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- *A B S T R A C T* -

This paper is concerned with a study of budget constrained Pareto efficient (B.C.P.E.) allocations, i.e. allocations which given a price system satisfy a given income distribution. We prove existence, show structural stability, and establish a sufficient condition for uniqueness of B.C.P.E. allocations. Those properties of the B.C.P.E. allocations are deduced from similar properties of Walrasian equilibria by a duality theory which is of independent interest.

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1. INTRODUCTION.

The purpose of this paper is to study within the framework of pure exchange economies those Pareto efficient allocations compatible with a given price system and with a given income distribution. Compatibility means that every consumer's budget equation with respect to the given price system and the income assigned to him is satisfied. Interest in the study of the budget constrained Pareto efficient (B.C.P.E.) allocations takes its origin primarily in those cases where prices do not exhibit enough flexibility to be considered as variable within a short enough time period. The scope of fixprice analysis (versus flexprice analysis) is generally limited to the short-run ; more precisely, short-run is even defined in that context as the time period during which prices can be considered as fixed with a sufficient level of accuracy. Recent microeconomic developments of fixprice

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analysis have emphasized the aggregate disequilibria resulting from having individual choices determined by rigid prices and therefore have focussed on the concept of equilibrium under price rigidity and quantity rationing (see mainly Benassy [2], Drèze [5], and Younès [9]. The so-called short-run period, however, may actually last a very long time as in the case of economies with centrally regulated prices ; therefore, besides the quantity rationing approach, it is natural to investigate efficiency properties of short-run allocations, for example whether Pareto efficiency can be consistent with the constraints resulting from price rigidities and fixed incomes. Evidently, efficient allocations, if they exist, won't be decentralized at the "official" prices and incomes. This situation is not new to economic theory ; Boiteux [3] was already concerned with (second-best) efficiency under exogeneous constraints, the final allocations being decentralized with the help of "shadow" prices instead of the official ones. Summarizing, instead of considering the price system as a tool to decentralize allocations, an aspect of the price system largely emphasized by microeconomic theory, we investigate whether Pareto efficiency is compatible with arbitrary distribution requirements expressed by way of a fixed price system and of a fixed income distribution. Compare with the traditionnal approach which establishes a clear-cut separation between Pareto efficiency and distribution considerations.

For the sake of simplicity, we consider only pure exchange economies. Given an arbitrary price system and an arbitrary income distribution, we prove the existence of allocations which simultaneously satisfy every consumer's budget equation and are Pareto efficient, namely B.C.P.E. allocations. This existence result cannot be considered as obvious ; its proof is as demanding as proving the existence of Walrasian equilibria. Then, we investigate the dependance of the B.C.P.E. allocations on the vector defined by the prices and the individual incomes. We show that for regular price-income vectors, i.e. price-income vectors taken outside closed set of measure zero, the number of B.C.P.E. allocations is odd and every B.C.P.E. allocation depends smoothly on the price-income vector. We also show uniqueness of the B.C.P.E. allocation when the imbalance between aggregate supply and aggregate demand for the given price-income vector is not too large. The central idea of this paper is that the theory of B.C.P.E. allocations is formally equivalent to Walrasian equilibrium theory ; this equivalence relies on a duality theory which is of independent mathematical interest. To simplify as far as possible this mathematical duality theory, we have felt free to use nice assumptions concerning consumption sets and preferences.

The paper is organized as follows. Definitions, assumptions and notation are gathered in section 2. The main properties of B.C.P.E. allocations occupy section 3. We develope the duality theory in section 4. The proofs via duality theory of the main properties of B.C.P.E. allocations appear in section 5 while section 6 concludes this paper with some remarks. Most parts of this paper, section 5 excepted, can be read with a mathematical knowledge not exceeding elementary calculus and linear algebra. Section 5, however, requires some familiarity with elementary differential topology by which we mean the content of the first five chapters of Milnor's book [10], another excellent reference being Dieudonné's book [7].

2. ASSUMPTIONS, DEFINITIONS, AND NOTATION.

PART A. *Prices, consumption sets, preferences, and demand.*

We consider pure exchange economies with ℓ commodities and m consumers. We choose the ℓ-th commodity as numeraire, i.e. we normalize the price vector by the convention $p_\ell = 1$. We assume that every price is strictly positive and we denote by S the set of strictly positive normalized price vectors, i.e.

$S = \{p = (p_1 , p_2 ,\dots, p_\ell) \in \mathbb{R}^\ell \mid p_1 > 0 , p_2 > 0 ,\dots, p_\ell = 1\}$.

We assume that every consumer's consumption set is equal to $\mathbb{R}^\ell$. This departs from more standard assumptions under which every consumption set is bounded from below in $\mathbb{R}^\ell$. This assumption, however, is not really restrictive as long as the boundaries of the consumption sets are not to be studied. We also assume that every consumer's preferences can be represented by a utility function $u_i : \mathbb{R}^\ell \to \mathbb{R}$ where i varies from 1 to m and where u_i satisfies the following properties :

1) u_i is smooth, i.e. differentiable to any order, and surjective ;
2) u_i is differentiably monotonic, i.e. $\frac{\partial u_i(x)}{\partial x^j} > 0$ for $j = 1 , 2 ,\dots, \ell$;
3) $u_i^{-1}([c, +\infty))$ is strictly convex for every $c \in \mathbb{R}$;
4) $u_i^{-1}([c, +\infty))$ is bounded from below for every $c \in \mathbb{R}$;

5) the Gaussian curvature of the hypersurface $u_i^{-1}(c)$ is everywhere $\neq 0$ for every $c \in \mathbb{R}$.

Surjectivity of u_i is used for convenience ; otherwise, assumptions (1), (2), and (3) are standard. Assumption (4) is intended to cope with the non-boundedness from below of the consumption sets. Let $p \in S$ and $w_i \in \mathbb{R}$ be given. Maximizing $u_i(x_i)$ under the constraint $p.x_i \leq w_i$ has a unique solution, denoted $f_i(p , w_i)$, which represents consumer i's demand. Walras law $p.f_i(p , w_i) = w_i$ is clearly satisfied for every $p \in S$ and every $w_i \in \mathbb{R}$. Assumption (5) is equivalent to smoothness of the individual demand mapping $f_i : S \times \mathbb{R} \to \mathbb{R}^\ell$. One checks easily that under assumptions (1) to (5) the individual demand mapping f_i is a diffeomorphism, i.e. a smooth bijection having a smooth inverse f_i^{-1}.

PART B. *Allocations and price-income vectors.*

Let $r \in \mathbb{R}^\ell$ denote the vector of total resources. This vector will be assumed constant throughout this paper. A (feasible) allocation is an m-tuple $x = (x_1 , x_2 ,\dots, x_m) \in (\mathbb{R}^\ell)^m$ such that $x_1 + x_2 +\dots+ x_m = r$. We denote by X the space of all these allocations.

We denote by $\hat{X}$ the subset of $S \times \mathbb{R}^m$ consisting of price-income vectors $\hat{x} = (p , w_1 , w_2 ,\dots, w_m)$ such that $p.r = w_1 + w_2 +\dots+ w_m$. In other words total income is equal to the value $p.r$ of total resources for the given price vector $p \in S$.

Note that $\hat{X}$ is a convex subset of an affine set of dimention $m + \ell - 2$.

