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GRADIENT PROJECTION ALGORITHMS FOR PROBLEMS IN PUBLIC FINANCE : A NOTE

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I - INTRODUCTION

The recent literature on theoretical public finance has shown an increased concern for the problems of gradual tax reform, putting a particular emphasis on the local analysis of the feasible and desirable moves of the system.

Desirability itself can be evaluated according to different criteria. Taking the example of the rather similar models studied in Dixit (1978), Diewert (1978), Guesnerie (1977),one sees that the Pareto criterion has been considered by the latter one, when the others advocate a local optimisation of the directions of tax changes[1]. Dixit looks for a "best small improvement" with respect to a social welfare index when Diewert considers an "optimal tax perturbation problem" associated with linearized objective function and constraints.

These procedures will remind the reader of the classical algorithm of search of the direction of steepest constrained ascent in applied mathematics. Such a direction is obtained by projecting the gradient of the objective function on the linear local approximation to the feasible set ; this is the gradient projection method or the gradient projection algorithm.

A further brief exploration of the interest of the gradient projection algorithm in theoretical public finance[2] is the main purpose of this note. Three different classical models will be considered and for each of them dynamic processes will be exhibited which define a continuous time[3] version of the gradient projection algorithms. At variance with Diewert who is mainly concerned with the existence of a solution for his optimal tax perturbation problem or with Dixit who concentrates on the desirability of production efficiency, we will focus the attention on explicit tax revision formulas and one will put the emphasis on developing intuition of their *economic meaning* (Section 3, 4,5). Moreover, we will check that under reasonable assumptions the processes which are considered have satisfactory existence and convergence properties (section 6).

The paper is precisely built as follows.
In section 2, notation common to all models is presented. In section 3, a model of the Diamond-Mirrlees type is considered, and in addition to the economic discussion of tax revision formulas, a brief discussion of the similarities and differences between the gradient projection method and Dixit's best small improvement problem is presented. In section 4, the model incorporates a public production sector and focuses on the analysis of the best investment policy of the public firm outside the optimum. Section 5 is concerned with the application of the method to a one consumer model with price distortions. Section 6 provides a more careful mathematical analysis of the processes and provides existence and convergence theorems.

II - PRELIMINARIES

Notation is close to Diamond-Mirrlees'one (1971) and similar to Guesnerie's one (1977)[4].
There are n commodities and $\mathcal{P}$ is the positive orthant $\mathring{\mathbb{R}}^n_+$.
Consumers are indexed by $h = 1,\ldots,m$. x_h is the consumption bundle of consumer h.
u_h being a utility function for h, q a consumption price vector, R_h an exogeneous income, consumer h solves the program Max $u_h(x_h) \mid q \cdot x_h \leq R_h$.
The solution of this program, supposed unique, is denoted $x_h(q,R_h)$ (when $R_h = 0$, which is the case of the models of section 3,4 , it will be denoted $x_h(q)$). x_h is supposed to be a C^2 function on the interior of its domain (and particularly for $q \in \mathcal{P}$, $R_h = 0$) . $V_h = u_h(x_h(q,R_h))$ is the indirect utility function. $x = \sum_h x_h$ is the total demand function.
$\overline{\partial X}$ denotes the (n×n) jacobian matrix associated with the partial derivatives of x with respect to prices. x_R is the vector of derivatives with respect to R.

There is one private production sector with production set Y, the production plan of which is denoted y . Faced with the production price system p, the private sector has a competitive supply denoted

$\eta(p)$ which is single valued and C^2 on $\mathcal{P}$. $\overline{\partial\eta}$ designates its jacobian matrix which will be supposed here of rank n-1 for any p in $\mathcal{P}$. Then, it is known that $\overline{\partial\eta}$ defines a one to one mapping between the vectors δu s.t. $p\cdot\delta u = 0$ and the vectors p such that $p\cdot\delta p = 0$. The inverse of $\overline{\partial\eta}$, (restricted to the appropriate δu) is denoted $\widetilde{\partial\eta}^{-1}$ and one notes $\delta p = \widetilde{\partial\eta}^{-1}\,\delta u$ (see Guesnerie (1977)).

