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# ON ECONOMIC GAMES WHICH ARE NOT NECESSARILY SUPERADDITIVE. SOLUTION CONCEPTS AND APPLICATION TO A LOCAL PUBLIC GOOD PROBLEM WITH FEW AGENTS

R. GUESNERIE*
C. ODDOU**

N° 7911

* R. GUESNERIE - C E P R E M A P

** C. ODDOU - G.R.A.S.C.E.

The modelling of economic situations in a game theoretical framework has resulted most of the time in considering superadditive games. Away the best known in the economic literature let us quote the market games associated with an exchange economy, the games associated with superadditive coalition production or with public good economies in a first best context.

We present in this brief note, an example of an economic game, which is not necessarily superadditive, associated with a second best problem in the theory of local public goods. The examination of various second best problems has convinced us that the absence of superadditivity in the associated economic games was not at all pathological. Non superadditive games have some unfamiliar characteristics, an appropriate reconsideration of standard concepts of efficiency and stability is required (see section I). The application of these modified concepts will prove fruitful for the local public good game with few agents (see section II).

## I. EFFICIENCY AND STABILITY IN GAMES WHICH ARE NOT NECESSARILY SUPERADDITIVE.

Through the concepts defined in this section, we want mainly to take into account two facts :

- Efficiency may require that the grand coalition $N$ breaks down.
- Stability requirements which have the same strength as those involved

in the core concept, may be obtained for some particularly appropriate grouping of agents even when the core stricto-sensu is empty.

Let us consider then a game in characteristic form : $N$ denote the set $\{1,2,\ldots,m\}$ , $\Xi$ denote the set of all non empty subsets of $N$ and $v$ denote a function from $\Xi$ to, non empty, compact subsets of $\mathbb{R}^N_+$ that satisfies : for all $S \in \Xi$ , if $\alpha \in v(S)$ and $\alpha^S = \beta^S$ , then $\beta \in v(S)$[1].

In the standard interpretation, the elements of $N$ are *players*, the elements of $\Xi$ *coalitions* and the elements of $\mathbb{R}^N$ *payoff* or *utility vectors*. The elements of $v(S)$ represent *feasible outcomes* for the coalition $S$ and we denote $\mathcal{D}(S)$ the set of outcomes that $S$ can *improve upon* i.e. the set of elements $v \in \mathbb{R}^N$ such that there is a vector $u \in v(S)$ for which $v^S \ll u^S$ (See Shapley (1972)).

DEFINITIONS :

A *structure* of coalitions is a partition $\Pi = (S_k)_{k \in K}$ of the set of agents. The set of partitions of $N$ is denoted $T(N)$ .For any $\Pi = (S_k)_{k \in K}$ , $\Pi \in T(N)$ we denote $v(\Pi) = \bigcap_{k \in K} v(S_k)$ .

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(1) Let $\mathbb{R}^N$ denote the n-dimensional euclidean space with coordinates indexed by the elements of $N$ , and for $S \in \Xi$ let $\mathbb{R}^S$ denote the corresponding $|S|$-dimensional subspace of $\mathbb{R}^N$ . If $\alpha \in \mathbb{R}^N$ and $S \in \Xi$ , then $\alpha^S$ will denote the projection of $\alpha$ on $\mathbb{R}^S$ , i.e., the restriction of $\alpha$ to the coordinates indexed by the elements of $S$ .

An *efficient outcome* of the game is a vector $\bar{u} \in \mathbb{R}^m_+$ such that :

a) there is $\Pi \in T(N)$ for which $\bar{u} \in v(\Pi)$

b) there is no $\Pi'$ and $u' \in v(\Pi')$ such that $u' \gg \bar{u}$ .

An *efficient structure* is a structure $\Pi$ such that there exists an efficient outcome $\bar{u} \in v(\Pi)$ . A structure $\Pi$ is *universally efficient* if any efficient outcome $\bar{u} \in v(\Pi)$ .

Game theoretical concepts of stability can now be introduced.