The set $A(\hat{x})$ of budget-constrained allocations associated with the price-income vector $\hat{x} = (p , w_1 , w_2 ,\dots, w_m) \in \hat{X}$ consists of those allocations $x = (x_1 , x_2 ,\dots, x_m)$ satisfying the equation system

$$\begin{cases} p.x_1 = w_1 , p.x_2 = w_2 , \dots , p.x_m = w_m , \\ x_1 + x_2 + \dots + x_m = r , \end{cases}$$

where $p , w_1 , w_2 ,\dots, w_m$, and r are fixed. The subset $A(\hat{x})$ of $\hat{X}$ is a linear manifold of dimension $(\ell-1)(m-1)$. Note that $A(\hat{x})$ has a unique equation system having the above form ; we call it the canonical equation of $A(\hat{x})$.

Let $\mathfrak{X}$ denote the set consisting of the sets $A(\hat{x})$ when $\hat{x}$ varies in $\hat{X}$, i.e.

$$\mathfrak{X} = \{A(\hat{x}) \mid \hat{x} \in \hat{X}\} .$$

PART C. *Pareto optima and price-income equilibria.*

We associate with the utility function $u_i : \mathbb{R}^\ell \to \mathbb{R}$ the function $v_i : X \to \mathbb{R}$ defined by the formula
$v_i(x_1 , x_2 ,\dots, x_m) = u_i(x_i)$.

DEFINITION 1. An allocation $x = (x_1 , x_2 ,\dots, x_m) \in X$ is Pareto efficient (with respect to the utility functions u_i) if there is no $x' = (x'_1 , x'_2 ,\dots, x'_m) \in X$ such that $v_i(x) \le v_i(x')$ for $i = 1 , 2 ,\dots, m$ with at least one strict inequality.

For convenience, we define Pareto efficiency in term of utility functions. This property, however, depends only on the preference preorderings represented by the utility functions. Let P denote the subset of X consisting of the Pareto efficient allocations. Let $g_i : \mathbb{R}^\ell \to S$ be the mapping defined by the formula

$$\left(g_i(x_i) = \frac{\partial u_i(x_i)}{\partial x_i^1} \bigg/ \frac{\partial u_i(x_i)}{\partial x_i^\ell} \ , \ \frac{\partial u_i(x_i)}{\partial x_i^2} \bigg/ \frac{\partial u_i(x_i)}{\partial x_i^\ell} \ , \ldots, \ 1 \right)$$

Clearly, $g_i(x_i)$ is the vector of S colinear with

$\text{grad } u_i = \left(\frac{\partial u_i(x_i)}{\partial x_i^1} , \frac{\partial u_i(x_i)}{\partial x_i^2} , \ldots, \frac{\partial u_i(x_i)}{\partial x_i^\ell} \right)$. Note that

$g_i(f_i(p , w_i)) = p$ where f_i is consumer i's demand function.

Recall that $x = (x_1 , x_2 , \ldots, x_m) \in X$ is Pareto efficient if and only if $g_1(x_1) = g_2(x_2) = \ldots = g_m(x_m)$; let $g(x)$ be this common value. We thus define a mapping $g : P \to S$ which associates with every Pareto optimum its supporting price vector. Finally, let $\phi : P \to \hat{X}$ be the mapping

$$x = (x_1 , x_2 , \ldots, x_m) \longmapsto (g(x) , g(x).x_1 , g(x).x_2 , \ldots, g(x).x_m)$$

which associates with every Pareto optimum $x \in P$ its supporting price vector and individual incomes. We recall the following properties of the set P.

<u>PROPOSITION 1</u>. <u>The set of Pareto optima</u> P <u>is a smooth submanifold</u> <u>of</u> X <u>diffeomorphic to</u> $\mathbb{R}^{m-1}$. <u>The mapping</u> $\phi : P \to \hat{X}$ <u>is an embedding</u>.

Recall that ϕ is an embedding if it is an immersion which maps homeomorphically P onto its image $\phi(P)$, [4] (16.8.4). Let $x = (x_1, x_2, \ldots, x_m) \in P$. Then $\phi(x) = (g(x), g(x).x_1, \ldots g(x).x_m)$ belongs to $\hat{X}$ and let $D(x) = A(\phi(x))$. In other words, D(x) is the linear manifold consisting of those allocations $x' = (x'_1, x'_2, \ldots, x'_m) \in X$ satisfying the equation system

$$\begin{cases} g(x).x'_1 = g(x).x_1 = w_1 \\ g(x).x'_2 = g(x).x_2 = w_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ g(x).x'_m = g(x).x_m = w_m \end{cases}$$

Let $\mathcal{P}$ be the subset of $\mathfrak{X}$ consisting of the sets D(x), when x varies in P , i.e.

$$\mathcal{P} = \{D(x) \mid x \in P\} .$$

A proof of proposition 1 is given in the appendix (App. 3.1).

Thinking of a price-income vector $\hat{x} = (p, w_1, w_2, \ldots, w_m) \in \hat{X}$ as a proposal for commodity prices and individual incomes in a planning process, then consumer i's demand is equal to $f_i(p, w_i)$ so that the sum $f_1(p, w_1) + f_2(p, w_2) + \ldots + f_m(p, w_m)$ represents the aggregate demand associated with $\hat{x} \in \hat{X}$. The proposal $\hat{x} \in \hat{X}$ is feasible if and only if aggregate demand is equal to total resource.

DEFINITION 2. A price-income vector $\hat{x} = (p, w_1, w_2, \ldots, w_m) \in \hat{X}$ is a price-income equilibrium if the equation

$$f_1(p, w_1) + f_2(p, w_2) + \ldots + f_m(p, w_m) = r$$

is satisfied.

In other words, $\hat{x}$ is a price-income equilibrium if and only if it is feasible. Let $\hat{P}$ denote the set of price-income equilibria. Note that the definition of $\hat{P}$ depends in fact only on the preference preorderings represented by the utility functions u_i.

The following proposition establishes a relationship existing between the set of Pareto optima P and the set of price-income equilibria $\hat{P}$.

PROPOSITION 2. We have $\hat{P} = \phi(P)$.

This proposition is also proved in the appendix (App. 3.2). It means that the set of price-income equilibria is generated by the prices and incomes supporting Pareto optima. Note that proposition 2 implies that $\hat{P}$, being the image of an embedding, cf [4] (16.8.4.), is a smooth submanifold of $\hat{X}$ diffeomorphic to P, hence to $\mathbb{R}^{m-1}$.

Summarizing, we have defined the sets X, $\mathfrak{X}$, P, and $\mathcal{P}$; they will enable us to reformulate the theory of Walrasian equilibrium and the theory of B.C.P.E. allocations in a more geometric way.

PART D. An example.

Let us first illustrate the above sets X , $\mathcal{X}$, P, and $\mathcal{P}$, in the Edgeworth box corresponding to the case of two consumers and of two commodities.

The set of feasible allocations

$X = \{x = (x_1 , x_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 + x_2 = r = \text{constant}\}$

is a plane. The set $\mathcal{X}$ consists of the lines in X which can be represented by an equation system of the form

$$\begin{cases} w_1 = p_1 \, x_1^1 + x_1^2 \\ w_2 = p_1 \, x_2^1 + x_2^2 \\ x_1^1 + x_2^1 = r^1 \\ x_1^2 + x_2^2 = r^2 \end{cases}$$

where r^1 , r^2 , p_1 , w_1 , and w_2 are fixed ; recall the normalization assumption $p_2 = 1$. Using the first consumer's coordinate axes, these lines are exactly those with negative slope.

If $x \in P$, i.e. x is Pareto efficient, then $D(x)$ is easily seen to be the common tangent to the two indifference curves passing through x. Therefore, $\mathcal{P}$ is the family of tangents $D(x)$ when x describes the set of Pareto optima.

