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THE MANAGEMENT AND

RENEWAL OF EXHAUSTIBLE RESOURCES (**)

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LA GESTION ET LE RENOUVELLEMENT DES RESSOURCES EPUISABLES

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RESUME.

Si les ressources épuisables posent toutes le problème de la gestion du stock disponible, il en est pour lesquelles ce problème ne peut être séparé de celui du renouvellement ; c'est le cas en particulier des sources d'énergie dont une gestion à long terme comporte le choix entre plusieurs techniques de renouvellement et celui des dates auxquelles on entreprend les investissements correspondants, compte tenu des délais de réalisation.

Nous proposons un modèle macroéconomique d'optimisation sur un horizon fini qui généralise les modèles classiques de croissance en introduisant ces choix supplémentaires. Ce type de modèle présente des particularités mathématiques nouvelles qui ne permettent pas de lui appliquer la forme classique du principe de Pontryagin. Ainsi nous avons été conduits à reformuler ce principe dans "le cas de plusieurs équations d'évolution possibles intervenant à des dates variables", Pour son application, nous avons retenu un modèle simple concernant la gestion des ressources énergétiques : le capital et l'énergie entrent dans la fonction de production globale comme facteurs substituables (mais le degré de substituabilité peut être faible). Les principaux résultats obtenus sont les suivants :

1°) sur les prix de l'énergie : à la différence du cas sans renouvellement, le prix actualisé de l'énergie qui est constant par palliers, décroît strictement au cours du temps ; d'un pallier au suivant lorsqu'un processus de reproduction particulier commence à fournir une énergie nouvelle. Le prix de l'énergie relatif à un processus nouveau est égal à son coût de développement.

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2°) Sur les dates : pour chaque processus nouveau, sa date de disponibilité coïncide avec celle où toutes les ressources précédemment disponibles sont épuisées : de plus cette date réalise le minimum du coût total de l'énergie (investissements et consommation),

3°) Sur le choix des techniques de renouvellement : le classement des différentes techniques est défini par leur coût total et leur niveau respectif réalise le maximum de leur bilan actualisé défini comme la différence entre la valeur de la consommation et le coût d'investissement correspondant.

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ABSTRACT.

If exhaustible resources all present the problem of the management of available stocks, there are some cases where this problem cannot be separated from that of their renewal; such is the case in particular with energy supplies whose long-term management entails the choice between several renewal techniques and that of the dates on which the corresponding investments are undertaken, with all due consideration for the delays in their realization.

We propose a macroeconomic optimization model on a finite horizon that generalizes the classical growth models while at the same time introducing these additional choices. This type of model presents new mathematical particularities which do not allow the application of the classical Pontryagin principle. Thus we have been led to the reformulation of this principle in " the case of the possibility of several evolution equations which intervene at dates which are not predetermined " . We will present a simple model of energy resource management : capital and energy enter into the production function as substitutable factors (but the degree of substitutability can be low).

The main results are the following :

- 1) For energy prices : in contrast with the non-renewal case, the present-value of the energy price, which is a step function, decreases in the strict sense from one step to the next at the time when a new source is available. The price of energy corresponding to a new process is equal to its development cost.
- 2) For the dates of availability : For each new process, this date is the time where all previously available resources are exhausted; moreover on this date the minimum total cost of energy (investment and consumption) is obtained.
- 3) For the choice of renewal techniques : their ordering is defined by the minimum of the total cost and their level by the maximum net result given by the difference between the consumption value and investment cost.

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I N T R O D U C T I O N

Long-term planning imposes constraints which do not appear too important in the short-run. Such in the case with resources that require significant efforts for their renewal.

Energy is an obvious example, but the same problem arises with all natural resources subject to intensive use, and which therefore can be exhausted in the foreseeable future. It is then clear that the management of these resources differs both from the non-renewal case (defletable resources, see for example [5]) and from the case of self-renewing resources (natural resources, see for example [1]) : a lot of studies have been carried out on these two cases (see the references of the survey in [4]). T. de Montbrial has also considered the choice among several energy resources ([3]).

We propose here a different approach that makes use of a macroeconomic optimization model on a finite horizon, and which generalizes the classical growth models by introducing the renewal problem. The exhaustible resource is here considered as energy (this is only an example). It is used by the productive system together with physical capital. The initial stock is fixed ; however, by investing during a given period, it is possible to get new amounts of energy. Constraints on the final value of capital and energy stocks ensure that expansion continues. The optimization criterion is the present-value of the utility of future consumption. The solution of the model will give us the optimal dates and levels of investments in the energy sector, the pricing policy, investment in the other sectors etc...

This article is presented in three sections. The first deals with the statement of the model and with the corresponding optimization problem. The second gives the new formulation of Pontryagin's principle, which is necessary, and the third presents the solution and its economic interpretation . The mathematical analysis and proofs are found in the appendix.

§ 1 - THE MODEL

1.1. Data and Assumptions.

Variables.

C : Consumption
 K : Capital (depreciation rate : m)
 E : Energy
 I : Gross Investment
 Y : Output

The initial values K_0 and S_0 of capital and energy stocks are given, and minimum final values that are positive, K_T and S_T are imposed, T being the horizon of the model.

Production Function.

$$Y(t) = F[K(t), E(t), t]$$

The following assumptions about F are made :

- 1a) For $K > 0$, $E > 0$ and $t \in [0, T]$: $F(K, E, t)$,
 $F'_K(K, E, t)$ and $F'_E(K, E, t)$ are continuous
- 1b) For $K > 0$ and $t \in [0, T]$: $F(K, 0, t) = 0$
- 1c) For $K > 0, E > 0$ and $t \in [0, T]$: $F'_E(K, E, t) > 0$

Renewal of the energy stock.

To renew the stock of energy, L processes are available. Each process $k (1 \leq k \leq L)$ is characterised by the length θ_k of the necessary investment and the investment function $R_k(t, N_k)$, $t \in [0, \theta_k]$, corresponding to the quantity of energy N_k . The date of availability of the new stock N_k is noted a_k , and its consumption is noted $E_k(t)$, $t \geq a_k$.

Assumptions concerning R_k

- 2a) $\theta_k < T$;
- 2b) $R_k(t, N) = 0$ for each t , iff $N = 0$;
- 2c) $R_k(t, N)$ is continuously differentiable, and zero for $t \notin]0, \theta_k[$ (this assumption of smoothness simplifies the mathematical study without modifying the essential conclusions) ;
- 2d) For $N > 0$: $\frac{\partial R_k}{\partial N}(t, N) \geq 0$ and non identically zero.

Objective function. For a trajectory of consumption $C(t), t \in [0, T]$, the corresponding utility is

$$\int_0^T e^{-\delta t} U(C(t)) dt, \quad \delta > 0.$$

Assumptions concerning U .

