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DYNAMIC PROCESSES FOR TAX REFORM THEORY (+)

BY

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This paper presents the study of procedure aimed at improving an indirect tax system. The procedure has the following features :

A - The procedure operates in an idealized economic world analogous to that described by Diamond-Mirrlees in their basic model of [5] : all commodities can be taxed, producers have a competitive behaviour, consumers have only a labour income. As it is supposed that there are no public goods, the tax system performs essentially a redistributive rather than a financing function.

- The principles of the procedure are borrowed from a preceding proposition of Guesnerie [7] which analyzes in the framework of the above model, the directions of tax reform with respect to feasibility and desirability criteria .

B - A dynamic process indexed by a continuous variable time is engendered through the linkage of the desirable infinetesimal tax changes. This dynamic process is built in such a way that it has the following characteristics of feasibility and monotonicity :

. feasibility : On all trajectory (if any) if one stops at any time t, the corresponding state of the economy is a feasible state (i.e an equilibrium with respect to taxes prevailing at this time)

. monotonicity: The welfare of all households in the economy -as measured by some utility level- is an increasing function of the variable time on all trajectory.

C - The specific purpose of the study of the dynamic process -once it has been defined-concerns the standard questions raised by dynamic systems which are existence and stability. It is worth noting that the differential system considered, by its nature, has a multivalued right hand side and that the recent mathematical and developments in the study of such systems provide us natural and appropriate tools .
(see Castaing [2], Champsaur Dreze Henry [3])

The reader will have noted that the preoccupations underlying this study have a narrow resemblance with those leading to the study of planning procedures concerning either the implementation of efficient allocation in the production sector, cf. [8] or the choice of efficient output levels for public goods [6] etc... This resemblance relies both on the definition of similar requirements for the dynamic processes- feasibility, monotonicity as defined by Malinvaud [11] and, on the similarity of the questions raised:existence, stability.

The interpretation of the system as a tâtonnement procedure aimed at allowing the center to implement efficient plans through an exchange of information with decentralized units- i.e in the implicit framework of an economic theory of socialism (cf. Heal [8] would suppose that the information gathered at any step by the center. concerns the individual consumption of households, elasticities of demand and elasticities of supply.

However a more natural interpretation of the procedure- and our assumptions are generally implicitly related to this view - consists in considering that the dynamic process describes the working of an algorithm used by the Government of a non centrally planned economy, for revising its indirect tax system. Such an algorithm which assumes the knowledge of demand and supply functions which may be provided by econometric estimates, would operate at some aggregate level (aggregation of commodities and households in classes).

1 MODEL AND NOTATION

1.A - The agents and the assumptions

The model we are considering was first explicitely introduced in the literature by Diamond-Mirrlees [5] who, in their seminal article, focused the attention on the derivation of optimality conditions for the tax system.

There are n commodities in the economy indexed by $k = 1 \ldots\ldots n$.

Two categories of commodities are considered. Commodities 1 to n_1 can only be consumed in negative quantities (or supplied) and commodities $n_1 + 1$ to n can only be consumed in positive quantities (or demanded), this being true for all consumers. An appropriate choice of consumption sets allows taking into account this assumption of specialized commodities.

Let the consumers be indexed by $h = 1 \ldots H$, and let Ω_h be the consumption set of agent h. The following assumption on Ω_h is made

$\boxed{H1}$ Ω_h is closed convex, bounded from below and included in $\mathbb{R}_-^{n_1} \times \mathbb{R}_+^{n-n_1}$ (*)

Each consumer h has preferences defined on Ω_h and represented by a utility function u_h which satisfies :

(*)Unless explicit contrary statement, the following conventions will hold thoughout the paper :

- consumption or production plans are column vectors
- price systems are line vectors
- $\overline{X}, \overset{\circ}{X}$, ri X, Fr.X, designate respectively the closure, the interior, the relative interior, and the frontier of the set X.
- ${}^tA$ denotes the transposed of matrix A

$\boxed{H\,2}$ u_h is positive, strictly quasi-concave, and continuously differentiable on ri(Ω h) with $\dfrac{\partial u_h}{\partial x_k} > 0 \quad \forall h, \forall k$

Faced with the price system Π (Π in $P = \overset{\circ}{\mathbb{R}}{}^n_+$) , consumer h determines his demand by solving the program :

$$\text{Max } \{ u_h (x_h) \quad / \quad x_h \in \Omega_h , \Pi . x_h \leq 0 \}$$

The reader will notice that, through this formulation, the consumer is supposed to have no other income than his labour income.

From H 1, H 2, this program has a unique solution X_h (Π) such that $\Pi . X_h (\Pi) = 0$

X_h is the demand function of consumer h.

$X = \sum_h X_h$ is the aggregate demand function.