Proposition 1 means that the set of Pareto optima is a smooth surve (the "contract" curve) ; proposition 2 says that the price-income vector $\hat{x} = (p_1 , 1 , w_1 , w_2)$ determined by the canonical equation system associated with $D(x)$ is feasible, i.e. a price-income equilibrium.

3. PROPERTIES OF B.C.P.E. ALLOCATIONS.

Taking the price-income vector $\hat{x} \in \hat{X}$ as given, the set of B.C.P.E. allocations associated with $\hat{x}$ is the intersection of the set of Pareto efficient allocations P with the set of budget-constrained allocations $A(\hat{x})$. The purpose of this paper amounts to studying the properties of the set $P \cap A(\hat{x})$ when $\hat{x}$ varies in $\hat{X}$.

The first result deals with an existence property of B.C.P.E. allocations $x = (x_1 , x_2 , \ldots, x_m) \in X$ associated with the budget constraints $p.x_1 \leq w_1$, $p.x_2 \leq w_2$, ..., $p.x_m \leq w_m$ where $\hat{x} = (p , w_1 , w_2 , \ldots, w_m) \in \hat{X}$ is given.

THEOREM 1. _There always exists a B.C.P.E. allocation_ $x \in X$ _associated with any given price-income vector_ $\hat{x} \in \hat{X}$.

Theorem 1 simply says that $P \cap A(\hat{x})$ is non-empty for any given $\hat{x} \in \hat{X}$. This result, however, is not sufficient for a study of the properties of B.C.P.E. allocations as functions of the price-income vectors, i.e. for a study of comparative statics. The next result establishes structural stability for B.C.P.E. allocations.

THEOREM 2. _There exists an open dense subset_ $\hat{\mathcal{R}}$ _of_ $\hat{X}$ _such that, for any_ $\hat{x} \in \hat{\mathcal{R}}$, _there exists an open neighborhood_ $U \subset \hat{\mathcal{R}}$ _and_ $2n + 1$ _smooth mappings_ $s_i : U \to \hat{X}$ _such that_ $\bigcup_{i=1}^{2n+1} s_i(\hat{y})$ _is the set of B.C.P.E. allocations associated with any_ $\hat{y} \in U$.

Theorem 2 means that, outside an exceptionnal set $\hat{\Sigma} = \hat{X} \setminus \hat{\mathcal{R}}$, the number of B.C.P.E. allocations is a locally constant odd number. Furthermore, B.C.P.E. allocations depend smoothly on $\hat{x}$ when $\hat{x}$ varies in $\hat{\mathcal{R}}$. The mapping $s_i : U \to \hat{X}$ is a smooth selection of B.C.P.E. allocation ; in other words, $s_i(\hat{y})$ represents a B.C.P.E. allocation associated with the price-income vector $\hat{y}$ which depends smoothly on $\hat{y}$. Note that this is true only for $\hat{y}$ belonging to some neighborhood of $\hat{x} \in \hat{\mathcal{R}}$.

The open dense subset $\hat{\mathcal{R}}$ of $\hat{X}$ is called the set of regular price-income vectors, its complement $\hat{\Sigma} = \hat{X} \setminus \hat{\mathcal{R}}$ the set of singular price-income vectors. One can show that $\hat{\Sigma}$ is closed with measure zero in $\hat{X}$. This gives a precise sense to the adjective exceptionnal used above.

We say that two points $\hat{x}$ and $\hat{y}$ in $\hat{\mathcal{R}}$ are arcconnected if there exists a smooth path in $\hat{\mathcal{R}}$ joining $\hat{x}$ and $\hat{y}$. This defines an equivalence relation of which equivalence classes are the connected components of $\hat{\mathcal{R}}$ ($\hat{\mathcal{R}}$ being open, arcconnectedness and connectedness are equivalent). The number of B.C.P.E. allocations, being locally constant, is therefore constant over every connected component of $\hat{\mathcal{R}}$. The next theorem provides a relationship between $\hat{P}$ and the connected components of $\hat{\mathcal{R}}$.

THEOREM 3. <u>The set of price-income equilibria $\hat{P}$ belongs to one connected component of $\hat{\mathcal{R}}$. Furthermore, there is only one B.C.P.E. allocation associated with every price-income vector $\hat{x}$ in that component.</u>

There is uniqueness of the B.C.P.E. allocation if the price-income vector $x = (p , w_1 , w_2 ,\dots, w_m)$ belongs to the connected component of $\hat{\mathcal{R}}$ containing $\hat{P}$, i.e. if the difference vector

$$f_1(p , w_1) + f_2(p , w_2) +\dots+ f_m(p , w_m) - r$$

is small enough. On the other hand, multiple B.C.P.E. allocations may be observed if the above difference vector is large. This, however, states only a possibility ; some difference vectors may be very large while uniqueness of B.C.P.E. allocations still holds. Anyway, when x varies while the difference vector remains large, one may observe catastrophes by which we designate the phenomenon occuring when two smooth selections of B.C.P.E. allocations vanish, a phenomenon which can be observed only when $\hat{x}$ crosses the set $\hat{\Sigma}$ of singular price-income vectors.

4. THE DUALITY THEORY.

Though proving directly the properties of B.C.P.E. allocations is possible, we prefer to develop an alternative method based on a duality between the theory of Walrasian equilibria and the theory of B.C.P.E. allocations. This duality theory can be viewed as an extension of Hotelling's and Roy's duality involving direct and indirect utility functions from the one-consumer case (see e.g. [7]) to any number of agents.

PART A. *Walrasian equilibrium theory and B.C.P.E. allocation theory reconsidered*.

The set X , $\mathfrak{X}$, P , and $\mathcal{P}$ being given, we define the abstract theories (I) and (W) as follows :

(I) : study of the set $P \cap A$, when A varies in $\mathfrak{X}$;

(W) : study of the set $\{A \in \mathcal{P} \mid x \in A$ where $x \in X$ is given$\}$,

when x varies in X.

The abstract theory (I) is just a reformulation of the theory of B.C.P.E. allocations. If in the abstract theory (W), one considers $x \in X$ as a vector of initial endowments, then, the price vector $p \in S$ associated with $A \in \mathcal{P}$ through its canonical equation system is an equilibrium price vector associated with x. Therefore, the abstract theory (W) appears to be equivalent to Walrasian equilibrium theory.

PART B. THE DUAL SPACE.

We reformulate the abstract theories (I) and (W) in the space $\hat{X}$. For a matter of convenience, we shall denote these theories $(\hat{I})$ and $(\hat{W})$ respectively.

We have already defined $\hat{P}$ and $\hat{X}$, let us define $\hat{\mathcal{X}}$ and $\hat{\mathcal{P}}$.

Definition of $\hat{\mathcal{X}}$: Let $\hat{\mathcal{X}}$ consist of the affine subspaces of $\hat{X}$ not perpendicular to S and having dimension $\ell-1$. Therefore, the affine subspace $\hat{A}$ of $\hat{X}$ belongs to $\hat{\mathcal{X}}$ if and only if it can be defined in $S \times \mathbb{R}^m$ by an equation system of the type

$$w_1 = p.x_1 \ , \ w_2 = p.x_2 \ , \ldots, \ w_m = p.x_m$$

where $w_1, w_2, \ldots, w_m$ and p are variable and $x_1, x_2, \ldots, x_m$ fixed. When it exists, such an equation system is unique which enables us to identify $\hat{\mathcal{X}}$ with X by the mapping $\hat{A} : X \to \hat{\mathcal{X}}$ where

$$\hat{A}(x) = \{\hat{x} = (p \ , w_1 \ , w_2 \ , \ldots, w_m) \in \hat{X} \mid p.x_1 = w_1 \ , \ p.x_2 = w_2 \ , \ldots, \ p.x_m = w_m\}$$

<u>Definition of</u> $\hat{\mathcal{P}}$: We define $\hat{\mathcal{P}}$ as the subset of $\hat{\mathfrak{X}}$ consisting of the spaces $\hat{A}(x)$ when x describes the set of Pareto optima P in X, i.e. $\hat{\mathcal{P}} = \hat{A}(P)$.