From an economic point of view, this assumption means that the private sector can be induced, through a small change of the production prices (and a unique change if p is normalized), to meet any small change in net demand which has a zero value measured with the production price system.

Finally, there is a public production sector controlled by the Government, the production plan of which is denoted z . z satisfies $g(z) \leq 0$ where g is a C^2 function. ∇g denotes the gradient of g .

For general comments on the policy assumptions in this framework the reader is referred to Diamond-Mirrlees (1971) or Guesnerie (1977).

III - THE GRADIENT PROJECTION METHOD IN A MODEL WITHOUT PUBLIC SECTOR.

In this section we assume $R_h = 0$, $\forall$ h .

Suppose that at time t the system is in a situation of tight equilibrium :

$$\sum_h x_h(q(t)) = \eta(p(t)) \tag{1}$$

Consider $p\cdot\overline{\partial X}(q) \stackrel{\text{def}}{=} \phi(p,q)$ (2)

An interpretation of ϕ can be proposed from the following remark : suppose that some policy change induces a change in aggregate consumption δX . If $p\cdot\delta X < 0$ (resp. > 0), meeting demand would require that the production plan shift into the interior (resp. outside) the production set. $- p\cdot\delta X$ can be considered a gain and $p\cdot\delta X$ a cost associated with the policy and measured with the production prices. They will be termed (equivalent) production gains or costs.

Hence $-\phi_\ell(p,q) = -\sum_k p_k \frac{\partial X_k}{\partial q_\ell}$ is the (equivalent) production gain or surplus associated with a unit increase of the price of commodity ℓ. Equivalently ϕ may be termed the vector of production costs of price changes.

Following Guesnerie (1977), a direction of price change $\left(\frac{dp}{dt},\frac{dq}{dt}\right)$ is tight equilibrium preserving i.f.f.

$$\phi(p,q)\cdot\frac{{}^t dq}{dt} = 0 \tag{3}$$

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p)\cdot\overline{\partial X}(q)\cdot\frac{{}^t dq}{dt} \tag{4}$$

Given some social welfare function (linear for simplicity) $W = \sum_h \alpha_h u_h$, one notes $W(t) = \sum_h \alpha_h u_h(x_h(q(t))$. It follows from our assumptions that $\frac{dW}{dt} = -\sum_h \beta_h(t)\frac{dq}{dt}\cdot x_h(q(t))$; $\beta_h = \alpha_h \frac{\partial V_h}{\partial R_h}$ being the social marginal value of h's income.

Elementary geometry indicates that the projection of the gradient of W, $-\sum \beta_h x_h$ on the tangent hyperplane to the set of feasible moves $(\{u|\phi\cdot u = 0\})$ is a vector of the form $-\sum \beta_h x_h - \mu\,{}^t\phi$ (see diagram 1). An immediate computation gives μ and

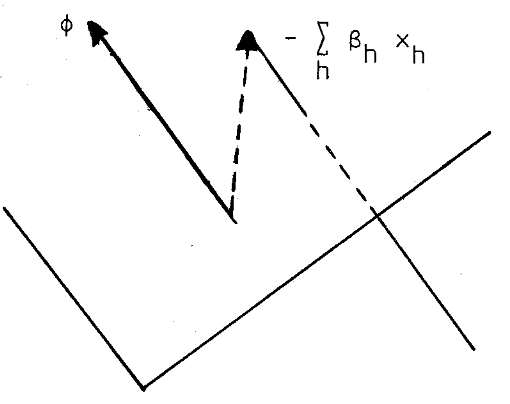

DIAGRAM 1.

PROPOSITION 1 :

The gradient projection method defines the following process

$$\frac{{}^t dq}{dt} = -\sum_h \beta_h(q)\, x_h(q) - \mu(p,q)\,{}^t\phi(p,q) \tag{5}$$

$$\text{with } \mu(p,q) = \frac{\phi(p,q)\cdot\left(-\sum \beta_h(q)\, x_h(q)\right)}{\|\phi(p,q)\|} \tag{6}$$ [5]

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p)\,\overline{\partial X}(q)\,\frac{{}^t dq}{dt} \tag{7}$$

First, let us make clear the relationship of the present approach and of Dixit's analysis (1978) which applies to a quite similar model[6]. Rephrasing Dixit's problem of finding the "best small improvement associated with the euclidean norm" for a general objective function $f(x)$ and a unique constraint $\psi(x) \leq 0$, one obtains the following program :

$$\text{Max } \nabla f(x_o)\cdot x \quad , \quad \nabla\psi(x_o)\cdot x \leq 0 \quad , \; \|x\| \leq 1$$

First order conditions give : $+\nabla f(x_o) - \lambda\, \nabla\psi(x_o) = \mu\, x^*$.