We will also consider in the following the indirect utility function $V_h (\Pi) = u_h \; (X_h(\Pi))$. Production possibilities are described through a production function G (which defines the production set $Y = \{ y \,/\, G (y) \leq 0 \}$) such that :

$\boxed{H\,3\,\alpha}$ G is strictly quasi convex, and monotonic :

$y > y' \Rightarrow G(y) > G(y')$; $\quad G(0) = 0$

$\boxed{H\,3\,\beta}$ The asymptotic cone of Y, A Y is $\mathbb{R}^n_-$

Faced with the production price system p (in P) the producer determines his supply by solving the following program :

$$\text{Max } \{ p . y , G (y) \leq 0 \}$$

It results from H 3 α, H 3 β that $\forall p \in P$, this program has a unique solution $\eta(p)$ such that $G(\eta(p)) = 0$

$\eta : P \to \mathbb{R}^n$ is the supply function (*)

The supply and demand functions will be supposed to fulfill the following requirements :

$\boxed{\text{H 4}}$ X_h is C^1 on P

$\boxed{\text{H 5}}$ η is C^1 on P (**)

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* If G is strictly quasi concave, and $AY = \mathbb{R}^n_-$, it follows from Artzner Neuefeind, [1] that $BY = \{p / \eta(p) \text{ is defined}\} = \mathring{\mathbb{R}}^n_+$

** The assumption that $\mathbb{R}^n_-$ is the asymptotic cone of Y does not play a decisive role in the following

In fact, il would be enough to consider throughout this paper that η which is from H 3 α and Artzner Neuefeind [1] defined on the interior of the polar of AY, is differentiable in this set.

Given H 4 et H 5 , one will denote $\partial\overline{\overline{\overline{X}}}(\Pi)$ the $n \times n$ matrix

$$\partial\overline{\overline{\overline{X}}}(\Pi) = \begin{array}{cc} & k \\ \ell & \left(\cdots \dfrac{\partial X^{\ell}}{\partial \Pi_k}(\Pi) \cdots \right) \end{array}$$

and $\partial\overline{\overline{\overline{\eta}}}(p)$ the $n \times n$ – matrix :

$$\partial\overline{\overline{\overline{\eta}}}(p) = \begin{array}{cc} & k \\ \ell & \left(\cdots \dfrac{\partial \eta^{\ell}}{\partial p_k}(p) \cdots \right) \end{array}$$

It is well known that : $p \ \partial\overline{\overline{\overline{\eta}}}(p) = 0$. So that $\partial\overline{\overline{\overline{\eta}}}(p)$ is at most of rank $n-1$. We shall assume precisely that :

$\boxed{\text{H 6}}$ $\partial\overline{\overline{\overline{\eta}}}(p)$ is of rank $(n-1)$, $\forall\, p \in P$

$\boxed{\text{H 7}}$ There does not exist $(p, \Pi) \in P \times P$ such that :

$$X(\Pi) = \eta(p) \qquad , \qquad p \, . \, \partial\overline{\overline{\overline{X}}}(\Pi) = 0$$

This is a kind of regularity assumption analogous to the regularity condition of Kuhn–Tucker type .

This property does not look very restrictive and it is argued in the footnote (*) that it is likely to be "generic" in the sense of differential topology.

H 7' is more restrictive :

$\boxed{\text{H 7'}}$ There does not exist $(p, \Pi) \in P \times P$ such that

$$X(\Pi) \leq \eta(p) \quad , \quad p . \overline{\overline{\partial X}}(\Pi) = 0$$

Let now $V(p) = \{ u \in \mathbb{R}^n \mid p . u = 0 \}$

H 5 - H 6 being given and according to lemma 1 in Guesnerie [7] $\partial \overline{\overline{\eta}}(p)$ defines a one to one correspondence from V(p) onto V (p). This correspondence will be denoted $\widetilde{\partial \eta}(p)$ and its inverse $\widetilde{\partial \eta}^{-1}(p)$

Let us briefly discuss the restriction in the range of production possibility sets implied by H 3α-H 3β - H 5 - H 6 . One may first remark that given H 3 α and β , H 5 and H 6 are mainly technical and it is likely that it could be proved that they are "generic" properties . H 3 β supposes some substituability between inputs and outputs and that marginal returns tend to vanish when the scale of production tends to infinity. On the one hand, it is not unreasonable at the aggregate level we are considering; on the other hand, this assumption is not necessary for the argument and is only intended to simplify the presentation .

(*) If one considers the set of equations $X(\Pi) = \eta(p)$ $p . \overline{\overline{\partial X}}(\Pi) = 0$, one sees that we have 2n -1 equations for 2n -2 variables. Hence, this suggests that they can only be satisfied for exceptionnal data , a point which could be confirmed by a more formal approach.

$H\,3\alpha$, even if it is not unusual in the literature, is perhaps less satisfactory since it rules out production sets where inputs and outputs can be completely distinguished. Actually $H3\alpha$ could be relaxed for allowing such production sets, but condition (A) below should then be reinforced (*) in such a way that $H3\alpha$ seemed an acceptable compromise between realism and simplicity .

Finally, a last assumption will be made which concerns demand behaviour

<u>Assumption (A)</u> :

$\alpha)\ X_h(\Pi) \in ri\ (\Omega_h)\ \ \forall\ \Pi \in P$

$\beta)$ If a sequence (Π_n) of vectors in P is such that :

- $\|\Pi_n\| = \|\Pi_o\|\ \forall\, n,\ \|\Pi_o\| \neq 0$ (**)
- $\Pi_n \xrightarrow[n \to +\infty]{} \overline{\Pi} \in \overline{P} / P$

Then ,

a) either $\|X(\Pi_n)\| \to +\infty$

b) or $\forall\ v > 0$, $\exists\ h \in (1 \ldots H)$ and N such that $n > N \Rightarrow u_h(X_h(\Pi_n)) < v$

(*) It should assure that :

1) If the price of one output tends to zero its supply tends to zero

2) All commodities are "essential" in the sense of p. 7
With $H\,3\alpha$ the things are simpler since the supply of one output tends to $-\infty$ when its price tends to zero.