Although the definitions of $\hat{P}$, $\hat{\mathfrak{X}}$, and $\hat{\mathcal{P}}$ may seem quite arbitrary for the time being, we can already define the abstract theories $(\hat{I})$ and $(\hat{W})$ under the following form :

$(\hat{I})$: study of the set $\hat{P} \cap \hat{A}$, when $\hat{A}$ varies in $\hat{\mathfrak{X}}$;

$(\hat{W})$: study of the set $\{\hat{A} \in \hat{\mathcal{P}} \mid \hat{x} \in \hat{A}$ where $\hat{x} \in \hat{X}$ is given$\}$, when $\hat{x}$ varies in $\hat{X}$.

We are now going to show that the sets P and $\hat{P}$, $\mathcal{P}$ and $\hat{\mathcal{P}}$ respectively, play similar roles in the spaces X and $\hat{X}$.

PART C. <u>THE DUALITY THEORY</u>.

The indirect utility function $\hat{u}_i : S \times \mathbb{R} \to \mathbb{R}^{\ell}$ associated with u_i is the composite mapping $\hat{u}_i = u_i \circ f_i$; in other words, $\hat{u}_i(p , w_i) = u_i(f_i(p , w_i))$ represents consumer i's utility for the demand vector $f_i(p , w_i)$. Let $\hat{v}_i : \hat{X} \to \mathbb{R}$ denote the mapping define by the formula

$$\hat{v}_i : (p , w_1 , w_2 ,\ldots, w_m) \longmapsto \hat{u}_i(p , w_i) .$$

The definition of Pareto efficiency extends to the case of price-income vectors in the following way :

<u>DEFINITION 3</u>. <u>The price-income vector</u> $\hat{x} \in \hat{X}$ <u>is a Pareto minimum</u> <u>(with respect to the indirect utility functions</u> $\hat{u}_i$) <u>if there is</u> <u>no</u> $\hat{y} \in \hat{X}$ <u>such that</u> $\hat{v}_i(\hat{y}) \leq \hat{v}_i(\hat{x})$ <u>with at least one strict inequality</u>.

Let $\hat{T}_i(\hat{x})$ denote the tangent hyperplane in $\hat{x}$ to the hypersurface $\{\hat{y} \in \hat{X} \mid \hat{v}_i(\hat{y}) = \hat{v}_i(\hat{x})\}$.

Taking into account the equality $g_i(f_i(p , w_i)) = p$, a straightforward calculation shows that the equation of the tangent hyperplane in $\hat{x} = (p , w_1 , w_2 ,\ldots, w_m)$ to the indifference hypersurface $\{\hat{x}' = (p' , w_1' , w_2' ,\ldots, w_m') \in \hat{X} \mid \hat{v}_i(\hat{x}) = \hat{v}_i(\hat{x}')\}$ takes the form

$$w_i' = p'.f_i(p , w_i)$$

(where p' and w_i' are variable).

The intersection of these tangent hyperplanes $\hat{D}(x) = \bigcap_{i=1}^{m} \hat{T}_i(\hat{x})$ is defined by the equation system

$$\begin{cases} p'.f_1(p , w_1) - w_1' = 0 \\ p'.f_2(p , w_2) - w_2' = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p'.f_m(p , w_m) - w_m' = 0 \\ p'.r - w_1' - w_2' - \ldots - w_m' = 0 \end{cases}$$

the last equation being the equation of $\hat{X}$.

The matrix defined by the coefficients of this equation system takes the form

$$M = \begin{vmatrix} f_1^1 & \dots & f_1^{\ell-1} & -1 & 0 & \dots & 0 \\ f_2^1 & \dots & f_2^{\ell-1} & 0 & -1 & \dots & 0 \\ . & \dots & . & . & . & \dots & . \\ f_m^1 & \dots & f_m^{\ell-1} & 0 & 0 & \dots & -1 \\ r^1 & \dots & r^{\ell-1} & -1 & -1 & \dots & -1 \end{vmatrix}$$

and $\dim \hat{D}(\hat{x}) = \dim (S \times \mathbb{R}^m) - \text{rank}(M)$.

Clearly, rank (M) is higher or equal to m, hence $\dim \hat{D}(\hat{x}) = \ell-1$ or $\ell-2$ (and codim $\hat{D}(\hat{x})$ in $\hat{X}$ is equal either to m-1 or m respectively).

<u>LEMMA 1</u>. $\hat{x}$ <u>is a Pareto-minimum if and only if</u> $\dim \hat{D}(\hat{x}) = \ell-1$.

This lemma characterizes Pareto minima through the usual first-order conditions. The strict quasi-concavity assumption concerning the utility functions $\hat{u}_i$ is, of course, essential for the sufficiency of the first-order conditions. The lemma will be proved simultaneously with the next theorem. Though $\hat{P}$ has been defined as the set of price-income equilibria, the next theorem establishes a formal equivalence between P and $\hat{P}$.

<u>THEOREM 4</u>. $\hat{P}$ <u>is the set of Pareto minima (with respect to the indirect utility functions</u> $\hat{u}_i$) <u>in</u> $\hat{X}$.

Proof. We shall prove theorem 4 and lemma 1 as follows :

$\hat{x}$ Pareto minimum $\Longrightarrow$ dim $\hat{D}(\hat{x}) = \ell-1 \Longrightarrow \hat{x}$ price-income equilibrium

$\Longrightarrow \hat{x}$ Pareto minimum.

1) Clearly, the necessary first-order conditions for Pareto minimality imply that, if $\hat{x}$ is Pareto minimal, codim $\hat{D}(x) \leq m-1$, or dim $\hat{D}(\hat{x}) \geq \ell-1$, hence dim $\hat{D}(\hat{x}) = \ell-1$.

2) Consider matrix (M). Then, rank (M) is equal to m if and only if its last line is the sum of the m first lines, in other words if we have the relationships

$$r^k = \sum_{i=1}^{m} f_i^k \qquad k = 1 , 2 ,\ldots, \ell-1 .$$

Walras law implies the equality

$$r = \sum_{i=1}^{m} f_i(p , w_i)$$

and $\hat{x}$ is therefore a price-income equilibrium.

3) To prove the last assertion, namely

$\hat{x}$ price-income equilibrium $\Longrightarrow \hat{x}$ Pareto minimum

we consider the mapping $\phi : P \rightarrow \hat{X}$ already used in proposition 1 and 2. By proposition 2, ϕ is a diffeomorphism between P and $\hat{P}$.

Let us consider $\hat{x} \in \hat{P}$ and assume that it is not Pareto minimal. Then there exists a Pareto minimum $\hat{y}$ such that $\hat{v}_i(\hat{y}) \leq \hat{v}_i(\hat{x})$ with at least one strict inequality. We already know that such a $\hat{y}$ belongs to $\hat{P}$, the set of price-income equilibria. Take the inverse image of $\hat{x}$ and $\hat{y}$ by s. Then, $\phi^{-1}(\hat{y})$ is a Pareto optimum since it belongs to P. However, the inequality

$$v_i(\phi^{-1}(\hat{x})) = \hat{v}_i(\hat{x}) \geq v_i(\phi^{-1}(\hat{y})) = \hat{v}_i(\hat{y})$$

shows that $\phi^{-1}(\hat{x})$ Pareto dominates $\phi^{-1}(\hat{y})$, hence a contradiction. Q.E.D.

