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SECOND BEST TAXATION AS A GAME

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I - INTRODUCTION

Second best theory usually lies in the realm of normative analysis. It focuses attention on the "*best*" solutions which still can be obtained in problems where constraints of various natures forbid the attainement of first best Pareto optimal situations. Its central concern is the characterization of such "*second best*" solutions, and a large part of the literature concentrates on the optimal design of taxes or of policies of public firms.

Taking a second best problem, we try in this paper, to shift the emphasis from "*normative*" to "*positive*" considerations. The solution we are interested in is not optimal with respect to some a priori given social welfare function embodying justice objectives; but it is supposed to reflect the power of the different agents in a negotiation process. Clearly, the conceptual tools required for such an analysis are to be found in game theory and it is actually a game theoretical approach that we follow here.

The model we are looking at is presented in section II. It formalizes one of the simplest second best problem that can be imagined. We have a two - good economy with one private good, and one public good. Agents have different wealth in private good and the public good is financed through a wealth tax. This latter fiscal system is clearly not flexible enough to adjust contributions with marginal willingnesses to pay, so that we actually face a second best situation. There is one decision variable in the society, the tax rate which determines the public good production, about which the agents have conflicting desires.

To this problem, we associate a game which provides a precise framework for the analysis of the power of the agents. The *"positive"* outcomes of the negotiation process are supposed to belong to the core of this game.

Sections III and IV concentrate on the analysis of the core. The results strongly contrats those obtained in the classical studies of the core of economies with public good in a first best context (see for example CHAMPSAUR [1975]). The core has no reason to be large. Furthermore, in order to assure non emptiness, conditions are required reflecting for example that the agents have similar enough opinions on the tax choice or that agents who disagree strongly with the average opinion do not own enough resources to have credible threats. Tax rates corresponding to intermediate opinions equally far from the extreme ones are shown to be more likely in the core.

We also recognize that the game under consideration has very unusual features, when compared with the classical economic games known from the literature. Particularly, it is not necessarly superadditive; when people from two disjoint coalitions have too much diverging feelings concerning the tax system, it may not be desirable in terms of economic efficiency that these coalitions merge putting all their resources in common. As a consequence, the core stricto sensu is empty in many economies without pathological features. For that case, we define in Section V a concept of stable structure wich can be considered as an extension of the core concept to non superadditive games. This concept describes a situation where the grand coalition breaks down but where some partition, which is stable in a strong sense, emerges. This latter fact can be related with the theory of local public goods which has its origin in *TIEBOUT* [*1956*] and which has recently known a growing interest from theorists (*PESTIEAU* [*1979*], *STIGLITZ* [*1977*], *WOODERS* [*1978*], *etc* ...). In fact, as in *WESTHOF* [*1977*], one can interpret the stable structure as an affectation of agents between different cities where the public good is locally produced. When the "*megalopolis*" (the grand coalition) is non viable, still may emerge a group of different cities which define a stable arrangement in the society. Section V consists in a tentative exploration of this latter point, extending a previous attempt of *GUESNERIE - ODDOU* [*1979*].

II - THE PUBLIC GOOD GAME

A. *The Basic Framework*

We consider a simple economy with the following characteristics. There is one pure public good and one private good. There are n consumers indexed by $i \in N \overset{\text{def}}{=} \{1,\ldots,n\}$. Consumer i has preferences on R_+^2, represented by a strictly quasi-concave utility function u_i: $u_i(x,q)$ is the utility level associated with the consumption of x units of private good and q units of public good. A constant returns to scale technology permits the transformation of one unit of private good into one unit of public good. Initial endowments are only in private goods and are privately owned. Initially, consumer i owns ω_i units of private good $(\omega_i > 0)$. There are no a priori restrictions either on the distribution of endowments or on preferences (but strict convexity).

Besides these intrinsic characteristics, we introduce in the description of our economy a basic *institutional assumption* : the public good can only be financed through a linear wealth tax. The positive tax rate $t > 0$ is constrained to be the same for all individuals; it permits the withdrawal of an amount of resources $t\left(\sum_{i=1}^{n} \omega_i\right)$ to produce $t\left(\sum_{i=1}^{n} \omega_i\right)$ units of public good. The quantity of private good remaining to individual i is hence $(1-t)\,\omega_i$.

The tax system so defined is not flexible enough to adjust the contributions of individuals in proportion to their marginal willingnesses to pay, as would be desirable to meet the Samuelson rules; the Lindhal equilibrium is attainable only in exceptionnal cases. Furthermore, any direct compensating transfer between agents is ruled out. From a normative point of view, we can only expect second best Pareto optimal allocations. The derivation of tax systems maximizing some a priori given social welfare function is outside the scope of this paper but the reader is invited to check that the characterization of the Rawlsian optimum (in case of identical utility functions) is particularly straightforward. As announced in the introduction, and in constrast with the optimal taxation tradition, we are interest in deriving a positive theory which relates the decision concerning the tax system with the power of the agents in the society. For that, it is natural to associate with our problem, a game.

The formalization of the game rests on three assumptions.

- First, the technology is available to any coalition each of which can therefore transform one unit of private good into one unit of public good.

- Second, the coalition alone cannot benefit from the public good produced by other coalitions. This assumption which assures that the game is orthogonal has always been made in the game theoretic study of public goods in a first best context. It is usually justified on the grounds that it reflects the maximin threat of a coalition. It will also fit well with a subsequent interpretation (Section IV), according to which agents split into cities (which can be thought of as geographically distinct) so that the pure public good is locally produced in a context *"à la Tiebout"*. (Cf. Introduction).

- Third, institutionnal constraints on tax schemes are the same within each coalition where the public good is financed through a linear wealth tax; the tax rate being the unique decision variable.

Considering jointly these three assumptions, which make our framework closely similar to that of WESTHOF [1977], we are in position to deduce the *characteristic form of the game*.

For any $S \subset N$, $v(S) = \{v \in R^n_+ , \exists t \in [0,1] , v_i \leqslant u_i \, [(1-t)\omega_i , \, t \sum_{i \in S} \omega_i] , \forall i \in S\}$ (★)

Before going further, we will give a first insight into this game in a simple case.

B. *A First Insight Into The Model*

Consider two agents A and B endowed respectively with a and b units of the private good. The feasible states of the coalition {A,B} are depicted in diagram I. When the tax rate is t, the consumption bundles of A and B are $E = ((1-t)\,a,q)$ and $F = ((1-t)\,b,q)$, their utility levels are u(E) and u(F). Suppose that A is a dictator in the coalition in the sense that he can choose the tax rate without modifying the basic tax law. Diagram I indicates that he would choose the consumption bundle G corresponding to $q^\star_A$ units of public good and then obtains an utility level $u^\star_A$. B as a dictator would choose H associated with $q^\star_B$ units of public good and an utility level $u^\star_B$.