3a) $U(C)$ is continuous for $C \geq 0$ and differentiable for $C > 0$;

3b) $U'(C) > 0$ for $C > 0$;

3c) $\lim_{C \rightarrow 0} U'(C) = +\infty$

1.2. Optimization problem.

We present the program in continuous time with the state variables, the controls, the parameters and the constraints.

a) State variables.

- Capital K

$$\dot{K}(t) = -m K(t) + I(t)$$

$$K(0) = K_0 \text{ and } K(T) \geq K_T$$

- Used-up energy $-Q_0, -Q_1, \dots, -Q_L$ of the respective initial stock S_0 and the newly obtained stocks $N_1, \dots, N_L$:

$$\dot{Q}_0(t) = -E(t) + \sum_{k=1}^L E_k(t) 1_{[a_k, T]}(t), \text{ and } Q_0(0) = 0$$

$$\dot{Q}_k(t) = -E_k(t) 1_{[a_k, T]}(t), \text{ and } Q_k(0) = 0, \text{ for } 1 \leq k \leq L$$

where $1_{[a_k, T]}(t)$ is the characteristic function of the interval $[a_k, T]$.

b) Controls and parameters.

- Consumption : $C \geq 0$

- Gross investment : $I \geq 0$

- Total energy consumed : $E \geq 0$

- Energy consumed out of the stock N_k : $E_k \geq 0$

- Stock of new energy obtained by the process k : $N_k \geq 0$

- Availability date of the process k : $a_k, \theta_k \leq a_k \leq T$.

c) Constraints.

- Resources - Uses of the output :

$$C(t) + I(t) + \sum_{k=1}^L R_k(t-a_k + \theta_k, N_k) \leq F(K(t), E(t), t) \text{ a.e.}$$

• Resources - Uses of energy :

$$Q_0(T) + S_0 \geq 0$$

$$Q_k(T) + N_k \geq 0 \quad \text{for } 1 \leq k \leq L$$

$$Q_0(T) + S_0 + \sum_{k=1}^L (Q_k(T) + N_k) \geq S_T .$$

Problem (P).

$$\text{Maximize } \int_0^T e^{-\delta t} U(C(t)) dt \text{ subject to}$$

$$(1) \begin{cases} \dot{K} = -mK + I, \text{ and } K(0) = K_0 \\ \dot{Q}_0 = -E + \sum_{k=1}^L E_k 1_{[a_k, T]}, \text{ and } Q_0(0) = 0, \\ \dot{Q}_k = -E_k 1_{[a_k, T]}, \text{ and } Q_k(0) = 0, \text{ for } 1 \leq k \leq L \end{cases}$$

$$(2) \begin{cases} K(T) - K_T \geq 0 ; Q_0(T) + S_0 \geq 0 ; Q_k(T) + N_k \geq 0 \text{ for } 1 \leq k \leq L \\ S_0 + \sum_{k=1}^L N_k + Q_0(T) + \sum_{k=1}^L Q_k(T) - S_T \geq 0 \end{cases}$$

$$(3) \quad F(K(t), E(t), t) \geq C(t) + I(t) + \sum_{k=1}^L R_k(t-a_k + \theta_k, N_k) \text{ a.e.}$$

$$(4) \begin{cases} \theta_k \leq a_k \leq T ; \text{ and } N_k \geq 0 ; \\ C(t) \geq 0, I(t) \geq 0, E(t) \geq 0, E_k(t) \geq 0 ; \\ \sum_{k=1}^L E_k(t) \leq E(t) . \end{cases}$$

One of the particularities of this system is that some of the controls are parameters (N_k and a_k) and it may have different evolutions according to the value of the parameters. Consequently, the Pontryagin principle which is needed for the optimality conditions has a new form that will be given below.

§ 2 - REFORMULATION OF THE PONTRYAGIN PRINCIPLE

Consider the following problem :

Maximize $f_0(x(T), h)$ subject to :

$$\frac{dx(t)}{dt} = F_0(t, x(t), h, u(t)) + \sum_{\ell=1}^L 1_{[a_\ell, T]} F_\ell(t, x(t), h, u(t)) \quad \text{a.e. in } [0, T]$$

$$1_{[a_\ell, T]} = \begin{cases} 0 & \text{if } t < a_\ell \\ 1 & \text{if } a_\ell \leq t \leq T \end{cases}$$

$$h \in C$$

$$x(0) = \varphi(h)$$

$$a_\ell = g_\ell(h) \quad 1 \leq \ell \leq L$$

$$0 \leq a_\ell \leq T \quad 1 \leq \ell \leq L$$

$$f_p(x(T), h) \geq 0 \quad 1 \leq p \leq P$$

$$f_p(x(T), h) = 0 \quad P+1 \leq p \leq Q$$

The controls $u(t)$ are piecewise continuous over $[0, T]$ and take their values in a Hausdorff space U .

The vector $x(t) = (x^1(t), \dots, x^n(t))$ is an element of the Euclidean space $\mathbb{R}^n$ and C any convex of $\mathbb{R}^m$.

The functions $F_\ell (0 \leq \ell \leq L)$, φ , $f_p (0 \leq p \leq Q)$ and $g_\ell (1 \leq \ell \leq L)$ are continuously differentiable with respect to x and h .

We will adopt the following notations :

$$a = (a_1, \dots, a_L)$$

$$F(t, x, h, u, a) = F_0(t, x, h, u) + \sum_{\ell=1}^L 1_{[a_\ell, T]} F_\ell(t, x, h, u)$$

$$F^-(t, x, h, u, a) = F(t-0, x, h, u, a)$$

$$= F_0(t, x, h, u) + \sum_{\ell=1}^L 1_{]a_\ell, T]} F_\ell(t, x, h, u)$$

The function F is continuous to the right of t , the function F^- is continuous to the left of t , for all t .

For $q \in E_n$, let :

$$H(t, q, x, h, u, a) = q \cdot F(t, x, h, u, a)$$

$$H^-(t, q, x, h, u, a) = q \cdot F^-(t, x, h, u, a)$$

$$H_\ell(t, q, x, h, u) = q \cdot F_\ell(t, x, h, u)$$

Let $(\hat{x}, \hat{u}, \hat{a}, \hat{h})$ be an optimal solution. Without loss of generality, we assume that $\hat{u}(t)$ is continuous to the right of t , for $t \in [0, T[$, and continuous to the left of T .

