs etc.) it is of universal theoretical validity.

In a second best world where the absence of markets or behavioural constraints prevents the attainment of first best Pareto optima, the prescriptions for optimal pricing policies lose both simplicity and universality. Simplicity, because prices should no longer equate marginal costs—even if marginal costs were computed on the basis of the social value of commodities rather than production prices—but should take into account other elements such as demand elasticities. Universality, because the difficulty of designing piecemeal policies defining rules valid for one sector independently of government action in other sectors has been constantly emphasized by second best theory. In particular, the pricing rules which are established from one theoretical model do depend in some sense on the whole set of policy and behavioural assumptions made in this model. Changing the policy tools available to the government not only changes the optimal prices which would emerge but also possibly the qualitative features of the optimal pricing rule and the type of information required for its implementation. It is then quite important for policy purposes to understand the logic of the derivation of pricing rules in order to evaluate their sensitivity to modifications of policy and behavioural assumptions.
We propose here a synthetical review of a class of second best pricing rules which places the emphasis on the analysis of their robustness to changes in the environment of the production unit under consideration. We successively focus attention on

1. **The principles governing the derivation of second best pricing rules in a general equilibrium setting.** The so-called second best pricing rules relate the prices quoted by the public firms under consideration and the shadow prices underlying their production decisions, around an optimal situation. They are themselves part of the characterization results obtained in the study of necessary conditions of second best Pareto optimality (or social welfare optimality). The analysis presented here attempts to provide a full and as intuitive as possible understanding of the derivation of such second best rules in different contexts. It relies heavily on a previous contribution of the author (Guesnerie (1979)) where general characterizations theorems for second best Pareto optimality are presented. These theorems are here translated into basic principles of intuitive economic appeal and are applied to the analysis of the set of models under consideration. This approach requires a preliminary investment which has a non-zero cost. Besides of external benefits due the potentially wide applicability of the principles, this investment has returns which ought to be valued positively: the derivation is based on reasoned argument rather than on blind computation; the loose although potentially rigorous argument is supported by economic intuition and develops it in return; the analysis of the process by which modifications of assumptions are reflected in results makes apparent the branching points of the theory.

2. **The comparative analysis of second best pricing rules under alternative assumptions.** Different models are considered which formalize different alternative assumptions on the set of policy tools available for public action: lump sum transfers, commodity taxation, linear or non-linear income tax are successively or simultaneously introduced. The paper gathers a rather broad set of alternative characterization results. The associated pricing rules can then be examined and compared from the point of view of their qualitative properties, economic interpretation, informational and computational requirements. Although most results are not new, it is hoped that put together they provide subjects of reflection on the theme of piecemeal economy policy and are a useful landmark for new advances in the understanding of the interaction between pricing and taxation policies.
Let us finally stress that the pricing problems to which we restrict our attention belong to what may be termed the Boiteux tradition. First they concern only linear pricing. Whether non linear pricing may be implemented or not depends on the nature of the commodity which is sold, but we exclude it from our field of reflection. Second we focus attention on cases in which the pricing policy is affected by some sectoral budget constraint (besides the general government budget constraint). Such sectoral budget constraints the study of which motivated the original Boiteux contribution, play a central role in the pricing problems considered here. Furthermore, for reasons which are explained below and which relate to the theoretical rather than operational point of view taken here, we adopt a formalization close to Boiteux's own formalization (1956).

The paper proceeds as follows.

In section I, general second best principles are presented. One of them (principle 2 corollary 1) which has immediate and important implications for our problem will be singled out.

In section II we focus attention on the original Boiteux model to which the principles of section I are applied.

Section III examines alternative assumptions. Taxation policies are combined with pricing policies in subsection III.B. Lump sum transfers are ruled out in subsection III.C where poll tax or/and commodity taxes are considered. Further topics are considered in subsection III.D: the effect on the design of pricing policies of the existence of a non linear income tax or the study of production decisions of firms subject to budget constraints producing only public goods.

Section IV illustrates the general formulas previously obtained for the case of a simple economy and brief conclusion are finally offered.
I. GENERAL SECOND BEST PRINCIPLES.

Let us consider an economy with \( n+1 \) commodities numbered from 0 to \( n \). Firms are indexed by \( j \in J = J_1 \cup J_2 \) and are associated with production sets \( Y_j \). Consumers are indexed by \( i \in I \) and have consumption sets \( X_i \) on which are defined preference preorderings.

Firms belonging to \( j \in J_1 \) are termed uncontrolled: they decide upon their production plan \( y_j \) according to the set of signals —denoted \( s \)— that they face. Firm \( j \)'s behaviour is formally described by its supply correspondence \( D_j : s \in S \rightarrow D_j(s) \subseteq \mathbb{R}^{n+1} \) (\( S \) is a subset of a Euclidean space where the vector of signals \( s \) lies). Consumers react to the same signals vector \( s \) according to a demand function \( \chi_i \): \( s \in S \rightarrow \chi_i(s) \in \mathbb{R}^{n+1} \). Firms belonging to \( J_2 \) are termed controlled; the government has the power to decide upon their production plans.

The concept of feasible states describes the states of the economy which can be attained through government intervention, given the autonomous behaviour of uncontrolled agents and the market clearing conditions.

Formally, a feasible state consists of consumption plans \( (x_i) \), production plans \( (y_j) \), signals \( s \) such that

\[
\begin{align*}
(1.1) \quad & x_i = x_i(s) \quad i \in I \\
(1.2) \quad & y_j \in D_j(s) \quad j \in J_1 \\
(1.3) \quad & y_j \in Y_j \quad j \in J_2 \\
(1.4) \quad & s \in \overline{S} \subseteq S \\
(1.5) \quad & \sum_{i} x_i \leq \sum_{j} y_j + \omega
\end{align*}
\]

where \( \omega \) is a vector of initial endowments and \( \overline{S} \) a closed subset of the space of signals.

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[1] A first difference with the abstract model of Guesnerie (1979) is that we immediately take for \( \chi_i \) a function and not a correspondence. Actually, in the more general framework, the treatment of producers and consumers is fully symmetrical. Particularly, one supposes that there exists a category of controlled consumers. However, these differences are unimportant for our analysis.
A second best Pareto optimum of the model is a feasible state \( (x^*_i, y^*_j, s^*) \) such that there does not exist another feasible state Pareto better than \( (x^*_i, y^*_j, s^*) \).

Assuming weak technical regularity conditions (on \( x_i, o_j, s \)) and possibly convexity \( (x_i, y_j, j \in J_2) \) second best Pareto optima are characterized in Guesnerie (1979) in this most general framework. Rather than reproducing here the precise statements (1) which have been obtained, we will state partly formally partly informally, the general principles of intuitive economic appeal which can be derived.

**PRINCIPLE 1**: Shadow prices of controlled firms and social value of commodities.

With any second best Pareto optimum is associated a vector \( p^* \) which can be considered (up to a constant) the vector of social values of commodities and which is the shadow price vector associated with the optimal production plan of any controlled firm.

Formally if \( y_j, j \in J_2 \) is convex, 
\[
\rho^* y_j \geq \rho^* y_j \quad \forall y_j \in Y_j
\]

In the case where the second best optimum under consideration is the maximum of some social welfare function, \( p^* \) corresponds (up to a multiplicative constant (2)) to the Lagrange multipliers which would emerge in the maximisation process, associated with the scarcity constraints (1.5).

Principle 1 indicates that production efficiency holds within the subset of controlled firms which should be given the instruction of using the shadow price vector \( p^* \). Furthermore, this vector \( p^* \) plays a central role in the following.

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(1) Strictly speaking these statements require, for technical reasons, that controlled consumer be taken into account (regularity condition are then simplified). This is a minor point for our purpose.

(2) In the following we will adopt the normalization rule \( p^*_0 = 1 \), a condition which cannot be imposed on Lagrange multipliers.
Principle 2 is concerned with the subset of uncontrolled firms for which the supply does not reduce to a single vector. These firms are termed partially uncontrolled since according to our definition of feasible states the government can choose any production plan \( y_j \) in \( D_j(s^*) \). The optimal rule of selection is quite simple.

**PRINCIPLE 2:** Optimal production plans of partially uncontrolled firms.

The social value of the optimal production plan of any partially uncontrolled firm is greater than the social value of any production plan that the firm would accept to implement; the signal vector being fixed at \( s^* \).

Formally

\[
(1.7) \quad \rho^* \cdot y^*_j \geq \rho^* \cdot y_j, \quad \forall y_j \in D_j(s^*), \forall j \in J_4
\]

This principle has important implications for constant returns to scale uncontrolled firms, for which necessarily \( \rho^* \cdot y_j = 0 \). We will concentrate here on the implications of this rule in the case where

\[
(1.8) \quad D_j(s^*) = M_j(s^*) \cap Y_j
\]

and where \( M_j(s) \) denotes some set depending on the signals vector. If \( M_j(s^*), Y_j \) are convex sets, (1.7) implies that the three convex sets \( M_j(s^*), Y_j \) and \( \{ y_j | \rho^* \cdot y_j > \rho^* \cdot y_j^* \} \) have an empty intersection. Calling respectively \( \psi^* \), \( \tilde{\rho}_j^* \) vectors normal to adequate supporting hyperplanes of \( M_j(s^*), Y_j \) in \( y_j^* \) it is classical that \( \psi^* \), \( \tilde{\rho}_j^* \) can be chosen such that

\[
(1.9) \quad \rho^* = \tilde{\rho}_j^* + \psi^*
\]

\( \rho^* \) is convex.

(2) Such a property was first emphasized in a different context by Diamond-Mirrlees (1975) who argued that it was general.

(3) A precise formulation of the well known separation theorem for \( n \) convex cones, of which this property is a particular case can be found in Guesnerie (1979). Mangasarian lemma, Farkas lemma often referred to in the second best literature (see Dievert (1978), Dixit (1979)) are closely related with this theorem.

(4) Even if the sets \( M_j(s^*), Y_j \) were not convex but only had a convex local approximation (which is a rather weak assumption) then (1.9) would hold, but \( \rho^* \), \( \psi^* \) would only be normal to the local approximations of the corresponding sets and \( \tilde{\rho}_j^* \) would only provide information for "local" decentralization.
The interest of (1.9) lies in the fact that $\tilde{p}_j^*$ is the shadow price vector which allows the decentralization of the second best optimal production plan of the partially uncontrolled firm.

The reader will have recognized that the Boiteux firm subject to a budget constraint can be considered a partially uncontrolled firm in the category defined by (1.8). Precisely, anticipating on the following, the equivalent of $M_j(s^*)$ in the Boiteux type models is the hyperplane associated with the budget constraint which has a normal in the direction of $- p^*$, $p^*$ being the production price vector. Hence if one supposes that both $p^*, \tilde{p}_j^*, p^*$ are normalized with the normalization rule $\tilde{p}_j^0 = p^0_0 = 1$ (1.9) becomes

$$\tilde{p}_j^* = \mu_j \hat{\beta}_j^* p^* - \beta_j, \beta_j \geq 0, \mu_j \geq 0, \mu_j - \beta_j = 1.$$

Or

$$\tilde{p}_j^* - p^* = \frac{1}{1+\beta_j}[p^* - p^*], \beta_j \geq 0$$

It is remarkable that this rule applies whatever the assumptions made on the rest of the economy. It will particularly hold in all the models considered in the following; for any firm subject to a budget constraint (1)

**PRINCIPLE 2 - Corollary 1 (21):** Shadow price system associated with optimal production plans for firms of the Boiteux type.