When x^* satisfies $\nabla\psi(x_o)\cdot x^* = 0$, $\lambda = + \dfrac{\nabla\psi(x_o)\cdot\nabla f(x_o)}{\|\nabla\psi(x_o)\|}$ and x^* is proportional to $+\nabla f(x_o) - \dfrac{\nabla\psi(x_o)\cdot\nabla f(x_o)}{\|\nabla\psi(x_o)\|}\nabla\psi(x_o)$, which is nothing else than the gradient projection in x_o .

However when the constraint $\nabla\psi\cdot x \leq 0$ is not binding, x^* is proportional to $\nabla f(x_o)$.

Then, the directions defined by formulas (5), (6) of proposition 1, may differ from Dixit's "best small improvement with the euclidean norm" essentially because we impose here that the above feasibility constraint (3) holds with equality, i.e. because we postulate efficiency in the private production sector, even when it is not desirable (see Guesnerie (1977)).

Actually, Dixit's conclusions on the desirability of productive efficiency have counterparts here : if $\mu(p,q) > 0$ production efficiency postulated here is shown to be desirable in Dixit whose formula (29) defines price changes colinear to those of formulas (5), (6) ; if $\mu(p,q) < 0$, production efficiency is no longer desirable and our tax revisions differ from Dixit's best small improvement .

Let us comment now on proposition 1 and stress the economic meaning of tax revision formulas.

Let us suppose first that $\mu(p,q) > 0$.

The direction of change $\frac{dq}{dt}$ is the combination of two opposite changes

- In the first one $(- \sum \beta_h\, x_h)$, consumption price of commodity ℓ is "reduced" in proportion to the sum of the quantities consumed by each consumer weighted by the social marginal utilities of incomes. Hence, the price of a consumption good $(x_{h\ell} > 0)$ should be reduced more if this good is much consumed or if it is demanded by people who have a high social marginal utility of income.

- In a second step, corrections should be made in order to fit feasibility. These corrections consist in increasing $(\mu > 0)$ the price of commodity ℓ in proportion of the production surplus generated by a unit increase of this price (if $-\phi_\ell > 0$) . Hence the price of a commodity whose price reduction is costly in terms of equivalent production should be increased more.

The appropriate weights of these two opposite changes --the first one which is the more socially desirable and the second one which gives the cheapest correction-- have to be inversely correlated with respectively their production cost $\phi \cdot (-\sum_h \beta_h x_h)$ and gain $\phi \cdot {}^t\phi$. Hence, the directions defined by formulas (5), (6).

μ is negative if the first step price reduction induces a positive production surplus $\phi \cdot (-\sum \beta_h x_h)$, i.e. if it releases, instead of consuming, resources. In this case, the second step, which is both costly and socially harmful, would not be desirable.

IV - THE GRADIENT PROJECTION METHOD AND THE BEST INVESTMENT POLICY OF A PUBLIC FIRM.

The model of this section differs from the preceding one in considering one public firm.

At time t, the prevailing equilibrium is tight and public production is efficient i.e.

$$\sum_h x_h(q(t)) = \eta(p(t)) + z(t) \tag{1}$$

$$g(z(t)) = 0 \tag{2}$$

In the same spirit as above, one can define a tight equilibrium preserving direction of prices and public production changes $\left(\frac{dq}{dt}, \frac{dp}{dt}, \frac{dz}{dt}\right)$ as follows :

$$\phi(p,q) \cdot \frac{{}^t dq}{dt} - p \cdot \frac{dz}{dt} = 0 \tag{3}$$

$$ {}^t\nabla g(z) \cdot \frac{dz}{dt} = 0 \tag{4}$$

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p) \left[\overline{\partial X}(q) \cdot \frac{{}^t dq}{dt} - \frac{dz}{dt}\right] \tag{5}$$