(**) $\|x\|$ is the euclidean norm of x

The idea underlying assumption(A) is that when a commodity price tends to zero, either it is a consumption good and its demand tends to infinity(a) or it is a type of labor and then the utility level of some household tends to zero (b).

Relatively to H 1 - H 6, (A) introduces additionnal restrictions on the preferences which are considered. For example, the reader will check that (A) will hold if all consumption commodities are supposed essential ($x_{ih} = 0 \Rightarrow u_h(x_h) = 0 \quad i = 1 \dots n_1, \; h = 1 \dots H$) and if each consumer is specialized (he can supply one and only one type of labour)

This brief analysis of assumption (A) suggests that it can be considered rather strong if one reasons at the disaggregate level. However, at a more aggregate level, which is an appropriate level for practical use of the algorithm proposed in the note, the assumption does not look unreasonable

I. B - Equilibria and the principles of an algorithm for tax reform

Let us give several definitions :

Definition 1 :

An equilibrium of the system consists in a couple (Π, p) of consumption price system and production price system such that :

$$\sum_{h=1}^{H} X_h(\Pi) \leq \eta(p)$$

Definition 2 :

The equilibrium is said to be tight if $\sum_{h=1}^{H} X_h(\Pi) = \eta(p)$. If the latter equality does not hold, the equilibrium is non tight or inefficient

Definition 1 describes an equilibrium with taxes : the disconnection between production and consumption price systems is supposed to be implemented through taxation. The formulation also supposes that the profit of the firms is completely taxed by the government. (For a comprehensive discussion of these assumptions see Diamond Mirrlees [5]

Definition 2 expresses a tightness condition which is the equivalent for our model of the efficiency property of Diamond-Mirrlees. If this condition is not satisfied, the total demand could be satisfied with an inefficient production plan, i.e a production plan in the interior of the total production set.

In this note, one will particularly be interested in special types of equilibria, the Pareto equilibria, defined as follows:

Definition 3 : A local Pareto equilibrium consists in a couple (Π, p) such that :

α) (Π, p) defines a tight equilibrium

β) $- p. \overline{\overline{\partial X}} (\Pi) = \sum_{h=1}^{H} \lambda_h X_h (\Pi)$ with $\lambda_h \geq 0$

Conditions (β) are necessary conditions for second best Pareto optimality (see Diamond-Mirrlees [ ]) and this fact motivates the vocabulary of local Pareto equilibria.

Actually, a local Pareto equilibrium may have different features:

- It may be a global second best Pareto optimum in the sense that there does not exist, in the set of all equilibria, one Pareto superior equilibrium. Moreover, under the conditions we consider all Pareto optima belong to the set of local Pareto equilibria.

- It may be a local Pareto optimum in the sense that there does not exist a neighbour Pareto superior equilibrium.
- It may be neither a local nor a global second best Pareto optimum. In this last category fall the saddle type Pareto equilibria on which we will come back later.

Let us now introduce some additional notation :

$$\mathring{K}(\Pi) = \{ a \in \mathbb{R}^n \mid a.\, X_h(\Pi) < 0 ,\ h = 1 \ldots H \}$$

$$Q(\Pi, p) = \{ a \in \mathbb{R}^n \mid p.\, \overline{\overline{\delta X}}(\Pi).\, {}^t a \leq 0 \}$$

$$Fr\, Q(\Pi, p) = \{ a \in \mathbb{R}^n \mid p.\, \overline{\overline{\delta X}}(\Pi).\, {}^t a = 0 \}$$

We are interested in designing an "algorithm of tax reform" ; i. e a procedure for modifying the tax system leading to an improvement of all consumers' welfare .

It has been argued in Guesnerie [7] that :

Such an algorithm can be implemented through the linkage of infinitesimal changes of taxes and prices, which, starting from a given equilibrium (Π, p), meet the following requirements :

$$(1) \begin{cases} \dfrac{d\Pi}{dt} \in \mathring{K}(\Pi) \cap Fr\, \{Q(\Pi, p)\} & (a) \\ {}^t\dfrac{dp}{dt} = \widetilde{\delta\eta}^{-1}(p).\, \overline{\overline{\delta X}}(\Pi).\, {}^t\dfrac{d\Pi}{dt} & (b) \end{cases}$$

An intuitive understanding of the above system can be got through the following remarks :

- The change in consumption prices is first astrained to be in $Fr\, Q(p, \Pi)$

i.e to induce a change in demand $\overline{\overline{\delta X}}(\Pi)\,{}^t\frac{d\Pi}{dt}$ whose value expressed with production prices is zero : $\left(p.\ \overline{\overline{\delta X}}(\Pi)\,{}^t\frac{d\Pi}{dt} = 0\right)$