If we normalize all prices such that the price of commodity zero equals one, the following holds:

The vector of shadow taxes of any Boiteux firm, which is the difference between its shadow price vector and the production price vector, is a fraction of the vector of social taxes, the difference between the social value of commodities and their production prices.

Let us come now to the third and last principle which focus attention on small moves of signals in the production and consumption sectors.

(1) Here, as in the following, such a firm will be referred to as a "Boiteux firm."
PRINCIPLE 3: Analysis of infinitesimal moves of signals and of consumption and production plans.

In any second best optimum, there exists vectors $\delta^*_1, \delta^*_2$ (of the same dimension as the vectors of signals) such that:

i) Any small move $(dy_1, ds)$ of total uncontrolled production and of the vector of signals which is compatible with the constraints induced by producer behaviour satisfies

$$ (1.12) \quad p^*\cdot dy_1 - \delta^*_1 \cdot ds \leq 0 $$

ii) Any small move $[\ldots dx_1 \ldots, ds]$ of the individual uncontrolled consumers and of the signals vector which is compatible with consumers behaviour and which is Pareto improving (i.e. making everyone better off) satisfies

$$ (1.13) \quad p^*\cdot dx_1 + \delta^*_2 \cdot ds \geq 0 $$

with $dx_1 = \sum_i dx_i$

iii) Any feasible $ds$ satisfies

$$ (1.14) \quad (\delta^*_1 + \delta^*_2) \cdot ds \leq 0 $$

It may be useful in order to illustrate principle 3 to consider specific cases. For example, if $s_k$, the $k^{th}$ component of the vector of signals is free, it results from (1.14) that $(\delta^*_1 + \delta^*_2)_k = 0$; considering a small move $ds_k$ it results from (1.12) and (1.13) that if $s_k$ only affects producer decisions (resp. consumers decisions), $p^*\cdot dy_{1i} \leq 0$ (resp. $p^*\cdot dx_{1i} \leq 0$). Appealing interpretations of these inequalities obtain when $p^*$ is considered as the vector of social values of commodities. Keeping these interpretations in mind, the reader will pursue the exercise when $\delta^*_1, k, \delta^*_2, k$ or both are different from zero. He will convince himself – as will also appear from subsequent statements - that $\delta^*_1, \delta^*_2, \delta^*_1 + \delta^*_2$ can be considered as vectors of social opportunity costs associated with the violation of constraints on signals.
Principle 3 applies whatever the local properties of \( O_j, x_i \).

However, it takes particularly simple forms if \( O_j, x_i \) are single valued and differentiable\(^{(1)}\). Let us then assume that around the optimum \( O_j, x_i \) are differentiable.

Calling \( \xi_i(\pi, R_i) \) the usual competitive demand function of the utility maximizing consumer faced with the price vector \( \pi \) and having an income \( R_i \)\(^{(2)}\), \( x_i \) can be written down if the preferences of \( i \) are smooth and convex.

\[
[1.16] \quad x_i(s) = \xi_i(\psi_i(s), m_i(s))
\]

where \( \psi_i(s) \) is a normalized vector (\( \psi_{i0} = 1 \)) of personalized "implicit" prices (or shadow prices) associated with the choice of consumer \( i \) and \( m_i(s) = \psi(s) \cdot x_i(s) \) a shadow income.

Consider one of the component of vector \( s \), call it \( \alpha \) (instead of \( s_k \)) and define:

\[
[1.17] \quad E_i(\alpha, \cdot) = \psi_i(\cdot, \frac{\partial x_i}{\partial \alpha})(\cdot)
\]

\( E_i(\alpha) \) is the gain of agent \( i \), or its surplus measured in quantity of numeraire, associated with a small move of signal \( \alpha \).

One can state.

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\(^{(1)}\) It may well happen that in economic problems without pathological features the response of an agent to changes of signals be discontinuous around the optimum. In that case, although conforming principle 3 first order optimality conditions are likely to be more complicated than in the differentiable case. The reader interested in this point should refer to Guesnerie-Laffont (1978) (which show that this issue is central for taxing price-makers) and Mirrlees (1977).

\(^{(2)}\) Formally \( \xi_i(\pi, R_i) \) is the solution of \( \text{Max } u_i(x_i) \left| \pi \cdot x_i \leq R_i \right. \), where \( u_i \) is the utility function of agent \( i \).
PRINCIPLE 3 - Corollary 1 (31): "Primary" conditions of second best Pareto optimality in the differentiable case.

In a second best Pareto optimum, besides the vectors $\rho^*, \delta_1^*, \delta_2^*$ already introduced, there exist positive numbers $\mu_i^* > 0$ such that:

1) (1.18)  
$$ \rho^* \left( \frac{\partial q}{\partial a} \right)_* = \delta_1^*, \alpha $$

2) (1.19)  
$$ \rho^* \left( \frac{\partial x}{\partial a} \right)_* = \sum \mu_i^* E_i(a,*) - \delta_2^*, \alpha $$

Particularly, if $\alpha$ is a free variable in the sense that it is unconstrained (by $S$) around the optimum, one has

3) (1.20)  
$$ \rho^* \left[ \left( \frac{\partial x}{\partial a} \right)_* - \left( \frac{\partial q}{\partial a} \right)_* \right] = \sum \mu_i^* E_i(a,*) $$

Both 1) and 2) are straightforwardly deduced from 1) and 2) in principle 3.

The $\mu_i^*$ appearing in 2) are the variables dual to the inequalities constraining the move considered in Principle 3 to be Pareto improving. They can be considered the social values of income (or surplus) of consumers. Then, the two conditions of principle 31 have an easy interpretation in terms of equality of social costs (the left hand side) with social benefits (the right hand side). All these conditions are termed primary because they may be seen as immediate consequences of the general heuristic principles.

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(1) $\psi_i \cdot dX_i \left[ = \frac{\partial x_i}{\partial a} = E_i(a) \right] \geq 0$

(2) When we compare these results with optimal taxation results let us remember that $\rho^*$ is normalized. Our $\mu_i^*$ identifies with the ratio of social value of i's income over the social value of the numeraire which relates to the social welfare function considered and hence has no reason to be equal to one.
successively applied to the different components of the vector of signals

By opposition, secondary conditions which combine primary conditions
derive from the application of principle 3 to simultaneous moves of
several variables. Going into that direction, consider

another coordinate $\beta$ of the signals vector and define:

$$U = C / \beta$$

(1.21) \[
\frac{\partial x_i}{\partial a} = \frac{\partial x_i}{\partial a} - \frac{E_i(a)}{E_i(\beta)} \frac{\partial \beta}{\partial a} U = C / \beta
\]

It is straightforward that $\psi_i \left( \frac{\partial x_i}{\partial a} \right) = 0$ so that $\frac{\partial x_i}{\partial a} U = C / \beta$

is associated with a change at fixed utility and defines a generalized
compensated derivative of demand\(^{(1)}\).

Let us also consider

(1.22) $v_i(a) = [v_i - \rho] \frac{\partial x_i}{\partial a}$

$v_i(a)$ can be interpreted as the net social marginal utility of policy $a$
for household $i$: it is equal to the gross social marginal efficiency of
the policy—the product of $E_i(a)$ by $\mu_i$, social value of $i$'s surplus—minus the social cost induced by $i$'s reaction $-\rho \frac{\partial x_i}{\partial a}$.

One can state

PRINCIPLE 3 - Corollary 2 (32) "Secondary" conditions of second best Pareto
optimality in the differentiable case.

In a second best Pareto optimum, we have

i) Whatever $\alpha, \beta$ components of the vector of signals\(^{(2)}\)

(1.23) $p \cdot \left( \frac{\partial x_i}{\partial a} \right) U = C / \beta = \sum_i v_i(\beta, *) \frac{E_i(a, *)}{E_i(\beta, *)} - \delta_{2,a}$

where $v_i(\beta)$, $E_i(a)$, $E_i(\beta)$ are defined by (1.22), (1.17) and

$$\left( \frac{\partial x_i}{\partial a} \right) U = C / \beta = \sum_i \frac{\partial x_i}{\partial a} U = C / \beta$$

Moreover if $\beta$ is a free variable which is not an effective
signal for the production sector\(^{(3)}\), we have

(1.24) $\sum_i v_i(\beta, *) = 0$

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(1) As the reader will easily notice, it coincides with the usual compensated
demand when $\alpha$ is one consumption price and $\beta$ the income level.

(2) Such that $E_i(\alpha), E_i(\beta) \neq 0 \forall i$

(3) i.e such that $\frac{\partial Q}{\partial \beta} = 0$
ii) (1.23) holds if one replaces $\beta$ by $\beta_i$ where the $\beta_i$ are individual signals $(E_i(\beta_i) \neq 0, E_i(\beta_j) = 0, i \neq j)$.

Principle 32 easily obtains from formulas (1.18), (1.19), (1.21), (1.22)

Although less immediately intuitive than the previous statements, principle 32 incorporates in the expression of principle 3 compendated derivatives which have been privileged in the discussions of the optimal taxation literature. Principle 32, (or a combination of 32 and 31) when applied with $\alpha$ as a consumption price and $\beta$ or $\beta_i$ as income of an usual utility maximizing consumer (whose shadow prices coincide with consumption prices\(^{(1)}\)) will result in rules often referred to as of Ramsey-type or of Ramsey-Boiteux-type.

The family air shown by such rules is in fact often better explained by their filiation from principle 32 rather that by a close resemblance with what Ramsey and Boiteux precisely asserted\(^{(2)}\).

\begin{itemize}
  \item \((1)\) This assumption is satisfied in most of the models of the optimal taxation literature, but for the models incorporating income tax. However it generally does not hold in second best models when markets are missing (insurance problems etc...).
  \item \((2)\) An original attempt to unify and give an intuitive understanding to such rules can be found in S.C. Kolm (1970). The geometric analysis used here may however look unfamiliar to economists more accustomed to the mathematics of optimization.
\end{itemize}
II. THE BOITEUX MODEL.

In this section, we focus attention on the original model of Boiteux (1956). Actually, in the whole paper we will keep some basic elements of the formalization chosen by Boiteux, even if we incorporate a variety of policy assumptions which he did not consider. Particularly as Boiteux we will suppose that the supply of the private sector is described by differentiable functions. This is a compromise which look adequate in view of our goals. On the one hand, the principles we have stated are powerful enough to deal with a more general formulation using multi-valued supply correspondences. But this would put a burden on notation which seems to be excessive. On the other hand, studies with pedagogical or operational concern (see Baumol-Bradford (1970), Feldstein (1973)) often have assumed that the global production set of the private sector is limited by an hyperplane. Such an assumption oversimplifies the analysis of the incidence of public pricing. It ignores interactions which are of genuine theoretical interest even if they can be dismissed on practical grounds in specific contexts\(^{(1)}\).

