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NUMBER AND DEFINITENESS OF ECONOMIC EQUILIBRIA

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1. - INTRODUCTION.

The main result of this paper is to establish a relationship between the number of equilibria and their definiteness (by which we mean the accurate knowledge of these equilibria). Loosely speaking, we shall prove that under suitable assumptions, especially smoothness of preferences, it is almost equivalent to know with accuracy the equilibria of every economy or simply to know just the number of equilibria of these economies. This result explains why so many properties of comparative statics depend so heavily on the number of equilibria. For example, competitive equilibria depend discontinuously on the parameters defining an economy if and only if multiple equilibria do exist. The transfer problem encountered in international trade theory provides another example of a problem of which solution depends on the number of equilibria.

We fix notations and discuss the main assumptions in section 2. We then state the main result in section 3, result valid for consumption sets equal to the whole commodity space. We follow an intuitive direct approach to the two-consumer-two-commodity case in section 4. An alternative (dual) approach solves the general case in section 5. Extensions to the case of consumption sets bounded from below are considered in section 6. A property of embedded manifolds needed in section 5 is

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proved in the appendix.

2. - NOTATIONS AND ASSUMPTIONS.

A. *Commodities and prices.* We consider pure exchange economies with ℓ commodities and m consumers. We choose the ℓ-th commodity as numeraire, i.e. normalize the price vector by the convention $p_\ell = 1$.

Every price is strictly positive ; let

$$S = \{p = (p_1 , p_2 ,\dots, p_{\ell-1} , p_\ell) \in \mathbb{R}^\ell \mid p_1 > 0 ,\dots, p_{\ell-1} > 0 , p_\ell = 1\}$$

denote the set of strictly positive normalized price vectors.

B. *Consumers.* We assume that consumption sets are equal to $\mathbb{R}^\ell$ and that the preference preordering of consumer i can be represented by a utility function $u_i : \mathbb{R}^\ell \to \mathbb{R}$ satisfying the following properties :

1) u_i is smooth, i.e. differentiable at any order ;
2) u_i is differentiably monotonic, i.e. $\partial u_i(x) / \partial x^j$ is > 0 for $j = 1, 2,\dots, \ell$ (the notation x^j represents the quantity x^j of commodity j) ;
3) $u_i^{-1}([c_1 + \infty))$ is strictly convex for every $c \in \mathbb{R}$;
4) $u_i^{-1}([c_1 + \infty))$ is bounded from below for every $c \in \mathbb{R}$;
5) the Gaussian curvature of the hypersurface $u_i^{-1}(c)$ is everywhere $\neq 0$ for every $c \in \mathbb{R}$.

Assumptions (1), (2), and (3) are standard. Assumption (4) is intended to cope with consumption sets which are not bounded from below.

Let $p \in S$ and $w_i \in \mathbb{R}$ be given, the problem of maximizing $u_i(x_i)$ under the constraint $p.x \leq w_i$ has a unique solution denoted $f_i(p , w_i)$ representing consumer i's demand. Walras law $p.f_i(p , w_i) = w_i$ is clearly satisfied. Note that assumption (5) is equivalent to the smoothness of the individual demand mapping $f_i : S \times \mathbb{R} \to \mathbb{R}^\ell$. Let $\mathcal{U}$ denote the set of utility functions $u_i : \mathbb{R}^\ell \to \mathbb{R}$ satisfying assumptions (1) to (5).

C. *Economies and equilibria.* An economy with m consumers and ℓ commodities is defined by :

1) m utility functions $u_i \in \mathcal{U}$
2) m vectors $x_i \in \mathbb{R}^\ell$ representing consumer i's initial endowments.

We denote by $u = (u_1 , u_2 ,\ldots, u_m) \in \mathcal{U}^m$ the m-tuple of utility functions and by $x = (x_1 , x_2 ,\ldots, x_m) \in (\mathbb{R}^\ell)^m$ the m-tuple of initial endowments. The case for resources just reallocated corresponding to fixed total resources $r = x_1 + x_2 +\ldots+ x_m$ leads us to introduce the space $X = \{x = (x_1 , x_2 ,\ldots, x_m) \in (\mathbb{R}^\ell)^m \mid x_1 + x_2 +\ldots+ x_m = r\}$.