The *Core* $C(v)$ is the set of $u^* \in v(N)$ which can be improved upon by no coalition -$C(v) = v(N) \setminus \bigcup_{S \in \Xi} D(S)$- .

A *Core like stable solution* or *C-stable solution* is a vector $u^* \in \mathbb{R}^m_+$ which satisfies the following properties :

- there is a $\Pi \in T(N)$ such that $u^* \in v(\Pi)$ ,
- for all $S \in \Xi$ , $u^* \notin D(S)$ .

Clearly a C-stable solution is an efficient outcome ; the core is a set of particular C-stable solutions ; if N is universally efficient any C-stable solution is necessarily in the core.

A *Stable structure* is a structure $\Pi$ such that there exists a C-stable solution $u^* \in v(\Pi)$ .

It is straightforward that :

LEMMA 1.

$\Pi = \{N\}$ is a stable structure if and only if the core is non empty.

For games which are not superadditive, $\{N\}$ is not necessarily universally efficient. In this case the concept of C-stable solution reflects the same ideas of stability as those underlying the core concept and appears as a natural generalization. To some extent, as made clear below, it is merely an adaptation of the core concept [1].

Let us define $\tilde{\tilde{v}}$ , the *smallest superadditive* game associated with $v$ :

$$\tilde{\tilde{v}}(S) = \bigcup_{\Pi \in T(S)} \bigcap_{T \in \Pi} v(T) \ .$$

$\tilde{v}$ will be intermediate between $v$ and $\tilde{\tilde{v}}$ and it is defined as follows :

$\tilde{v}(N) = \tilde{\tilde{v}}(N)$

$\tilde{v}(S) = v(S)$ for all $S \neq N$ .

Then :

LEMMA 2.

$u^*$ is a C-stable solution of $v$ if and only if it belongs to the core of $\tilde{\tilde{v}}$ and if and only if it belongs to the core of $\tilde{v}$ .

Proof.

Take $u^*$ a strongly stable solution, there is a $\Pi \in T(N)$ such that $u^* \in v(\Pi)$ so $u^* \in \tilde{v}(N) = \tilde{\tilde{v}}(N)$ , if $u^*$ don't belong to the core of $\tilde{v}$ (resp. $\tilde{\tilde{v}}$ ) there is $S \in \Xi$ and $\overline{u} \in \tilde{v}(S)$ (resp. $\overline{u} \in \tilde{\tilde{v}}(S)$ ) such that $\overline{u}^S \gg u^{*S}$ . For $\tilde{v}$ , if $S \neq N$ $u^* \in D(S)$ ; we obtain a contradiction.

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(1) In particular, we discovered that a similar concept was used by Shaked (1978) who called it simply "core". Even if lemma 2 gives some justification for this usage, it seems to us that it would be a source of confusion not to distinguish core from stable solutions, at least in the context of the second best games we are looking at.

If $S = N$, or for $\tilde{\tilde{v}}$, there is a $\Pi \in T(S)$ such that $\bar{u} \in \bigcap_{T \in \Pi} v(T)$ and for some $T$ belonging to $\Pi$ $u^* \in D(T)$, which is impossible.

On the other hand, if $u^* \in C(\tilde{v})$ (resp $u^* \in C(\tilde{\tilde{v}})$ ), $u^* \in \tilde{v}(N) = \tilde{\tilde{v}}(N)$ so there is a $\Pi \in T(N)$ such that $u^* \in v(\Pi)$ .

If there is an $S \in \Xi$ such that $u^* \in D(S)$ , then there is an $\bar{u} \in v(S)$ with $\bar{u}^S \gg u^{*S}$ . $\bar{u} \in v(S) \subset \tilde{v}(S) \subset \tilde{\tilde{v}}(S)$ , so $S$ can improve upon $u^*$ and we obtain a contradiction.