Following textbooks, the gradient projection is obtained as

$$\begin{pmatrix} \frac{{}^t dq}{dt} \\ \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} -\sum\limits_h \beta_h x_h \\ \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \\ \nabla g \end{pmatrix} + \nu \begin{pmatrix} {}^t\phi \\ \\ -{}^t p \end{pmatrix} \tag{6}$$

λ and ν being computed for meeting constraints (3) and (4). It comes :

<u>PROPOSITION 2</u> :

Revisions associated with the gradient projection are the following :

$$\frac{{}^t dq}{dt} = -\sum \beta_h(q)\, x_h(q) - \nu(p,q,z)\ {}^t\phi(p,q) \tag{7}$$

$$\frac{dz}{dt} = \nu(p,q,z)\ \mathrm{Proj}_{H(z)}\, p \tag{8}$$

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p) \left[\overline{\partial X}(q) \cdot \frac{{}^t dq}{dt} - \frac{dz}{dt}\right] \tag{9}$$

where $$\nu(p,q,z) = \frac{\phi(p,q) \cdot \left[-\sum \beta_h(q) x_h(q)\right]}{\|\phi(p,q)\| + \|p\| - \frac{[p \cdot \nabla g(z)]^2}{\|\nabla g(z)\|}} \tag{10}$$

and $$\mathrm{Proj}_{H(z)}\, p = -\left[\frac{p \cdot \nabla g(z)}{\|\nabla g(z)\|}\right] \nabla g(z) + {}^t p \tag{11}$$

is the projection of p on the hyperplane

$H(z) = \{u \mid {}^t\nabla g(z) \cdot u = 0\}$.

The most remarkable characteristic of the process exhibited in proposition 2 is given by (8) : supposing $\nu > 0$, one sees that *at each time the "best" direction of public production change is obtained by projecting* p *the production price system on the tangent hyperplane to the public production set, and this conclusion holds irrespective of the specific social weights* β_h . Hence, the production price system which is an adequate reference shadow price system in any second best optimum of this economy, is also an appropriate reference, in the sense just mentioned, outside the optimum all along a process directed to the steepest ascent.
However this latter condition would fail to be true if $\nu(p,q,z) < 0$ where the right direction of change of public production would be the *opposite* of $\mathrm{Proj}_{H(z)}p$. Noting that in formula (10), the denominator is necessarily positive, $\nu < 0 \iff \phi \cdot \left[- \sum \beta_h x_h \right] < 0$. Referring to the discussion of the preceding section, $\nu < 0$ occurs in the following circumstances : if production were blocked in the public sector, production efficiency in the private sector alone would cease to be desirable. Then, although production inefficiency in the private sector remains forbidden, a public production change into the direction of $-p$ provides a substitute for the desirable inward move to the interior of the private production set.

Let us now recapitulate and give an intuitive economic understanding of formulas (7) to (11) when $\nu > 0$. The best move is obtained as the addition of three moves :

- First, a move of public production in the direction $\mathrm{Proj}_{H(z)}p$, with magnitude α . This move generates a social surplus $\alpha \left[- \frac{(p \cdot \nabla g)^2}{\|\nabla g\|} + \|p\| \right]$ (positive if $\alpha > 0$).
- Second, a move of the consumption price system in the direction of $-\phi(p,q)$, *with the same magnitude* α . The social surplus generated is $\alpha \|\phi\|$.
- Third and finally a move of the consumption price system in the direction $-\sum \beta_h x_h$, with magnitude δ . The cost associated is $\Delta \left\{ {}^t\phi \cdot (- \sum \beta_h x_h) \right\}$.

Observing that the first and second step surplus has to be integrally consumed during the third step, α and δ have to be proportional respectively to ${}^t\phi\cdot(-\sum \beta_h x_h)$ and $-\frac{(p\cdot\nabla g)^2}{\|\nabla g\|} + \|p\| + \|\phi\|$. This explains ν in formula (7), (8).