. The change in demand induced by the change in consumption prices just considered can then be matched by a change of production prices defined by formula (b)

. In order to be Pareto improving the change in consumption prices must be such that the value of all consumption bundles decreases, a condition which is expressed in formalized terms as $\frac{d\Pi}{dt} \in \overset{\circ}{K}(\Pi)$

More precisely, the interest of system 1, lies in the following precise property (which is Corollary III in Guesnerie [7])

Proposition I

If there exists a solution of system (1) $(\Pi(t), p(t))$ defined on $[0, T]$ and starting from $(\Pi(0), p(0))$ such that $\eta(p(0)) = X(\Pi(0))$, then

$(X_h(\Pi(t)), \eta(p(t)), \Pi(t), p(t))$ define a tight equilibrium $\forall\, t \in [0, T]$

$V_h(\Pi(t))$ is a strictly increasing function of t

The rest of the paper is devoted to a comprehensive study of dynamical processes governed by system 1 or by similar systems. This study raises the types of questions that are usually considered in the literature on planning procedures (Heal [8], Malinvaud [11]) and which are twofold :

(1) Do there exist solutions of the system of differential equations which make it meaningful ?

(2) Given a solution path of the system, does it converge and to which points ?

These two problems -existence problems and convergence problems - will be examined for a system derived from system I in section II. It will be shown that some limit points of paths have undesirable properties. A more complicated system which excludes such undesirable limit points will then be considered in section III. All proofs are in the appendix

II - A FIRST DYNAMICAL PROCESS

System I will be slightly modified in order to be defined for all $(\Pi, p) \in P \times P$. For that, one will assume that prices remain constant as soon as the second member of equation (a) becomes empty. This gives system (1')

$$(1')\left\{\begin{array}{l} \dfrac{d\Pi}{dt} \in \overset{\circ}{K}(\Pi) \cap \mathrm{Fr}\ Q(\Pi, p) \quad \text{if } \overset{\circ}{K}(\Pi) \cap \mathrm{Fr}\ Q(\Pi, p) \neq \emptyset \\ \qquad = 0 \ \text{ otherwise} \\ {}^{t}\dfrac{dp}{dt} = \partial \tilde{\eta}^{-1}(p) \,.\, \overline{\overline{\partial X}}(\Pi) \,.\, {}^{t}\dfrac{d\Pi}{dt} \end{array}\right.$$

II-A Existence problems

Has system (1'), which is a system of multivalued differential equations, a solution ?

The answer to this question rests on mathematical results sometimes directly initiated by problems in economic theory (cf. Cl. Henry [10]). A view of these results is given in Champsaur Dreze Henry [3].

Here, a way of approaching existence is to consider an auxiliary system obtained by building an upper hemi -continuous convex compact valued correspondence M which is extracted from $\overset{\circ}{K}(\Pi) \cap \mathrm{Fr}\ Q(\Pi, p)$ when it is non empty.

To this end let us consider :

$$\varphi(\Pi, p) = \{ a \in \mathbb{R}^n / p\, \overline{\overline{\partial X}}(\Pi) \, . \, {}^t a = 0 \, , \; \Pi \, . \, {}^t a = 0 \, , \; \|a\| \leq 1 \}$$

$$f(\Pi, p) = \operatorname{Max} \{ \operatorname{Min}_h (- a \, . \, X_h (\Pi)) / a \in \varphi(\Pi, p) \}$$

Let us notice that $f(\Pi, p) = 0$ if and only if $\overset{\circ}{K}(\Pi) \cap Fr\, Q(\Pi, p) = \emptyset$

$$M(\Pi, p) = \{ a \in \varphi(\Pi, p) / f(\Pi, p) = \operatorname{Min}_h (-a \, . \, X_h (\Pi)) \}$$

(2) is :

$$(2) \begin{cases} \dfrac{d\Pi}{dt} \in M(\Pi, p) \\ {}^t\dfrac{dp}{dt} = \partial \tilde{\eta}^{-1}(p) \, . \, \overline{\overline{\partial X}}(\Pi) \, . \, {}^t\dfrac{d\Pi}{dt} \end{cases}$$

From an economic point of view, system (2) has the following characteristics :

. the speed of change of consumption prices is bounded : $\|a\| \leq 1$

. the norm of the consumption price system remains constant along any solution path (as does the norm of p) .

. the change in prices are designed in order to maximize the smallest speed of decrease in the value of agents' consumption bundles.

The following theorem holds :

<u>Theorem 1 :</u>

Under assumptions H1 - H7 and (A), for all $(\Pi°, p°) \in P \times P$ such that $\eta(p°) = X(\Pi°)$, there exists a solution $(\Pi(t), p(t))$ of system (2), defined on $[0, +\infty[$ and starting from $(\Pi°, p°)$.

<u>Corollary 1</u>

The same statement is true for system (1')

In other words with vocabulary on general dynamical processes there exists a trajectory for systems 1' et 2 . Let us recall that according to proposition 1, along this trajectory utilities increase and equilibria remain tight .