To develop the duality theory, we now consider the mappings A and $\hat{A}$. Recall that we have the bijections

$$A : \hat{X} \to \mathfrak{X} \quad \text{where } A(\hat{x}) = \{x = (x_1 , x_2 , \dots , x_m) \in X \mid w_1 = p.x_1 , w_2 = p.x_2 , \dots , w_m = p.x_m\}$$

$$\hat{A} : X \to \hat{\mathfrak{X}} \quad \text{where } \hat{A}(x) = \{\hat{x} = (p , w_1 , w_2 , \dots , w_m) \in \hat{X} \mid w_1 = p.x_1 , w_2 = p.x_2 , \dots , w_m = p.x_m\}.$$

LEMMA 2. The inclusion $x \in A(\hat{x})$ is equivalent to the inclusion $\hat{x} \in \hat{A}(x)$.

Proof. Obvious.

LEMMA 3. We have $\hat{A}(P) = \hat{\mathcal{P}}$ and $A(\hat{P}) = \mathcal{P}$.

Proof. The first equality is just the definition of $\hat{\mathcal{P}}$. Now, let $\hat{x} \in \hat{P}$. The vector

$$f(\hat{x}) = (f_1(p , w_1) , f_2(p , w_2) , \dots , f_m(p , w_m))$$

belongs to X since we have chosen $\hat{x}$ in $\hat{P}$ (i.e.

$$f_1(p , w_1) + f_2(p , w_2) + \dots + f_m(p , w_m) = r).$$

Furthermore, $f(\hat{x})$ is the Pareto optimum supported by $p , w_1 , w_2 , \dots , w_m$. Therefore, the space $D(f(\hat{x}))$ is defined by

$$D(f(\hat{x})) = \{x = (x_1 , x_2 , \dots , x_m) \in X \mid p.x_1 = w_1 , p.x_2 = w_2 , \dots , p.x_m = w_m\}$$

which proves that $D(f(\hat{x})) = A(\hat{x})$ for any $\hat{x} \in \hat{P}$. Therefore, we have proved the equality $A(\hat{P}) = \hat{\mathcal{P}}$. Q.E.D.

We now state the duality theorem.

THEOREM 5. <u>(I) and $(\hat{W})$, (W) and $(\hat{I})$ respectively, are isomorphic by the mappings A and $\hat{A}$.</u>

Proof. Using the equality $\mathcal{X} = A(\hat{X})$, (I) becomes the study of the set $P \cap A(\hat{x})$ when $\hat{x}$ varies in $\hat{X}$. From lemma 2 $x \in A(\hat{x})$ is equivalent to $\hat{x} \in \hat{A}(x)$; clearly, $x \in P$ is equivalent to $\hat{A}(x) \in \hat{A}(P) = \hat{\mathcal{P}}$. We restate (I) as the study of the set $\{\hat{A}(x) \in \hat{\mathcal{P}}$ such that $\hat{x} \in \hat{A}(x)\}$, when $\hat{x}$ varies in $\hat{X}$. Therefore,(I)becomes the study of $\{\hat{A} \in \hat{\mathcal{P}}$ such that $\hat{x} \in \hat{A}\}$, when $\hat{x}$ varies in $\hat{X}$, which is theory $(\hat{W})$. A similar proof applies for (W) and $(\hat{I})$. Q.E.D.

The identity between (I) and $(\hat{W})$, respectively (W) and $(\hat{I})$, provides a duality theory between (W) and (I). As a consequence, a property established for (W) yields a dual property for (I) and vice-versa. This viewpoint has been already encountered in equilibrium analysis where several results on the number of equilibria and on singular economies, i.e. properties of (W), have been established with the help of their formulation through $(\hat{I})$. Furthermore, any statement or proof of (I), resp. (W), has a formal analogue in $(\hat{I})$, resp. $(\hat{W})$. For example, Walrasian equilibrium theory, i.e. (W), provides a list of properties of $(\hat{W})$ by easy transcriptions and the duality theorem enables us to deduce from $(\hat{W})$ a list of properties of (I), i.e. of B.C.P.E. allocations.

5. PROOFS OF THEOREM 1 TO 3.

Theorems 1 to 3 deal with properties of the theory (I) ; by the duality theorem, they can be reformulated as properties of the theory $(\hat{W})$. Now, the theory $(\hat{W})$ is built like the theory of Walrasian equilibria (W), the underlying spaces being different . Therefore, to prove properties of the theory $(\hat{W})$, it is sufficient fo adapt to the case of $(\hat{W})$ proofs already established in the context of (W). This is the approach we are going to follow to prove theorems 1 to 3.

A. - The differential setting.

Equilibrium manifold. Let $\hat{E}$ be the subset of the Cartesian product $P \times \hat{X}$ consisting of the pairs $(x , \hat{x})$ such that $x \in A(\hat{x})$. The relationship $x \in A(\hat{x})$ being equivalent to $\hat{x} \in \hat{A}(x)$, we have

$$\hat{E} = \{(x,p,w_1,w_2,\dots,w_m) \in P \times \hat{X} \mid w_1 = p.x_1 \ , \ w_2 = p.x_2 \ ,\dots, \ w_m = p.x_m\}.$$

In other words, $w_1 \ , w_2 \ ,\dots, w_m$ are smooth functions of x and p and is E smoothly parametrized by $x = (x_1 \ , x_2 \ ,\dots, x_m) \in P$ and by $p \in S$. Therefore $\hat{E}$ is a smooth submanifold of $P \times \hat{X}$ diffeomorphic to $P \times S$, hence to $\mathbb{R}^{\ell+m-2}$. This manifold $\hat{E}$ corresponds in $(\hat{W})$ to the equilibrium manifold of Walrasian equilibrium theory (W).

Debreu mapping. The "dual" Debreu mapping $\hat{\tilde{\pi}} : \hat{E} \to \hat{X}$ is obtained by restricting the natural projection $(x , \hat{x}) \longmapsto \hat{x}$ to the equilibrium manifold $\hat{E}$. This mapping is smooth as the restriction of a smooth mapping (the natural projection) to a submanifold of $P \times \hat{X}$. The relationship

$$\hat{\tilde{\pi}}^{-1}(\hat{x}) = \{(x , \hat{x}) \in P \times \hat{X} \mid x \in A(\hat{x})\} = \{P \cap A(\hat{x})\} \times \{\hat{x}\}$$

reminds us that the study of B.C.P.E. allocations amounts to the study of $\hat{\tilde{\pi}}$.