Hence, the mechanisms of revisions are quite similar in proposition 1 of section III and in proposition 2 above : in each case, a "reduction" of prices in the direction of the gradient of the social welfare function is combined with a correction in the direction of the vector of social costs of price changes. However the weights of the two basic moves are modified, as compared to the previous section : the weight of the correction is smaller thanks to the additional surplus generated by the public investment.

V - THE GRADIENT PROJECTION METHOD IN A ONE CONSUMER ECONOMY WITH INITIAL DISTORTIONS.

We consider in this section a one consumer economy in which a price distortion exists and where income can be transferred from the government to the consumer in a lump sum way(7). For such a model the desirability of a partial reduction of price distortion from the distorted initial situation is an issue which has often being considered in the literature. The problem has actually a reasonably simple structure and strong and appealing results have been obtained (see in particular Dixit (1975) for the derivation of the most important ones, and a more comprehensive set of references).

It seems worth exhibiting what is the best infinitesimal change defined in this model by the gradient projection method. We will do that here, while letting the reader compare the results more carefully with classical ones.

Dropping index h , we define, as precedently a tight equilibrium preserving change $\frac{dq}{dt},\frac{dp}{dt},\frac{dR}{dt}$ as satisfying

$$\phi(p,q,R)\cdot{}^t\frac{dq}{dt} + r(p,q,R)\frac{dR}{dt} = 0 \qquad (1)$$

with $\phi(.) = p \cdot \overline{\partial X}(q,R)$ (2) , $r = p \cdot x_R(q,R)$ (3)

$$\frac{{}^t dp}{dt} = \partial \tilde{\eta}^{-1} [\ \overline{\partial X} \cdot \frac{{}^t dq}{dt} + x_R \frac{dR}{dt}] \qquad (4)$$

Considering $\psi(.) = p \cdot [\overline{\partial X}(q,R)]^{U=Cste}$ where $(\overline{\partial X})^{U=Cste}$ is the matrix of compensated derivatives ; putting $\frac{dR'}{dt} = \frac{dR}{dt} - \frac{dq}{dt} \cdot x$ and using Slutsky relations, (1) becomes

$$\psi(p,q,R) \cdot \frac{{}^t dq}{dt} + r(p,q,R) \frac{dR'}{dt} = 0 \qquad (5)$$

With respect to the new set of variables , the price changes $\frac{{}^t dq}{dt}$ and real income change $\frac{dR'}{dt}$ being proportional to the utility change, it is straightforward that the gradient projection algorithm will select a direction proportional to $-\psi$. Considering $t = q - p$, the vector of taxes, it comes

PROPOSITION 3 :

The feasibility conditions being rewritten as in (5), the gradient method will select the following consumer prices and real income revisions[8] rules.

$$\frac{{}^t dq}{dt} = r \quad t \cdot (\overline{\partial X})^{U=Cste} \qquad (6)$$

$$\frac{dR'}{dt} = \| t \cdot (\overline{\partial X})^{U=Cste} \| \qquad (7)$$

Reminding the symmetry of $(\overline{\partial X})^{U=Cste}$, $[t \cdot (\overline{\partial X})^{U=Cste}]_h$ has a remarkably simple interpretation : it is the reduction of consumption of h if taxes are increased proportionally to their actual values i.e. if the "intensity" of the tax system increases. By analogy with Mirrlees (1976) who calls $\frac{[t \cdot (\overline{\partial X})^{U=Cste}]_h}{x_h}$ the index of discouragement of commodity h , we may call $[t \cdot (\overline{\partial X})^{U=Cste}]_h$ the absolute index of discouragement of h . Stipulations of (6) and (7) can then be summarized as follows (when $r > 0$). *Consumption prices of commodity* h *should be reduced proportionally to their absolute index of discouragement.* It is only in the case where such indices are perfectly correlated with distortions that the direction of steepest ascent actually coïncides with a proportional reduction of distortions.

Actually, following a line of reasoning inspired of section 3 and 4, one can argue that the case $r < 0$ corresponds to a situation where the (postulated) efficiency of private production is no longer desirable[9].

VI - EXISTENCE OF TRAJECTORIES AND CONVERGENCE FOR GRADIENT ALGORITHMS.