A solution $\Pi(t)$, $p(t)$ on $[0, +\infty[$ of system 2, if any, allows to define a solution $\tilde{\Pi}(t)$, $\tilde{p}(t)$ of system 1' on $[0, +\infty[$ in the following way : From $t = 0$ until the first time T where $f(\Pi(T), p(T)) = 0$, $\tilde{\Pi}(t) = \Pi(t)$ $\tilde{p}(t) = p(t)$, $t \in [0, T]$; for $t > T$ $\tilde{\Pi}(t) = \tilde{\Pi}(T)$, $\tilde{p}(t) = \tilde{p}(T)$ (*)

This allows to prove Corollary I

II B - Quasi - Stability

Let us recall first some definitions relative to a general dynamical process (P) governed by the system of multivalued differential equations :

$$\frac{dx}{dt} \in F(x) \qquad (S)$$

$\bar{x}$ is an equilibrium of (P) if $0 \in F(\bar{x})$.

A trajectory of (P) is a solution of (S) defined on $[0, +\infty[$.

We will say that $\bar{x}$ is a limit point of a trajectory $x(t)$ if there exists a sequence $t_n \underset{n \nearrow +\infty}{\longrightarrow} +\infty$ such that : $x(t_n) \underset{n \nearrow +\infty}{\longrightarrow} \bar{x}$

Process (P) is quasi stable iff any limit-point of a trajectory is an equilibrium .

For the systems we consider, the following propositions which are proved in the appendix hold :

Proposition 2 :

System 2 is quasi-stable

(*) In system 2, we obtained the upper hemi-continuity of M when it passes through 0, to the cost of introducing vectors $\frac{d\Pi}{dt}$ which were not allowed by system 1'.

Theorem 2 :

For any trajectory of system 1' such that : $\forall h\ (1 \ldots H)$ $\forall t \geq 0$, $-\frac{d\Pi}{dt} \cdot X_h(\Pi(t)) \geq k\, f(\Pi(t), p(t))$, where k is a strictly positive number (smaller than one), every limit point is an equilibrium .

Remark :

An obvious requirement for a limit point of a trajectory to be an equilibrium for systems like (1') and (2) is that the speed of change of prices do not tend to zero "too fast" on the trajectory.

This requirement is automatically satisfied on a trajectory of system (2) - which then can be proved quasi-stable - but not on any trajectoryof system 1'. So, the condition given in theorem 2 is intended to assure that the speed of change of prices does not become too small.

Let us now consider a trajectory of system (1') where the speed meets the requirement $-\frac{d\Pi}{dt} \cdot X_h(\Pi(t)) \geq k\,(f(\Pi(t), p(t)))$ - such a trajectory exists from theorem 1 . Let $\bar{\Pi}$, $\bar{p}$ be a limit point of this trajectory . According to theorem 2, $\bar{\Pi}$, $\bar{p}$ is an equilibrium of system (1') i.e :

$$\mathring{K}(\bar{\Pi}) \cap \mathrm{Fr}\ Q(\bar{\Pi}, \bar{p}) = \emptyset$$

What can be said about such an equilibrium ?

An answer is provided by proposition 3 , which can be seen as a corollary of proposition 4 in Guesnerie [7]

Proposition 3 :

If $\bar{\Pi}$, $\bar{p}$ is a limit point of system (1')

- Either it is a local Pareto equilibrium
- or $\mathring{K}(\bar{\Pi}) \cap Q(\bar{\Pi}, \bar{p}) \neq \emptyset$

Thus, we are faced with four types of possible limit points :

α - $(\overline{\Pi}, \overline{p})$ is a global second best optimum

β - $(\overline{\Pi}, \overline{p})$ is a local Pareto optimum

γ - $(\overline{\Pi}, \overline{p})$ is a saddle type Pareto equilibrium. It is such that there exist Pareto superior points in any neighbourhood of $(\overline{\Pi}, \overline{p})$ but however the necessary conditions of second best Pareto optimality are satisfied. Our algorithm stops in these points because it does not allow, even temporarily, a null speed of increase of one agent's utility.

δ - If $\mathring{K}(\overline{\Pi}) \cap Q(\overline{\Pi}, \overline{p}) \neq \emptyset$, $(\overline{\Pi}, \overline{p})$ is not a local Pareto equilibrium in the sense of definition 3. And generally there exist Pareto superior equilibria in any neighbourhood of $(\overline{p}, \overline{\Pi})$, but they are (or may be) non tight so that the process associated to system (1'), which is constrained to remain in the set of tight equilibria, stops. With the vocabulary of Guesnerie [7] in which this phenomenon -which may look strange - has been studied, there exist strictly Pareto improving directions of price changes but they lead (or at least tend to lead) to non tight equilibria. In other words a time path of price changes inducing a monotonic increase of all utilities can be extended only if temporary (*) inefficiencies are allowed, which is not the case for systems 1, 1' and 2.

Limit points corresponding to δ are particularly unsatisfactory We would try to rule them out, by considering in section III a more complicated system which will allow temporary inefficiencies.

(*) Inefficiencies are only temporary, in the sense that the attainment of a global second best optimum would remove them.