Let us describe now the Boiteux model while referring the reader interested in more details to the original article as well as to the subsequent work by Drèze (1964) and Rees (1968).

\(^{(1)}\) Furthermore two points should be stressed.

- The constant returns to scale case can be obtained from the differentiable case as a limit when supply elasticities tend to infinity. (see section IV).
- Any mixture of differentiability and constant returns to scale can also be directly studied, by taking into account principle 2 (see section IV).
In this model, consumers and producers are faced with the same price system that we will denote \( p \) (\( p \in \mathbb{R}_{+}^{n+1} \)). The Government can through lump sum transfers control the individual income \( R_i \) of consumer \( i \) (who has the usual utility maximizing behaviour). Firms are of two types. Private firms \( (j \in J_{11}) \) have convex production sets; they maximize their competitive profit in a standard way. Faced with the production price system \( p \), firm \( j \) chooses a competitive production plan (supposed to be unique) \( n_j(p) \).

Public firms subject to a budgetary constraint only implement production plans \( y_j \), \( j \in J_{12} \), which assure an amount of profit \( p.y_j \) equal to the value of an exogeneous bundle of commodity \( v_j \) i.e. \( p.y_j \geq p.v_j \).

It is not our purpose to discuss in detail here the rationale for this constraint. For Boiteux as well as for the more recent literature on "natural monopoly" (see for example Baumol (1977)), it originates in the existence of increasing returns: marginal cost pricing would imply a deficit which is not acceptable either because the Government requires a budget balance for the public firm or because the "natural monopoly" is exploited by a private firm which has to make profit. The precise shape of the production sets \( Y_j \) (of firms \( j \) subject to a budgetary constraint) will not play an important role in

\[ v_j \] can be considered a numeraire specific to the \( j \)th firm but it would coincide with the numeraire of the economy, only if it consisted of one (or several) units of commodity zero and zero units of the other commodities, a case which will be often considered in the following.

(2) These justifications are highly questionable in a world of perfect information where the technological possibilities of firms would be known to the Center. Taking into account the imperfect information aspects would lead to an analysis outside the scope of this paper.
the following. However some of our comments will implicitly assume that non convexities only derive from the existence of fixed costs (so that the corresponding production sets is the union of the origin and of a convex set).

We are now in position to define a feasible state of the Boiteux model as associated with vectors \((x_i, y_j, p)\) and numbers \(R_i\) such that:

\[
\begin{align*}
(2.1) & \quad x_i = \xi_i(p, R_i) \\
(2.2) & \quad y_j = \eta_j(p) \\
(2.3) & \quad p \cdot y_j \geq p \cdot y_j \\
(2.4) & \quad y_j \in Y_j \\
(2.5) & \quad \sum_i x_i \leq \sum_j y_j + \omega
\end{align*}
\]

(1) There are several ways of identifying the Boiteux model with the abstract model of section I.

We will define \(s = (p, R_i, i \in I, z_j, j \in J_{12})\) where \(z_j\) are vectors of \(\mathbb{R}^{n+1}\) and will replace (2.3) and (2.4) by:

\[
\begin{align*}
(2.6) & \quad y_j = z_j \\
(2.7) & \quad p \cdot z_j - p \cdot v \geq 0 \\
(2.8) & \quad z_j \in Y_j
\end{align*}
\]

Clearly (2.1), (2.2), (2.6), (2.7), (2.8), (2.5) defines a model which fits the general framework of section II, (2.7) and (2.8) defining jointly the set \(S\) of equation (1.5). Noting that now supply and demand functions are differentiable functions of the signals, one can apply different versions of principle 3.

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(1) Actually in the following for the sake of notational simplicity we will often consider a simplified version of this model with one aggregate private sector with supply function \(\eta(p)\) and an aggregate financial constraint on the public sector measured with the numeraire vector \(v\).

(2) Note that in order to derive principle 21 we implicitly defined the signal vector in another way.
Small moves \( dp_k \) compensated with adequate \( dR_i \) define Pareto improving changes in the sense of principle 3 ii). So it comes out with obvious notation

\[
\rho^* \left( \frac{\partial \xi}{\partial p_k} \right)^c + \delta^*_{2,p_k} = 0
\]

\( (2.9) \) can also be viewed as an immediate consequence of Principle 3 i), where

\( (1.24) \) implies \( v_i(R_i) = 0 \) (since \( R_i \) is a free variable).

Considering small moves \( dp_k \) and principle 3i) (or equivalently corollary 3i) with \( \alpha = p_k \) it comes out

\[
(2.10) \quad \rho^* \left( \frac{\partial \eta}{\partial p_k} \right) - \delta^*_{1,p_k} = 0
\]

In the same way considering small moves of \( z_{j,k} \), it comes out with straightforward notation

\[
(2.11) \quad \rho^* = \delta^*_{1,z_j} , \quad j \in J_{12}
\]

Reminding that the normalized shadow price vector (i.e. if the production set is limited by a smooth surface, a vector defining the marginal rates of substitution between commodities and the numeraire) of firm \( j \), \( j \in J_{12} \) is denoted \( \bar{\rho}_j^* \), one knows that if the Budget constraint is tight at the optimum, locally \( \bar{S} \) can be described by

\[
(2.12) \quad - dp_*(z_j^* - v_j) - \rho^* dz_j \leq 0 , \quad \forall j \in J_{12}
\]

\[
(2.13) \quad \bar{\rho}_j^* dz_j \leq 0
\]

Since the only vectors which can be "normal" to \( \bar{S} \) are convex combinations of vectors which are "normal" to the sets the intersection of which defines \( \bar{S} \), heuristic principle 3 iii) implies :

\[
(2.14) \quad \delta_{1,p_k}^* + \delta_{2,p_k}^* = - \sum_{j \in J_{12}} \beta_j (z_j^* - v_j) , \quad \beta_j \geq 0
\]

\[
(2.15) \quad \delta_{1,z_j}^* = - \beta_j \rho^* + \nu_j \bar{\rho}_j^* , \quad \nu_j \geq 0
\]

\( \frac{\partial \xi_i}{\partial p_k} \) is the usual compensated derivative which obviously identifies with \( \left( \frac{\partial \xi}{\partial p_k} \right)^c \) of section II. \( \left( \frac{\partial \xi}{\partial p_k} \right)^c \) \( \text{def} \sum_i \left( \frac{\partial \xi}{\partial p_k} \right)^c \cdot \delta^*_{i,p_k} \) is the component associated with signal \( p_k \) in the vector \( \delta^*_1 \).
Taking into account the normalization rule, (2.15) joined to (2.11) give:

\[ \rho^* = - \beta_j p^* + (1 + \beta_j) \tilde{\rho}_j^* \]

i.e.

\[ (2.16) \quad \rho^* - p^* = (1 + \beta_j) (\tilde{\rho}_j^* - p^*) \quad \forall j \in J_{12} \]

(2.16) is nothing else than a restatement of the property (1.10) (obtained here with another proof). We already stressed in Principle 21 that such a property was very general.

Summing (2.9) and (2.10) and taking into account (2.14) leads to:

\[ (2.17) \quad \rho^* \left( \frac{\partial \xi}{\partial p_k}\big|_c \big) - \frac{\partial \eta}{\partial p_k}\big|_c \right) \right] = \sum_{j \in J_{12}} \beta_j (y_j^* - v_j^*) \]

The symmetry properties of matrices of derivatives of compensated demand and supply imply:

\[ (2.18) \quad (\rho^* - p^*) \left( \frac{\partial \xi_k}{\partial p}\big|_c \big) - \frac{\partial \eta_k}{\partial p}\big|_c \right) \right] = \sum_{j \in J_{12}} \beta_j (y_j^* - v_j^*) \]

(2.18) together with (2.16) are the central Boiteux results. As commented earlier, (2.16) implies that the vectors of shadow taxes \((\tilde{\rho}_j^* - p^*)\) specific to different firms subject to a Budget constraint are all proportional.

(2.18) can be commented as follows. Consider \(\rho^* - p^*\) the difference between social values and prices as shadow social taxes. The left side of (2.18) corresponds to a change in compensated excess demand of commodity \(k\) corresponding to an intensification of shadow taxation (small moves of taxes proportional to their actual values). As argued by Kolm (1971), the right-hand side crucially depends on the choice of the exogeneous bundle (here the \(v_j\)) serving to the evaluation of Budget surplus. However the interpretation of the right hand side simplifies when there is only one aggregate Budget constraint with the requirement that the profit exceeds the value of a certain amount of a single commodity, here commodity zero, the numeraire. In that case, a simple version of the Boiteux rule (2.18) obtains which can be stated as follows.

An "intensification" of shadow taxation i.e. a small increase of shadow taxes in proportion to their actual values would lead to a change in compensated excess demand which is proportional for any commodity but the numeraire to the actual optimal net supply of the Boiteux firm.
Although because of the demand interdependancies this rule is in general rather complex, it takes in some circumstances more appealing and intuitive forms. As an example, let us consider the case where the Boiteux firm produces two outputs (commodity n and n-1) from a single input, commodity zero. Let us also suppose that all cross elasticities (n,h) (n-1,h) (h ≠ 0) are negligible.

It is immediate to see that formulas (2.16) and (2.18) imply:

\[ \frac{p_h - \tilde{p}_h}{\hat{p}_h} = \frac{\beta}{1 + \beta \varepsilon_h}, \quad h = n, n-1 \text{ with } \varepsilon_h = -\frac{\hat{p}_h}{\hat{y}_h} \left( \frac{\partial \xi_h}{\partial p_h} \right)_c \]

As with the simple production function considered \( \tilde{p}_n \) and \( \tilde{p}_{n-1} \) are nothing else than marginal costs in terms of numeraire, (2.19) reads: the ratio price minus marginal cost over price is proportional for each produced commodity to the inverse of the elasticities of compensated demand.

In case where the cross elasticities between n and n-1 are no longer small, although other cross elasticities are negligible, the reader will easily obtain formulas extending (2.19). He will find a skillful diagrammatical illustration of this latter case in A. Philips (1979).

Another application of formula (2.18) can be found in subsection III.B where the Boiteux firm produces one commodity which has a substitute produced by the private sector. The determination of the ratio price minus marginal cost over price involves cross elasticities in a formula much less intuitively appealing that (2.19) (formula 3.21).

The more complex situation considered in section IV is still analytically tractable and the formulas obtained reflect more clearly the interactions the knowledge of which is relevant to the computation of shadow prices of the Boiteux firm.
III. PRICING RULES UNDER ALTERNATIVE ASSUMPTIONS.