We recall that $p \in S$ is an equilibrium price vector associated with the economy $(u , x) \in \mathcal{U}^m \times X$ if and only if the equality

$$\sum_i f_i(p , p.x_i) = \sum_i x_i$$

is satisfied. We denote by $W(u , x)$ the set of equilibria associated with (u , x) and by $N(u , x) = \# W(u , x)$ the number, possibly infinite, of equilibria. Therefore, we have defined : 1) a correspondence $W : \mathcal{U}^m \times X \to S$; 2) a mapping $N : \mathcal{U}^m \times X \to \mathbb{N} \cup \{\infty\}$.

3. - THE RIGIDITY THEOREM.

Let $u \in \mathcal{U}^m$ and $u' \in \mathcal{U}^m$ be two m-tuples of utility functions. The rigidity property relates the correspondences $W(u\ ,\ .)$ and $W(u'\ ,\ .)$ from X into S to the mappings $N(u\ ,\ .)$ and $N(u'\ ,\ .)$ from X into $\mathbb{N} \cup \{\infty\}$ respectively.

RIGIDITY THEOREM. If the equality $N(u\ ,\ .) = N(u'\ ,\ .)$ holds and if there exists an $x \in X$ such that $N(u\ ,\ x) \neq 1$, then the correspondences $W(u\ ,\ .)$ and $W(u'\ ,\ .)$ are equal.

It is obvious that the rigidity theorem is not true for arbitrary correspondences ; its validity is specific of the economic definition of the correspondences $W(u\ ,\ .)$ and $W(u'\ ,\ .)$. We can also reformulate the rigidity property by saying that the number of equilibria, i.e. the mapping $N(u\ ,\ .) : X \to \mathbb{N} \cup \{\infty\}$, characterizes the equilibrium set correspondence (provided there exists an economy having several equilibria). The characterization through the rigidity theorem, however, is non-constructive in the sense that it tells nothing on how to deduce $W(u\ ,\ .)$ from $N(u\ ,\ .)$. A refinement of the proof of the rigidity theorem (see appendix) will enable us to give an explicit construction of $W(u\ ,\ .)$ knowing $N(u\ ,\ .)$.

In the case $N(u\ ,\ .) = 1$, i.e. if equilibrium is unique then the equilibrium prices are constant, property which replaces the rigidity property :

PROPOSITION 1. Assume $N(u\ ,\ .) = 1$; then $W(u\ ,\ .)$ is a constant mapping, i.e. there exists $p \in S$ such that $W(u\ ,\ x) = \{p\}$ for every $x \in X$.

In other words, the proposition says that if $W(u\ ,\ .)$ is a mapping, then it is constant. Given any price vector $p \in S$, one can readily find m-tuples of utility functions $u \in \mathcal{U}^m$ such that $W(u\ ,\ .) = \{p\}$. Therefore, no restriction exists on the values $W(u\ ,\ .)$ in the set of price vectors.

4. - DIRECT APPROACH TO THE CASE $(\ell, m) = (2\ ,\ 2)$.

Let us see what the above properties really mean in the Edgeworth box, i.e. the case $\ell = 2$ and $m = 2$. The set X of initial endowments is a plane. Associating with every Pareto optimum the tangent to the two indifference curves passing through the given Pareto optimum generates a one-parameter family $\mathcal{F}$ of lines in the plane. It is equivalent to determine the equilibria associated with the vector of initial endowments $x \in X$ or to find the lines belonging to $\mathcal{F}$ and passing through the point $x \in X$. Studying $W(u\ ,\ .)$ is therefore equivalent to studying the family $\mathcal{F}$.

Consider now the mapping $N(u\ ,\ .)$. Let Σ be the subset of X consisting of the points where $N(u\ ,\ .)$ is not locally constant, i.e. $\Sigma = \{x \in X \mid N(u\ ,\ .)$ restricted to any neighborhood of x is not constant$\}$. It results from Debreu's (1970) theorem on the constancy of the number of equilibria in small neighborhoods of regular economies that Σ is contained in the set of singular economies. Vice versa, it is straightforward, either from the envelope viewpoint or from the dual geometric viewpoint (Balasko 1978 , 1979) that the number of equilibria is not constant in any neighborhood of a singular economy. Therefore, Σ coincides with the set of singular economies. Furthermore, Σ is the envelope of the one-parameter family of lines $\mathcal{F}$.