The Debreu mapping is also proper, i.e. the inverse image of every compact set is compact. To prove this property, let K be a compact subset of $\hat{X}$. Consider $\hat{\tilde{\pi}}^{-1}(K) = \{(x , \hat{x}) \in \hat{E} \mid \hat{x} \in K$ and $x \in A(\hat{x}) \cap P\}$. Clearly, $\hat{\tilde{\pi}}^{-1}(K)$ is closed in $P \times \hat{X}$. To prove that $\hat{\tilde{\pi}}^{-1}(K)$ is bounded, we just need to show that the set $A(K) \cap P$ (where $A(K) = \bigcup_{\hat{x} \in K} A(\hat{x})$) is bounded since we have the inclusion

$$\hat{\tilde{\pi}}^{-1}(K) \subset K \times (A(K) \cap P).$$

It results from (App. 3.4) that we just need to show that the image of $A(K) \cap P$ by the mapping $u : P \to \mathbb{R}^m$ where

$$u(x) = (u_1(x_1) , u_2(x_2) ,\dots, u_m(x_m))$$

is also bounded. Let us define

$$\bar{u}_i = \sup_{\hat{x} \in K} u_i(f_i(p , w_i)).$$

We have $\bar{u}_i \geq u_i(f_i(p , w_i))$ whenever $\hat{x} \in K$. When x belongs to $A(\hat{x})$, we have $p.x_i = w_i$, hence $u_i(f_i(p , w_i)) \geq u_i(x_i)$, and therefore $u(x) \leq (\bar{u}_1 , \bar{u}_2 ,\dots, \bar{u}_m)$ for every $x \in A(K) \cap P$, which proves that $A(K) \cap P$ is bounded from above.

Now let $\hat{x}(i)$ be the unique Pareto optimum such that $u_k(\bar{x}(i)) = \bar{u}_k$, $k \neq i$ (App. 3.3). It results from the definition of $\bar{u}_k$ that any $x \in A(K) \cap P$ satisfies the inequalities

$$u_k(x_k) \leq \bar{u}_k = u_k(\bar{x}(i)) \quad \text{with} \quad k \neq i.$$

This implies the inequality $u_i(x_i) \geq u_i(\bar{x}(i))$ since, otherwise, $\bar{x}(i)$ would Pareto dominate x, which is itself a Pareto optimum, hence a contradiction. Therefore, we have proved that any $x \in A(K) \cap P$ satisfies the inequality

$$u(x) \geq (u_1(\bar{x}(1)), u_2(\bar{x}(2)), \ldots, u_m(\bar{x}(m))) , \quad \text{and}$$

$u(A(K) \cap P)$ is bounded from below.

B. - Applications of elementary differential topology.

We can now apply to $\hat{\tilde{\pi}}$ standard methods of elementary differential topology (see e.g. Milnor [6]). Note that the properties established in [6] with the assumption that the source manifold is compact are valid in the context considered here, the mapping $\hat{\tilde{\pi}}$ being proper.

Finite covering of $\hat{\mathcal{R}}$ defined by $\hat{\tilde{\pi}}$.

Consider the set $\hat{\Sigma}$ of singular values of $\hat{\tilde{\pi}}$. It has measure zero by Sard's theorem and is closed as the image of a closed subset by a proper mapping. Its complement $\hat{\mathcal{R}} = \hat{X} \setminus \hat{\Sigma}$, in other words, the set of regular values is open dense and $\hat{\tilde{\pi}}$ defines a finite covering of $\hat{\mathcal{R}}$ (see [6], p. 8). As a consequence, $\hat{\tilde{\pi}}^{-1}(\hat{x})$ is finite for every $\hat{x} \in \hat{\mathcal{R}}$ and constant over every connected component of $\hat{\mathcal{R}}$.

Uniqueness of the inverse image of a price-income equilibrium.

Let $\hat{x} \in \hat{P}$ and let us prove that $\hat{\tilde{\pi}}^{-1}(\hat{x})$ has just one element.

Since $\hat{x} \in \hat{P}$, the allocation $f(\hat{x}) = (f_1(p , w_1) , f_2(p , w_2) ,\ldots,$ $f_m(p , w_m))$ is feasible, i.e. $f_1(p , w_1) + f_2(p , w_2) + \ldots +$ $f_m(p , w_m) = r$. Furthermore, $f(\hat{x})$ is clearly a Pareto optimum. We deduce from $f(\hat{x}) \in A(\hat{x})$ that $(f(\hat{x}) , \hat{x})$ belongs to $\hat{\tilde{\pi}}^{-1}(\hat{x})$. Let now $(x' , \hat{x}) \in \hat{\tilde{\pi}}^{-1}(\hat{x})$. By definition, we have

$$p.x'_1 = w_1 , p.x'_2 = w_2 , \ldots , p.x'_m = w_m$$

so that $u_1(x'_1) \leq u_1(f_1(p , w_1)) , u_2(x'_2) \leq u_2(f_2(p , w_2)) ,\ldots,$ $u_m(x'_m) \leq u_m(f_m(p , w_m))$. Since x' is Pareto efficient by definition, we necessarily have the equality $u_i(x'_i) = u_i(f_i(p , w_i))$ where $i = 1 , 2 ,\ldots, m$, and, therefore, $x' = f(\hat{x})$, which proves that

$$\hat{\tilde{\pi}}^{-1}(\hat{x}) = \{(f(\hat{x}) , \hat{x})\}.$$

$\hat{P}$ <u>belongs to one connected component of</u> $\hat{\mathcal{R}}$.

We already know (propositions 1 and 2) that $\hat{P}$ is diffeomorphic to $\mathbb{R}^{m-1}$, hence connected. To show that $\hat{P}$ belongs to one connected component of $\hat{\mathcal{R}}$, we need to prove that $\hat{P}$ is included in $\hat{\mathcal{R}}$. Let us check that no point of $\hat{\tilde{\pi}}^{-1}(\hat{P})$ is critical for $\hat{\tilde{\pi}}$.

- 1) We know that $\hat{\tilde{\pi}}^{-1}(\hat{P}) = \{(f(\hat{x}) , \hat{x}) \mid \hat{x} \in \hat{P}\}$, hence $\hat{\tilde{\pi}}^{-1}(\hat{P})$ is also the set

$$\{(x , \phi(x)) \mid x \in P\}$$

where we recall that $\phi(x) = (g(x) , g(x).x_1 ,\ldots, g(x).x_m)$, $g(x)$ being the price vector supporting the Pareto optimum $x \in P$. Therefore, we just need to check that no point $(x , \phi(x))$, where $x \in P$, is critical for $\hat{\tilde{\pi}}$.

- 2) We now give a geometrical characterization of regularity for $\hat{\tilde{\pi}}$. Namely, $(x , \hat{x}) \in \hat{E}$ is regular (resp. critical) for $\hat{\tilde{\pi}}$ if and only if the tangent space $T_x(P)$ to P at x considered as a subset of X and the subset
$A(\hat{x}) = \{x = (x_1 , x_2 , \ldots , x_m) \in X \mid p.x_1 = w_1 , \ldots , p.x_m = w_m\}$
of X are transverse (resp. not transverse).
To prove this property, parametrize E by $x \in P$ and $p \in S$. Then, $\hat{\tilde{\pi}}$ becomes the mapping
$(x , p) \longrightarrow (p , w_1 = p.x_1 , w_2 = p.x_2 , \ldots , w_m = p.x_m)$.
This mapping is defined from $X \times S$ into $\hat{X}$.
An easy calculation shows that its tangent mapping in $(x , \hat{x})$ restricted to the tangent space $T_x(P)$ is surjective if and only if $T_x(P)$ and $A(\hat{x})$ are transverse.
- 3) To prove that $(x , \phi(x))$ is regular for $\hat{\tilde{\pi}}$, we just need to check the transversality at a Pareto optimum $x \in P$ of the tangent space $T_x(P)$ and of the set $A(\phi(x)) = \{x' = (x'_1 , x'_2 , \ldots , x'_m) \in X \mid p.x'_1 = w_1 , \ldots , p.x'_m = w_m\}$ where p is the price vector supporting the Pareto optimum x , and where $w_i = p.x_i$ with $i = 1 , 2 , \ldots , m$. This property is well-known (see e.g. Smale [8], proposition 4,).