Even if one accepts the line of approach, several objections can be addressed to the solution proposed by Boiteux and briefly recalled in the preceding section.

First, there remains at the optimum wedges between social values of commodities and production prices which justify the use of additional policy tools. For example commodity taxation which is useless in a first best context could be utilized in conjunction with the pricing rules to reduce the remaining distorsions, either directly through taxation and subzidization of some of the commodities produced by the Boiteux firm or indirectly through taxation of other commodities.

Second, because of the assumption of existence of lump sum transfers, distributional problems and allocational questions can be in some sense separated. The effect of equity considerations on the design of pricing policies should be evaluated in a more realistic framework.

In order to discuss these questions, we will introduce in this section several variants of the model of section II. In subsection III.B, the assumption of existence of lump sum transfers is maintained and we focus attention on the role of commodity taxation as an additional device for correcting the distorsions introduced by budget constraints. In subsection III.C1 we consider the Boiteux model of section II where individualized lump sum transfers are ruled out: distributional problems are explicitly reflected in pricing policies. In subsection III.C2 lump sum transfers are still ruled out but commodity taxation in taken into account as in III.B and plays both allocational are distributional roles. Finally in III.D, we examine the qualitative modifications of our results if more sophisticated policy tools are used as a (non necessarily optimal) income tax. We also focus attention on Boiteux firms producing only public goods. Most analytical résultats are preliminary derived from the general principles of section I, in subsection III.A.
III.A - PRELIMINARIES.

In this section the production price system (which will still be denoted \( p \)) is possibly allowed to differ from the consumption price system (which will be denoted \( \pi \)). Lump sum transfers \( R_i \) will be either considered or ruled out. And for the sake of notational simplicity we will only consider one aggregate Budget constraint in the partially uncontrolled sector (denoted 12 here) and an aggregate controlled sector (denoted 2).

Most of the models of subsections III.B to III.D are particular cases of the model the feasible states of which are given by the following relationships:

\[
\begin{align*}
(3.1) & \quad x_i = \xi_i(\pi, R_i) \\
(3.2) & \quad y_{11} = \eta(p) \\
(3.3) & \quad p \cdot y_{12} \geq p \cdot v \\
(3.4) & \quad y_{12} \in Y_{12} \\
(3.5) & \quad y_2 \in Y_2 \\
(3.6) & \quad (p, \pi) \in \Delta \\
(3.7) & \quad (...R_i...) \in D \\
(3.8) & \quad \sum_i x_i \leq \sum_j y_j + \omega
\end{align*}
\]

The notation just presented conforms the general notation previously adopted. However for the sake of simplicity, and as there is no ambiguity, we will simplify it in the following by putting \( y_{12} = z, y_{11} = y_1 \).

- The Boiteux model of Section II corresponds to the case \( \Delta = \{(p, \pi) | p = \pi\} \), (production prices equal consumption prices), \( D = \mathbb{R}^m \) (free lump sum transfers).
- In subsection III.B, lump sum transfers are still free (\( D = \mathbb{R}^m \)) and commodity taxation is allowed and \( \Delta \) is either a subset of \( \mathbb{R}^{2n+2}_+ \) or \( \mathbb{R}^{2n+2}_+ \) if there is no a priori restriction on commodity taxation.
- In subsection III.C1 only uniform lump sum transfers are allowed: \( D = (...)R_i...) | R_i = R \) and commodity taxation is not considered (\( p = \pi \)).

(1) As discussed in Bernard (1976) it is natural and reasonable to suppose that the Budget constraint is written down with production prices.
In subsection III.C2 commodity taxation is again introduced \( (\Delta = \mathbb{R}_{+}^{2n+2}) \); lump sum transfers are ruled out \( (R_{i} = 0) \) or only uniform lump sum transfers are allowed \( \mathcal{D} = \{ (\ldots R_{i} \ldots) | R_{i} = R, \forall i \} \).

Now, the machinery of section II applied to the model as rewritten in (2.6), (2.7), (2.8), will give us very efficiently the characteristics of second best optima corresponding to any alternative set of assumptions.

Principle 3i) applies as in section II to small moves \( dp_{k} \) of the component \( p_{k} \) of the vector of signals, so that (2.10) still holds. Let us rewrite it here as (1)

\[
(3.9) \quad \rho \frac{\partial \eta}{\partial p_{k}} - \delta_{1} p_{k} = 0
\]

The application of principle 3ii) falls in 2 different cases

1°) The consumption and production prices of commodity \( k \), \( \pi_{k} \) and \( p_{k} \) are rigidly linked, either because \( \pi_{k} = p_{k} \) is imposed or because the constraint on taxation of commodity \( k \) is binding at the optimum (for example \( \pi^{*} = p^{*} + \bar{t}_{k} \) where \( \bar{t}_{k} \) is the maximal tax on commodity \( k \)). In that case, principle 3ii) applies to small moves \( dp_{k} = dp_{k} \)

1.1) If lump sum transfers are allowed, these moves can be compensated by income in order to be (weakly) Pareto improving; formula (2.9), rewritten here still holds.

\[
(3.10) \quad \rho \left[ \frac{3 \xi}{3 \pi_{k}} \right]_{\pi_{k}} + \delta_{2} p_{k} = 0
\]

1.2) If lump sum transfers are not allowed, principle 3ii) takes the less intuitive form of formula (1.23) in corollary 32.

\[
\rho \left[ \frac{3 \xi}{3 \pi_{k}} \right] = \sum_{i} v_{i}(R_{i}) \frac{E_{i}(\pi_{k})}{E_{i}(R_{i})} - \delta_{2} p_{k}
\]

But given the utility maximizing behaviour of the consumers one knows that

\[
E_{i}(\pi_{k}) = -x_{ik}, \quad E_{i}(R_{i}) = 1. \quad \text{Hence}
\]

\[
(1) \quad \text{From now, we will drop the stars indicating that derivatives are taken at the optimum.}
\]

(2) since according to (1.17), \( E_{i}(\pi_{k}) = \pi \cdot \frac{3 \xi}{3 \pi_{k}}, \quad E_{i}(R_{i}) = \pi \cdot \frac{3 \xi}{3 R_{i}} \).
\[(\text{3.11}) \quad \rho \cdot \left[ \frac{\partial \xi}{\partial \pi_k} \right]^C = - \sum_i \nu_i(R_i) x_{ik} - \delta_2 p_k \]

with

\[(\text{3.12}) \quad \nu_i(R_i) = \left( \nu_i \cdot \pi - \rho \right) \left[ \frac{\partial \xi}{\partial \pi_i} \right] = \nu_i - \rho \left[ \frac{\partial \xi}{\partial \pi_i} \right] \]

\[2°) \text{ If } \pi_k \text{ and } p_k \text{ are disconnected around the optimum, principle } 3i11 \text{ applies to small compensated moves of } d\pi_k \text{ of the free variable } \pi_k.\]

\[2.1) \text{ In case of lump sum transfers,} \]

\[(\text{3.13}) \quad \rho \cdot \left[ \frac{\partial \xi}{\partial \pi_k} \right]^C = \theta \]

\[2.2) \text{ If lump sum transfers are ruled out:} \]

\[(\text{3.14}) \quad \rho \cdot \left[ \frac{\partial \xi}{\partial \pi_k} \right]^C = - \sum_i \nu_i(R_i) x_{ik} \]

where \(\nu_i(R_i)\) is defined as in (3.12)

- With respect to section II, nothing is changed to the local analysis of the sets \(S\) defining constraints on signals. Simplified versions of (2.14) and (2.15) still hold so that

\[\text{in the case where } \pi_k \text{ and } p_k \text{ are rigidly linked, summing (3.9), (3.11) and taking into account (2.14) gives} \]

\[(\text{3.15}) \quad \rho \cdot \left[ \left[ \frac{\partial \xi}{\partial \pi_k} \right]^C - \frac{\partial \eta_k}{\partial p_k} \right] = \beta \left( z_k - v_k \right) - \sum_i \nu_i(R_i) x_{ik} \quad (1) \]

a formula which reduces when lump sum transfers are allowed to

\[(\text{3.16}) \quad \rho \cdot \left[ \left[ \frac{\partial \xi}{\partial \pi_k} \right]^C - \frac{\partial \eta_k}{\partial p_k} \right] = \beta \left( z_k - v_k \right) \]

If one takes into account the symmetry of demand derivatives the left hand side of (3.15) and (3.16) becomes

\[\rho \cdot \left[ \left[ \frac{\partial \xi}{\partial \pi_k} \right]^C - \frac{\partial \eta_k}{\partial p_k} \right] \quad \text{So we obtain (3.15')} (3.16') \]

- in the case where \(\pi_k\) is a free variable, (3.9) together with (2.14) imply

\[(\text{3.17}) \quad \rho \cdot \left[ \frac{\partial \eta_k}{\partial p_k} \right] = \rho \left[ \frac{\partial \eta_k}{\partial p_k} \right] = \left( \rho - \rho \right) \left[ \frac{\partial \eta_k}{\partial p_k} \right] = \beta \left( z_k - v_k \right) \]

- Finally from (2.15) or as a consequence of the general principle 21:

\[(\text{3.18}) \quad \rho - \rho = \left( 1 + \beta \right) \left( \rho - \rho \right) \]

\[\---------------------------\]

\[(1) \text{ The reader will have noticed that, until now, the assumption that the consumer demand is purely competitive has been used only for deriving} \]

\[E_1(\pi_k), E_1(R_i). \text{ Hence, for a general demand function (3.15) holds with} \]

\[E_1(\pi_k) / E_1(R_i) \text{ instead of } -x_{ik}. \text{ However (3.15')} \text{ would not generally hold.}\]
III.B PRICING POLICIES WITH COMMODITY TAXATION AND LUMP SUM TRANSFERS.

Supposing that lump sum transfers can still be used by the government and that some commodities can be taxed or subsidized freely, we immediately draw from the preceding analysis the characterization results relevant for the discussion of pricing policies of the firm subject to a budget constraint.

\[(3.17) \ (p - p) \left[ - \frac{\partial \pi_k}{\partial p} \right] = \beta(z_k - v_k) \text{ if } \pi_k \text{ is a free variable} \]

\[(3.13) \ p \cdot \left( \frac{\partial \xi_k^c}{\partial \pi} \right) = 0 \text{ if } \pi_k \text{ is a free variable} \]

\[(3.16') \ p \left[ \frac{\partial \xi_k^c}{\partial \pi} - \frac{\partial \eta_k}{\partial p} \right] = \beta(z_k - v_k) \ \forall k \]

\[(3.18) \ \tilde{\rho} - p = \frac{1}{1 + \beta} (p - p) \]

Three remarks are in order.