These considerations enable us to give intuitive proofs of both the ridigity and the constancy properties.

1. - Assume first $\sum = \phi$, assumption clearly equivalent to $N(u\ ,\ .) =$ constant. For x Pareto optimum, $N(u\ ,\ x)$ is equal to 1 ; therefore, we have $N(u\ ,\ .) = 1$. Conversely, note that $N(u\ ,\ .) = 1$ implies $\sum = \phi$. It results from $N(u\ ,\ .) = 1$ that the lines of the family $\mathcal{F}$ are all parallel (otherwise, let x be an intersection point of two distinct lines, then one would have $N(u\ ,\ x) \geq 2$, a contradiction). Therefore, the price vector perpendicular to the set of parallel lines $\mathcal{F}$ is the unique and constant equilibrium price vector associated with every $x \in X$.

2. - Assume $\sum = \sum' \neq \phi$ (where $\sum'$ denotes the set of points where $N(u'\ ,\ .)$ is not locally constant). The families $\mathcal{F}$ and $\mathcal{F}'$ being the sets of tangents to $\sum$ and $\sum'$ respectively, it results from the equality $\sum = \sum'$ that the sets $\mathcal{F}$ and $\mathcal{F}'$ are equal to the set of tangents to the set $\sum = \sum'$. We get from $\mathcal{F} = \mathcal{F}'$ the final equality $W(u\ ,\ .) = W(u'\ ,\ .)$.

The lack of rigor of this intuitive proof consists in the identification between $\mathcal{F}$ (resp. $\mathcal{F}'$) and the set of tangents to $\sum$ (resp. $\sum'$). All this amounts to knowing what is a tangent to a curve with singularities like $\sum$ or $\sum'$. It is also implicit in this intuitive proof that it is equivalent for $\sum$ to be the envelope of the family $\mathcal{F}$ and for $\mathcal{F}$ to be the set of tangents to $\sum$. These properties are often taken as granted in old geometry textbooks (for example in relationship with the tangential equation of a curve). They need, however, proofs satisfying modern standards of rigor.

This can be achieved without too much effort for planar curves exhibiting sufficient regularity like algebraic curves or curves with normal crossings, i.e. curves of which singularities are not too much degenerated. It is not obvious whether such results can be extended to $\sum$ and $\sum'$, curves of which singularities may be very much degenerated .

5. - THE ALTERNATIVE DUAL APPROACH.

The difficulties encountered in the previous section on the rigorous foundations underlying the relationships between $\mathcal{F}$ and $\sum$ (resp. $\mathcal{F}'$ and $\sum'$) make highly desirable an alternative approach that could avoid such considerations. The alternative approach we are going to follow is based on the observation that it is equivalent to study the equilibrium set correspondence W(u , .) or to study the intersection of a smooth manifold embedded in some Euclidean space with a family of affine spaces (Balasko (1979)). More precisely, if $x \in X$ represents the initial endowments, then W(u , x) is the set of solutions $p \in S$ of the vector equation

$$(*) \qquad \sum_i f_i(p\ ,\ p.x_i) = \sum_i x_i = r\ .$$

The equation (*) is equivalent to the equation (**) where the unknowns are now $p \in S$ and the real numbers $w_1\ ,\ w_2\ ,\ldots,\ w_m$:

$$(**) \qquad \begin{cases} \sum_i f_i(p\ ,\ w_i) = \sum_i x_i = r \\ \\ p.x_i = w_i \qquad\qquad i = 1, 2,\ldots, m. \end{cases}$$

Let $\hat{X}$ be the affine space defined by the equation $p.r = w_1+w_2+\ldots+w_m$ where r is fixed. Let $\hat{P}$ be the subset of $\hat{X}$ defined by the equation $\sum f_i(p , w_i) = r$ and $\alpha(x)$ by the equations $p.x_i = w_i$ where $i = 1,2,\ldots,m$. The equation (**) can be geometrically interpreted as representing the intersection of the set $\hat{P}$ with $\alpha(x)$.