Constraint qualification conditions :

The following linear system admits the unique solution

$$b_p = 0 \quad (0 \leq p \leq Q) \quad \text{and} \quad c_\ell = 0 \quad (1 \leq \ell \leq L)$$

$$\left\{ \begin{array}{l} b_p \geq 0 \quad \text{for } 0 \leq p \leq P \quad \text{and} \quad b_p f_p(\hat{x}(T), \hat{h}) = 0 \quad \text{for } 1 \leq p \leq P \\ \sum_{p=0}^Q b_p \frac{\partial f_p}{\partial x_i}(\hat{x}(T), \hat{h}) = 0 \quad \text{for } 1 \leq i \leq n \\ \hat{a}_\ell = 0 \Rightarrow c_\ell \leq 0 ; \hat{a}_\ell = T \Rightarrow c_\ell \geq 0 ; 0 < \hat{a}_\ell < T \Rightarrow c_\ell = 0 \\ \sum_{i=1}^m (h_i - \hat{h}_i) \left(\sum_{p=0}^Q b_p \frac{\partial f_p}{\partial h_i}(\hat{x}(T), \hat{h}) - \sum_{\ell=1}^L c_\ell \frac{\partial g_\ell}{\partial h_i}(\hat{h}) \right) \leq 0 \quad \text{for all } h \in C \end{array} \right.$$

THEOREM. If the constraint qualification conditions are satisfied, and if $(\hat{x}, \hat{u}, \hat{h}, \hat{a})$ is an optimal solution, then there exist real numbers b_p ($0 \leq p \leq Q$) and c_ℓ ($1 \leq \ell \leq L$), which are not all zero, and a non-zero continuous function $q(t)$ such that

- for $0 \leq p \leq P$: $b_p \geq 0$; and for $1 \leq p \leq P$: $b_p f_p(\hat{x}(T), \hat{h}) = 0$
- $q(t)$ verifies :

$$\frac{dq_i(t)}{dt} = - \frac{\partial H}{\partial x_i}(t, q(t), \hat{x}(t), \hat{h}, \hat{u}(t), \hat{a}) \quad 1 \leq i \leq n ;$$

$$q_i(T) = - \sum_{p=0}^Q b_p \frac{\partial f_p}{\partial x_i}(\hat{x}(T), \hat{h}) \quad 1 \leq i \leq n .$$

- for all $t \in [0, T[$ (respectively $t \in]0, T]$), the function $u \rightarrow H(t, q(t), \hat{x}(t), \hat{h}, u, \hat{a})$ (respectively $u \rightarrow H^-(t, q(t), \hat{x}(t), \hat{h}, u, \hat{a})$) attains its maximum $M(t)$ (respectively $M^-(t)$) over the space U at $\hat{u}(t)$ (respectively $\hat{u}(t-0)$).
- for each $h \in C$:

$$\sum_{i=1}^m (h_i - \hat{h}_i) \left[q(0) \cdot \frac{\partial \varphi(\hat{h})}{\partial h_i} + \sum_{p=0}^Q b_p \frac{\partial f_p}{\partial h_i}(\hat{x}(T), \hat{h}) + \int_0^T \frac{\partial H}{\partial h_i}(t, q(t), \hat{x}(t), \hat{h}, \hat{u}(t), \hat{a}) dt \right]$$

$$\leq \sum_{i=1}^m \sum_{\ell=1}^L (h_i - \hat{h}_i) c_{\ell} \frac{\partial g_{\ell}}{\partial h_i}(\hat{h})$$

• for each ℓ , $1 \leq \ell \leq L$:

$$\text{if } \hat{a}_{\ell} = 0 : H_{\ell}(0, q(0), \hat{x}(0), \hat{h}, \hat{u}(0)) > c_{\ell}$$

$$\text{if } \hat{a}_{\ell} = T : H_{\ell}(T, q(T), \hat{x}(T), \hat{h}, \hat{u}(T)) \leq c_{\ell}$$

$$\text{if } 0 < \hat{a}_{\ell} < T : H_{\ell}(\hat{a}_{\ell}, q(\hat{a}_{\ell}), \hat{x}(\hat{a}_{\ell}), \hat{h}, \hat{u}(\hat{a}_{\ell})) > c_{\ell}$$

$$\text{and } M(\hat{a}_{\ell}) - M^{-}(\hat{a}_{\ell}) \leq \sum \{c_{\tau} ; \hat{a}_{\tau} = \hat{a}_{\ell}\}$$

Proof : see [2].

§ 3 - SOLUTION AND ECONOMIC INTERPRETATION OF THE MODEL

3.1. Solution.

Let $\bar{K}(t), \bar{Q}_k (0 \leq k \leq L), \bar{C}(t), \bar{I}(t), \bar{E}(t), \bar{E}_k(t) (1 \leq k \leq L), \bar{a}_k$ and $\bar{N}_k (1 \leq k \leq L)$ be an optimal solution. We make the following assumptions :

Assumptions.

4a) There exists a feasible trajectory with non-zero consumption

4b) At no time is gross production fully used up for energy investment, i.e.

$$\forall t, F(\bar{K}(t), \bar{E}(t), t) > \sum_{k=1}^L R_k(t - \bar{a}_k + \theta_k, \bar{N}_k)$$

4c) The function $\pi(t) = e^{-\delta t} U'(\bar{C}(t))$ decreases with respect to t over $[0, T]$.

Remarks. i) For 4c) to be satisfied, it is sufficient that

$$\forall t, F'_K(\bar{K}(t), \bar{E}(t), t) - m > 0$$

(positive net marginal productivity of the capital)

Proof. For $t_1 < t_2$, the decrease in $\bar{C}(t)$ by $\varepsilon > 0$ over $[t_1, t_1 + \lambda[$ allows an increase in $\bar{C}(t)$ over $[t_2, t_2 + \lambda[$ by ε' , where, for order 1 :

$$\varepsilon' = \varepsilon \exp \int_{t_1}^{t_2} (F'_K(\bar{K}(t), \bar{E}(t), t) - m) dt > \varepsilon$$

(this follows from the theorem of differentiability of the solution of a differential equation with respect to the initial conditions).

It follows that the corresponding increase in $\int_0^T e^{-\delta t} U(\bar{C}(t)) dt$ which is non-positive, is equal, for order 1, to

$$\begin{aligned} & - \lambda \varepsilon e^{-\delta t_1} U'(\bar{C}(t_1)) + \lambda \varepsilon' e^{-\delta t_2} U'(\bar{C}(t_2)) \\ & > \lambda \varepsilon [\pi(t_2) - \pi(t_1)] ; \end{aligned}$$

Consequently : $\pi(t_2) < \pi(t_1)$ ■

ii) At any point t where $\pi(t)$ is differentiable, assumption 4c) is equivalent to

$$\delta - \frac{d}{dt} \text{Log } U'(\bar{C}(t)) > 0$$

This means that, along the optimal path, the rate of discount for the economy is positive.

Properties. It is possible to state the problem by eliminating $C(t)$ and then applying to it the necessary conditions of § 2. (see appendix I and II).