1. The just written formulas display significant qualitative alterations when compared with the Boiteux formulas of section II. Taking into account the proportionality of shadow taxes of the Boiteux firm \((\tilde{\rho} - p)\) and of social taxes \((p - p)\) (recalled in (3.16)) we can rewrite (3.17) and (3.18) as

\[(3.19) \ (\tilde{\rho} - p) \left[ - \frac{\partial \eta_k}{\partial p} \right] = \frac{\beta}{1 + \beta} (z_k - v_k) \text{ if } k \text{ is taxed optimally} \]

\[(3.20) \ (\tilde{\rho} - p) \left[ \frac{\partial \xi_k^c}{\partial \pi} - \frac{\partial \eta_k}{\partial p} \right] = \frac{\beta}{1 + \beta} (z_k - v_k) + (\pi - p) \left( \frac{\partial \xi_k^c}{\partial \pi} \right) \]

An intensification of shadow taxation has two effects: (with \(v_k = 0, \forall k\))

- The competitive excess supply of any optimally taxed commodity is increased in proportion of minus the net production of the Boiteux firm.

- The compensated excess demand of any other commodity varies in proportion of the production of the Boiteux firm corrected by a term \(\frac{1 + \beta}{\beta} (\pi - p) \left( \frac{\partial \xi_k^c}{\partial \pi} \right) \).

This latter term reflects the effect of an "intensification" of real taxes on the compensated demand of the considered commodity.

2. The information needed by the firm for the verification of conditions (3.19), (3.20) (which loosely speaking determine pricing and "investment" rules) is weaker than in the Boiteux case.
In particular the knowledge of demand elasticities relative to commodities which are taxed optimally is certainly irrelevant to the decision of the firm. This point is more completely illustrated in section IV. We can here consider the following simple example. There are three commodities 0, 1, 2; the Boiteux firm produces commodity 2 from a single input commodity zero, the numeraire; commodity 1 is a substitute of commodity 2 produced from the numeraire by a competitive firm. Let us call $E_{ij}$ (resp. $N_{ij}$) the compensated demand derivative (resp. the supply derivative) of commodity $i$ with respect to the price of commodity $j$.

A simple calculation shows that in the case of section II, we obtain

$$\bar{p}_2 - p_2 = \frac{\beta}{1 + \beta} \gamma_2 \left[ \frac{1}{(E_{11}-N_{11}) E_{22}-E_{12}^2} \right]$$

And if the tax on commodity 1 is optimized

$$\bar{p}_2 - p_2 = \frac{\beta}{1 + \beta} \gamma_2 \frac{1}{E_{22}}$$

Clearly, less information is required if the firm is instructed to use (3.22) instead of (3.21). It should nevertheless be noted that the firm ignoring that the tax on commodity 1 has been optimized would find with (3.21) the result it would have found with (3.22). However, if the firm assumes wrongly that the tax on commodity 1 has been optimized it will find using (3.22) results which may be very different from correct ones.

3. It should be kept in mind that commodity taxation, when it can be used widely, does not only affect the form of the pricing rules but also in extreme case the nature of the problem under consideration. Consider for example a Boiteux firm which produces one final good on which it has a monopoly (commodity n). With (3.17) it comes out

$$\frac{\partial n}{\partial p} \bigg|_o = 0 = \beta \gamma_n \implies \beta = 0$$

Taking into account (3.13) and (3.16') we obtain $p_h = p_n \ (h \neq n)$ $p_n = \pi_h$, $\forall h$ : first best Pareto optimality is restored.

From an economic point of view, the result is not surprising: the firm takes advantage of its monopoly power to solve its budget problem and an
adequate subsidy suppress the distortion\(^{(1)}\).

However, if the Boiteux firm competes with private firms for selling its output, a second best problem still obtains even if all commodities can be taxed. Then, (3.13) holds whatever \( k \) and (if the matrix of compensated demand derivative is of rank \( n \) ) this implies \( p = \pi \) and (3.16') and (3.18) reduce to\(^{(2)}\).

\[(3.23) \quad (\pi - p) \frac{\delta n_k}{\delta p} = \beta (v_k - z_k) \]
\[(3.24) \quad (\tilde{\rho} - p) = \frac{1}{1 + \beta} (\pi - p) \]

The nature of the conclusions are significantly affected by the assumption of unconstrained commodity taxation: shadow taxes simply are a fraction of the (observable) real taxes. The determination of real taxes relies heavily on (3.23), the interpretation of which would refer to the effect of "intensification" of real taxes on excess supply.

\section{IIIC - Pricing Policy Without Lump Sum Transfers.}

\subsection{III.C1 Absence of commodity taxation}

We will consider two cases:

a) In the first one, the model of III.A is specified as follows: \( p = \pi \), \( R_i = R \), \( \forall i \). There is no taxation and the Government can implement a uniform lump sum transfer. The demand function reads \( \xi(p,R) \) and does not depend on the profit of private producers. Hence, we implicitly suppose that they are taxed by the Government. In that case, taking into account the classical properties of the jacobian of supply and demand, the characterization results of subsection III.A can be written again\(^{(3)}\).

\[(3.15') \quad (\rho - p) \left[ \left( \frac{\delta x_k}{\delta p} \right)_{p, \pi} - \frac{\delta n_k}{\delta p} \right] = \beta z_k - \sum_i \nu_i(R_i) x_{ik}, \forall k \]
\[(3.18) \quad \tilde{\rho} - p = \frac{1}{1 + \beta} (\rho - p) \]
\[(3.19) \quad \nu_i(R_i) = \nu_i - \rho \frac{\delta x_i}{\delta R_i} \]

\footnotesize{(1) For related argument with private monopolies see Guesnerie-Laffont (1976)}

\footnotesize{(2) These results can be found in Guesnerie (1975) or Bernard (1976). The property \( \pi = p \) is called CC-efficiency in Guesnerie (1979) where its range of validity is evaluated.}

\footnotesize{(3) For the sake of notational simplicity we suppose here that the budget constraint has the form \( p \cdot y > 0 \).}
(3.15") appears as a generalization of the simple Boiteux rule taking into account distribution problems:

An intensification of shadow social taxation \((\rho - \rho)\) leads to an increase of the compensated excess demand of any commodity \(k\) (but possibly the numeraire) equal to the sum of a term proportional to the net supply of the Boiteux firm plus the inner product of the vectors of net social marginal efficiency of income and of the consumptions of commodity \(k\).

The same holds with shadow taxation of the firm instead of social taxation because of (3.18).

Distributional considerations explicitly appear in pricing formulas through the additional term \(\sum \nu_i(R_i) x_{ik}\). With the terminology of M. Feldstein (1973) one could refer to it as the distributional characteristic of commodity \(k\). In order to take optimal decisions, the Boiteux firm needs to know distributional characteristics of the commodities that it produces and also of a certain range of others commodities (see section IV).

As we have supposed that a uniform lump sum transfer is implemented by the government \((R_i = R)\) and actually optimized, one has \(\sum \nu_i(R_i) = 0\).

Then, \(\sum \nu_i(R_i) x_{ik} = \sum \nu_i(R_i) (x_{ik} - \bar{x}_k)\) with \(\bar{x}_k = \frac{1}{m} \sum x_{ik}\). The distributional characteristic is then the covariance of the net social marginal efficiency of income and of the consumption of commodity \(k\).

Its sign cannot be unambiguously predicted, in general. The simple case of a two consumers economy illustrates the theoretical indeterminacy(1).

\[\text{(1) In this case } \nu_1(R) = - \nu_2(R).\]

Call \(\rho_k - \rho_k = t_k\) and multiply (3.15") by \(t_k\) and sum over \(k\).

The matrix of compensated excess demand being negative semi definite one has: \(\beta \left( \sum t_k y_{12k} \right) + \nu_1(R) \left( \left( \sum t_k x_{2,k} \right) - \left( \sum t_k x_{1,k} \right) \right) \leq 0\)

But \(\sum t_k y_{12k} = (1+\beta) \left( \sum \rho_k y_{12k} - \sum p_k y_{12k} \right)\) is positive if the production set of the Boiteux firm is convex and contains the origin. In that case, \(\nu_1(R)\) is negative if \(\sum t_k x_{2,k} - \sum t_k x_{1,k} > 0\).

This conclusion is formally similar to that obtained by Mirrlees (1975), studying commodity taxation in a two class economy. But contrary to what happens in this latter paper, we cannot here on economic grounds predict the sign of this expression.
However, in some problems approximate methods of evaluation may be thought as reasonably accurate for operational purposes \(1\).

b) For the second case where still \(p = \pi\), we rule out the poll tax policy and suppose that the profit of the private firm is entirely distributed between consumers according to the keys \(\rho_i\). Then the demand of consumer \(i\) is \(\xi_i(p, \rho_i, p \cdot \pi(p)) \overset{\text{def}}{=} \xi_i(p)\). Strictly speaking the model corresponding to this assumption is not a particular case of the model of section IV.A. However, from footnote p. 20, we know that a modified version of (3.15) holds with \(E_{ik}\) instead of \(-x_{ik}\) and \(\frac{\partial E_i}{\partial p_k}\) instead of \(\frac{\partial \xi_i}{\partial p_k}\) where

\[
E_{ik} = -x_{ik} + \rho_i y_{1k}\quad \text{and}\quad \frac{\partial E_i}{\partial p_k} = \sum_i \frac{\partial \xi_i}{\partial p_k} \quad \text{where according to (1.21)}
\]

\[
\frac{\partial \xi_i}{\partial p_k} = \frac{\partial \xi_i}{\partial p_k} + (x_{ik} - \rho_i y_{1k}) \frac{\partial \xi_i}{\partial p_k} = \frac{\partial \xi_i}{\partial p_k} - \rho_i y_{1k} \frac{\partial \xi_i}{\partial p_k}
\]

(3.25) 

\[
\beta z_k - \sum_i \nu_i(R_i) \left( x_{ik} - \rho_i y_{1k} \right) - \left[ \frac{\partial \xi_i}{\partial p_k} \right] y_{1k}
\]

Furthermore \(\nu_i(R_i)\) is an in (3.12) and (3.18) still hold.

Compared with (3.15'), (3.25) takes into account the distributional effects of the price change of commodity \(k\) induced by the change in profit:

\[x_{ik} - \rho_i y_{1k}\] replaces \(x_{ik}\) in a fairly intuitive way. However, the effect of an intensification of shadow social taxation on excess demand of commodity \(k\) involves in the right hand side of (3.25) a less intuitive additional term.

\[\text{---------------------------}\]

\[1\) Write \(\nu_i(R_i) = \mu_i - \rho \frac{\partial \xi_i}{\partial R_i} = \mu_i - 1 + (\rho - \pi) \frac{\partial \xi_i}{\partial R_i}\).

If one ignores the last term (which may be more or less satisfactory according to the problem under consideration [see section IV]), \(\nu_i(R_i) = \mu_i - 1\).

Assumptions on the distribution of the marginal utilities of income allow to give an operational measure of \(\nu_i\) (see M. Feldstein (1973)).
III.C2 Commodity taxation, no lump sum transfer

First, we suppose here that $\pi$ and $p$ can be disconnected and that $R_i = 0$. Consumer $i$ has no other income than his labour income and his demand function $\xi_i(\pi, 0)$ still does not depend on the profit of private producers, which are 100% taxed by the Government.