III - A dynamic process with temporary inefficiencies

Defining $Q(\Pi, p, \lambda) = \{ a \in \mathbb{R}^n \, / \, p.\overline{\overline{\delta X}}(\Pi).\, {}^t a \leq \lambda \|p\| \}$ where λ is a parameter belonging to $\mathbb{R}$, we will be interested in this section by the following system (3)

$$
(3)\quad \begin{cases}
\dfrac{d\Pi}{dt} \in \mathring{K}(\Pi) \cap Q(\Pi, p, \lambda) \text{ if this set is not empty .} \\[2mm]
\dfrac{d\Pi}{dt} = 0 \quad \text{otherwise} \\[2mm]
\dfrac{{}^t dp}{dt} = \delta \tilde{\eta}^{-1}(p) \left[\overline{\overline{\delta X}}(\Pi) . \dfrac{{}^t d\Pi}{dt} + \dfrac{d\lambda}{dt} \, {}^t\Pi + \lambda \dfrac{{}^t d\Pi}{dt} \right] \\[2mm]
\dfrac{d\lambda}{dt} = - \dfrac{p. \overline{\overline{\delta X}}(\Pi) + \lambda p}{p. {}^t\Pi} . \dfrac{{}^t d\Pi}{dt} \\[2mm]
\left\| \dfrac{d\Pi}{dt} \right\| \leq 1
\end{cases}
$$

Solutions of this system (if any), have the monotonicity property that we expect and possibly display temporary inefficiencies, as stated in

Proposition 4 (*)

If there exists $\Pi(t)$, $p(t)$, $\lambda(t)$, a solution of system (3) starting from $\Pi(0)$, $p(0)$, $\lambda(0)$ and defined on $[0, T]$ with $\lambda(0)\, {}^t\Pi(0) = \eta(p(0)) - X(\Pi(0))$ and $\lambda(0) \geq 0$ then :

- $\lambda(t) \geq 0 \quad \forall\, t \in [0, T]$
- $X(\Pi(t)) + \lambda(t)\, {}^t\Pi(t) = \eta(p(t)), \forall\, t \in [0, T]$
- $V_h(\Pi(t))$ is a stricly increasing function of t,

$\forall\, h = 1 \ldots H$, for all t where $\mathring{K}(\Pi(t)) \cap Q(\Pi(t), p(t), \lambda(t)) \neq \emptyset$

The proof of proposition 4 is given in the appendix and rests upon the fact that $\lambda(t) \leq 0$ would imply $(\frac{d\lambda}{dt})(t) \geq 0$ (which implies $\lambda(t)$ cannot become negative)

(*) The reader will fruitfully compare this proposition with corollary 3 in Guesnerie [7]

As in section II we introduce an auxiliary system. Let be :

$$\varphi(\Pi, p, \lambda) = \left\{ \begin{array}{l} a \in R^n : p_\circ \overline{\overline{\delta X}}(\Pi).\, {}^t a \leq \lambda \| p \| \\ \Pi\, {}^t a = 0 \\ \|a\| \leq 1 \end{array} \right\}$$

$$f(\Pi, p, \lambda) = \max \{ \min_h (-a.\, X_h(\Pi)),\ a \in \varphi(\Pi, p, \lambda) \}$$

$$M(\Pi, p, \lambda) = \{ a \in \varphi(\Pi, p, \lambda) : f(\Pi, p, \lambda) = \min_h (-a_\circ X_h(\Pi)) \}$$

System (4) is the following :

$$(4) \quad \begin{cases} \dfrac{d\Pi}{dt} \in M(\Pi, p, \lambda) \\ {}^t\dfrac{dp}{dt} = \delta\tilde{\eta}(p)^{-1}.\, [\, \delta\overline{\overline{X}}(\Pi)_\circ\, {}^t\dfrac{d\Pi}{dt} + \dfrac{d\lambda}{dt}\, {}^t\Pi + \lambda\, {}^t\dfrac{d\Pi}{dt}\,]_\circ \\ \dfrac{d\lambda}{dt} = -\dfrac{p_\circ\, \overline{\overline{\delta X}}(\Pi) + \lambda p}{p_\circ\, {}^t\Pi}\; {}^t\dfrac{d\Pi}{dt} \end{cases}$$

Theorems similar to those of section II can be stated

Theorem 3 :

Under assumptions H1-H6-H7' and A for all $(\Pi(0)\ p(0)) \in P \times P$ such that $\eta(p(0)) = X(\Pi(0))$, there exists a solution defined on $[\, 0, +\infty\, [$ starting from $(\Pi(0), p(0), 0)$ for systems (3) and (4) .

Theorem 4 :

System (4) is quasi-stable

For any trajectory of (3) such that : $\forall\, h\ (1 \ldots H)$ $\forall\, t \geq 0,\ -\dfrac{d\Pi}{dt}\, X_h(\Pi(t)) \geq k\, f(\Pi(t), p(t), \lambda(t)),\ k \in\,]0, 1]$ every limit point is an equilibrium.

Corollary 2

> For any trajectory of (3) meeting the above speed condition, every limit point is a local Pareto equilibrium.

Thus with the vocabulary of sectionII., we proved that the limit points of a trajectory of(3) meeting the speed requirements are either second best Pareto optima (α) or local second best Pareto optima (β) or saddle type Pareto equilibria (γ).

Obviously, one would wish to design process for which limit points fall in case (α). It is clear from the basic non convexity of the set of equilibria that such a property cannot be expected for processes which only consider local information on the feasible states.