• The multipliers corresponding to equations (1) verify :

$$\begin{aligned} \dot{p}(t) &= - \pi(t) F'_K(\bar{K}(t), \bar{E}(t), t) + m p(t) \\ \dot{q}_k(t) &= 0, \text{ for } 0 \leq k \leq L. \end{aligned}$$

(Appendix, relations 7)

• The hamiltonian maximum conditions are

$$\begin{aligned} & p(t) \leq \pi(t), \text{ and if } \bar{I}(t) > 0 : p(t) = \pi(t) ; \\ \text{if } \bar{E}_k(t) > 0 : & \pi(t) F'_E(\bar{K}(t), \bar{E}(t), t) = q_k \quad (1 \leq k \leq L) ; \\ \text{if } \bar{E}(t) > \sum_{k=1}^L \bar{E}_k(t) : & \pi(t) F'_E(\bar{K}(t), \bar{E}(t), t) = q_0. \end{aligned}$$

(Appendix, relations 9 and 10)

3.2. Interpretation.

We can use the various mathematical results found in the appendix to give a general economic interpretation. This interpretation is related to

the system of prices, the renewal dates, and the selection of techniques.

1. Prices.

- We can interpret $\pi(t) = e^{-\delta t} U'(\bar{C}(t))$ as the discounted shadow price of consumption, and $p(t)$ as that of capital.

We have (Appendix, relation 9)

$$\pi(t) \leq p(t) \quad , \quad \text{and if } \bar{I}(t) > 0 : \pi(t) = p(t).$$

For $\bar{I}(t) > 0$, the discount rate for the economy is

$$a(t) = \delta - \frac{d}{dt} \text{Log } U'(\bar{C}(t)) = - \frac{\dot{p}(t)}{p(t)} = F'_K - m$$

as classical.

- We can interpret $q(t) = \pi(t) F'_E(\bar{K}(t), \bar{E}(t), t)$ as the discounted shadow price of energy (marginal productivity). This shadow price has some important properties :

i) It is a step function which decreases from one step to the next (Appendix property 11) ; the dates on which it changes are the dates when a new stock becomes available : the new price is the energy price of the new stock (Appendix, relation 18).

ii) If the k-type energy is used, then the price of this energy is equal to its marginal development cost (Appendix, relation 21) :

$$\text{for } \bar{N}_k > 0 : q_k = \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) \frac{\partial R_k}{\partial N_k}(t - \bar{a}_k + \theta_k, \bar{N}_k) dt ;$$

if not, then the price of the energy $q(t)$ is less than the marginal development cost of the k-type energy. (Appendix, relation 20).

iii) The final stock is made up of whatever may remain out of the most recent type k_s and also of one or more stocks developed especially to satisfy the final energy constraint ; all these energy processes have exactly the same development cost.

2. Renewal Dates.

The optimal dates on which the different energies become available verify the following properties :

- i) As we have seen (property i) for prices) each of these dates corresponds to a change in the energy price and moreover, it coincides with the date on which the previous stocks become exhausted. (Appendix, property 8).

ii) The choice of the optimal dates minimizes the total amount of energy expenditures. This amount is equal to the sum of the consumption and investment expenditures .

iii) If we note the amount of the type k investment expenditure as D_k , the multiplier c_k corresponding to the optimal date $\bar{a}_k$ (theorem § 2) may be interpreted as " the marginal cost " of this date.

$$(c_k = \left| \frac{\partial D_k}{\partial a_k} \right|, \text{ Appendix, property 15}).$$

3. Choice of Techniques : ordering and level.

i) The energy investments are decided at the time $\bar{a}_{k_i} - \theta_{k_i}$, defined by the availability dates $\bar{a}_{k_i}$ (the length θ_{k_i} being fixed). The order corresponds to the decreasing steps of the discounted price of energy (Property i) for prices) ; the optimal dates $\bar{a}_{k_i}$ correspond to the minimum of the total energy expenditures . (Property ii) for Dates).

ii) Assuming that the functions $R_k(t, N_k)$ are convex with respect to N_k , the optimum level $\bar{N}_k$ of the process k maximizes the net result of this process :

$$B_k = q_k N_k - \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) R_k(t - \bar{a}_k + \theta_k, N_k) dt$$

(Appendix, property 17).

This model may be generalized in many respects. In particular :

- the choice between importing raw materials and developping national resources.
- the introduction of several sectors of production and several rare resources.

A P P E N D I X

MATHEMATICAL ANALYSIS OF THE MODEL

I - An equivalent formulation of the problem.

PROPERTY 1.

With assumptions 3 and 4, the optimal solution of problem (P) verifies :
 $\bar{C}(t) > 0$ for each t in $[0, T]$.

Proof. Let $\bar{E}_k$, $\bar{E}$, $\bar{N}_k$, $\bar{a}_k$ be fixed and consider the following problem

$$\left\{ \begin{array}{l} \text{maximize} \quad \int_0^T e^{-\delta t} U((1-s)G(K, t)) dt \\ \dot{K} = sG(K, t) - mK, \quad K(0) = K_0, \quad K(T) \geq K_T \\ s \in [0, 1] \end{array} \right.$$

where

$$G(K, t) = F(K, \bar{E}(t), t) - \sum_{k=1}^L R_k(t - \bar{a}_k + \theta_k, \bar{N}_k).$$

This program admits the optimal solution :

$$\bar{K}(t) \quad \text{and} \quad \bar{s}(t) = \frac{\bar{I}(t)}{G(\bar{K}(t), t)}.$$

By Pontryagin's maximum principle, there exists a nonzero pair $(p_0, p(t))$ such that $p(T) \geq 0$ and :

$$H = p_0 e^{-\delta t} U((1-s)G(\bar{K}(t), t)) + p(t)(sG(\bar{K}(t), t) - m\bar{K}(t))$$

is maximum over $[0, 1]$ at $\bar{s}(t)$.

p_0 is different from zero : if not, then $p(t) > 0 \quad \forall t$, and $\bar{s}(t) = 1 \quad \forall t$ (assumption 4b) ; that is $\bar{C}(t) = 0 \quad \forall t$, which is impossible (assumption 4a).

Consequently, $p_0 > 0$ and $\forall t$, the maximum of H cannot be obtained at $s = 1$, because

$$\lim_{s \rightarrow 1} \frac{\partial H}{\partial s} = -\infty \quad (\text{assumption 3c}) \quad \blacksquare$$

PROPERTY 2.

The optimal problem (P) is equivalent to the following.