From subsection III.A we know that the following characterization results obtain:

\[(3.15) \quad \rho \left[ 1 - \frac{\partial \xi_k^C}{\partial \pi} + \frac{\partial \eta_k}{\partial p} \right] = \beta z_k - \sum_i v_i(R_i) x_{ik}, \quad \forall k \neq 0\]

Furthermore if the consumption price of commodity $k$ can be chosen freely

\[(3.14') \quad (\rho - \pi) \left[ \frac{\partial \xi_k^C}{\partial \pi} \right] = - \sum_i v_i(R_i) x_{ik}\]

\[(3.17) \quad (\rho - p) \left[ \frac{\partial \eta_k}{\partial p} \right] = \beta z_k\]

Finally

\[(3.18) \quad (\tilde{\rho} - p) = \frac{1}{1 + \beta} (\rho - p)\]

Several observations are in order

1) The modifications brought into the characterization formulas when compared to the formulas of subsection III.C1 are similar to the modifications introduced when passing from the Boiteux model of section II to the model with commodity taxation and lump sum transfers of subsection III.B. As in subsection III.B we can rewrite (3.15) (3.17), taking into account (3.18)

\[(\tilde{\rho} - p) \left[ \frac{\partial \xi_k^C}{\partial \pi} - \frac{\partial \eta_k}{\partial p} \right] = \frac{\beta}{1 + \beta} z_k, \quad \text{if } k \text{ is optimally taxed}\]

\[(\tilde{\rho} - p) \left[ \frac{\partial \xi_k^C}{\partial \pi} - \frac{\partial \eta_k}{\partial p} \right] = \frac{\beta}{1 + \beta} z_k - \sum_i v_i(R_i) x_{ik} + (\pi - p) \left[ \frac{\partial \xi_k^C}{\partial \pi} \right] \]

The consequence of a small increase of shadow taxes in proportion with their optimal values, on the compensated excess demand of a non optimally taxed commodity, depends on three terms: the supply of the Boiteux firm, the covariance net social marginal efficiency of income and consumption and the additional term $(\pi - p) \left[ \frac{\partial \xi_k^C}{\partial \pi} \right]$. The rule simplifies greatly when the tax on commodity $k$ is optimal.
ii) Formulas (3.14') are loosely speaking taxation formulas. As in (3.15") the sign of $\sum_i v_i(R_i) x_{ik}$ is a priori indeterminate. If the model is modified in order to consider uniform lump sum transfers $\sum_i v_i(R_i) = 0$ so that the right hand side of (3.14') is the covariance of the net social marginal efficiency of income transfers and of consumption. (3.14') is a variant of the Ramsey many person tax rule of Diamond (1975)\(^{(1)}\) a rule generalized here to the case where the vector of social value of commodities $p$ does not coincide with the production price vector $\tilde{p}$.

iii) If full taxation is allowed, phenomena paralleling those mentioned in subsection III. appear. Particularly if the Boiteux firm has the monopoly of production of some final good (n) the reader will prove that $p_h = \bar{p}_h \quad \forall \ h \neq n$. Production efficiency holds and the optimum is similar to what would obtain in a Diamond-Mirrlees economy with a production sector entirely competitive. If however the firm subject to budget constraint has no monopoly, pricing problems and taxation problems are still in some sense separated. No explicit redistributional considerations appear in pricing formula which reduce to

\[
(3.26) \quad (p - \tilde{p}) \frac{\partial \tilde{\pi}_k}{\partial p} = \frac{\beta}{1 + \beta} z_k
\]

III-D: FURTHER INSIGHTS INTO THE MODEL.

In this subsection we will give brief further insights into the model. First we will analyse the consequences of the introduction of a more sophisticated policy tool as a non linear income tax. Second we will consider the case where the firm subject to a budget constraint produces a pure Samuelsonian public good.

III-D1. Suppose that the Government can implement, besides other policy tools, a non linear income tax : given a pre tax income $R$, the after tax income is $\phi(R)$ where $\phi$ is a function depending possibly on several parameters, which we consider here as fixed. What are the consequences of the introduction of this new policy tool on the pricing formulas obtained

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(1) See Atkinson-Stiglitz (1979). Deaton (1979) studies the solutions of equations of type (3.14'). Note that in this model coexist a formula of "Ramsey type" (3.14') and a formula of "Boiteux type" (3.17).
in subsection III.C? To answer this question we will consider an economy where
there exists only one type of labour, which will be numbered as commodity
zero; the different endowments of agents in labour possibly reflect diffe-

erences in skill\(^{(1)}\); as in IV.C1a and IV.C2 agents have no other income than
their labour income.

The formulation of our problem is only affected by this new

assumption through the definition of the consumer demand function: faced

with the set of consumption prices \( \pi = (1, \tilde{\pi}) \) and with an exogeneous income

\( R \), consumer \( i \) chooses a consumption bundle \( \bar{x}_i(\pi, R) \) solution of

\( \bar{\pi} \cdot \bar{x}_i \leq \phi(-x_{10}) + R \) where \(-x_{10}\) denotes his labour supply and \( \bar{x}_i \) his

consumption bundle truncated to commodities 1 to \( n \). As we noted in

footnote p 20, the analysis of the characteristics of second best Pareto

optimality with this demand function (which formally differs from the

competitive one \( \xi_i \) ), only slightly differs from the analysis of section III.A

and instead of (3.15) we have

\[
(3.27) \quad \rho \left[ \left( \frac{\partial x}{\partial \pi} \right)^{C/R_1} - \frac{\partial \pi}{\partial p_k} \right] = \beta(y_{i2k} - v_k) - \sum_i v_i(R_i) \frac{E_i(\pi_k)}{E_i(R_i)}
\]

But \( v_i(R_i) \), \( E_i(\pi_k) \), \( \left( \frac{\partial x}{\partial \pi} \right)^{C/R_1} \) have to be computed from formulas

(1.22), (1.17), (1.21) taking into account that the vector of marginal

erates of substitution between labour and commodities does not coïncide

with the consumption price vector. Here, the normalized shadow price faced

by consumer \( i \) is \( \psi_i = (1, ..., \frac{\pi_k}{\phi}, ...) \) where \( \phi \) designates the derivative

of \( \phi \) (around the optimal consumption bundle).

It is easy to see\(^{(2)}\) that \( E_i(\pi_k) = -\frac{x_{ik}}{\phi}\), \( k = 1, ..., n \).

\-----------

(1) Different preferences may reflect both differences in skill and diffe-

rences in tastes.

(2) \( \tilde{x} \) denoting truncated vectors in \( \mathbb{R}^n \), one has \( \tilde{\pi} \cdot \tilde{x}(\tilde{\pi}, R) - \phi(-x_{10}(\tilde{\pi}, R)) \equiv R \).

Deriving with respect to \( \pi_k \) and \( R \), gives \( \phi' \psi \frac{\partial x}{\partial \pi_k} + x_k = 0 \),

\( \phi' \psi \frac{\partial x}{\partial R} = 1 \) .
Lowering of one unit the consumption price of commodity \( x_{ik} \) is worth \( \frac{x_{ik}}{\phi} \) units of numeraire, one more unit of income is worth \( \frac{1}{\phi} \) units of numeraire. Hence \( \frac{E_1(R_i)}{E_i(p_k)} = -x_{ik} \) is unchanged compared with the previous section. Still, \( \left( \frac{\partial x_{ih}}{\partial \pi_k} \right) U = C/R_i = \frac{\partial x_{ih}}{\partial \pi_k} + x_{ik} \frac{\partial x_{ih}}{\partial R_i} \) and for \( h, k=1, \ldots, n \), the nxn associated square matrix is not longer necessarily symmetric (1). Also, the net social marginal efficiency of income \( v_1(R_i) \) previously equal to \( \mu_i - \rho \left( \frac{\partial E_1}{\partial R_i} \right) \) takes now the expression of

\[
(3.28) \quad v_1(R_i) = \frac{\mu_i}{\phi_i} - \rho \frac{\partial E_i}{\partial R_i}
\]

Taking into account \( \psi_i \left( \frac{\partial x_{i1}}{\partial \alpha} \right) U = C/\beta \), (3.27) can be written down

\[
(3.30) \quad \sum_{i} (\rho - \psi_i) \left( \frac{\partial x_{ih}}{\partial \pi_k} \right) - (\rho - \rho) \frac{\partial n}{\partial \rho_k} = \beta (y_{12k} - v_k) - \sum_{i} v_1(R_i) x_{ik}
\]

Now we let to the reader to obtain the full relevant set of results corresponding to the combination of the assumption of existence of a non linear income tax with any set of assumptions considered in subsection III.C. Two qualitative modifications of the pricing rule deserve emphasis. First, the interpretation of the left hand side of equations of type (3.30) as the effect of an intensification of some shadow tax system, which is the essence of the so called Ramsey-Boiteux conditions, becomes increasingly less natural. On the one hand, the demand compensated derivatives

---

(1) Take the identity \( x_{ih}(\pi, R) = \xi_{ih}(\pi, \phi, R + \phi', \chi_{i10}(\pi, R)) + \phi(\chi_{i10}(\pi, R)) \)

It follows from the derivation that \( \left( \frac{\partial x_{ih}}{\partial \pi_k} \right) U = C/R_i = \left( \frac{\partial E_1}{\partial R_i} \right) C \) for \( h = 0, \ldots, n \), \( k = 1, \ldots, n \).
which have to be taken into account are no longer necessarily symmetric (but with strong separability assumptions and with one type of labor).

On the other hand, the shadow price system implicit to consumer's decision differs from one consumer to another and the left hand side of (3.30) involves a complete set of personalized shadow taxes $(\rho - \psi_1)$.

Second, the social value of income which serves to the computation of the net social marginal efficiency of income has to be corrected in order to take into account the relative distortions in the arbitrage consumption-leisure introduced by the non linearity of the income tax schedule. Strong progressivity which implies the existence of significant heterogeneity of marginal rates of substitution across consumers might have practical incidence on pricing formulas.

III-D2. Until now, we suppose that the firm astrained to a budgetary constraint was only involved in the production of private goods. What happens if it produces one public good?

In that case, pricing problems strictly speaking disappears and the concern goes to the production decisions of the firms (inputs and public goods levels). It is of particular interest to know how the Samuelsonian rule are modified.