In this section we will consider existence and convergence problems for the system of differential equations (7) (8) (9) of section IV i.e. for the model with one public firm. Very similar results would apply to the more specific problem of section III. Similar methods would apply to the process of section V.

First we have to check that a solution of the process has actually the properties expected from its design. This is easily done in proposition 1.

PROPOSITION 4 :

A solution $p(t)$, $q(t)$, $z(t)$ of the system of differential equations (7) (8) (9) of section IV defined on $[0,T]$ and starting from $p(0)$, $q(0)$, $z(0)$ which is a tight equilibrium has if it exists, the following properties

1) At each time t, $\{p(t),q(t),z(t)\}$ define a tight equilibrium and $\|p(t)\| = \|p(0)\|$, $\|q(t)\| = \|q(0)\|$.
2) W is a non decreasing function of t .

The proof of the first part of 1) is straightforward. The fact that the euclidean norm of prices is preserved, results from the integration of ${}^{t}p\cdot\frac{dp}{dt} = 0$, which as in Guesnerie (1977) is a consequence of the definition of $\partial\tilde{\eta}^{-1}$ and ${}^{t}q\cdot\frac{dq}{dt} = 0$ which is easily shown from (7) .

For proving the second part, let us put $D(t) = \|\phi\| + \|p\| - \frac{[p\cdot\nabla g]^2}{\|\nabla g\|}$ .One has

$$\frac{dW}{dt} = \frac{1}{D}\left[\|\sum \beta_h x_h\| \; \|\phi\| - [\phi\cdot\sum \beta_h x_h]^2 + \|\sum \beta_h x_h\| \left(\|p\| - \frac{[p\cdot\nabla g]^2}{\|\nabla g\|}\right)\right]$$

Then $\frac{dW}{dt}$ is strictly positive unless

a) $p(t) = \lambda \nabla g(z(t)) \quad \lambda > 0 \quad , \quad {}^t\phi(p(t),q(t)) = -\sum_h \beta_h(q(t)x_h(q(t))$

b) $p(t) = \lambda \nabla g(z(t)) \quad \lambda > 0 \quad , \quad {}^t\phi(p(t),q(t)) = \sum_h \beta_h(q(t)x_h(q(t))$

In the following, the corresponding points will be said either *of type* a) or *of type* b).

The next question to be raised concerns existence of solutions to the system. Actually, local existence would be assured by the differentiability assumptions as soon as $D > 0$. But we need more, existence of trajectories i.e. of solutions on $[0,+\infty)$. For that, we will introduce the following assumptions :

H1) There does not exist p,q,z and $\lambda > 0$ s.t. $\sum x_h(q) = \eta(p) + z$, $\phi(p,q) = 0$, $p = \lambda \nabla g(z)$, $g(z) = 0$(10).

Actually H1) assures that D is always positive. This assumption has to be compared with assumption H7) in Fogelman-Guesnerie-Quinzii (1978). As they do, one can argue here that H1) is likely to be generically true.

H2) Let $A = \{x_h, y, z \mid \sum x_h = y + z \; , x_h \in X_h \; , \; y \in Y \; , \; g(z) = 0\}$, the set of "physical feasible states". A is compact.

This is a standard assumption. It assures that any trajectory would remain in the compact set $K \cap \mathrm{Proj}_z A$ where $K = \{(p,q) \text{ s.t. } \|p\| = \|p(0)\|, \|q\| = \|q(0)\|\}$.

For avoiding problems of the boundary of $\mathcal{P}$, we will add two other assumptions which should be compared with A and H3) in Fogelman-Guesnerie-Quinzii, to which the reader is referred for justifications.

H3) Let q_n be a sequence of consumption prices such that $q_n \to \bar{q}$, some (but not all) $\bar{q}_\ell = 0$. Then either $\|x(q_n)\| \to +\infty$ or $W(\ldots,x_h(q_n),\ldots) < W(\ldots,x_h(q(0)),\ldots)$ for n large enough.

H4) Let p_n be a sequence such that $p_n \to \overline{p}$, some $\overline{p}_\ell = 0$ then $\|\eta(p_n)\| \to +\infty$.

One has :

PROPOSITION 5 :

Whatever the initial tight equilibrium $p(0), q(0), z(0)$, the system (7) (8) (9) has a solution defined on $[0,+\infty)$.