(P') maximize $Z(T)$ such that

$$(1) \left\{ \begin{array}{l} \dot{K} = -mK + I, \text{ and } K(0) = K_0 ; \\ \dot{Z} = e^{-\delta t} U \left[F(K, E, t) - I - \sum_{k=1}^L R_k(t-h_k+\theta_k, N_k) \right] \text{ and } Z(0) = 0 ; \\ \dot{Q}_0 = -E + \sum_{k=1}^L E_k l[a_k, T], \text{ and } Q_0(0) = 0 ; \\ \dot{Q}_k = -E_k l[a_k, T], \text{ and } Q_k(0) = 0, \text{ for } 1 \leq k \leq L. \end{array} \right.$$

$$(2) \left\{ \begin{array}{l} K(T) - K_T \geq 0 ; Q_k(T) + N_k \geq 0 \text{ for } 0 \leq k \leq L ; \\ \sum_{k=0}^L (Q_k(T) + N_k) - S_T \geq 0 . \end{array} \right.$$

$$(3) \quad F(K, E, t) - I - \sum_{k=1}^L R_k(t-h_k+\theta_k, N_k) > 0, \text{ a.e.}$$

$$(4) \left\{ \begin{array}{l} a_k = h_k, \quad h_k \geq \theta_k, \quad 0 \leq a_k \leq T, \text{ for } 1 \leq k \leq L ; \\ N_0 = S_0 ; N_k \geq 0, \text{ for } 1 \leq k \leq L ; \\ I \geq 0 ; E_k \geq 0 \text{ for } 1 \leq k \leq L ; E \geq \sum_{k=1}^L E_k . \end{array} \right.$$

This form is obtained from (P) with the notation

$$Z(t) = \int_0^t e^{-\delta s} U(C(s)) ds$$

and the elimination of C ■

PROPERTY 3. The optimal solution of (P) defines with

$$\bar{Z}(t) = \int_0^t e^{-\delta s} U(\bar{C}(s)) ds$$

a local optimal solution of problem (P'') obtained from (P') by leaving out the constraint (3) .

This results from property 1 : the constraint (3) is never saturated ■

II - Application of the necessary conditions (§ 2)

PROPERTY 4. The constraint qualification conditions are satisfied.

Proof. The linear system, which define the constraint qualification, is the following :

$$\left\{ \begin{array}{l} \alpha \geq 0, b' \geq 0, b_k \geq 0 \text{ for } 0 \leq k \leq L, b \geq 0 \\ b'(\bar{K}(T) - K_T) = 0; b_k(\bar{Q}_k(T) + \bar{N}_k) = 0 \text{ for } 0 \leq k \leq L \\ b\left(\sum_{k=0}^L (\bar{Q}_k(T) + \bar{N}_k) - S_T\right) = 0 \\ b' = 0, \alpha = 0, b_k + b = 0 \text{ for } 0 \leq k \leq L \\ \text{if } \bar{a}_k = T: c_k \geq 0; \text{ if } \bar{a}_k < T: c_k = 0, \text{ for } 1 \leq k \leq L \\ \forall h_k > \theta_k, c_k(h_k - \bar{h}_k) \geq 0 \text{ for } 1 \leq k \leq L \\ \forall N_k > 0, (N_k - \bar{N}_k)(b_k + b) \leq 0 \text{ for } 1 \leq k \leq L. \end{array} \right.$$

In this system, $\alpha = 0 = b'$, and $b = 0 = b_k$ ($0 \leq k \leq L$);

if $\bar{a}_k > \theta_k$, then $c_k = 0$; if not, then $\bar{a}_k < T$ (assumption 2a) and

$c_k = 0$. The system admits only the trivial solution 0 and the constraint qualification conditions are verified ■

PROPERTY 5. From the theorem § 2, there exist real numbers α, b', b_k ($0 \leq k \leq L$), b, c_k ($1 \leq k \leq L$), and functions p, r, q_k ($0 \leq k \leq L$) not all zero, such that

$$(5) \quad \alpha \geq 0, b' \geq 0, b_k \geq 0 \text{ } (0 \leq k \leq L), b \geq 0;$$

$$(6) \quad \left\{ \begin{array}{l} b'(\bar{K}(T) - K_T) = 0, b_k(\bar{Q}_k(T) + \bar{N}_k) = 0 \text{ } (0 \leq k \leq L); \\ b\left(\sum_{k=0}^L (\bar{Q}_k(T) + \bar{N}_k) - S_T\right) = 0. \end{array} \right.$$

Let

$$H_0 = p(-mK + I) + r e^{-\delta t} U [F(K, E, t) - I - \sum_{k=1}^L R_k(t - h_k + \theta_k, \bar{N}_k)] - q_0 E;$$

$$H_k = (q_0 - q_k) E_k;$$

$$H = H_0 + \sum_{k=1}^L H_k 1_{[a_k, T]}.$$

$$(7) \begin{cases} \dot{p} = -\frac{\partial H}{\partial K} = pm - r e^{-\delta t} U' F'_K ; \\ \dot{r} = -\frac{\partial H}{\partial Z} = 0 ; \\ \dot{q}_k = -\frac{\partial H}{\partial Q_k} = 0 \quad (0 \leq k \leq L) \end{cases}$$

$$(8) \quad p(T) = b', \quad r(T) = \alpha, \quad q_k(T) = b_k + b \quad (0 \leq k \leq L).$$

The maximum of the Hamiltonian with respect to I gives :

$$(9) \quad p - r e^{-\delta t} U'(\bar{C}(t)) \leq 0, \text{ and } = 0 \text{ if } \bar{I}(t) > 0.$$

Let $E = E_0 + \sum_{k=1}^L E_k$; H is maximum over the set of the $E_k \geq 0$

$(0 \leq k \leq L)$; one obtains :

$$(10) \begin{cases} \frac{\partial H}{\partial E_0} = r e^{-\delta t} U' F'_E - q_0 \leq 0 ; \text{ and } = 0 \text{ if } \bar{E}_0(t) > 0 ; \\ \frac{\partial H}{\partial E_k} = \begin{cases} \text{for } t < \bar{a}_k : r e^{-\delta t} U' F'_E - q_0 \leq 0 ; \\ \text{for } t \geq \bar{a}_k \neq T : r e^{-\delta t} U' F'_E - q_k \leq 0, \text{ and } = 0 \text{ if } \bar{E}_k(t) > 0 \end{cases} \end{cases}$$