We will give here an answer which relies on a formula giving the social value of one public good in the general model of section I. This formula, obtained in Guesnerie (1979) from a reinterpretation of Principle 3 in an economy with public goods (corollaries 1 and 2) writes down \(^{(1)}\):

\[
(3.31) \quad \rho_L = \sum_i \psi_{i,q} + \sum_i \psi_i(q) \frac{\psi_{i,q}}{E_i(\beta_i)} + \sum_i (\psi_i - \rho) \left[ \frac{\delta x_i}{\delta q} \right] U^{C/\beta_i}
\]

where $\rho_L$ is the social value of the public good, $\rho$ the normalized vector of social value of private goods, $\psi_{i,q}$ i's marginal willingness to pay (in terms of numeraire) for the public good, $\psi_i$ the normalized

\(^{(1)}\)Actually this formula requires that the consumers be CSI regular deviant with the terminology of Guesnerie (1979) a condition which is always trivially satisfied in the following.
vector of marginal rates of substitution of consumer $i$, $v_i(\beta_i), E_i(\beta_i)$, the net marginal efficiencies and individual surplus associated with variable $\beta$ and \( \left( \frac{\partial x_i}{\partial q} \right)^{U=C/\beta} \defeq \frac{\partial x_i}{\partial q} = \frac{\psi_i(q)}{E_i(\beta)} \frac{\partial x_i}{\partial R} \), a compensated effect of the change of the public good level on demand.

Taking $\beta_1 = R_1$, this formula applies to all models of subsection III.B, III.C where furthermore $\psi_1 = \pi$. But we are interested in the shadow value $\tilde{\rho}_L$ of the public good for the Boiteux firm; we can notice that the separation argument of page 6 applies whatever the nature of the commodities under consideration. Principle 21 here implies:

\[ (3.32) \quad \tilde{\rho}_L = \frac{1}{1 + \beta} \rho_L \]

\[ (3.18) \quad \tilde{\rho} - p = \frac{1}{1 + \beta} (\rho - p) \]

Hence the shadow price to be given to the Boiteux firm is:

- in the Boiteux model of section II

\[ (3.32) \quad \tilde{\rho}_L = \frac{1}{1 + \beta} \left( \sum_i \psi_{i,q} \right) - (\tilde{\rho} - p) \left( \frac{\partial x_i}{\partial q} \right)^{U=C/R_1} \]

- in the model of subsection III.B, in case of full taxation

\[ (3.33) \quad \tilde{\rho}_L = \frac{1}{1 + \beta} \left( \sum_i \psi_{i,q} \right) \]

- in the model of subsection III.C1 a)

\[ (3.34) \quad \tilde{\rho}_L = \frac{1}{1 + \beta} \left[ \sum_i \psi_{i,q} - \sum_i v_i(R_1) \psi_{i,q} + \sum_i (\rho - p) \left( \frac{\partial x_i}{\partial q} \right)^{U=C/R_1} \right] \]

- in the model of subsection III.C2, same formula with $\pi$ instead of $\rho$.

Formulas giving social values of private commodities and shadow prices of inputs for the Boiteux firm remain similar to those previously derived.

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(1) An analogous formula is derived in Drèze-Marchand (1975) (formula (2.13)).
IV. PRICING AND PRODUCTION DECISIONS IN THE BOITEUX FIRM UNDER ALTERNATIVE POLICY ASSUMPTIONS: AN EXAMPLE.

This section illustrates the abstract formulas precedentely derived for the computation of the shadow prices of the firm subject to a Budget constraint. The economy considered is simple enough to allow analytical derivations. But it is complex enough in order that the basic interactions which have to be taken into account in the pricing decision, be reflected in the formulas obtained.

The economy has six commodities:

Commodity zero is the numeraire which can be interpreted as labour.
Commodity one is a final consumption good produced by a private firm from two inputs, goods 0 and 4. Commodity 2 is a final good, also produced by a private firm from two inputs, goods 0 and 3. Commodity 3 is an intermediate good serving to the production of commodities 2 and 5 and produced by a private firm. (We can, for example, think of commodity 3 as a capital good.)
Commodity 4 is an intermediate good produced from labour by the public firm subject to the budget constraint. As already said, it serves to the production of commodity 1. Finally commodity 5, also produced by the Boiteux firm, is a final good, the production of which requires two inputs, labour and good 3.

We suppose that commodity 1 has negligible cross elasticities with respect to commodities 2 and 5, but commodities 2 and 5 are together close substitutes.

Although the picture remains simple, the linkages between the Boiteux firm and the rest of the economy are not oversimplified. The Boiteux firm produces for final as well as for intermediate consumption, it competes with
the private sector for the use of some input, (different from the numeraire) and it produces a final commodity which has a close substitute produced by the private sector.

Taking into account our assumptions and the general formulas of the preceding sections, we know that the shadow prices of the Boiteux firm can be derived through formulas of the following type.

\[
\begin{align*}
(\bar{p}_1 - p_1)A_{11} + (\bar{p}_4 - p_4)A_{14} &= B_1 \\
(\bar{p}_2 - p_2)A_{22} + (\bar{p}_3 - p_3)A_{23} + (\bar{p}_5 - p_5)A_{25} &= B_2 \\
(\bar{p}_2 - p_2)A_{32} + (\bar{p}_3 - p_3)A_{33} &= B_3 \\
(\bar{p}_1 - p_1)A_{41} + (\bar{p}_4 - p_4)A_{44} &= B_4 \\
(\bar{p}_2 - p_2)A_{52} + (\bar{p}_5 - p_5)A_{55} &= B_5
\end{align*}
\]

We put \( a_{ij} = \frac{p_i}{y_j} A_{ij} \) and \( b_i = \frac{B_i}{y_i} \) where by definition \( y_i \) will be a positive number equal to the total production of commodity \( i \) by the firm producing it. (This number is different from the total net production of commodity \( i \) for \( i = 3, 4 \).) After routine computations we obtain as solutions of the system (1) to (5),

\[
\begin{align*}
\frac{\bar{p}_5 - p_5}{p_5} &= \frac{b_5}{a_{55}} \\
\left[ 1 + \frac{b_2}{b_5} \frac{a_{33}a_{52}}{a_{23}a_{32} - a_{33}a_{22}} - \frac{b_3}{b_5} \frac{a_{52}a_{33}}{a_{23}a_{32} - a_{33}a_{22}} \right] \\
&= \frac{1 + \frac{a_{25}a_{52}a_{33}}{(a_{23}a_{32} - a_{33}a_{22})a_{55}}}{1 - \frac{b_1}{b_4} \frac{a_{14}a_{41}}{a_{44}a_{22}}}
\end{align*}
\]

\[
\begin{align*}
\frac{\bar{p}_4 - p_4}{p_4} &= \frac{b_4}{a_{44}} \\
\left[ 1 - \frac{b_1}{b_4} \frac{a_{14}a_{41}}{a_{44}a_{22}} \right]
\end{align*}
\]
These formulas can now be specified for the different models studied in the preceding sections. Let us first consider the model of section II. We put \( y_3' = \lambda_3 y_3 \) where \( y_3' \) is the positive number measuring the quantity of commodity 3 used as input by the Boiteux firm. We designate by

\[
\varepsilon_{ij} = \frac{p_i}{y_i} \left( \frac{\partial \varepsilon_i}{\partial p_j} \right) \text{C}
\]

the elasticities of compensated demand and by \( \eta_{ij} = \frac{p_i}{y_i} \frac{\partial \eta_i}{\partial p_j} \); for \( i \neq 3 \), \( \eta_{ij} \) coincides with the actual supply elasticities of the firm producing commodity \( i \), but for \( i = 3 \), the private supply of commodity 3 is the sum of the net supply of firms 2 and 3 (\( \eta_{32} = \eta_3^3 + \eta_3^2 \)) so that \( \eta_{32} = (1 - \lambda_3) \eta_{32}^2 \) and \( \eta_{33} = \eta_{33}^3 + (1 - \lambda_3) \eta_{33}^2 \) (with straightforward notation).

Reminding these facts we have

\[
\begin{align*}
    a_{11} &= -\varepsilon_{11} - \eta_{11}, \
    a_{14} &= -\eta_{14}, \
    a_{22} &= -\varepsilon_{22} - \eta_{22}, \
    a_{23} &= -\eta_{23}, \
    a_{25} &= -\varepsilon_{25}, \
    a_{32} &= -\eta_{32}, \
    a_{33} &= -\eta_{33}, \
    a_{41} &= -\eta_{41}, \
    a_{44} &= -\eta_{44}, \
    a_{52} &= -\varepsilon_{52}, \
    a_{55} &= -\varepsilon_{55}, \
    b_4 &= \frac{\beta}{1 + \beta}, \
    b_5 &= \frac{\beta}{1 + \beta}, \
    b_3 &= -\frac{\beta}{1 + \beta} \lambda_3.
\end{align*}
\]

It comes out,

\[
\frac{p_5 - \tilde{p}_5}{p_5} = \frac{\beta}{1 + \beta} \frac{1}{\varepsilon_{55}} \left[ 1 + \lambda_3 \frac{\varepsilon_{52} \eta_{32}}{\eta_{23} \eta_{32} - \eta_{33} (\varepsilon_{22} + \eta_{22})} \right] - \frac{1}{\varepsilon_{55}} \left[ \frac{1 + \lambda_3 \frac{\varepsilon_{52} \eta_{32}}{\eta_{23} \eta_{32} - \eta_{33} (\varepsilon_{22} + \eta_{22})}}{1 + \eta_{33} \varepsilon_{25} \varepsilon_{52}} - \frac{1}{\varepsilon_{55} (\eta_{23} \eta_{32} - \eta_{33} (\varepsilon_{22} + \eta_{22}))} \right]
\]
These formulas illustrate the following facts:

- The decision of the Boiteux firm does not only concern pricing, the rules of which are associated with formulas (9) and (10) but also the input levels (or "investment" decisions) the determination of which relies as formula (11).

- Compared with the simple formula which would obtain for the ratio \( \frac{p_5 - \rho_5}{p_5} \) if cross elasticities between commodities 2 and 5 were zero (the ratio would then be proportional to the inverse of the compensated elasticity \( \varepsilon_{55} \)) formula (9) introduces corrections which have no reason to be small or negligible. Analogous remarks apply to formulas (10) and (11).

- The computations to be made by the Boiteux firm require information on the demand elasticities of the consumption goods which are in direct competition with its own products on the final markets (\( \varepsilon_{22}, \varepsilon_{25} \)), on the demand elasticities of the consumption goods produced by firms which use a good produced by the firm as an input (\( \varepsilon_{11} \)), on the supply elasticities of the sectors providing its inputs (\( \eta_{33}, \eta_{32} \)) on the supply and possibly demand elasticities of the sector using the same inputs (\( \eta_{32} \)). It is left to the reader to characterize in a more general way the indirect connection between goods which assure...
that elasticities relative to a given good have to be taken into account by the Boiteux firm.

It is fair to mention now that (9), (10), (11) may significantly simplify when any or all of the three first commodities are produced with simple constant returns to scale and fixed factors coefficients techniques.