We associate with $u' = (u'_1 ,\ldots, u'_m) \in \mathcal{U}^m$ the manifold $\hat{P}'$ defined by the equation $\sum_i f'_i(p , w_i) = r$ where f'_i is the demand function associated with the utility function u'_i.

Proof of the rigidity property.

Given $u \in \mathcal{U}^m$ and $u' \in \mathcal{U}^m$, assume $N(u , x) = N(u' , x)$ for every $x \in X$ and $N(u , x) \geq 2$ for some $x \in X$. When x varies in X, the affine space $\alpha(x)$ describes the set $\mathcal{A}$ consisting of the dimension $\ell-1$ affine subspaces of $\hat{X}$ not perpendicular to S ($\hat{X}$ being identified to $S \times \mathbb{R}^{m-1}$ through the parametrization $(p , w_1 ,\ldots, w_{m-1})$).

Let us show that $\hat{P}$ and $\hat{P}'$ are not contained in a dimension m-1 affine subspace of $\hat{X}$ perpendicular to S : such spaces are necessarily equal to $\{p\} \times \mathbb{R}^{m-1}$ where $p \in S$; assume $\hat{P} \subset \{p\} \times \mathbb{R}^{m-1}$, then any affine space $\alpha(x) \in \mathcal{A}$ will intersect $\hat{P}$ in at most one point, which contradicts the assumption $N(u , x) \geq 2$ for some $x \in X$.

The rest of the proof is a straightforward application of the theorem in the appendix : if $\hat{P}$ and $\hat{P}'$ were distinct, there would exist $\alpha(x) \in \mathcal{A}$ (hence $x \in X$) such that the number of intersection points $\#(\hat{P} \cap \alpha(x))$ and $\#(\hat{P}' \cap \alpha(x))$ would be unequal, leading to a contradiction with the equality $N(u , x) = N(u' , x)$. Q.E.D.

Proof of the constancy property.

Assume $N(u, x) = 1$ for every $x \in X$. Since $\dim \hat{P} = m-1$ is larger than or equal to 1, we can find two distinct points b and b' in $\hat{P}$. Any affine subspace of $\hat{X}$ containing the line bb' intersects $\hat{P}$ in at least these two points. It results from $N(u, x) = 1$ that the space $\alpha(x)$ intersects $\hat{P}$ in only one point when $\alpha(x)$ describes $\mathcal{A}$. Therefore, the line bb' cannot be a subset of any affine space belonging to $\mathcal{A}$. This implies that any affine subspace of $\hat{X}$ having dimension $\ell-1$ and containing the line bb' is perpendicular to S. Therefore, the line bb' itself is perpendicular to S. Taking $b \in \hat{P}$ fixed, let b' vary in $\hat{P} \setminus \{b\}$. The line bb' is always perpendicular to S so that $\hat{P}$ is necessarily contained in the dimension (m-1) affine subspace of $\hat{X}$ perpendicular to S passing through $b = (p, w_1, w_2, \ldots, w_m)$. Therefore, we have $W(u, x) = \{p\}$ for every $x \in X$. Q.E.D.

6. - CONSUMPTION SETS $\neq \mathbb{R}^{\ell}$.

We restrict this discussion to consumption sets bounded from below and, for the sake of simplicity, we assume they are equal to the positive orthant $\mathbb{R}^{\ell}_{+}$. We also assume that preferences satisfy assumptions (1), (2), (3), (5) of section 2, and that the indifference hypersurfaces do not intersect any coordinate hyperplane. Denote by $\precsim_i$ such a preference preordering on $\mathbb{R}^{\ell}_{+}$. Let now I_i be an indifference hypersurface and take $y_i \in \overset{\circ}{\widehat{\mathrm{Conv}(I_i)}}$ where $\mathrm{Conv}(I_i)$ is the convex hull of I_i. Clearly, y_i belongs to the interior of $\mathrm{Conv}(I_i)$ if and only if y_i is strictly preferred to any commodity bundle of the indifference hypersurface I_i. Extend $\precsim_i$ restricted to $\mathrm{Conv}(I_i)$ by considering on $\mathbb{R}^{\ell}$ the following family of hypersurfaces :