All the parameters h_k and N_k are independent ; it results that, for $1 \leq k \leq L$:

$$(11) \quad \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} r e^{-\delta t} U'(\bar{C}(t)) \frac{\partial R_k}{\partial t}(t - \bar{a}_k + \theta_k, \bar{N}_k) dt \leq c_k, \text{ and } = c_k \text{ if } \bar{a}_k > \theta_k$$

$$(12) \quad \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} r e^{-\delta t} U'(\bar{C}(t)) \frac{\partial R_k}{\partial N_k}(t - \bar{a}_k + \theta_k, \bar{N}_k) dt \geq b_k + b, \text{ and } = b_k + b \text{ if } \bar{N}_k > 0.$$

$$(13) \begin{cases} \text{if } \bar{a}_k = T : (q_0 - q_k) \bar{E}_k(\bar{a}_k) \leq c_k \\ \text{if } \bar{a}_k < T : (q_0 - q_k) \bar{E}_k(\bar{a}_k) \geq c_k \\ \text{and } M(\bar{a}_k) - M^-(\bar{a}_k) \leq \sum \{c_\ell ; \bar{a}_\ell = \bar{a}_k\} \end{cases}$$

where

$$M(\bar{a}_k) = p(\bar{a}_k) (-m\bar{K}(\bar{a}_k) + \bar{I}(\bar{a}_k)) + r e^{-\delta \bar{a}_k} U(\bar{C}(\bar{a}_k)) - q_0 \bar{E}(\bar{a}_k) + \sum_{\{\ell ; \bar{a}_\ell \leq \bar{a}_k\}} (q_0 - q_\ell) \bar{E}_\ell(\bar{a}_k) ;$$

$$\begin{aligned} M^-(\bar{a}_k) &= p(\bar{a}_k) (-m\bar{K}(\bar{a}_k) + \bar{I}(\bar{a}_k - 0)) + r e^{-\delta \bar{a}_k} U(\bar{C}(\bar{a}_k - 0)) \\ &\quad - q_0 \bar{E}(\bar{a}_k - 0) + \sum_{\{l; \bar{a}_l < \bar{a}_k\}} (q_0 - q_l) \bar{E}_l(\bar{a}_k - 0). \end{aligned}$$

III - Conséquences

PROPERTY 6. The multipliers verify the following properties :

$$(14) \quad \begin{cases} r = \alpha > 0 ; & q_k = b_k + b \geq 0 \quad (0 \leq k \leq L) ; \\ p(t) > 0 \quad \forall t ; & q_k > 0 \quad (0 \leq k \leq L) \end{cases}$$

Proof.

- r and q_k are constant (relation 7) and equal to their final values (relation 8).
- If $r = 0$, then $p(t) \leq 0 \quad \forall t$ (relation 9) and $p(t) = 0 \quad \forall t$ (relation (7) and $p(T) = b' \geq 0$) ; relations (5) and (12) give : $q_k = b_k + b \geq 0$, and ≤ 0 ; then all the multipliers are zero, which is impossible.
- If $p(t_0) = 0$, then $\dot{p}(t_0) < 0$ (relation 7), $\dot{p}(t) < 0$ for $t \geq t_0$, and $p(T) < 0$, which is impossible.
- $q_k > 0$ results from relation (10) if $\bar{a}_k < T$; for $\bar{N}_k = 0$, $\bar{a}_k$ is arbitrary in $[0, T]$ and can be chosen $< T$; for $\bar{a}_k = T$ and $\bar{N}_k > 0$, $q_k > 0$ results from relation (12) and assumption 2 d. ■

Notation. We can take $\alpha = r = 1$. Let

$$(15) \quad \pi(t) = e^{-\delta t} U'(\bar{C}(t))$$

It results from (9) that, for each t ,

$$(16) \quad p(t) \leq \pi(t) \quad \text{and if } \bar{I}(t) > 0 : p(t) = \pi(t).$$

PROPERTY 7. For each k such that $\bar{N}_k > 0$, the multiplier c_k verifies : $c_k > 0$.

Proof. It results from (11), assumptions 2 and 4c, that :

$$\begin{aligned} c_k &\geq \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) \frac{\partial R_k}{\partial t} dt, \\ \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) \frac{\partial R_k}{\partial t} dt &= - \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} R_k d\pi > 0 \quad \blacksquare \end{aligned}$$

PROPERTY 8. A new energy stock is available only when all previous stocks are exhausted. Let

$$\bar{a}_{k_1} \leq \bar{a}_{k_2} \leq \dots \leq \bar{a}_{k_s} < T$$

the optimal values of a_k such that $\bar{a}_k < T$ and $\bar{N}_k > 0$; then necessarily :

$$(17) \quad \bar{a}_{k_1} < \bar{a}_{k_2} < \dots < \bar{a}_{k_s} < T ;$$

$$(18) \quad \left\{ \begin{array}{l} \text{for } 0 \leq t < \bar{a}_{k_1}, \quad \pi(t) \frac{\partial F}{\partial E}(\bar{K}(t), \bar{E}(t), t) = q_0 \\ \text{for } \bar{a}_{k_i} \leq t < \bar{a}_{k_{i+1}}, \quad \pi(t) \frac{\partial F}{\partial E}(\bar{K}(t), \bar{E}(t), t) = q_{k_i} \text{ and } \bar{E}(t) = \bar{E}_{k_i}(t) \\ \hspace{15em} (1 \leq i \leq s-1) \\ \text{for } \bar{a}_{k_s} \leq t \leq T, \quad \pi(t) \frac{\partial F}{\partial E}(\bar{K}(t), \bar{E}(t), t) = q_{k_s} \text{ and } \bar{E}(t) = \bar{E}_{k_s}(t) ; \end{array} \right.$$

$$(19) \quad \bar{Q}_0(\bar{a}_{k_1}) + S_0 = 0 \text{ and } \bar{Q}_{k_i}(\bar{a}_{k_{i+1}}) + \bar{N}_{k_i} = 0 \quad (1 \leq i \leq s-1).$$

Proof. Let us assume that the previous stocks $S_0, \bar{N}_{k_j}$ ($1 \leq j \leq i-1$)

are not exhausted at the time $\bar{a}_{k_i}$; then it is possible to choose $E_{k_i}(\bar{a}_{k_i}) = 0$

with the same values of $\bar{E}(t), \bar{C}(t), \bar{I}(t)$: we obtain an optimal trajectory for which the necessary condition (13) gives $c_k \leq 0$; this is in contradiction with property 7.