Suppose for example that the production of one unit of commodity 2 requires \( k_{20} \) units of labour and \( k_{23} \) units of commodity 3. According to principle 2, \( \rho_2 \) the social value of commodity 2 satisfies

\[
\rho_2 = k_{20} + k_{23} \rho_3 \quad \text{so that} \quad \rho_2 - p_2 = k_{23} (\rho_3 - p_3)
\]

(12)

In the equations (1) to (5) determining \( \bar{\rho} \), taking into account principle 2 and principle 3 (corollary 32), (2) and (3) have to be replaced by

\[
(\bar{\rho}_3 - p_3) A'_{33} + k_{23} \left[ (\bar{\rho}_2 - p_2) A'_{22} + (\bar{\rho}_5 - p_5) A_{25} \right] = B_3 + k_{23} B_2
\]

(2')

Taking into account (12) and combining with (5) it comes out

\[
\bar{\rho}_5 - p_5 = \frac{B_3 \left[ k_{23} A'_{22} + \frac{A'_{33}}{k_{23}} \right] - B_3 A_{52} - k_{23} B A_{52}}{-k_{23} A_{52} + A_{55} \left[ k_{23} A'_{22} + \frac{A'_{33}}{k_{23}} \right]}
\]

(13)*

* One can check that this expression is actually the limit of the corresponding expression drawn from the original equations (1) to (5) by noticing that when the technique "tends" to a Leontieff technique \( A'_{33}/A_{32} \rightarrow -k_{23} \) etc ... However the computation is rather tedious because both numerator and denominator tend to zero.
Finally in the Boiteux model we obtain

\[
\frac{p_5 - \tilde{p}_5}{p_5} = \frac{\beta}{1 + \beta} \frac{1}{\epsilon_{55}} \left[ 1 + \frac{\lambda \epsilon_{52}}{\eta_3 (k_{23})^{-1} \frac{p_2}{p_3} - k_{23} \epsilon_{22} \frac{y_2}{y_3}} \right] \tag{14}
\]

If both commodities 2 and 3 were produced with technologies of Leontief type, it immediate that \( p_2 = p_2 \) and \( p_3 = p_3 \).

From equation (5), taking into account corollary 21 (of principle 2) we have

\[
\frac{p_5 - \tilde{p}_5}{p_5} = \frac{\beta}{1 + \beta} \frac{1}{\epsilon_{55}} \tag{15}
\]

(15) is clearly the limit of (14) when \( \eta_{33} \to \infty \).

Consider now the case where commodity one is produced with a simple Leontief technology. As above one can argue that

\[
\frac{\rho_1 - p_1}{p_1} = k_{14} (p_4 - p_4)
\]

\[
(p_1 - p_1) \left( \frac{\partial \xi_1}{\partial \rho_1} \right)^c \left( k_{14} \right)^{1} = \beta y_4
\]

and finally

\[
\frac{\rho_4 - p_4}{p_4} = - \frac{\beta}{1 + \beta} \frac{1}{k_{14} \epsilon_{11}} \frac{p_1 y_4}{p_4 y_1} \tag{16}
\]

Still we can check that this expression obtains as a limit to (10)

We will now give up the model of section III and will suppose that as in section III.B, commodity 2 can be optimally taxed.

According to formulas (3.16'), (3.17), (3.18) we have now to introduce the following modifications in formula (6) : \( a_{22} = \eta_{22} \), \( a_{25} = 0 \).

It comes out
\[ \frac{p_5 - \tilde{p}_5}{p_5} = \frac{\beta}{1 + \beta \varepsilon_5} \left[ 1 + \frac{\lambda_3 \varepsilon_{52} \eta_{32}}{\eta_{32} \eta_{33} - \eta_{23} \eta_{32}} \right] \] (17)

For the application of (14) which is much simpler than (9), \( \varepsilon_{22} \) can be ignored but \( \varepsilon_{52} \) has to be known. However, we must notice that

\[ \varepsilon_{52} = \frac{\rho_2}{\lambda_5} \left( \frac{\delta \varepsilon_5}{\delta \lambda_5} \right) \text{C} \]

now differs from the exact cross elasticity of demand by a factor of one plus the tax rate on commodity 2.

If commodity 1 is optimally taxed, it is also easy to see that (10) has to be replaced by

\[ \frac{p_4 - \tilde{p}_4}{p_4} = \frac{\beta}{1 + \beta} \frac{1}{\eta_{44}} \left[ \frac{1}{1 - \eta_{44} \eta_{41}} \right] \] (18)

Still remaining in the case where lump sum transfers are allowed and supposing that commodities 2 and 5 are taxed but not necessarily optimally taxed, we note that with respect to the Boiteux model, the coefficients \( B_i \) have to be modified as follows:

\[ B_2 = (\pi - p) \left( \frac{\delta \varepsilon_2}{\delta \pi} \right) \text{def} d_2 y_2 \]
\[ B_5 = \frac{\beta}{1 + \beta} y_5 + (\pi - p) \left( \frac{\delta \varepsilon_5}{\delta \pi} \right) \text{def} \frac{\beta}{1 + \beta} y_5 + d_5 y_5 \]

so that

\[ \frac{b_2}{b_5} = \frac{d_2}{d_5} = \frac{1}{1 + \beta} \]

Then, \( \frac{p_5 - \tilde{p}_5}{p_5} \) obtains from a modified version of formula (9) where

\[ \frac{b_2}{b_5} \left[ \frac{\eta_{33} \varepsilon_{52}}{\eta_{32} \eta_{33} - \eta_{23} \eta_{32}} \right] \] would have been added to the numerator and where the \( \varepsilon_{ij} \) would be cross elasticities multiplied by one plus the tax rate.

An additional relationship between \( B_2 \) and \( B_5 \) would still have to be considered if for example a unique tax rate but optimal for the aggregate were applied to commodities 2 and 5.
Let us now come to the cases where the assumptions of lump sum transfers are given up and let us consider precisely the model of subsection IIIC.1. The coefficients of formulas (6) to (8), precedentely introduced for the model of section III have to be modified as follows.

\[ b_1 = \sum v_i(R_i) \frac{x_{11}}{y_1}, \quad b_2 = \sum v_i(R_i) \frac{x_{12}}{y_2}, \quad b_5 = \frac{\beta}{1+\beta} + \sum v_i(R_i) \frac{x_{15}}{y_5} \]

All other coefficients are as in the Boiteux model.

One obtains for example from (6)

\[ \frac{p_5 - \bar{p}_5}{p_5} = \frac{b_5}{\varepsilon_{55}} \left[ 1 + \frac{b_2}{b_5} \frac{\varepsilon_{33} n_{52}}{n_{23} n_{32} - n_{33} (\varepsilon_{22} + \varepsilon_{22})} - \frac{b_3}{b_5} \frac{\varepsilon_{52} n_{32}}{n_{23} n_{32} - n_{33} (\varepsilon_{22} + \varepsilon_{22})} \right] \]

The redistributive considerations are reflected through the values of \( b_2, b_5, b_3 \) which have just been written down. These coefficients (the distributional characteristics) affect in a rather complex way the pricing rules (even for commodity 4).

Introduce now taxes in the latter model as done in subsection III.C.2. If taxes are non zero for commodities 2 and 5, although not separately optimal, terms analogous to \( d_2 \) and \( d_5 \) defined above would have to be added to \( b_2 \) and \( b_5 \). If commodity 2 were taxed optimally, \( b_2 \) would again be zero and supposing commodity 5 untaxed, the numerator of formula (9) would become

\[ 1 + \frac{1+\beta}{\beta} \left( \sum v_i(R_i) \frac{x_{15}}{y_5} \right) \left[ 1 + \frac{\lambda}{n_{23} n_{32} - n_{33} n_{22}} \right] \frac{\varepsilon_{52} n_{32}}{\varepsilon_{55}} \]

It is rather a mechanical exercise to go on exhibiting the optimal pricing formulas under the other assumptions which have been considered in
the paper. It is also left to the reader to see what are the limit of the formulas already obtained when simple Leontief techniques are used in sectors 1, 2, 3. The exercise illustrates the qualitative changes associated with changes in policy assumptions and stresses then the importance of a careful coordination between pricing policies and general economic policies.

A last question will now be briefly examined: until now we have not tried to take into account the fact that the sector of firms between which the interactions are not negligible, may be itself small with respect to the rest of the economy. Such a fact may crucial for deriving operational rules, and to discuss it we will suppose that instead of commodity zero we have \( n+1 \) commodities numbered \( 0, -1, \ldots, -n \). Suppose that this subsystem is closed in the sense that each such commodity is produced from the numeraire (0) and the other commodities with negative index with Leontief technologies. It results from principle 2 that the social values of commodities (with negative indices) equal their production prices (here their labour values). As argued by M. Feldstein (1973) it is likely that income effects be small and if consumption prices equal production prices \( v_\downarrow(R_\downarrow) = \mu_\downarrow - 1(1) \). However, it should be stressed that this latter simplifications strongly relies on the equality of consumption and production prices for most commodities \(-1, \ldots, -n\), and may not hold (even approximately) if taxes are levied (whether optimal or not). In that case, only the fact that the sign of the bias between \( v_\downarrow(R_\downarrow) \) and \( \mu_\downarrow - 1 \) is often not known can justify the simplified operational prescription just mentioned.

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(1) M. Feldstein (1973) proposes an operational measure of this coefficient under reasonable assumptions on income distribution and social welfare.
CONCLUSION.

We have adopted here a full general equilibrium approach with the goal of obtaining an accurate theoretical understanding of the economic interdependencies which should be taken into account in the design of optimal pricing rules. Such interdependencies are often partly ignored either by partial equilibrium derivations of normative pricing rules based on more or less sophisticated versions of the consumer surplus, or by general equilibrium approaches with operational concern which rely on simplifications only acceptable in special problems.

A method of derivation of results which goes from the general to the particular, contrasting the dominant tendency in the optimal taxation literature, has been presented. According to this option second best analysis is not only characterized by the methods it uses (the mathematics of optimization) but some general principles of wide applicability can be identified. These general principles provide a basic frame around which the argument can be developed with a minimum of calculation. It is hoped that the reader will have been convinced that such an approach has some merits: It allowed us to stress (with principle 21) that search of the shadow prices of the Boiteux firm was not different in nature from the search of social values of commodities, (a central concern in any second best problem). Also, the derivation and understanding of results associated with many variant has been made efficient and intuitive.

The comparative analysis of second best rules has emphasized their different informational requirements and their sensitivity to the general policy assumptions which are made. This is not surprising and only reflects the fact that the effect of a change in public prices in the economy does
not only depend on this change but also on the accompanying changes of other policy tools. This confirms the inadequacy of piecemeal policies, a point often analyzed in different contexts (see Boadway-Harris (1977)). Without being too much pessimistic, since many potential difficulties are actually insignificant in specific contexts, one should however, mention the great care which should be taken in the design of operational rules adapted to practical problems.

Actually, if this study may have remained too much academic it is not mainly because we were unconcerned with operational rules. But rather because the way we have discussed the coordination of taxation and pricing policies is too much simplistic: In practice, there exist only a few numbers of differentiated tax rates associated with a few numbers of groups of commodities. Taxation concern aggregate commodities when pricing policies are concerned with elementary goods. This feature which has been ignored in this paper, should be incorporated in further studies on the coordination of pricing and taxation policies.
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