in $Conv(I_i)$, the indifference hypersurfaces of $\precsim_i$; in $\mathbb{R}^\ell \setminus Conv(I_i)$, the hypersurfaces resulting from I_i by dilatations having center y_i. Clearly, the extended preordering $\precsim_i$ can be represented by a utility function $\bar{u}_i : \mathbb{R}^\ell \to \mathbb{R}$ satisfying the assumptions (1) to (5) of section 2. Note that $u_i = \bar{u}_i \mid Conv(I_i)$ represents the preordering $\precsim_i$ on $Conv(I_i)$.

Let (u , x) define an economy where $u = (u_1 , \dots, u_m)$. Denote by $(\bar{u} , x)$ the extended economy.

Proposition 2. There exists a neighborhood V of $x \in X$, an extension $\bar{u} = (\bar{u}_1 , \dots, \bar{u}_m)$ of $u = (u_1 , \dots, u_m)$ such that $W(\bar{u} , .)$ and $W(u , .)$ restricted to V are equal.

Let $x = (x_1 , \dots, x_m) \in X$ be such that $x_i \in \overset{\circ}{\mathbb{R}}{}^\ell_+$. Let I_i be any indifference hypersurface in $\mathbb{R}^\ell_+$ such that $x_i \in \widehat{Conv(I_i)}$ and let V_i be an open neighborhood of x_i in $Conv(I_i)$. Define $V = V_1 \times \dots \times V_m \cap X$. Let us fix some $\bar{x} \in V$ and construct $\bar{u}_i$ with $i = 1, 2 , \dots, m$ associated with $\bar{x}$ as above. Then, it is sufficient to prove that $W(u , .)$ and $W(\bar{u} , .)$ restricted to V are equal.

Associate with every Pareto optimum $y = (y_1 , \dots, y_m) \in X$ the affine subspace of X defined by the equations $p.y_i = w_i$ where $i = 1,2,\dots, m$, the price vector p being the unique one supporting the Pareto optimum y. Let $\mathcal{F}$ be the family of these affine spaces. Finding the $p \in W(u , x)$ is equivalent to finding the elements of $\mathcal{F}$ containing x (this remark underlies the envelope theoretic viewpoint in section 4 and in Balasko , 1978) .

Actually, it is sufficient to consider the elements of $\mathcal{F}$ associated with those Pareto optima which are Pareto superior to $x = (x_1, \ldots, x_m)$. Define similarly the family $\bar{\mathcal{F}}$ associated with $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$. The Pareto optima associated with $\bar{u}$ which are Pareto superior to x belong to $\mathrm{Conv}(I_1) \times \ldots \times \mathrm{Conv}(I_m) \cap X$; therefore, they are also Pareto optima for $u = (u_1, \ldots, u_m)$. Conversely, every Pareto optimum associated with $u = (u_1, \ldots, u_m)$ in $\mathrm{Conv}(I_1) \times \ldots \times \mathrm{Conv}(I_m) \cap X$ is obviously a Pareto optimum associated with $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$. This proves the identity between the elements of $\mathcal{F}$ and the elements of $\bar{\mathcal{F}}$ containing any $x \in V$. Therefore, we have proven that $W(u, .)$ and $W(\bar{u}, .)$ restricted to V are equal. Q.E.D.

Proposition 2 shows that in order to study the equilibrium set correspondence locally, i.e. in some open subset of X, we may consider consumption sets equal to the whole commodity space. Note, however, that the process of extending an economy to having consumption sets equal to $\mathbb{R}^{\ell}$ is not unique. Therefore, we could obtain distinct equilibrium set correspondences $W(\bar{u}, .)$ and distinct mappings $N(\bar{u}, .)$ though these mappings (resp. correspondences) coincide on the open subset V of X.

Without using proposition 2, one can also ask whether the results of section 3 remain true in the case of consumption sets $\neq \mathbb{R}^{\ell}$. Once more, let us consider the simple $(\ell, m) = (2,2)$ case.