Consequently we obtain (19) and for $t \in [\bar{a}_{k_i}, \bar{a}_{k_{i+1}}[$, the previous stocks being exhausted, $\bar{E}_{k_i}(t)$ is equal to $\bar{E}(t) > 0$, and relation (10) implies :

$$\pi(t) \frac{\partial F}{\partial E}(\bar{K}(t), \bar{E}(t), t) = q_{k_i}.$$

One obtains similar relations for the intervals $[0, \bar{a}_{k_1}[$ and $[\bar{a}_{k_s}, T]$ ■

PROPERTY 9. Let

$$L_1 = \{k ; \bar{N}_k = 0\},$$

$$L_2 = \{k ; \bar{N}_k > 0 \text{ and } \bar{a}_k < T\} = \{k_1, k_2, \dots, k_s\}$$

$$L_3 = \{k ; \bar{N}_k > 0 \text{ and } \bar{a}_k = T\};$$

1) for $k \in L_1$ and $t \in [\theta_k, T]$

$$(20) \quad \pi(t) \frac{\partial F}{\partial E}(\bar{K}(t), \bar{E}(t), t) \leq \int_{t-\theta_k}^t \pi(s) \frac{\partial R_s}{\partial N_k}(s-t+\theta_k, 0) ds ;$$

2) for $k \in L_2 \cup L_3$

$$(21) \quad q_k = \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) \frac{\partial R_k}{\partial N_k}(t - \bar{a}_k + \theta_k, \bar{N}_k) dt ;$$

3) for $k \in L_3$:

$$(22) \quad q_k = b \quad \text{and} \quad q_k \leq q_\ell \quad \forall \quad 1 \leq \ell \leq L .$$

Proof. 1) for $k \in L_1$, $\bar{N}_k = 0$ and $\bar{a}_k$ can be chosen anywhere in $[\theta_k, T]$; for $\bar{a}_k = t$, one obtains with property 6, and relations (10) and (12) :

$$\pi(t) \frac{\partial F}{\partial E}(\bar{K}(t), \bar{E}(t), t) \leq q_k \leq \int_{t-\theta_k}^t \pi(s) \frac{\partial R_k}{\partial N_k}(s-t+\theta_k, 0) ds$$

2) for $\bar{N}_k > 0$, relation (21) results from relation (12) ;

3) for $k \in L_3$, $\bar{a}_k = T$, $\bar{Q}_k(T) = 0$ and $\bar{Q}_k(T) + \bar{N}_k > 0$;

from relation (6), $b_k = 0$, $q_k = b$ and $q_k \leq b + b_\ell = q_\ell$ ($1 \leq \ell \leq L$) ■

PROPERTY 10. We have :

$$(23) \quad b > 0 \quad \text{and} \quad \sum_{k=0}^L (\bar{Q}_k(T) + \bar{N}_k) - S_T = 0$$

Proof. If $\bar{Q}_{k_s}(T) + \bar{N}_{k_s} > 0$, then $b_{k_s} = 0$, and $b = q_{k_s} > 0$ (property 6).

If $\bar{Q}_{k_s}(T) + \bar{N}_{k_s} = 0$, then there exists $k \in L_3$ and $q_k = b > 0$: this

results from the constraint

$$\sum_{k=0}^L (Q_k(T) + N_k) \geq S_T > 0 .$$

Finally $b > 0$ implies $\sum_{k=0}^L (\bar{Q}_k(T) + \bar{N}_k) - S_T = 0$ (relation 6). ■

PROPERTY 11. The sequence $q_0, q_{k_1}, \dots, q_{k_s}$ is strictly decreasing.

Proof. From property 7, $c_{k_i} > 0$ ($1 \leq i \leq s$) and from relation (13)

$q_0 > q_{k_1}$. For each i , let us consider the optimality problem from $\bar{a}_{k_i}$ to T with initial stock $\bar{N}_{k_i}$ and other optimal values at $\bar{a}_{k_i}$:

for this problem, the corresponding multipliers verify : $q'_{k_i} > q'_{k_{i+1}}$,

and the values of the multipliers q'_j ($j \geq i$) are identical to q_{k_j} from relation (18). ■

Let us note : $q_0 = q_{k_0}$.

PROPERTY 12. For $1 \leq i \leq s$:

$$(24) \quad c_{k_i} > (q_{k_{i-1}} - q_{k_i}) \bar{E}(\bar{a}_{k_i} - 0) .$$

Proof. We have

$$M(\bar{a}_{k_i}) = H_0(\bar{E}(\bar{a}_{k_i}), \bar{I}(\bar{a}_{k_i}), \bar{a}_{k_i}) + \sum_{j=1}^i (q_0 - q_{k_j}) \bar{E}_{k_j}(\bar{a}_{k_i}) .$$

The hamiltonian is maximum with respect to E , I and E_k ; this implies

for $E_{k_j} \geq 0$ such that $\sum_{j=1}^i E_{k_j} \leq \bar{E}(\bar{a}_{k_i} - 0)$:

$$M(\bar{a}_{k_i}) \geq H_0(\bar{E}(\bar{a}_{k_i} - 0), \bar{I}(\bar{a}_{k_i} - 0), \bar{a}_{k_i}) + \sum_{j=1}^i (q_0 - q_{k_j}) E_{k_j} .$$

Particularly, for $E_{k_i} = \bar{E}(\bar{a}_{k_i} - 0)$ and $E_{k_j} = 0$ ($j \neq i$), we obtain :

$$M(\bar{a}_{k_i}) \geq H_0(\bar{E}(\bar{a}_{k_i} - 0), \bar{I}(\bar{a}_{k_i} - 0), \bar{a}_{k_i}) + (q_0 - q_{k_i}) \bar{E}(\bar{a}_{k_i} - 0)$$

Moreover this inequality is strict : if not, then H attains its maximum at $\bar{a}_{k_i}$ with respect to E and I for $E = \bar{E}(\bar{a}_{k_i} - 0)$ and $I = \bar{I}(\bar{a}_{k_i} - 0)$

and necessarily

$$\pi(\bar{a}_{k_i} - 0) F'_E(\bar{K}(\bar{a}_{k_i}), \bar{E}(\bar{a}_{k_i} - 0), \bar{a}_{k_i}) = q_{k_i} ;$$

but relation (18) implies that

$$\pi(\bar{a}_{k_i} - 0) F'_E(\bar{K}(\bar{a}_{k_i}), \bar{E}(\bar{a}_{k_i} - 0), \bar{a}_{k_i}) = q_{k_{i-1}}$$

and this is impossible because $q_{k_{i-1}} \neq q_{k_i}$ (property 11).

We have also

$$M^-(\bar{a}_{k_i}) = H_0(\bar{E}(\bar{a}_{k_i}-0), \bar{I}(\bar{a}_{k_i}-0), \bar{a}_{k_i}) + (q_0 - q_{k_{i-1}})\bar{E}(\bar{a}_{k_i}-0)$$

and from relation (13)

$$c_{k_i} \geq M(\bar{a}_{k_i}) - M^-(\bar{a}_{k_i}) > (q_{k_{i-1}} - q_{k_i})\bar{E}(\bar{a}_{k_i}-0) \quad \blacksquare$$

PROPERTY 13. For $1 \leq i \leq s$:

$$(25) \quad \int_{\bar{a}_{k_i}-\theta_{k_i}}^{\bar{a}_{k_i}} \pi(t) \frac{\partial R_{k_i}}{\partial t}(t - \bar{a}_{k_i} + \theta_{k_i}, \bar{N}_{k_i}) dt \leq (q_{k_{i-1}} - q_{k_i})\bar{E}(\bar{a}_{k_i}) .$$

PROOF. Let us consider the optimality problem with the modified energy stock evolution equations :

$$\begin{aligned} \dot{Q}_0 &= -E + E1_{[a_{k_1}, T]} \\ \dot{Q}_{k_i} &= -E1_{[a_{k_i}, T]} + E1_{[a_{k_{i+1}}, T]} \quad (1 \leq i \leq s-1) \\ \dot{Q}_{k_s} &= -E1_{[a_{k_s}, T]} \end{aligned}$$

and the constraint $0 < a_{k_1} < a_{k_2} < \dots < a_{k_s} < T$.