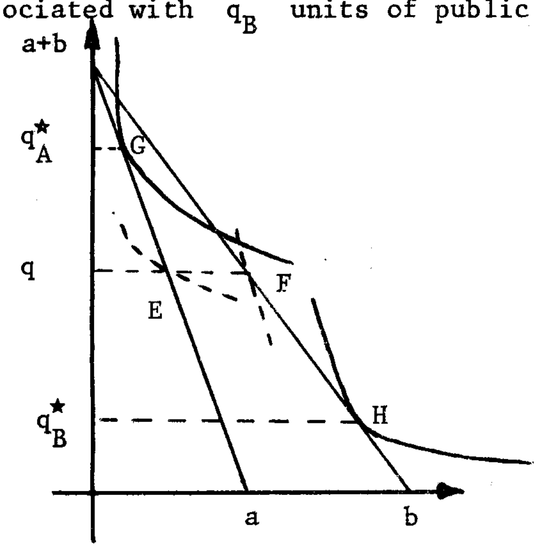

DIAGRAM I

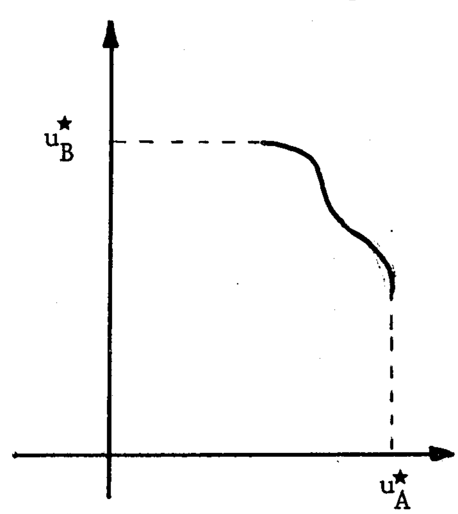

DIAGRAM II

(★): *Following the notation of SHAPLEY [1972] , we associate with an agent who does not belong to the coalition any positive utiliy level.*

Hence, diagram II which depicts the set of pareto optimal feasible utilities, for coalition {A,B} is obtained when q varies in $[q_A^\star, q_B^\star]$.

Let us consider now what A can do when he is alone. From diagram III, he will refuse any tax rate for the coalition {A,B} which would lead to a level of public good outside $\Delta_A = [\underline{q}_A, \bar{q}_A]$ since such public good levels give him an utility level smaller than $\bar{u}_A$.

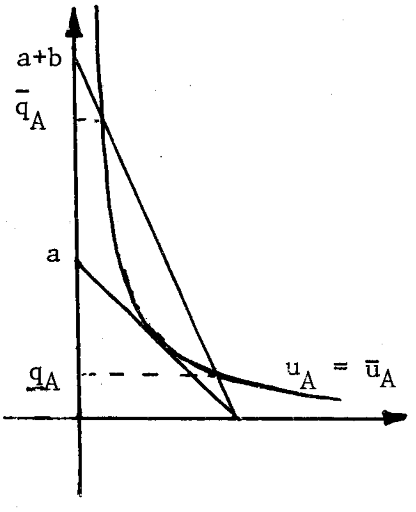

DIAGRAM III

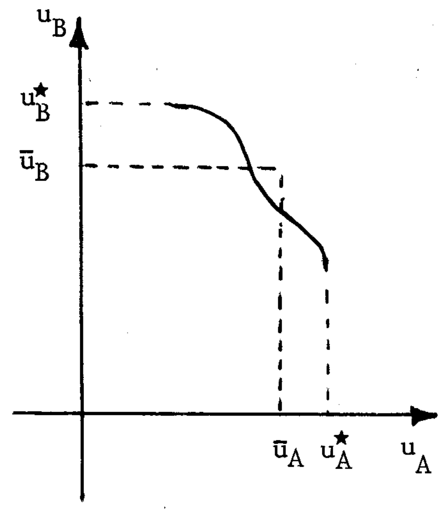

DIAGRAM IV

Similarly B will refuse any level of public good outside $\Delta_B = [\underline{q}_B, \bar{q}_B]$ which would give him an utility level less than $\bar{u}_B$. If the intersection of Δ_A and Δ_B is empty (and the reader will easily convince himself from the diagram that this may actually accur) A and B will have no interest in cooperating.

Diagram IV illustrates several facts which are crucial for the analysis.

α) The efficient points of the grand coalition do not necessarily coïncide with the efficient points of the game. In other words, from the efficiency point of view, it may not be desirable that the grand coalilition forms. It is equivalent here, to say that the game v is not necessarily superadditive, a property which strongly contrasts this game with most games previously studied in the economic literature.

β) The core stricto sensu, i.e. the set of allocations attainable by the grand coalition and blocked by no coalition, may be empty. In this simple two-person game, efficiency through the grand coalition, superadditivity and non emptines of the core obtain simultaneously when they are in general independant properties. However they have connections which will be established in section III devoted to the study of the core of superadditive games. Section IV will consider the core of non necessarily superadditive games.

γ) Although the core is empty, a stable arrangement consisting of A and B in isolation, emerges from diagram IV. In the language of the theory of local public goods, the grand city made of A and B will split and two "*small*" cities A and B will emerge each one producing locally its own public good. The case where the core is empty and the emergence of stable structures, which can be interpreted as independant cities, is analyzed in Section V.

III - SUPERADDITIVITY, EFFICIENCY OF THE GRAND COALITION AND EXISTENCE OF THE CORE.

We have just noticed that the formation of the grand coalition was not necessarily desirable from the efficiency point of view. In order to take into account this fact more precisely, we introduce the following concepts :

A structure S = $(S_1, S_2, \ldots, S_P)$ is a partition of the set of agents into distinct coalitions $S_1, S_2, \ldots, S_P$.

<u>*An efficient outcome*</u> of the game is a vector $\bar{u} \in R^n_+$ such that :

1) $\bar{u} \in \bigcap_{k \in K} v(S_k)$ where $S = (S_k)_{k \in K}$ is a structure,

2) $\nexists u \in \bigcap_{\ell \in L} v(S_\ell)$ where $S' = (S_\ell)_{\ell \in L}$ is a structure and where $u \gg \bar{u}$. (*)

<u>*An efficient structure*</u> $S = (S_k)_{k \in K}$ is a structure such that there is $\bar{u} \in \bigcap_{k \in K} v(S_k)$ which is an efficient outcome.

A structure S is <u>*universally efficient*</u> if any efficient outcome $\bar{u}$ belongs to $\bigcap_{S \in S} v(S)$.

In other words, when a structure is efficient, it should be implemented when some specific arbitrage between consumers welfare prevails i.e. for some individualistic social welfare function. A structure is universally efficient if its implementation is desirable whatever the social welfare function chosen or whatever the specific arbitrage scheme between consumers welfare. The reader is invited to illustrate these definitions with the example of the preceding section. He will also notice that in the model (**) of this paper, the structure constituted by the grand coalition alone $\{N\}$ is always efficient, so that $\{N\}$ is the only possible candidate for universal efficiency. Let us also notice that when the game is superadditive, $(v(P \cup T) \supset v(P) \cap v(T)$, $\forall$ P,T s.t. $P \cap T = \emptyset)$, the grand coalition is actually universally efficient, but the converse is not true.