## II. A LOCAL PUBLIC GOOD GAME WHICH IS NOT NECESSARILY SUPERADDITIVE.

Consider the following simple economy : there are two goods, the first is a private good, the second is a pure public good. There are $m$ consumers indexed by $i = 1,\dots,m$ having preferences represented by an utility function $U_i$ defined on $\mathbb{R}^2_+$ $(U_i(0,0) = 0)$.
Moreover, consumers are initially endowed with $\omega_i$ $(\omega_i > 0)$ units of the private good. Preferences and initial endowments being given, we define the rules of the game as follows : each coalition $S$ which is a subset of $N = (1,\dots,m)$ can use the same simple constant returns to scale technology which transforms one unit of private good into unit of public good. The public good can be financed only through a wealth tax : each coalition $S$ can choose a rate of tax $t \in [0,1]$ , which applies to all members of the coalition. When the tax rate chosen is $t$ , the tax revenue raised is $t \sum_{i \in S} \omega_i$ and the amount of public good produced is $q(t,S) = t \sum_{i \in S} \omega_i$ . Given these assumption on the technology of production

and on the institutional constraints for the financing of the public good one can associate the problem of the allocation of the public good in this simple economy with a game, whose characteristic function is v(S) defined as follows :

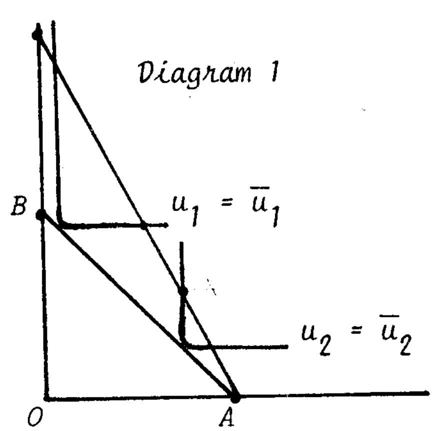
$$v(S) \;=\; \{u \in \mathbb{R}^N_+,\ \exists\ t \in [0,1],\ u^S \;\leq\; u(t,S)\}$$

where $u(t,S)$ is the vector of $\mathbb{R}^S$ whose components are :

$$u_i(t,S) \;=\; u_i\{(1-t)\omega_i, q(t,S)\} \quad , \quad i \in S \ .$$

The assumption concerning the financing of the public good is here crucial. If coalitions were able to raise personal taxes and hence to relate exactly marginal willingnesses to pay and individual contributions, we would be in a framework, within which the existence of core solutions has been explored. In particular it is well known in this case that under standard assumptions the core is non-empty and even, in some sense, large. But with a wealth tax, the contributions of people to the financing of the public good can hardly be correlated with the marginal willingnesses to pay. We have thus a second best problem.

Diagram 1 below shows that in this case, as stated in the introduction, superadditivity does not necessarily hold.



Two individuals have the same initial endowments. When they are isolated, the agent 1 obtains $\bar{u}_1$ and the agent 2 obtains $\bar{u}_2$. When they are together the attainable states are associated with the segment AC and it is clear that $(\bar{u}_1, \bar{u}_2)$ is not realisable.

The questions which are raised through the model under consideration are closely connected with the topics of the theory of local public goods, partly initiated by Tiebout (1956). Although Tiebout himself does not consider a pure public good, several contributions are centered on the study of the organization of communities facing the problem of the production of a pure public good. In particular our model is similar to that of Westhof (1975). The concepts we have just introduced above, look appealing in the sense that they allow some hope of obtaining in some cases an endogeneous explanation of the formation of cities.

## III. STABLE SOLUTIONS AND STABLE STRUCTURES IN A PUBLIC GOOD ECONOMY WITH A SMALL NUMBER OF AGENTS.

In this section preferences are only assumed to be convex and initial endowments of private goods can be distributed among agents in any way.

The analysis is based on the application of Scarf's theorem according to which any balanced game has a non empty core. We show that when the number of agents is small $(m \leq 4)$ the systematic use of Scarf's sufficient condition allows us to make conclusive statements. Precisely we have :

PROPOSITION 1 :

When the number of agents is smaller or equal to three, $m \leq 3$, the game $v$ has a C-stable solution.

Proof.