The degree concept.

The topological degree of $\hat{\tilde{\pi}}$ is an invariant defined as the parity of the number of elements of $\hat{\tilde{\pi}}^{-1}(\hat{x})$ when $\hat{x}$ is a regular value (see e.g. Milnor [6], p. 24). Therefore, by taking $\hat{x} \in \hat{P}$, one sees that the number of B.C.P.E. allocations associated with a regular price-income vector $\hat{x} \in \mathcal{R}$ is odd. Furthermore, this implies that $\hat{\tilde{\pi}}$ is surjective, i.e. that there always exists a B.C.P.E. allocation associated with any $\hat{x} \in \hat{X}$ (otherwise, assume that $\hat{\tilde{\pi}}$ is not surjective ; if $\hat{x}$ does not belong to Im $\hat{\tilde{\pi}}$, then $\hat{x}$ is a regular value and $\# \, \hat{\tilde{\pi}}^{-1}(\hat{x}) = 0$, a contradiction).

The above developments parallel the methods of equilibrium analysis from the differential viewpoint.
It is clear, however, that a property of Walrasian equilibria, i.e. (W), need not be formulated in a differential setting to provide a dual statement, i.e. a property of B.C.P.E. allocations. The differential setting was chosen here both for a matter of convenience and for proving

6. CONCLUSION.

The main purpose of this article has been a study of B.C.P.E. allocations from a static viewpoint in pure exchange economies where preferences satisfy rather strong assumptions like smoothness, strict convexity, etc ... Therefore, this study calls for direct extensions in at least two directions. One consists in weakening the various assumptions concerning the pure exchange economies. The results of this paper seem to be robust enough to be valid in a more general framework. The method of proof, however, namely the duality theory would become more delicate to handle in such a setting. Another direction for further research consists in introducing production, public goods, etc ..., in the analysis of B.C.P.E. allocations. Once again, it seems that a duality theory will hold, but in a more complicated form than for the pure exchange case.

APPENDIX

This appendix contains a unified treatment of some now well-known properties of Pareto optima which, however, seem to lack convenient references.

We recall that consumers' preferences (more precisely, their utility functions) satisfy the properties 1 to 5 of section 2. Let Q be the subset of $(\mathbb{R}^{\ell})^m$ consisting of the Pareto optima associated with a given $r \in \mathbb{R}^{\ell}$, when the vector r varies in $\mathbb{R}^{\ell}$. Thus,

$$Q = \left\{ x = (x_1 , x_2 , \ldots , x_m) \in (\mathbb{R}^{\ell})^m \;\middle|\; \begin{array}{l} \text{there is no } x' = (x_1' , x_2' , \ldots , x_m') \\ \text{such that } \sum x_i' = \sum x_i \text{ and} \\ u_i(x_i') \geq u_i(x_i) \text{ with at least one} \\ \text{strict inequality.} \end{array} \right\}$$

1 - STRUCTURE OF Q.

Let $f : S \times \mathbb{R}^m \to (\mathbb{R}^{\ell})^m$ be defined by the formula

$$f(p , w_1 , w_2 , \ldots , w_m) = (f_1(p , w_1) , f_2(p , w_2) , \ldots , f_m(p , w_m))$$

1.1. The mapping $f : S \times \mathbb{R}^m \to (\mathbb{R}^{\ell})^m$ is an embedding. Its image $f(S \times \mathbb{R}^m)$ is equal to Q.

Recall that an embedding is an immersion which defines a homeomorphism between the domain and its image ([4], (16.8.4.).

Clearly, $f(p , w_1 , w_2 , \ldots , w_m)$ is Pareto efficient for the total resources $r = f_1(p , w_1) + f_2(p , w_2) + \ldots + f_m(p , w_m)$. Therefore, we have $f(S \times \mathbb{R}^m) \subset Q$. The mapping

$\phi : x \longmapsto (g(x) , g(x).x_1 , \ldots , g(x).x_m)$ where $g(x)$ is the price vector supporting the Pareto optimum x is defined on Q and takes its values in $S \times \mathbb{R}^m$. By composing ϕ and g , one immediately checks that ϕ is a continuous inverse of f. Therefore, we have proved that $f(S \times \mathbb{R}^m) = Q$ and that f is a homeomorphism between $S \times \mathbb{R}^m$ and Q.

1.2. The set Q is a smooth submanifold of $(\mathbb{R}^\ell)^m$ diffeomorphic to $\mathbb{R}^{\ell+m-1}$.

It results from (1.1) that Q is the image of an embedding. Hence, by [4] (16.8.4.) , Q is a submanifold of $(\mathbb{R}^\ell)^m$ diffeomorphic to $S \times \mathbb{R}^m$, hence to $\mathbb{R}^{\ell+m-1}$.

1.3. The mapping $\phi : Q \to S \times \mathbb{R}^m$ is a diffeomorphism.

We already know that ϕ is a homeomorphism. It is a diffeomorphism as the inverse of the diffeomorphism $f : S \times \mathbb{R}^m \to Q$.

2 - A FUNDAMENTAL DIFFEOMORPHISM.

Let $t : Q \to \mathbb{R}^\ell \times \mathbb{R}^{m-1}$ be the restriction to Q of the mapping

$$x = (x_1 , x_2 ,\ldots, x_m) \longmapsto \left(\sum_{i=1}^{m} x_i , u_1(x_1) ,\ldots, u_{m-1}(x_{m-1})\right).$$

2.1. The mapping t is a diffeomorphism.

Let $Z = \mathbb{R}^\ell \times \mathbb{R}^{m-1}$. The mapping t is obtained by composing $a : Q \to Z \times \mathbb{R}$ where

$$a(x) = \left(\sum x_i , u_1(x_1) , u_2(x_2) ,\ldots, u_m(x_m)\right)$$

with the natural projection

$$b : Z \times \mathbb{R} \to Z$$

The proof proceeds in two steps. In the first one, we show that t is a continuous bijection. In the second one, we show that the jacobian determinant of t (in a suitable coordinate system) is $\neq 0$.

Step 1.

a is an injection. Assume $a(x) = a(y)$, in other words $\sum x_i = \sum y_i$ and $u_i(x_i) = u_i(y_i)$ where $i = 1, 2, \dots, m$. Let $z = (x + y)/2$; if $x_i \neq y_i$ for some i, then $u_i(z_i) > u_i(x_i) = u_i(y_i)$ by the strict quasi-concavity of u_i (Assumption 3 in section 2). This contradicts the Pareto optimality of x and y.

b restricted to $a(Q)$ is injective. Let x and y in Q be such that $b(a(x)) = b(a(y))$, i.e. $\sum x_i = \sum y_i$ and $u_i(x_i) = u_i(y_i)$ where $i = 1, 2, \dots, m-1$. Assume that $u_m(x_m) \neq u_m(y_m)$, for example $u_m(x_m) > u_m(y_m)$; this contradicts the Pareto optimality of y. Therefore, we have $u_m(x_m) = u_m(y_m)$, i.e. $a(x) = a(y)$.

t is a bijection. We already know that $t = b \circ a$ is an injection. Let us show that t is surjective. Let $z = (r, u_1, \dots, u_{m-1}) \in Z$ be fixed and consider the optimization problem

Find $x = (x_1, \dots, x_m)$ which maximizes $u_m(x_m)$ subject to the constraints

$$\begin{cases} u_i(x_i) \leq u_i & i = 1, \dots, m-1 \\ \sum x_i \leq r . \end{cases}$$

Given the assumptions on preferences, existence of a solution is straightforward ; by the monotonicity property of preferences the constraints are binding hence $u_i(x_i) = u_i$ and $\sum x_i = r$. It is straightforward that a solution x is a Pareto efficient.