If limit points of type β cannot be excluded, can one atleast rule out case (γ) which is particularly unsatisfactory, by defining an appropriate process ?

In the state of art, it does not seem clear that such processes can be designed without looking at second order conditions. This is certainly a provisional conclusion, which let open a door for future research .

APPENDIX I

Section I

Proof of proposition 1

$(\Pi(t), p(t))$ being a solution of system (1), one has :

$$ {}^t\frac{dp}{dt} = \partial\tilde{\eta}^{-1}(p).\ \partial\overline{\overline{X}}(\Pi)\ {}^t\frac{d\Pi}{dt} $$

hence $\partial\overline{\overline{\eta}}(p).\ {}^t\frac{dp}{dt} = \partial\overline{\overline{X}}(\Pi)\ .\ {}^t\frac{d\Pi}{dt}$

And thus $\forall\ t \in [0, T]$,$\eta(p(t)) = X(\Pi(t))$ (because of the initial condition)

i.e the equilibium remains tight.

If V_h is the indirect utility function :

$$ \frac{d V_h(\Pi(t))}{dt} = \sum_{k=1}^{n} \frac{\partial V_h}{\partial \Pi_k} \frac{d\Pi_k}{dt} $$

and a classical calculus shows that there exist $\alpha_h > 0$, $h=1\ldots H$ such that :

$$ \frac{dV_h(\Pi(t))}{dt} = \sum_{k=1}^{n} - \alpha_h(\Pi(t))\, X_h^k(\Pi(t)) \frac{d\Pi_k}{dt} = -\alpha_h(\Pi(t)).X_h(\Pi(t)).\ {}^t\frac{d\Pi}{dt}\ . $$

but : $\forall\ h:1\ \ldots\ H \quad X_h(\Pi(t))\ .\ {}^t\frac{d\Pi}{dt} < 0$.

and thus V_h is strictly increasing $\forall\ h\ (1\ldots H)$.

Section II

Proof of theorem 1

The proof has two steps : in the first step, we prove that there is a local solution ; in the second step, the solution is extended to $[0, +\infty[$

Step 1

System (2) can be written $\left(\frac{d\Pi}{dt}, {}^t\frac{dp}{dt}\right) \in F(\Pi, p)$

where $F(\Pi,p) = \{(a, \partial\tilde{\eta}^{-1}(p) \,.\, \partial\overline{\overline{X}}(\Pi) \,.\, {}^t a),\ a \in M(\Pi,p)\}$

To prove local existence, we will refer to Castaing's theorem stated below (see appendix II), which requires that F be a compact convex valued upper hemi continuous correspondence and adequately bounded.

Let be $T = \{(\Pi,p) \in P \times P \mid p.\, \partial\overline{\overline{X}}(\Pi) = 0\}$

$S = P \times P \setminus T$

One can prove :

1 - F is a compact convex valued upper hemi continuous correspondence on S . We note that φ is continuous (it is upper hemi continuous as intersection of upper hemi continuous correspondences, and lower hemi continuous by an ad hoc argument, cf. theorem 5 in appendix II) and that $\underset{h}{\mathrm{Min}}\{-a.\, X_h(\Pi)\}$ is a continuous function in (Π, p, a). The maximum theorem then implies that M is upper hemi continuous and compact valued.

Let be $g : (\Pi,p) \in S \to \partial\tilde{\eta}^{-1}(p).\ \partial\overline{\overline{X}}(\Pi)$

$\Phi : (x, A) \in \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^n) \to (x,\ A^t x)$ (where $\mathcal{L}(\mathbb{R}^n)$ is the set of all linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$)

g and Φ are continuous and $F = \Phi_\circ (M, g)$ implies that F is upper hemi continuous and compact valued.

The fact that F is convex valued results from the concavity of the maximized function $\underset{h}{\mathrm{Min}}\{-a.\, X_h(\Pi)\}$.

2- F is bounded on any compact set K of S, as a consequence of the upper hemi continuity of F .

Hence, from Castaing's theorem one can infer that :

For all compact $K \subset S$ there exists $T_K > 0$, such that for all $(\Pi^\circ, p^\circ) \in K$, there is a solution $(\Pi(t)\ p(t))$ of system (2) starting from (Π°, p°) and defined on $[0, T_K]$. (*)

(Take an open set containing the compact K and apply the theorem).

Step 2 :

Let (Π°, p°) be such that $\eta(p^\circ) = X(\Pi^\circ)$ and let us consider a non decreasing sequence of compact sets C^k such that $S = \bigcup_k C^k$.

Let $C^{k(0)}$ be the smallest compact set of the family containing (Π°, p°). $\exists\ T_{k(0)} > 0$ and $\Pi_\circ(t)$, $p_\circ(t)$ a solution of system (2) starting from (Π°, p°) and defined on $[0, T_{k(0)}]$

Let be $\Pi_\circ(T_{k(0)}) = \Pi^1$, $p_\circ(T_{k(0)}) = p^1$, and let $C^{k(1)}$ be the smallest compact set of the family containing (Π^1, p^1) ; etc ...