The statement is trivial for $m = 1$, and straightforward for $m = 2$. However it is not at all obvious for $m = 3$. We show in this case that the game $\tilde{v}$ is balanced in the sense of Scarf : i.e. we want to prove that if :

$$A = (S_i)_{i \in I}$$

is a balanced family and if $u \in \tilde{v}(S_i)$ for all $i \in I$, then $u \in \tilde{v}(N)$.
Actually there are six minimal balanced families which are the partitions $T_1 = (\{A\},\{B\},\{C\})$, $T_2 = (\{A\},\{B,C\})$, $T_3 = (\{A,B\},\{C\})$, $T_4 = (\{A,C\},\{B\})$, $T_5 = (\{A,B,C\})$ and the family of two member coalitions :

$$T_6 = (\{A,B\},\{B,C\},\{C,A\}) \quad .$$

Scarf conditions are immediately verified for $T_1$ to $T_5$. For $T_6$ we have to prove if :

$(u_A,u_B) \in \tilde{v}(\{A,B\})$, $(u_B,u_C) \in \tilde{v}(\{B,C\})$, $(u_A,u_C) \in \tilde{v}(\{A,C\})$,
then :

$$(u_A,u_B,u_C) \in \tilde{v}(\{A,B,C\}) \quad .$$

It is immediate that this statement will be proved if it is proved for $v$ instead of $\tilde{v}$.
We then notice the following basic property : if $\bar{u}^1$ and $\bar{u}^2$ are utility levels obtained by agent $i$ respectively in coalition $S_1$ and $S_2$ when

these coalitions choose the public good levels $\overline{q}_1$ (resp. $\overline{q}_2$), then when the level of public good chosen by coalition $N$ is $q$, between $\overline{q}_1$ and $\overline{q}_2$, the agent $i$ obtains at least $\mathrm{Min}\ (\overline{u}^1, \overline{u}^2)$ .

We denote by $q_{AB}$, $q_{BC}$, $q_{CA}$, the levels of public goods in $\{A,B\}$, $\{B,C\}$, $\{A,C\}$ assuring respectively $(u_A,u_B)$, $(u_B,u_C)$, $(u_A,u_C)$. Suppose that $q_{AB} \leq q_{BC} \leq q_{CA}$.
$\overline{q} = q_{BC}$ belongs to the intersection of the segments $[q_{AB},q_{CA}]$, $[q_{AB},q_{BC}]$, $[q_{BC},q_{CA}]$, then the choice of $\overline{q}$ in $N$ would assure at least the utility vector $[u_A,u_B,u_C]$. This terminates the proof.

From lemma 1, one can also assert.

Corollary : For $m \leq 3$, if $v$ is superadditive, then it has a non empty core.

In this case there is a C-stable solution $u^*$ corresponding to a partition $\Pi$. Superadditivity implies $u^* \in v(N)$ and then $\{N\}$ is a C-stable structure and from lemma 1 the core is non empty.

We can actually prove that a similar result holds for $m = 4$ by using the same techniques.

PROPOSITION 2 :

For $m = 4$, if $v$ is superadditive, it is Scarf balanced and has a non empty core.

Proof.

According to Shapley (1967) the minimal balanced families of a 4 players-game are apart from the partitions, the following families :

$T_1 = (\{1\ 2\ 3\},\{1\ 2\ 4\},\{3\ 4\})$ , $T_2 = (\{1\ 2\},\{1\ 3\},\{4\})$ ,

$T_3 = (\{1\ 2\ 3\},\{1\ 4\},\{2\ 4\},\{3\})$ , $T_4 = (\{1\ 2\ 3\},\{1\ 4\},\{2\ 4\},\{3\ 4\})$

$T_5 = (\{1\ 2\ 3\},\{1\ 2\ 4\},\{1\ 3\ 4\},\{2\ 3\ 4\})$

and all families obtained by permutation. The game being superadditive we can prove a theorem similar to theorem 3 of Shapley (1967). It is thus sufficient to check that the Scarf condition is verified for proper minimal balanced families. For these families each agent at least belongs to two coalitions. Using a similar argument to that of the preceding proposition and Helly's theorem in 1 dimension (cf. Berge (1966) page 173) the proof is straightforward.