Assume $\sum = \sum' \neq \phi$. The intuitive approach of section 4 still holds and therefore the family $\mathcal{F}$ (resp. $\mathcal{F}'$) contains the tangents to $\sum$ (resp. $\sum'$). Therefore, $W(u,x)$ and $W(u',x)$ have some elements in common, namely the lines passing through x and tangent to $\sum$ (resp. $\sum'$). The family $\mathcal{F}$ (resp. $\mathcal{F}'$), however, contains lines which are not tangent any more to $\sum$ (resp. $\sum'$) (these lines would correspond to tangents to $\sum$ (resp. $\sum'$) outside of the Edgeworth box taken stricto sensu) since some elements of $\mathcal{G}$ (resp. $\mathcal{G}'$) can pass through x without being tangent to Σ (resp. Σ'). In conclusion, some equilibria may coincide though in general the sets of equilibria will be distinct.

<u>Remark</u> * : Note also that if one allows initial endowments which do not belong necessarily to the consumption sets ($\neq \mathbb{R}^{\ell}$), then the results of section 3 are also true.

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* I owe this remark to K. Vind.

APPENDIX

In the Euclidean space $\mathbb{R}^{s+t}$, let $\mathcal{A}^t$ be the family of dimension t affine subspaces of $\mathbb{R}^{s+t}$. One can identify $\mathcal{A}^t$ with a Grassmann manifold naturally equipped with a measure : Santaló (1976, p. 199). Let $\mathcal{A}$ be the open dense subset of $\mathcal{A}^t$ consisting of these dimension t subspaces of $\mathbb{R}^s \times \mathbb{R}^t$ not perpendicular to $\mathbb{R}^s \times (0)$.

Let now M be a dimension s submanifold of $\mathbb{R}^{s+t}$; let $\mathcal{M}$ be the submanifold of $M \times \mathcal{A}$ consisting of pairs $(x,A) \in M \times \mathcal{A}$ such that $x \in A$. We denote by $\pi : \mathcal{M} \to \mathcal{A}$ the restriction of the natural projection $(x,A) \longmapsto A$ to the submanifold $\mathcal{M}$ of $M \times \mathcal{A}$.

Definition. The manifold M is properly embedded in $\mathbb{R}^{s+t}$ relatively to $\mathcal{A}$ if the mapping $\pi : \mathcal{M} \to \mathcal{A}$ is proper.

A mapping is proper if the inverse image of every compact set is compact. The following lemma provides an example of a properly embedded manifold which eventually leads to a proof of theorem 1.

LEMMA. The manifold $\hat{P}$(section 5) is properly embedded in $\hat{X}$ (identified to $S \times \mathbb{R}^{m-1}$) relatively to the set $\mathcal{A}$ of dimension ℓ-1 affine subspaces of $\hat{X}$ not perpendicular to S.

Proof : This is the corollary to theorem 5 by Balasko , 1979 . This lemma is equivalent to the properness of the Debreu mapping which associates with equilibrium $(p,x) \in S \times X$ its natural projection $x \in X$ (see Balasko, 1978). The Debreu mapping corresponds in the current context to $\pi : \mathcal{M} \to \mathcal{A}$.

Let now M and M' be two dimension s manifolds properly embedded in $\mathbb{R}^{s+t}$ relatively to $\mathcal{A}$. We also assume that neither M nor M' are embedded in a dimension affine space not belonging to $\mathcal{A}$. Then, we have :

<u>THEOREM</u>. <u>If for every</u> $A \in \mathcal{A}$, <u>the number of intersection points</u> $\#(M \cap A)$ <u>and</u> $\#(M' \cap A)$ <u>are equal, then</u> $M = M'$.

A related property dealing with curves with finite length was first mentioned by Steinhaus (1954) without proof. Steinhaus'idea was to define a distance on the set of curves with finite length within the set-up of integral geometry ; see Santaló (1976) for further details. Sulanke (1966) gave a rigorous proof of a particular case of the theorem. Sulanke's proof deals with planar curves which can be stratified by points and convex arcs. This includes the case of smooth curves with normal crossings.