In this section, we will first focus attention on the problems of superadditivity of the game and of universal efficiency of the grand

(*) $u \gg v \Leftrightarrow u_i > v_i$, $\forall i$.

(**) *But not necessarily in other models. See for example SHAKED [1978.]*

coalition for which we will derive, in subsection III A. necessary and sufficient conditions which are economically meaningful. We will present a strong result on the existence of the core in games which have a universal efficient structure in subsection III B.

III - A. *Necessary and Sufficient Conditions for Superadditivity and Universal Efficiency*

Let us first introduce some piece of notation.

Given a coalition S and an agent $i \in S$, we will call $u_i(t,S)$ the utility level obtained by agent i when the tax level in coalition S is t.

$$u_i(t,S) = u_i(\ (1-t)\omega_i\ ,\ \ t \sum_{i\in S} \omega_i) \ .$$

Similarly, we will denote by $u_i(q,S)$ the utility obtained by i when the public good level in S is q; it is only defined for $0 \leqslant q \leqslant \sum_{i\in S} \omega_i$.

$$u_i(q,S) = u_i(\ (1- \frac{q}{\sum_{i\in S} \omega_i})\ \omega_i\ ,\ q) \ .$$

Let $u_i^{\star}(S)$ denote the utility level which i would be able to obtain in coalition S if he had the power to dictate the tax rate of coaltion S, for short the dictatorial utility level of i in S.

$$(3.1.)\ u_i^{\star}(S) = \underset{t\in[0,1]}{\text{Max}}\ u_i(t,S) = u_i(t_i^{\star}(S),\ S) = u_i(q_i^{\star}(S),S)$$

u_i being continuous and strictly quasi concave, $t_i^{\star}(S)$ and $q_i^{\star}(S)$ are defined without ambiguity.

Let us also define, associated with an utility level a

$$(3.2)\ \Delta(i,a,S) = \{q \leqslant \sum_{i\in S} \omega_i,\ u_i(q,S) \geqslant a\} \ .$$

It straightforward to check that $\Delta(i,a,S)$ is a closed interval in $[0, \sum_{i\in S} \omega_i]$ and that $a<b$ leads to

(3.3) $\Delta(i,a,S) \supset \Delta(i,b,S)$.

Let us also remark that $\Delta(i, u_i^\star(S), S) = \{q_i^\star(S)\}$

Now let us come back to our public game of Section II and let us suppose that it is superadditive. In that case, the simple example of subsection II suggests that, loosely speaking the agents do not have too much divergent opinions concerning the tax system. As an illustration of this fact, consider two disjoint coalitions P,T $(P \cap T \neq \emptyset)$ and a couple of agents $i \in P$, $j \in T$. Suppose that i and j respectively obtain their dictatorial utility levels $u_i^\star(P)$, $u_j^\star(T)$. It is straightforward that superadditivity requires that $u_i^\star(P)$, $u_j^\star(T)$ are feasible in $S = P \cup T$. In other words, i and j being dictators in P and T would find an agreement for merging P and T favourable to each of them.

Formally, this can be expressed as Condition α)

Condition α) for coalition S : Existence of Bilateral Merging Agreements

For all P and T such that $P \cup T = S$ and $P \cap T = \emptyset$, for all $i \in P$, for all $j \in T$, there is $t \in [0,1]$ such that :

(3.4.) $u_i^\star(P) \leqslant u_i(t,S)$, $u_j^\star(T) \leqslant u_j(t,S)$

Equivalently :

(3.5.) $\Delta(i,u_i^\star(P),S) \cap \Delta(j,u_j^\star(T),S) \neq \emptyset$

Condition α) can still be explained in economic terms as follows : take two agents i and j belonging to two distinct coalitions; suppose that these two agents have the power to decide, in their self interest, on whether the two coalitions to which they belong should merge and at which conditions ; condition α) says that such a bilateral merging agreement is always possible, whatever the couple of coalitions whatever the couple of agents, and whatever their initial situations in the coalitions (and particularly if each of them had a dictatorial utility level).

It is remarkable that condition α) which is necessary for superadditivity is also sufficient.

Proposition I :

The public good game is superadditive if and only if condition α) holds for any coalition S.

Proof:

To prove that condition α) is sufficient, we have to show that for any S and for any two-partition of S , $\{P,T\}$, we have :

$$v(S) \supset v(P) \cap v(T) \cdot$$

This is true if and only if for any $\bar{v} \in v(P) \cap v(T)$

(3.6.) $\bigcap_{l \in S} \Delta(l, \bar{v}_l, S) \neq \emptyset$ ·

But when a finite number of intervals is such that any two of them have a non empty intersection, all of them have a non empty intersection : this is *HELLY*'s theorem (★) for one dimensionnal spaces (see for example *BERGE* [*1959*]). Hence, *(3.6.)* would be a consequence of *(3.7.)*

(3.7.) $\Delta(l,\bar{v}_l,S) \cap \Delta(k,\bar{v}_k,S) \neq \emptyset$ for any l, k.

To prove *(3.7.)*, we note first that if $l \in P$ and $k \in T$, $\bar{v}_l \leq u^\star_l(P)$, $\bar{v}_k \leq u^\star_k(T)$. *(3.7.)* then is a consequence of *(3.5.)* in condition α) and of *(3.3.)*.

(★) :*The reference to Helly's theorem makes clear that extension of the result could be obtained, if instead of one public good, we had considered any number of public goods. It is left to the reader to find a generalization of condition α) and of Proposition I in an economy with p public goods.*

If $1 \in P$ (resp T), $k \in P$ (resp T), it is straightforward that $\bar{q}$ the public good level which allows the realization of $\bar{v}$ in P (resp T) belongs to the intersection defined in *(3.7.)*. The conclusion follows.

Roughly speaking, the idea underlying the proof of proposition *I* is that the existence of a global agreement for the realization of $\bar{v}$ in S results from the existence of a set large enough of bilateral merging agreements within S. A similar idea applies to the characterization of games with N as a universal efficient structure. A result still simpler obtains

Proposition II :

N is universally efficient if and only if condition α) holds for N.

Proof

Necessity is still straightforward. For sufficiency, let us consider a partition $S = (S_k)$, $k \in K$, and $v \in \bigcap_{k \in K} v(S_k)$. We have to prove that v is dominated by some $u \in v(N)$ or equivalently that

(3.8.) $\bigcap_{i \in N} \Delta(i, v_i, N) \neq \emptyset$.

As in proposition I and for the same reasons, it is enough to prove that if $1 \in S_k$, $1' \in S_{k'}$, $k \neq k'$.

(3.9.) $\Delta(1', v_{1'}, N) \cap \Delta(1, v_1, N) \neq \emptyset$.

But if we denote $q_{k'}$ the level of public good associated with v in $S_{k'}$ (i.e. such that $v_i = u_i(q_{k'}, S_{k'})$ $\forall\, i \in S_{k'}$), one has from *(3.2.)* :

$$\Delta(1', v_{1'}, N) \supset \Delta(1', u_{k'}(q_{k'}, N/S_k), N) \supset \Delta(1', u^{\star}_{k'}(N/S_k), N)$$

(3.9.) follows then from condition α) and the proof terminates as in Proposition I.