This problem admits the same optimal solution and the necessary conditions (§ 2) give :

$$\begin{aligned} \int_{\bar{a}_{k_i}-\theta_{k_i}}^{\bar{a}_{k_i}} \pi(t) \frac{\partial R_{k_i}}{\partial t} dt &\leq c'_{k_i} \leq (q'_{k_{i-1}} - q'_{k_i})\bar{E}(\bar{a}_{k_i}) \\ q'_{k_i} &= \int_{\bar{a}_{k_i}-\theta_{k_i}}^{\bar{a}_{k_i}} \pi(t) \frac{\partial R_{k_i}}{\partial N_{k_i}} dt = q_{k_i} \end{aligned}$$

and we obtain conclusion (25). $\blacksquare$

PROPERTY 14. The total expenses of energy consumption

$$(26) \quad \mathcal{Z} = \int_0^{a_{k_1}} q_0 \bar{E}(t) dt + \sum_{i=1}^{s-1} \int_{a_{k_i}}^{a_{k_{i+1}}} q_{k_i} \bar{E}(t) dt + \int_{a_{k_s}}^T q_{k_s} \bar{E}(t) dt$$

admits with respect to a_{k_i} a derivative to the left of $\bar{a}_{k_i}$:

$$(27) \quad \frac{\partial \mathcal{L}}{\partial a_{k_i}} (\bar{a}_{k_i} - 0) = (q_{k_{i-1}} - q_{k_i}) \bar{E}(\bar{a}_{k_i} - 0)$$

and a derivative to the right of $\bar{a}_{k_i}$:

$$(28) \quad \frac{\partial \mathcal{L}}{\partial a_{k_i}} (\bar{a}_{k_i} + 0) = (q_{k_{i-1}} - q_{k_i}) \bar{E}(\bar{a}_{k_i}) .$$

Proof. For a piecewise continuous function $\varphi(t)$,

$$\frac{1}{h} \left(\int_a^{b+h} \varphi(t) dt - \int_a^b \varphi(t) dt \right) = \frac{1}{h} \int_b^{b+h} \varphi(t) dt$$

admits as its limit $\varphi(b+0)$ (respectively $\varphi(b-0)$) when h converges to 0 by positive (respectively negative) values ; to obtain the conclusion it is sufficient to apply this result to

$$\int_{a_{k_{i-1}}}^{a_{k_i}} q_{k_{i-1}} \bar{E}(t) dt - \int_{a_{k_{i+1}}}^{a_{k_i}} q_{k_i} \bar{E}(t) dt \quad \blacksquare$$

PROPERTY 15. The investment expenditure for the energy stock of type $k \in L_2 \cup L_3$:

$$(29) \quad D_k = \int_{a_k - \theta_k}^{a_k} \pi(t) R_k(t - a_k + \theta_k, \bar{N}_k) dt$$

is derivable with respect to a_k and its derivative at the optimal time $\bar{a}_k$ verifies :

$$(30) \quad - \frac{\partial D_k}{\partial a_k} (\bar{a}_k) \leq c_k \text{ and } \bar{a}_k > \theta_k \Rightarrow \frac{\partial D_k}{\partial a_k} (\bar{a}_k) = - c_k .$$

Proof. From assumption 2c ,

$$\frac{\partial D_k}{\partial a_k} = - \int_{a_k - \theta_k}^{a_k} \pi(t) \frac{\partial R_k}{\partial t}(t - a_k + \theta_k, \bar{N}_k) dt ;$$

and relation (30) results from relation (11).

PROPERTY 16. For $1 \leq i \leq s$, the sum of the investment and the consumption expenditures of energy admits with respect to a_{ki} a derivative to the right of $\bar{a}_{ki}$ which is nonnegative and, if

$\bar{a}_{ki} > \theta_{ki}$, a derivative to the left of $\bar{a}_{ki}$ which is negative.

Proof. The derivative to the right of $\bar{a}_{ki}$ is equal to

$$(q_{k_{i-1}} - q_{k_i}) \bar{E}(\bar{a}_{k_i}) - \int_{\bar{a}_{k_i} - \theta_{k_i}}^{\bar{a}_{k_i}} \pi(t) \frac{\partial R_k}{\partial t} dt$$

which is non negative (relation 25).

If $\bar{a}_{ki} > \theta_{ki}$, the derivative to the left of $\bar{a}_{ki}$ is equal to

$$(q_{k_{i-1}} - q_{k_i}) \bar{E}(\bar{a}_{k_i} - 0) + \frac{\partial D_{k_i}}{\partial a_{k_i}} (\bar{a}_{k_i}) = (q_{k_{i-1}} - q_{k_i}) \bar{E}(\bar{a}_{k_i} - 0) - c_{k_i} < 0$$

(relation 24) ■

Remark. Properties 15 and 16 give an interpretation of the multipliers c_k and the necessary conditions (13) :

c_k is equal to $|\frac{\partial D_k}{\partial a_k}(\bar{a}_k)|$; and the properties $\varphi'(\bar{a}_k - 0) \geq 0$ and

$\varphi'(\bar{a}_k + 0) \leq 0$ and $\varphi'(\bar{a}_k + 0) \leq 0$ are necessary conditions for φ to attain its minimum at $\bar{a}_k$.

PROPERTY 17. Let us assume that $R_k(t, N_k)$ is convex with respect to N_k .

Then the net result obtained by energy of type k :

$$(31) \quad B_k = q_k N_k - \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) R_k(t - \bar{a}_k + \theta_k, N_k) dt$$

is nonnegative and maximum for $N_k = \bar{N}_k$.

Proof. It results from relation (12) that

$$\frac{\partial B_k}{\partial N_k}(\bar{N}_k) \leq 0 \text{ and } = 0 \text{ if } \bar{N}_k > 0 ;$$

and from convexity of R_k :

$$B_k = \int_{\bar{a}_k - \theta_k}^{\bar{a}_k} \pi(t) [N_k \frac{\partial R_k}{\partial N_k} - R_k] dt \geq 0 \quad \blacksquare$$

R E F E R E N C E S

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