We shall prove the theorem in the following form : assume M and M' distinct ; then, we shall show that there exists an open set $\mathcal{U}$ consisting of affine spaces in $\mathcal{A}^t$ such that the number of intersection points $\#(A \cap M)$ and $\#(A \cap M')$ are distinct for $A \in \mathcal{U}$. Taking A belonging to the non-empty intersection $\mathcal{A} \cap \mathcal{U}$ gives $A \in \mathcal{A}$ such that $\#(M \cap A) \neq \#(M' \cap A)$.

In a first step, let us show that there exists $A \in \mathcal{A}$ transverse to M and M' such that $A \cap M \neq A \cap M'$ (Note that we do not claim yet that $\#(A \cap M) \neq \#(A \cap M')$). Since $M \neq M'$, take $x_1 \in M$ such that x_1 does not belong to M' ; let $U_{x_1} \subset M$ be an open neighborhood of x in M such that $U_{x_1} \cap M' = \phi$. The set $\mathcal{V}$ of $A \in \mathcal{A}$ transverse to U_{x_1} is open ;

it is non-empty since M is not embedded in a dimension t affine space belonging to $\mathcal{A}^t \setminus \mathcal{A}$. Therefore, we have $A \cap M \neq A \cap M'$ for any A in this set. Now, it is a straightforward consequence of Sard's theorem applied to the mapping $\pi : \mathcal{M} \to \mathcal{A}$ that the set consisting of $A \in \mathcal{A}$ not transverse to M has measure 0 ; it is also closed since M is properly embedded relatively to $\mathcal{A}$. The same reasonning applies to the set of $A \in \mathcal{A}$ not transverse to M'. Therefore, the set of $A \in \mathcal{A}$ not transverse either to M or to M' is closed with measure 0. Its complement is therefore open dense, its intersection with the open set $\mathcal{V}$ is therefore non-empty. This yields $A \in \mathcal{A}$ such that A is transverse both to M and M' and $x_1 \in A \cap M$ while $x_1 \notin A \cap M'$. Transversality implies that $A \cap M$ (resp. $A \cap M'$) is discrete ; these sets are finite since, M (resp. M') being properly embedded, they are compact. Therefore, we have $A \cap M = \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_k\} \neq A \cap M' = \{x_1'\} \cup \{x_2'\} \cup \ldots \cup \{x_k'\}$. (Note that we take $k = \#(A \cap M) = \#(A \cap M')$ the only case which requires a proof).

Take now a point $x \notin A$; it determines with A a dimension $t+1$ affine subspace of $\mathbb{R}^{s+t}$. Let $\mathcal{A}^{t+1}$ be the set of the dimension $t+1$ affine subspaces of $\mathbb{R}^{s+t}$. We now define the mapping $\rho : M \setminus A \cap M \to \mathcal{A}^{t+1}$ in the following way : ρ associates with $x \neq x_1, x_2, \ldots, x_k$ the affine space (x, A). Clearly $A^{t+1} \in \mathcal{A}^{t+1}$ is a regular value of ρ if and only if A^{t+1} is transverse to M. Define similarly $\rho' = M' \setminus A \cap M' \to \mathcal{A}^{t+1}$. Let $S(\rho)$ (resp. $S(\rho')$) denote the set of singular values of ρ (resp. ρ'). The complement of $S(\rho) \cup S(\rho')$ in $\mathcal{A}^{t+1}$ being open dense by an easy consequence of Sard's theorem is not empty : take A^{t+1} in this set. It results from $x_1 \in A \cap M$ and $x_1 \notin A \cap M'$ that $x_1 \in M \cap A^{t+1}$ and $x_1 \notin M' \cap A^{t+1}$; therefore, $M \cap A^{t+1}$ is distinct from $M' \cap A^{t+1}$. It results from transversality that $M \cap A^{t+1}$ and $M' \cap A^{t+1}$ are smooth curves embedded in the Euclidean space A^{t+1} which can be identified to $\mathbb{R}^{t+1}$. Let $(\mathcal{A}| A^{t+1})$ consist of the spaces $A \in \mathcal{A}$ contained in A^{t+1}. Clearly $M \cap A^{t+1}$ and $M' \cap A^{t+1}$ are properly embedded in A^{t+1} relatively to $\mathcal{A}| A^{t+1}$. Furthermore, neither $M \cap A^{t+1}$ nor $M' \cap A^{t+1}$ are embedded in a hyperplane of A^{t+1} not belonging to $\mathcal{A}$, hence not belonging to $\mathcal{A}| A^{t+1}$.