Finally, one has obtained necessary and sufficient conditions for superadditivity and universal efficiency which are rather surprisingly simple. The verification of condition α) calls for the following remarks.

I - For a coalition S with s elements the verification of conditions α) requires the implementation of (at most) $C_s^2 \; 2^{s-2}$ differents *"experiments"*, each experiment consisting in the search of a bilateral agreement for a couple of agents and a two partition of S. $\sum_{s=2}^{n} C_s^2 \, C_n^s \, 2^{s-2}$ such experiments are sufficient in theory to check that the game is superadditive. In fact, as will be clear in the following, the most relevant property for our study is universal efficiency of N, the verification of which only requires $C_n^2 \; 2^{n-2}$ *"experiments"*.

II - It would be possible to state conditions bearing on the characteristics of individual preferences (in terms of income and price elasticities of the public good demand) and on the distribution of endowments which do imply condition α). Besides the fact that such conditions would be rather intricate, they would be in practice more difficult to verify than the existence of bilateral merging agreements.

III - B. The existence Of The Core In Superadditive Games

As usual, we define here the *Core* as the set of $u^{\circ} \in v(N)$ which are blocked by no coalition i.e. such that : $\nexists$ S and $u \in v(S)$ such that : $u \gg u^{\circ}$.

By a straightforward extension, we often refer in the following to allocations, public good levels or tax rates belonging to the core or blocked by a coalition : for example, we say that the public good level q is blocked by coalition S if the utility vector $u_i(q, N))_{i \in N}$ is blocked by S.

Let us now introduce some additionnal piece of notation. Without loss of generality we can suppose that the agents are ranked as their dictatorial public good level (defined in *(3.1.)*) in the grand coalition, i.e. that $\forall$ i, j $i<j$ implies $q_i^\star \overset{\text{def}}{=} q_i^\star(N) \leq q_j^\star(N) = q_j^\star$. This being done, taking some public good level q belonging to $]q_1^\star, q_n^\star[$ we denote by $i^+(q)$ the smallest i such that $q_i^\star > q$ and by $i^-(q)$ the greatest i such that $q_i^\star < q$. And we define :

$$I(q) = \{1, \ldots, i^-(q)\} \quad , \quad J(q) = \{i^+(q), \ldots, n\}$$

I(q) and J(q) are respectively the set of agents which in the grand coalition would desire less or more public good than q (in terms of their dictatorial public good levels). We can remark that $I(q)$, $J(q)$, $\{i/q_i^\star = q\}$, form a partition of N.

Now, what about the core of our public good game ? To prove that the core is non empty, a standard strategy of proof consists in showing that the game is balanced in the sense of Scarf . Using this latter sufficient condition, we actually proved in *GUESNERIE - ODDOU [1979]* the following :

Proposition III

For $n < 5$, if the public good game is superadditive, then it is Scarf balanced and hence has a non empty core.

As we already noticed in Section II, the property is straightforward in the case of n = 2. For n = 3 and n = 4, the proof proceeds by inspection : balanced families are examined one by one and the proof that they are Scarf balanced rests on ad hoc specific arguments.

However we were not able to extend the proof beyond. Furthermore, it is very likely that public good games may not be Scarf balanced from $n = 6$.

However, a more direct argument not relying on Scarf condition shows that public good games which are superadditive have actually a non empty core, even if they may be not *Scarf* balanced. More generally, we have :

Proposition IV

If N is universally efficient, then the public good game has a non empty core.

Proof

Let $q \in [q_1^\star, q_n^\star]$ the set of second best Pareto optimal levels of public good.

Let Q^- be the set of $q \in [q_1^\star, q_n^\star]$ which are blocked by some coalition $P \subset I(q)$.

Let Q^+ be the set of $q \in [q_1^\star, q_n^\star]$ which are blocked by some coalition $T \subset J(q)$.

Our definition of blocking referring to strict inequalities, the sets Q^+ and Q^- are open in $[q_1^\star, q_n^\star]$. Moreover Q^+ and Q^- are intervals of the form $Q^- =]q^-, q_n^\star]$, $Q^+ = [q_1^\star, q^+[$.

This comes from the fact that $q \in Q^-$ (resp. $q \in Q^+$) implies $[q, q_n^\star] \subset Q^-$ (resp. $[q_1^\star, q] \subset Q^+$).

Let us now state :

Assertion I : For any $q \in [q_1^\star, q_n^\star]$, there is no blocking coalition $S = P \cup T$ such that $\emptyset \neq P \subset I(q)$, $\emptyset \neq T \subset J(q)$ and no blocking coalition contains an i such that $q_i^\star = q$.

Assertion II : There is no $q \in [q_1^\star, q_n^\star]$ simultaneously blocked by two coalitions $P \subset I(q)$ and $T \subset J(q)$.

If both assertions were true, one would have $q^- \geqslant q^+$ (from assertion *2*) and any q in $[q^-, q^+] \neq \emptyset$ would be from assertion *1* in the core of the game.

It remains to prove assertions *1* and *2*.

α) If S is a blocking coalition, there exists a $q°$ such that for every $i \in S$, $u_i(q°,S) > u_i(q,N)$. But if $i \in I(q)$, necessarily $q° < q$ and if $i \in J(q)$, $q° > q$; the two are simultaneously impossible which proves assertion *1*. The second part of the assertion is trivial.

β) Suppose the contrary. Let $\bar{u}_i$ be the utility levels of the agents $i \in P \cup T$ in the corresponding blocking coalitions. From universal efficiency, there is $\bar{q}$ s.t. $u_i(\bar{q},N) \geqslant \bar{u}_i > u_i(q,N)$, $\forall i \in P \cup T$. But this is impossible, since from q, one cannot increase in N the welfare of two agents having a $q_i^\star$ on each side of q.

Proposition IV has two corollaries. The first one already has been mentioned.

Corollary 1 :

If the public good game is superadditive, then it has a non empty core.

The second one combines proposition *II* and *IV*.

Corollary 2 :

If condition α) *concerning the existence of bilateral merging agreement holds for the grand coalition N, then the public good game has a non empty core.*

This last statement is particularly interesting since he gives a criterion for the non emptiness of the core simple enough to be checked through a procedure based on a finite number of experiments as those defined at the end of Section III A.

In summary, the mains conclusions to be drawn from this section are the following :

- *When the formation of the grand coalition is certainly desirable*, in terms of efficiency (certainly in the sense that it not contingent on the specific social welfare arbitrage which is made), *it can be implemented in a stable way*, i.e. there exists a choice of tax rate which is unblocked.

- The property of universal efficiency of the grand coalition, which guarantees the non emptiness of the core, is itself equivalent to a simple condition which reflects that the agents do not have too much diverging opinions on the tax system, and which relates to the existence of the set of potential bilateral merging agreements as expressed in condition α).