Proof : Thanks to H3), H4) and H2), any trajectory remains in the compact subset of feasible states N contained in $\{K \cap \mathcal{P} \times \mathcal{P}\} \times \mathrm{Proj}_z A$. Because of H1), there exists an open set $\mathcal{O}$ containing N and such that the second member of equations (7), (8), (9) is continuously differentiable. Hirsch-Smale theorem (1974) gives the conclusion (p. 196)

Finally we have

PROPOSITION 6 :

Every trajectory converges either to points of type a) or to points of type b).

Proof : We use an extension of Liapounov theorem, with the Liapounov function $V = \underset{N}{\mathrm{Max}}\, W(p,q,z) - W(p(t),q(t),z(t))$

V is non increasing on trajectory and being bounded below from zero has a limit $\overline{V} \leq \underset{N}{\mathrm{Max}}\, W - W(p(0),q(0),z(0))$

Let Γ^+ be the limit set of the process i.e. the set of points which are accumulation points of $p(t_n), q(t_n), z(t_n)$ for infinite sequences t_n , $t_n \to +\infty$. One can see that on Γ^+ , $V = \overline{V}$ and $\frac{dV}{dt} = 0$. Then according to Hurewicz (1958), any trajectory (which is bounded) converges to Γ^+ and therefore to the set of points a) and b) which contains Γ^+ .

Points of type a) are local social welfare extrema in the sense that the necessary conditions of optimality are satisfied : they may be either global optima or local optima or saddle points.

Points of type b) are points in which an increase of social welfare is impossible if one is constrained, as supposed here, to have an efficient public production. Hence in such points temporary production inefficiencies in the public sector are desirable.

CONCLUSION.

The investigation reported in the present note had two aims :

First, it attempted to show that the gradient projection provided rules of tax revisions which are economically meaningful. From this point of view, two economic insights on the models under consideration deserve a particular emphasis : the appealing public production revision rule (section IV) the meaning of which should be deepened and the tax revision rule of section V which could be more systematically compared with previous literature.

Second, it tried to demonstrate that the gradient projection defined processes which had rather satisfactory properties. The technical analysis of section VI makes clear that these properties can rigorously be established at reasonable additional cost.

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FOOTNOTES.

(1) See also Weymark (1979) for a comparison of the papers by Guesnerie and Diewert.

(2) Let us notice that the use of gradient process in economics goes back at least to Arrow-Hurwicz. A systematic analysis of the usual planning processes of the first best theory in light of the gradient projection method has been recently attempted by d'Aspremont-Tulkens (1978).

(3) Gradient projection algorithms are usually defined for processes operating in discontinuous time (see Rosen (1961)). They can be adapted for processes in continuous time which only will be considered here.

(4) Throughout the paper, quantity vectors x_h, y, z will be column vectors, price vectors will be line vectors. ${}^t x$ will designate the transposed of x .

(5) $\| \ \|$ designates the square of the euclidean norm : $\|\phi\| = \phi \cdot {}^t\phi$.
It should be noted, that the direction defined by formulas (5), (6) —which is the only relevant information— is not modified if all β_h are doubled ; i.e. does not depend upon the cardinal representation of W (W could be changed in $U(W)$, U monotone increasing).

(6) This issue was clarified thanks to conversations with A. Dixit.

(7) We do not consider the public firm.

(8) By using (1) as local equations, different formulas would have been obtained.There is no mystery in that if one reminds from section III that the gradient direction is the best small improvement with respect to some particular norm, the euclidean one, and if one remarks that the euclidean norm puts different constraints on the vector

$\left(\frac{dq}{dt}\ \frac{dR}{dt}\right)$ or on the vector $\left(\frac{dq}{dt}, \frac{dR'}{dt}\right)$. For the same reason, the gradient direction is not invariant with respect to the choice of units of measurement, as shown in particular by formula (6).

(9) Let us note however, that the desirability of production inefficiency is much less likely than previously. When all commodities are not inferior $r > 0$ (see Dixit (1975) for very closely related remarks p. 107).

(10) The reader will check that there is a close connection between H1) and the fact that $D > 0$ and the constraints qualifications of the optimisation literature.

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