We are therefore reduced to proving the theorem in the case $s = 1$ for any t , i.e. for curves in a Euclidean space. We shall only give a sketch of this part of the proof. Assume $M \neq M'$, and M not embedded in a proper affine subspace of $\mathbb{R}^{t+1}$ (this assumption is not really restrictive but one has then to adapt the subsequent arguments to the subspace instead of $\mathbb{R}^{t+1}$). Assume that there exists $x \in M$ such that the curvature of the curve M at x is $\neq 0$, that the tangent T to M at x intersects M only at a finite number of points, i.e. $\# (T \cap M) < + \infty$. Let us show that x belongs to M'. Assume the contrary, i.e. $x \notin M'$. There exists a (relatively) open set of hyperplanes H tangent to M at x and trasverse to $M \setminus \{x\}$ (hint : apply Sard's lemma to the mapping which associates with every points $y \in M \setminus \{x\}$ the hyperplane determined by y , the tangent T , and suitably chosen points $y_1 , y_2 , \dots , y_{t-2}$ in $\mathbb{R}^{t+1}$). Such a hyperplane H admits arbitrarily close hyperplanes H' and H'' such that the finite number of intersection points $\# (H' \cap M)$ and $\# (H'' \cap M)$ differs by exactly two units (hint : use transversality to $M \setminus \{x\}$ and the local form of M near x taking into account the non-zero curvature assumption). Therefore, H being a "bifurcation" hyperplane for the mapping $H \longmapsto \# (H \cap M) = \# (H \cap M')$, this implies that H cannot be transverse to M'. By having H vary, x and T being fixed, we obtain that T must be tangent to M' (hint : use a one-parameter family of hyperplanes H ; then M' must satisfy a differential equation whose solution leads to the fact that T is tangent to M' ; this is analogue to the fact that a point is the envelope of the pencil of lines passing through it). Let x' be the contact point of T with M'.

We have $x \neq x'$ since by assumption $x \notin M'$. By repeating a previous argument, there exists a hyperplane H transverse to $M \setminus \{x\}$ and to $M' \setminus \{x'\}$ and containing $T = xx'$. Let H' be arbitrarily close to H. Then, we have $\# (H' \cap M \setminus \{x\}) = \# (H \cap M \setminus \{x\}) < +\infty$ and respectively $\# (H' \cap M' \setminus \{x'\}) = \# (H \cap M' \setminus \{x'\}) < +\infty$; by considering the local form of M and of M' at x and x' respectively, one can make H vary in a way such that $\# (H' \cap M) = \# (H \cap M) + 1$ and $\# (H' \cap M') = \# (H \cap M') - 1$; this yields $\# (H' \cap M') = \# (H' \cap M) - 2$, hence a contradiction. Now, if there does not exist $x \in M$ satisfying the above assumptions, then we are reduced to considering segments of straight lines of M and M' ; in such cases, the final result is obvious.

<u>Remark</u>. Beyond the rigidity theorem, it may be interesting to construct the manifold M knowing the mapping $A \longmapsto \# (M \cap A)$. A simple adaptation of the above proof provides a solution. It is sufficient to construct $M \cap A^{t+1}$ for any dimension t+1 affine subspace of $\mathbb{R}^{s+t}$ and make A^{t+1} vary. One is then reduced to considering a family of hyperplanes ; actually, considering the above proof, $M \cap A^{t+1}$ is the envelope of the family of hyperplanes H such that $H \longmapsto \# (M \cap H)$ is not locally constant at H. A way to construct M is therefore to project M on a plane and to consider the hyperplanes parallel to the projection. Then one is reduced to consider the classical problem of finding the envelope of a family of lines in a plane. By taking a sufficient number of distinct projections, one can reconstruct M knowing its projections. Note that this construction can be considerably simplified by considering nice (generic) intersections $M \cap A^{t+1}$ and then nice (generic) projections. Summarizing, and back to our economic problem, we are entitled to say that knowing the function number of equilibria determines the equilibrium set correspondence.

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