IV - THE CORE OF THE PUBLIC GOOD GAME WHEN IT IS NOT NECESSARILY SUPERADDITIVE

We will still focus in this section on the study of the core of the public good game, but with an approach different in two respects of the approach of Section III.

The field covered is distinct : the results apply to a class of games which are not necessarily superadditive; it does not include or is included in the class of superadditive games.

The nature of the results differs : when in Section III, we only established existence theorems, the propositions given in this section either exhibit some public good level which is in the core (Proposition *V*) or give some interval of public good levels where one can find at least one element in the core (Proposition *VI*).

A. *Testing Whether a "Median" Level of Public Good Is In The Core*

The proof of proposition *IV* suggests that good candidates for the core are public good levels which are in an intermediate average position between $q_1^\star$ and $q_n^\star$. We will consider here in particular a resource-weighted median level of public good defined as follows :

q^m is a resource-weighted median level of public good if and only if:

$$(4.1.)\quad \sum_{i \in I(q)} \omega_i \leqslant \frac{1}{2} \sum_{i \in N} \omega_i \overset{\text{def}}{=} \frac{1}{2}\,\omega(N)\ ,\quad \sum_{i \in J(q)} \omega_i \leqslant \frac{1}{2}\,\omega(N)$$

(4.1.) defines one q^m which is unique but for exceptional cases .

In the following we will denote $\Gamma^-(q) = \sum_{i \in I(q)} \omega_i$ (resp. $\Gamma^+(q) = \sum_{i \in J(q)} \omega_i$) the total resources of agents who desire less (resp. more) public good than q (in the sense that their dictatorial level in the grand coalition is greater or smaller than q).

In the definition of q^m, *(4.1.)* can be written

$$(4.2.)\quad \Gamma^-(q^m) \leqslant \frac{1}{2}\,\omega(N)\quad ,\quad \Gamma^+(q^m) \leqslant \frac{1}{2}\,\omega(N).$$

Loosely speaking, q^m is a R.W.M. level of public good if both people who want more than q^m and people who want less than q^m in the grand coalition own less than half of total resources.

We will now give a criterion assuring that q^m is in the core of our public good game. This criterion rests on the analysis of the answer of the agents of the economy to a simple question about which a poll could be organized in the society. The question is the following :

"Suppose that you have to choose between accepting q^m in N or being a dictator in a coalition which owns some fraction α of the total resources of the society $\omega(N)$. From which minimum level $\bar{\alpha}$ would you prefer the second solution ?"

Let $\bar{\alpha}_i$ be the answer of consumer i. Supposing that individual wealths ω_i are known, for every $\gamma \in [0,1]$ one can infer from the answers $\bar{\alpha}_i$ the proportion of total wealth owned by people whose answer is smaller than γ.

Let $\beta(\gamma, q^m)$ be this number.

Formally, if we denote

(4.3.) $A(\gamma, q^m) = \{i \in N, \; u_i^\star(\gamma\omega(N)) > u_i(q^m, N)\}$

where $u_i^\star(\lambda) = \underset{t \in [0,1]}{\text{Max}} \; u_i((1-t)\omega_i, \lambda)$ (★)

(4.4.) $\beta(\gamma, q^m)\,\omega(N) = \sum_{i \in A(\gamma, q^m)} \omega_i$

If furthermore, the dictatorial levels of agent i in the grand coalition where known, one could deduce from the poll

(4.5.) $\bar{A}(\gamma, q^m) = \{i \in J(q^m), \; u_i^\star(\gamma\omega(N)) > u_i(q^m, N)\}$,

$\underline{A}(\gamma, q^m) = \{i \in I(q^m), \; u_i^\star(\gamma\omega(N)) > u_i(q^m, N)\}$,

and :

(4.6.) $\bar{\beta}(\gamma, q^m)\,\omega(N) = \sum_{i \in \bar{A}(\gamma, q^m)} \omega_i$, $\underline{\beta}(\gamma, q^m)\,\omega(N) = \sum_{i \in \underline{A}(\gamma, q^m)} \omega_i$

$\bar{\beta}$, $\underline{\beta}$ and $\beta = \bar{\beta} + \underline{\beta}$ are non decreasing steps functions of γ.

In order to find some coalition blocking q^m, it is necessary to find some number γ such that either $\underline{\beta}(\gamma, q^m) > \gamma$ or $\bar{\beta}(\gamma, q^m) > \gamma$.

(★) : *Actually, we defined else where $u_i^\star(S)$. But there is no risk of confusion between the two notations and $u_i^\star(S) = u_i^\star(\omega(S))$ with $\omega(S) = \sum_{i \in S} \omega_i$.*

We immediatly obtain.

Proposition V

If the following condition holds :

(4.7.) $\beta(\gamma, q^m) \leq \gamma$ and $\bar{\beta}(\gamma, q^m) \leq \gamma$ for all $\gamma \in]0, \frac{1}{2}]$, *then* q^m *the weighted resource median level of public good belongs to the core.*

Particularly (4.7.) is implied by :

(4.8.) $\beta(\gamma, q^m) \leq \gamma$ for all $\gamma \in]0, \frac{1}{2}]$

Clearly, the fact that we have taken q^m rather than another public good level does not play a crucial role in the analysis. For any q belonging to $[q_1^\star, q_n^\star]$ the criterion of proposition *V* would only be slightly modified (the interval where γ varies being greater) and the method proposed to check it would remain valid.

Let us also remark that as the indirect preferences of agents are here single peaked, q^m the RWM level is the *Condorcet* winner of a majority voting procedure when all agents have the same resources (or when votes are distributed in proportion of wealth). Hence, it comes out :

Corollary I

If all agents have the same resources, and if condition (4.7.) or (4.8.) of proposition V is satisfied, the Condorcet winner public good level belongs to the core.

B. *Conditions Assuring that "Intermediate" Public Good Levels Are in the Core*

In this section, we will consider some interval $[a,b]$ included in $]q_1^\star, q_n^\star[$ the set of second best Pareto optimal public good levels and we will exhibit a condition assuring that some public good level in $[a,b]$ belongs to the core of the public good game.

Actually, reminding that $\Gamma^-(b) = \sum_{i \in I(b)} \omega_i$ and $\Gamma^+(a) = \sum_{i \in J(a)} \omega_i$, and defining $\Gamma = \max\{\Gamma^-(b), \Gamma^+(a)\}$, the condition we will consider can be introduced :

Condition β (with respect to the interval $[a,b]$) consists in the two following requirements :

(4.9.) $\Delta(i, u_i^\star(\Gamma), N) \cap [a,b] \neq \emptyset \quad \forall\, i \in N$

(4.10.) $\Delta(j, u_j^\star(\Gamma), N) \cap \Delta(i, u_i^\star(\Gamma), N) \neq \emptyset \quad \forall\, j \in J(a), \forall i \in I(b).$

(4.9.) means that for every i, $\exists\, q^i \in [a,b]$ such that $u_i(q_i, N) \geq u_i^\star(\Gamma)$; there is some public good level q_i in $[a,b]$ which is better for i when he is in N than his dictatorial level in a coalition which has an amount of resources Γ .