The mapping t is smooth as the restriction of a smooth mapping to a submanifold. Being a bijection, t is going to be a diffeomorphism if and only if it has a smooth inverse, which we prove in step 2.

Step 2.

Using the diffeomorphism $f : S \times \mathbb{R}^m \to Q$, we just have to show that $t \circ f : S \times \mathbb{R}^m \to Z$ has everywhere a non-zero Jacobian determinant. From the relationships grad $u_i(f_i(p\ ,\ w_i)) = \lambda_i\ p$; $p.\frac{\partial f_i}{\partial p_k} = - f_i^k$ and $p\ \frac{\partial f_i}{\partial w_i} = 1$ (the last two equalities are deduced from Walras law $p.f_i(p\ ,\ w_i) = w_i$ by taking suitable derivatives), we have to show that the following determinant is $\neq 0$:

$$\begin{vmatrix}
\sum_i \frac{\partial f_i^1}{\partial p_1} & \cdots & \sum_i \frac{\partial f_i^1}{\partial p_{\ell-1}} & \frac{\partial f_1^1}{\partial w_1} & \cdots & \frac{\partial f_{m-1}^1}{\partial w_{m-1}} & \frac{\partial f_m^1}{\partial w_m} \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\sum_i \frac{\partial f_i^\ell}{\partial p_1} & \cdots & \sum_i \frac{\partial f_i^\ell}{\partial p_{\ell-1}} & \frac{\partial f_1^\ell}{\partial w_1} & \cdots & \frac{\partial f_{m-1}^\ell}{\partial w_{m-1}} & \frac{\partial f_m^\ell}{\partial w_m} \\
- f_1^1 & \cdots & - f_1^{\ell-1} & 1 & \cdots & 0 & 0 \\
\cdot & \cdots & \cdot & & \cdots & \cdot & \cdot \\
- f_{m-1}^1 & \cdots & - f_{m-1}^{\ell-1} & 0 & \cdots & 1 & 0
\end{vmatrix}$$

After multiplying line K by p_k where $k = 1, 2, \ldots, \ell$ and adding up, one obtains the line

$$(- \sum_i f_i^1\ ,\ - \sum_i f_i^2\ , \ldots ,\ - \sum_i f_i^{\ell-1}\ ,\ 1\ \ldots\ 1).$$

We do not change the value of the Jacobian determinant by replacing the ℓ-th line by the above one. Now, substracting from the ℓ-th line the last (m-1) lines, one obtains the line

$$(- f_m^1 , - f_m^2 , \dots , - f_m^{\ell-1} , 0 , \dots , 0 , 1).$$

Putting this line from the ℓ-th position to the last one, one obtains the determinant

$$\begin{vmatrix} \sum_i \frac{\partial f_i^1}{\partial p_1} & \dots & \sum_i \frac{\partial f_i^1}{\partial p_{\ell-1}} & \frac{\partial f_1^1}{\partial w_1} & \dots & \frac{\partial f_{m-1}^1}{\partial w_{m-1}} & \frac{\partial f_m^1}{\partial w_m} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot \\ \sum_i \frac{\partial f_i^{\ell-1}}{\partial p_1} & \dots & \sum_i \frac{\partial f_i^{\ell-1}}{\partial p_{\ell-1}} & \frac{\partial f_1^{\ell-1}}{\partial w_1} & \dots & \frac{\partial f_{m-1}^{\ell-1}}{\partial w_{m-1}} & \frac{\partial f_m^{\ell-1}}{\partial w_m} \\ - f_1^1 & \dots & - f_1^{\ell-1} & 1 & \dots & 0 & 0 \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot \\ - f_{m-1}^1 & \dots & - f_{m-1}^{\ell-1} & 0 & \dots & 1 & 0 \\ - f_m^1 & \dots & - f_m^{\ell-1} & 0 & \dots & 0 & 1 \end{vmatrix}$$

which is equal up to a + or - sign to the Jacobian determinant of t.

The new determinant takes the form $\begin{vmatrix} A & B \\ C & I \end{vmatrix}$

where I the mxm identity matrix. We are now going to cancel our the elements of B by suitable combinations of lines. Thus, multiply line

$(\ell-1) + j$ by $\dfrac{\partial f_j^k}{\partial w_j}$. Substract the result from line k and perform these operations for all k and j. One gets a determinant of the form $\begin{vmatrix} M & 0 \\ * & I \end{vmatrix}$, equal to det (M), where M is a $(\ell-1) \times (\ell-1)$ matrix which actually is the sum of the Slutsky matrices of every consumer, i.e.

$$M = \sum M_i(p\ ,\ w_i)$$

where

$$M_i(p\ ,\ w_i) = \left(\frac{\partial f_i^j(p\ ,\ w_i)}{\partial p_k} + f_i^k(p\ ,\ w_i) \frac{\partial f_i^j(p\ ,\ w_i)}{\partial w_i} \right)_{k,j}$$

and $1 \leq k\ ,\ j \leq \ell - 1$.

The Slutsky matrices $M_i(p\ ,\ w_i)$ being symmetric and negative definite with our assumptions concerning consumers' preferences, M is symmetric and negative definite. Therefore, det (M) is $\neq 0$, which proves the step 2.

3 - THE SET OF PARETO OPTIMA P.

Let now $r = \sum\limits_{i=1}^{m} x_i$ be fixed. Recall that

$$X = \{x = (x_1\ ,\ x_2\ ,\ldots,\ x_m) \in (\mathbb{R}^{\ell})^m \mid \sum x_i = r\} \quad \text{and} \quad P = Q \cap X$$

denotes the set of Pareto optima in X.

3.1. The set P of Pareto optima in X is a smooth submanifold of X diffeomorphic by the mapping $x \longmapsto (u_1(x_1)\ ,\ u_2(x_2),\ldots,\ u_{m-1}(x_{m-1}))$ to $\mathbb{R}^{m-1}$.

Note that P is the inverse image of $\{r\} \times \mathbb{R}^{m-1}$ by the diffeomorphism t.

3.2. The set P is diffeomorphic to $\hat{P}$ by the mapping

$\phi : x \longmapsto (g(x), g(x).x_1, \ldots, g(x).x_m)$.

This is a straightforward consequence of (3.1) and of the proof of (1.1).

3.3. Let $u_1, u_2, \ldots, u_{m-1}$ be arbitrarily given real numbers. Then, there exists a unique Pareto optimum $x \in P$ such that $u_i(x_i) = u_i$ where $i = 1, 2, \ldots, m-1$.

This results from (3.1.).

Note that (3.3.) extends to the case where the utility levels of any (m-1) consumers (not necessarily the first (m-1) ones) are given.

3.4. A closed subset of the set of Pareto optima P is compact if and only if its image by the mapping : $x \longmapsto (u_1(x_1), u_2(x_2), \ldots, u_m(x_m))$ is bounded.

Let $K \subset P$ be closed. If K is compact then u(K) is compact, hence bounded in $\mathbb{R}^m$. Vice versa, assume that u(K) is bounded. Then the image of K by the mapping $x \longmapsto (u_1(x_1), \ldots, u_{m-1}(x_{m-1}))$ is also bounded. From (3.1.), this set being homeomorphic to K is closed, hence compact, and therefore K itself is compact.

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