Corollary : If $m = 4$ if all agents have the same resources, there is a C-stable solution and hence a stable structure.

Using the proof of proposition 2, it is enough to prove that Scarf's condition is verified for the families $T_2$ and $T_3$ relative to the game $\tilde{v}$ .

If :

$$u^* \in v(\{1,2\}) \cap v(\{1,3\}) \cap v(\{2,3\})$$

Proposition 1 shows that :

$$u^* \in v(\{1,2,3\})$$

so :

$$u^* \in v(\{1,2,3\}) \cap v(\{4\}) \subset \tilde{v}(\{1,2,3,4\}) .$$

We have now only to prove that if :

$$u^* \in v(\{1,2,3\}) \cap v(\{1,4\}) \cap v(\{2,4\}) \cap v(\{3\}) ,$$

then :

$$u^* \in \tilde{v}(\{1,2,3,4\}) .$$

Let $q_{1,4}$ , $q_{2,4}$ , $q_{1,2,3}$ , $q_3$ be the levels of public good assuring in coalitions $\{1,4\}$ , $\{2,4\}$ , $\{1,2,3\}$ , $\{3\}$ the utility levels $u^*$ . Let $\bar{q}$ be the median level of public good in :

$$\{q_{1,4} , q_{2,4} , q_{1,2,3}\} .$$

We prove that via $\bar{q}$ the coalition $\{1,2,4\}$ gives at its members better utility levels than $u^*$ .

i) If $\bar{q} = q_{1,2,3}$ the agents 1 and 2 have in coalition $\{1,2,4\}$ the same initial resource that in coalition $\{1,2,3\}$ . So for $\bar{q}$ they obtain the same utility in the two coalitions. $\bar{q}$ is between $q_{1,4}$ and $q_{2,4}$ so the agent 4 obtains in $\{1,2,4\}$ with $\bar{q}$ something better than in $\{1,4\}$ or in $\{2,4\}$ with $q_{1,4}$ or $q_{2,4}$ .

ii) If $\bar{q} = q_{1,4}$ (resp. $\bar{q} = q_{2,4}$ ) the agents 1 and 4 (resp. 2 and 4 ) are with this quantity of public good in $\{1,2,4\}$ better off than in $\{1,4\}$ with $q_{1,4}$ (resp. in $\{2,4\}$ with $q_{2,4}$ ) . The agent 2 (resp. 1 ) has a level of public good intermediate between $q_{2,4}$ and $q_{1,2,3}$ , in a coalition which has more or at least as much initial resources than $\{2,4\}$ , (resp. $\{1,4\}$ ) $\{1,2,3\}$ ; so he obtains a better utility level. In conclusion $u^* \in v(\{1,2,4\}) \cap v(\{3\}) \cap \tilde{v}(\{1,2,3,4\})$ . This completes the proof.

We have considered in a simple model the existence problem for the core and its most obvious generalization when the game is not superadditive the C-stable solution.

We attacked existence problems with a method which looks natural, trying to prove that the game, or its superadditive extension, is Scarf balanced. We obtained significantly positive results for $m \leq 4$. One would expect that the approach remains fruitful when a greater number of agents is involved. One might hope for example that proposition 2 holds for any $m$. Unfortunately this is not true. For $m = 6$, the reader will convince himself that we can construct non pathological preferences associated with the repartition of endowments $\omega_1 = 100$, $\omega_2 = \omega_3 = \omega_4 = \omega_5 = 1$, $\omega_6 = 100$ such that, for the balanced family $\{1,2,3\}$, $\{1,4,5\}$, $\{2,4,6\}$, $\{3,5,6\}$ Scarf condition does not hold. Hence for $m = 6$, the game $v$ may not be Scarf balanced even if it is super additive.

However we prove in a forthcoming paper, using quite different techniques that whatever $m$ superadditive local public goods games have non empty core, although as we just argued they might not be Scarf balanced. The model under consideration provides an example of a class of economically meaningful games which have a non empty core and in general violate Scarf condition for large values of $m$.

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