(4.10) means that if one considers one agent j who would desire in the grand coalition more public good than a, and an agent i who wants less public good than b, then there exists $\tilde{q}$ such that $u_j(\tilde{q}, N) \geq u_j^\star(\Gamma)$ and $u_i(\tilde{q}, N) \geq u_i^\star(\Gamma)$. j and i dictators in coalitions with an amount of resources Γ would find profitable to join the grand coalition where would prevail some mutually agreed level of public good.

The amount of resources Γ which is considered in condition β) is the maximum of the amount of resources respectively owned by the agents who "*desire*" less public good than b and by the agents who "*want*" more public good than a. One sees that when $[a,b]$ is a very small interval around q^m, Γ is approximatively equal to $\frac{\omega(N)}{2}$, it increases when the size of the interval increases. For that reason, condition *(4.10.)* is more likely to be satisfied when the size of the interval is small; in counterpart, condition *(4.9.)* is more likely to be satisfied when the interval is large. One will actually see that the most interesting statement will be obtained for an intermediate size of the interval.

Instead of condition β) one could have stated condition β') which, as the reader will easily check is equivalent.

Condition β')

$\forall\, j \in J(a), \forall\, i \in I(b)$,

(4.11.) $[a,b] \cap \Delta(j, u_j^{\star}(\Gamma), N) \cap \Delta(i, u_i^{\star}(\Gamma), N) \neq \emptyset$

In condition β', the requirement corresponding to *(4.9.)* is suppressed; in counterpart, j and i should agree on a public good level in N which belongs to $[a,b]$.

We are now in position to prove :

Proposition VI

If for an interval $[a,b]$, condition β (or condition β') is satisfied, then the public good game has a non empty core which has a non empty intersection with $[a,b]$.

Proof

Let us consider the following game W.

$W(S) = v(S)$ if S is a subset of I(b) or of J(a)

$W(N) = \{v \in \mathbb{R}_+^n \;/\; \exists\, q \in [a,b],\ v_i \leq u_i(q, N),\ \forall\, i \in N\}$

$W(S) = \{v \in \mathbb{R}_+^n,\ v_i = 0,\ \forall\, i \in S\}$ for other S.

We will prove first that this game has a non empty core. For that, let us consider $\mathcal{C}$ a balanced family of coalitions and let

$\bar{u} \in \bigcap_{S \in \mathcal{C}} W(S)$, or we have to prove that $\bar{u} \in W(N)$, or equivalently

$\bigcap_{l \in N} \Delta(l, \bar{u}_l, N) \cap [a,b] \neq \emptyset$.

From the one dimensionnal version of Helly's theorem, it is ennough to prove :

(4.12.) $\Delta(l, \bar{u}_l, N) \cap \Delta(k, \bar{u}_k, N) \cap [a,b] \neq \emptyset$, $\forall$ l, $\forall$ k.

Let us remark that if we take i and S a coalition of $\mathcal{C}$ to which i belongs, we have, (given the definition of W):

$\bar{u}_i \leqslant u_i^\star(S) \leqslant u_i^\star(\Gamma)$.

Then :

- if $l \in I(b)$, $k \in J(a)$, the left hand side of *(4.7.)* contains $\Delta(l, u_l^\star(\Gamma), N) \cap \Delta(k, u_k^\star(\Gamma), N) \cap [a,b]$ which according to condition β' is non empty.

- if $l \notin I(b)$, $k \notin I(b)$, one has simultaneously $\Delta(l, \bar{u}_l, N) \cap [a,b] \neq \emptyset$ (condition *(4.9.)*) and $b \leqslant q_l^\star$.

It follows that $b \in \Delta(l, \bar{u}_l, N)$. Similarly $b \in \Delta(k, \bar{u}_k, N)$ and *(4.7.)* holds .

- if $l \notin J(a)$, $k \notin J(a)$, an argument similar to the preceding one (with a instead of b) applies .

The game W is Scarf balanced and hence has a non empty core. Let $\bar{q}$ (belonging to [a,b]) be in the core of W. We assert that $\bar{q}$ is also in the core of the game v ; the only possible blocking coalitions should be subsets either of $I(\bar{q})$ or of $J(\bar{q})$ (see the proof of proposition III) and hence of I(b) or of J(a), contradicting the fact that $\bar{q}$ is in the core of W.

Q.E.D.

In the general case, conditions (β) or (β') have a rather complicated statement. They simplify for the following appropriate choice of $[a,b]$, $a \overset{def}{=} q(1/3)$ and $b \overset{def}{=} q(2/3)$ are defined by :

$$\sum_{i \in J(q(1/3))} \omega_i \leq \frac{2}{3}\, \omega(N) \;, \qquad \sum_{i \in I(q(1/3))} \omega_i \leq \frac{1}{3}\, \omega(N)$$

$$\sum_{i \in I(q(2/3))} \omega_i \leq \frac{2}{3}\, \omega(N) \;, \qquad \sum_{i \in J(q(2/3))} \omega_i \leq \frac{1}{3}\, \omega(N)$$

It should be noticed that $q_{1/3}$ and $q_{2/3}$, as defined here, are in general (but not always), unique. Although the definition is rather complicated, it is designed such that the three groups of agents -those who want less than $q_{1/3}$, those who want more than $q_{2/3}$ and those who have a dictatorial public good level between $q_{1/3}$ and $q_{2/3}$- have roughly one third of total resources. $[q_{1/3}, q_{2/3}]$ is then a kind of median interval in $[q_1^\star, q_n^\star]$.

Then, as a corollary of proposition VI, we have :

<u>Corollary VI.1.</u>

Suppose that the following holds :

β1) : Every agent prefers some q in $[q_{(1/3)}, q_{(2/3)}]$ in the grand coalition rather than to be a dictator in a coalition owning 2/3 of total resources. Formally :

$$\Delta(i, u_i^\star(\tfrac{2}{3}\, \omega(N)) \;, N) \cap [q_{(1/3)}, q_{(2/3)}] \neq \emptyset \;.$$

β2) Any couple of agents who are dictators in coalitions, each of one has 2/3 of total resources, would agree on some common level of public good in the grand coalition. Formally :

$$\forall\, i,j \;, \quad \Delta(i, u_i^\star(\tfrac{2}{3}\, \omega(N)), N) \cap \Delta(j, u_j^\star(\tfrac{2}{3}\, \omega(N)), N) \neq \emptyset$$

Then, the public good game has a non empty core with some element in $[q_{(1/3)}, q_{(2/3)}]$

Among the criteria assuring the non emptiness of the core which have been exhibited here, corollary VI. 1. is one of the most attractive. Particularly the number of experiments (in the sense of Section III) required for its verification is (when n is large) of the order of magnitude of n^2 which has to be compared with 2^n, the order of magnitude associated with the criterion of universal efficiency. In counterpart, the criterion is somewhat more demanding, since the two agents considered in $\beta 2)$ own together $\frac{4}{3}\,\omega(N)$ and not $\omega(N)$ as in condition $\alpha)$ of Section III.

Still corollary VI.1. can be considered as another illustration of the idea according to which the existence of the core relates with converging (or not too much diverging) opinions of the agents concerning the tax system.

V - THE PUBLIC GOOD GAME WHEN THE CORE IS EMPTY

When the core is empty, two different directions of reflections are open. The first one leads to examine restrictions on the formation of coalitions which would guarantee again the existence of the core (Section V.A.), the second one turns the attention to other possible solutions; so in Section V.B., we will discuss the possibility of emergence of *stable structures* associated with *stable solutions*, a generalization of the concept of core.

V.A. Restrictions On The Formation of Coalitions

Let us suppose that only coalitions with $n-1$ agents are allowed. Let us consider γ the game built from v when taking into account this latter assumption :

$\gamma(N) = v(N)$, $\gamma(S) = v(S)$ if $|S| = n-1$, $\gamma(S) = \{v/v_i = 0, i \in S\}$ for others S.

One can assert :

Proposition VII

The public good game γ where only coalitions with n-1 agents are allowed to block is Scarf balanced and has a non empty core.

Proof

Consider γ . Scarf condition is trivially verified for all balanced families but

$$\Phi = (\{1,2, \ldots, n-1\}, \{2,3, \ldots, n\}, \{3,4, \ldots, n,1\}, \ldots)$$

Consider $\bar{u} \in \bigcap_{S \in \Phi} \gamma(S) = \bigcap_{S \in \Phi} v(S)$.

For every S in Φ, there exists q^S such that

$$\forall\, i \in S \;,\; \bar{u}_i = u_i(q^S, S)$$

We can define : $\underline{q}_i = \min \{q^S, S \in \Phi, i \in S\}$

$$\bar{q}_i = \max \{q^S, S \in \Phi \;\; i \in S\}$$

Clearly, $\Delta(i, \bar{u}_i, N) \supset [\underline{q}_i \;, \bar{q}_i]$.

Now, Φ is such that for every couple i,j , there exists an $S \in \Phi$ such that $i \in S$, $j \in S$.

Hence $q^S \in [\underline{q}_i, \bar{q}_i] \cap [\underline{q}_j, \bar{q}_j]$ and $\Delta(i, \bar{u}_i, N) \cap \Delta(j, \bar{u}_j, N) \neq \emptyset$

The standard argument already used in proposition *I*, *II*, *VI*, implies $\bar{u} \in \gamma(N)$. Hence, the conclusion.

■

The method of proof uses the fact that the restricted game γ is Scarf-balanced. We remind from previous attempts in Section III that the utilization of Scarf condition for proving non emptiness does not reveal very convenient in our problem. Actually, renouncing to prove that the restricted game γ is Scarf balanced, we will obtain a proposition stronger than proposition VII.

Let us suppose that we associate with each consumer i a number $\theta_i > 0$ with $\sum_{i=1}^{n} \theta_i = 1$. ($\theta_i$ could be interpreted as a number of votes and could for example, be chosen such that $\theta_i = \frac{1}{n}$).

We will restrict blocking coalitions to the family

$S = \{S \,/ \sum_{i \in S} \theta_i > \frac{1}{2}\}$.

Then, we have:

Proposition VIII

If the blocking coalitions are restricted to belong to the just defined family S, then the core is non empty.

Proof

Let us come back to the proof of Proposition *III* and particularly consider assertion *I*.

Take a $\bar{q}$ defined by : $\sum_{i \in I(\bar{q})} \theta_i \leq \frac{1}{2}$ and $\sum_{i \in J(\bar{q})} \theta_i \leq \frac{1}{2}$ (such a $\bar{q}$ exists but is not necessarly unique).

It is left to the reader to show that such a $\bar{q}$ is in the core of the restricted game.

An obvious corollary obtains :

Corollary VIII.1.

If blocking coalitions are restricted to coalitions with strictly (★) more than $\frac{m}{2}$ members, then the core is non empty.

(★) :*The strict inequality in the definition of S is necessary : consider the example of a two agents economy of Section III with an empty core.*

In some sense, the above analysis emphasizes that in the public good game, coalitions of small size have an important blocking power.

We will finally mention, for the sake of completeness, another problem suggested by the classical studies on the core of economies with many agents. Consider an r- replica of our original economy; the per-unit cost of the public good in the grand coalition decreases steadily. For that reason, in a first best context, the core of the public good economy becomes very large. One could have expected that in our second best context, the core would have ceased to be empty for a large enough value of r.

Actually, this is wrong :

Proposition IX

There are economies such that whatever r, the r replica of the public good game of Section II has an empty core.

We will briefly describe an example where such a phenomenon occurs :

The agents have the same initial endowments :

$\omega_{ij} = 1 \quad , \quad i = 1, 2 \quad ; \quad j = 1, \ldots, r$

Preferences are COBB-DOUGLAS $\quad u_1(x, y) = x^{\alpha} y^{\beta}$,

$$u_2(x, y) = x^{\beta} y^{\alpha} \quad (\alpha + \beta = 1).$$

For a given t, agents of type 1 and 2 respectively obtain

$$u_1(t) = r^{\beta}\, 2^{\beta}\, t^{\beta} \quad (1-t)^{\alpha}$$

$$u_2(t) = r^{\alpha}\, 2^{\alpha}\, t^{\alpha} \quad (1-t)^{\beta}$$

The coalition consisting of all agents of type 1 (resp. 2) can guarantee to his members $\bar{u}_1 = r^{\beta}\, \beta^{\beta}\, \alpha^{\alpha}$ (resp. $\bar{u}_2 = r^{\alpha}\, \beta^{\beta}\, \alpha^{\alpha}$).

The tax rate t belongs to the core, $\forall\, r$, if

$t \in [\alpha,\beta]$, $g(t) = \left(\frac{2t}{\beta}\right)^{\beta}\left(\frac{1-t}{\alpha}\right)^{\alpha} > 1$ and $f(t) = \left(\frac{2t}{\alpha}\right)^{\alpha}\left(\frac{1-t}{\beta}\right)^{\beta} > 1$

For α small enough, these two inequalities are contradictory ; the core is empty whatever r.

V.B. Existence of Stable Structures

When the core is empty no agreement on the tax rate which meets the stability requirement of being unblocked, can be expected within the grand coalition. There is a tendancy for the agents to split. However, as noted in Section II, there might emerge a partition of N, with different groups of agents with different tax rates and different levels of public good, which meets stability requirements very similar to those underlying the concept of core. We will introduce here the concept of stable structure (see the concept of structure introduced at the beginning of Section III) which formalizes this idea (see also *GUESNERIE - ODDOU* [1979]).

A C- stable solution is a vector $u^\circ \in R^n_+$ which satisfies the following properties :

$\exists \mathcal{S} = (S_k)_{k\in K}$ such that $u^\circ \in \bigcap_{k\in K} v(S_k)$ where $\mathcal{S}$ is a structure.

$\nexists S$ and $u \in v(S)$ such that $u \gg u^\circ$.

A *stable structure* is a structure $\mathcal{S}$ such that there is a stable solution u° belonging to $\bigcap_{S\in\mathcal{S}} v(S)$.

Several straightforward remarks are in order :

- A C- stable solution is necessarly an efficient outcome.

- An utility vector in the core is a C - stable solution associated with the stable structure made of {N} alone.

- If N is universally efficient, the only possible stable structure is {N} itself.

This latter remark makes clear that the concept of stable solution is useless for superadditive games but suggests that it might reveal appropriate for our problem.

Let us briefly emphasize the conceptual connections between the notions of core and of C - stable solution. In some sense, the C-stable solution generalizes the concept of core, but it can also be viewed as merely an adaptation of the core concept (*). To make this remark precise, let us define :

$\tilde{\tilde{v}}$: the smallest superadditive game associated with v is defined

$$\tilde{\tilde{v}}(S) = \bigcup_{\mathcal{C} \in \mathcal{P}(S)} \bigcap_{T \in \mathcal{C}} v(T)$$ where $\mathcal{P}(S)$ is the set of all partitions of S.

$\tilde{v}$: is intermediate between v and $\tilde{\tilde{v}}$; $\tilde{v}$ is defined as follows :

$$\tilde{v}(S) = v(S) \text{ if } S \neq N \text{ and } \tilde{v}(N) = \tilde{\tilde{v}}(N) .$$

The interest of these definitions relies in the following property, proved in *GUESNERIE - ODDOU [1979]* :

v has a strongly stable solution if and only if $\tilde{\tilde{v}}$ has a non empty core ; $\tilde{\tilde{v}}$ has a non empty core if and only if $\tilde{v}$ has a non empty core.

Let us now comment the meaning of the notion of C - stable solution and stable structure in our public good game.

A stable structure defines a partition of agents between different subsets. Different tax rates prevail in different groups; one group can benefit from its own public good but not of the public good produced by the other groups. The outcome cannot be blocked by the grand

(*) : *Particularly SHAKED (1978) uses a similar concept without distinguishing it from the concept of core, but this appears natural in the context of his model.*

coalition (since it is efficient) as well as by no other coalition.

Actually, what is called here a structure is exactly analogous to a set of cities in WESTHOF [1977] , and a stable structure can be interpreted as a stable structure of different cities(★). As in WESTHOF, the exclusion of use of the public good between cities could be justified by the nature of the public good or by spatial considerations. It has anyway to be considered as an interesting polar case in the theory of local public goods.

What can be said about the existence of stable structures ?

A first result can be given :

<u>*Proposition X*</u>

For $m < 4$, *the public good game has a C-stable solution.*

For $m = 4$, *if all agents have the same resources, there is a C-stable solution* .

The proof can be found in GUESNERIE - ODDOU [1979] where it is shown that in these cases, $\tilde{\tilde{v}}$ is Scarf balanced.

We will establish an extension of this last result for economics with different types od agents. For that, let us consider an economy ξ with n types of agents and r_i agents in each type. To the economy ξ let us associate the economy $\tilde{\xi}$ with n agents:each agent i represents a type, has the initial resources $r_i\omega_i$, and has a utility function $\tilde{u}_i(x,y) = u_i(\frac{x}{r_i}, y)$ where ω_i and u_i respectively are the initial resources and preferences of an agent of type i in ξ.

We can prove the following lemma :

(★): *Actually, our stability requirements are considerably stronger than those of WESTHOF.*

Lemma

Consider the public good game. If for the economy $\tilde{\xi}$, there exists a stable structure then the same obtains for the economy ξ.

Proof

We first remark that in any C- stable solution in ξ, two agents of the same type obtain the same utility level (straightforward). Then, we notice that if a coalition S gives to his members an utility vector u_i, $i \in S$, a coalition gathering agents of S and agents who are not in S but who are of a type who has representants in S, can give to his members of type i more than u_i. Hence, we can restrict ourselves, to consider in ξ blocking coalitions which contain all the agents of the types which are represented in S. It follows that a C- stable solution in $\tilde{\xi}$ corresponds a C- stable solution in ξ since any blocking coalition in ξ would result in a blocking coalition in $\tilde{\xi}$.

Proposition XI

If the economy is made of three different types of agents, the public good game has a stable structure.

If the economy is made of four different types of agents, the total resources of each type being the same, then the public good game has a C-stable structure.

We will give a last statement concerning the existence of stable structures. This statement does not rely on basically new arguments but takes advantage of the knowledge we already acquired.

Proposition XII

Let $S = (S_k)_{k \in K}$ be a structure which has the following properties :

1) The subgame associated with the subset of agents who are in S_k has S_k as an universally efficient structure.

2) Any subgame associated with a subset of agents which do not coincide with one of the S_k has an empty core.

Then, S is a stable structure.

Proof

From Proposition *III*, the subgame associated with S_k has a non empty core. Let q_k be a public good level belonging to the core of this subgame. We will prove that the sequence of $(\ldots, q_k, \ldots)$ defines a C- stable solution.

Suppose that this allocation is blocked by some coalition $S^{(o)}$, for some public good level $q^{(o)}$. Because of (1), $S^{(o)}$ cannot be a subset of some S_k. But from (2), $q^{(o)}$ in $S^{(o)}$ is blocked by some subcoalition $S^{(1)} \subset S^{(o)}$ associated with some $q^{(1)}$. But $S^{(1)}$ cannot be a subset of some S_k ... Then, the argument can be repeated for $S^{(2)}$, $q^{(2)}$, ..., $S^{(p)}$, $q^{(p)}$ ... As p cannot tend to infinity, a contradiction obtains.

Q.E.D.

Proposition *XII* can be expressed in another way. Let us say that a coalition is *internally stable* if the associated subgame has a non empty core. *Then a structure is stable if its elements are the only internally stable groups in the society*. Such a condition is clearly rather restrictive.

VI - CONCLUSION

Let us first mention some possible extensions of the analysis presented in this paper.

As mentioned in footnote (page 11), several public goods could have been incorporated in the analysis of superadditivity and universal efficiency, and actual generalizations of the corresponding results could have been obtained without major difficulties. Actually, Helly's theorem provides a guideline for extension of most of the statements where it intervenes.

The reader will also have noticed that the assumption that the wealth tax is linear only serves the purpose of assuring the connectedness of the sets $\Delta(i, q, S)$. Many results of the paper would remain true with a non linear tax schedule, under the condition that it belongs to some appropriate class of functions and that it only depends of one parameter.

At contrary, the extension of the analysis to an economy with several public goods, where for example the wealth tax would be replaced by transaction taxes would raise difficult problems. In that case, the set of feasible states would have a much more complicated mathematical structure.

Let us notice finally some open problems. Whether the existence of stable structures in the sense of Section V is "frequent", or at contrary, unlikely is still unclear and we have not even been able to exhibit a counterexample to the existence of stable structures (counterexample which at least requires 4 agents or 5 agents with the same initial resources).

On the other hand, the assumption that "coalitions" or "cities" do not cooperate once the public good has been produced (or do not take into account this possible cooperation before), can only be considered as formalizing a polar case. Its relaxation would raise the difficult conceptual problems associated with non orthogonal games.

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