Thus we build sequences $C^{k(n)}$, $T_{k(n)}$, $\Pi_n(t)$, $p_n(t)$ such that $C^{k(n)}$ is the smallest compact set containing $(\Pi_{n-1}(T_{k(n-1)}), p_{n-1}(T_{k(n-1)}))$, and $(\Pi_n(t), p_n(t))$ is a solution path of system (2) defined on $[0, T_{k(n)}]$ and starting from $\Pi_{n-1}(T_{k(n-1)})$, $p_{n-1}(T_{k(n-1)})$.

(*) - One must notice that the theorem gives more than a local existence statement which alone could be obtained by more elementary method cf. Guesnerie [7] but would be insufficient for the following.

We have then got a solution on $[0, \sum_n T_{k(n)}]$ starting from $\Pi°, p°$ and such that : 1) $\|p_n(t)\| = \|p°\|$, $\|\Pi_n(t)\| = \|\Pi°\|$,

2) $X(\Pi_n(t)) = \eta(p_n(t))$ (from proposition 1)

It remains to prove that $\sum T_{k(n)}$ is a divergent series.

From $(\Pi_n(T_{k(n)}), p_n(T_{k(n)}))$ one can extract a subsequence (Π_n, p_n) converging to $(\bar{\Pi}, \bar{p})$.

We will prove that $(\bar{\Pi}, \bar{p}) \notin \bar{S} \setminus S$.

If $p_n \to \bar{p} \in \bar{P}$, it would follow from H 3, that $\|\eta(p_n)\| \to +\infty$ (cf. Artzner Neuefeind [1] theorem 1) But H 1 - H3 - imply that the set of tight equilibria of the model is contained in a compact set (cf. Debreu [4]) which leads to a contradiction.

If $\Pi_n \to \bar{\Pi} \in \bar{P}$, assumption (A) would imply either that $\|X(\Pi_n)\| \to +\infty$, -which is impossible for the reason just stated- or $U_h(X_h(\Pi_n)) \to 0$ for some h, which contradicts the strict monotonicity of the process in terms of utility.

If $(\Pi_n, p_n) \to (\bar{\Pi}, \bar{p}) \in T$, $(X(\Pi_n), \eta(p_n)) \to (X(\bar{\Pi}), \eta(\bar{p}))$ such that $X(\bar{\Pi}) = \eta(\bar{p})$ which contradicts H 7.

Hence, there is a compact set $C^{\bar{k}}$ containing an infinity of points of the sequence $(\Pi_n(T_{k(n)}), p_n(T_{k(n)}))$.

Let $\bar{T} = \mathrm{Min}(T°, \dots T^{\bar{k}}) > 0$ which ensures that for an infinity of $n : T_{k(n)} > \bar{T}$. This terminates the proof of this step.

<u>Proof of Proposition 2 and Theorem 2 :</u>

<u>Step 1</u>

Let be $E(k_1, k_2, r_1, \dots r_H) = \{(\Pi, p) \in P \times P \;\; \|\Pi\| = k_1, \|p\| = k_2, \eta(p) - X(\Pi) = 0,$
$U_h(X_h(\Pi)) \geq r_h, r_h > 0, \forall h\}$.

Let us consider a sequence (Π_n, p_n) in E. One can extract a subsequence converging to $(\bar{\Pi}, \bar{p})$ (because of the boundedness of the norm). An argument similar to that of step 2 above shows that $(\bar{\Pi}, \bar{p}) \in P \times P$ and hence, by the continuity of η , X, U_h, to E. Hence E is compact.

Let now $\Pi(t)$, $p(t)$ be a trajectory of system (2), starting from $(\Pi^\circ, p^\circ) \in P \times P$. Let (Π^*, p^*) be a limit point.
(Π^*, p^*) is an equilibrium if and only if :
$0 \in F(\Pi^*, p^*) \Leftrightarrow 0 \in M(\Pi^*, p^*) \Leftrightarrow f(\Pi^*, p^*) = 0$

For proving the statement, we shall show that $f(\Pi^*, p^*) > 0$ is impossible .

Step 2

Let $f(\Pi^*, p^*) = \varepsilon$ be strictly positive. An easy argument shows that from a time $t_\circ$ on, there exist $r_h > 0$, such that the trajectory lies in the compact set $E(\|\Pi^\circ\|, \|p^\circ\|, (r_h)) \overset{\text{def}}{=} K$, on which f is uniformly continuous. Furthermore $(\Pi^*, p^*) \in K$.

The continuity of f implies :
$\exists \eta > 0 : \|(\Pi, p) - (\Pi^*, p^*)\| < \eta \rightarrow f(\Pi, p) > \frac{\varepsilon}{2}$.

Let $\Pi(t_n)$, $p(t_n)$ be a sequence of points of the trajectory converging to (Π^*, p^*) , when $t_n \to \infty$

$$\exists n_\circ : \forall n > n_\circ , \|(\Pi(t_n), p(t_n)) - (\Pi^*, p^*)\| < \eta$$

Hence : $f(\Pi(t_n), p(t_n)) > \frac{\varepsilon}{2}$ for $n > n_\circ$

On the other hand, f being uniformly continuous on the trajectory $(\Pi(t), p(t))$ (for $t \geq t_\circ$)

$$\exists \delta > 0 : \forall n \geq n_\circ, \|(\Pi(t), p(t)) - (\Pi(t_n), p(t_n))\| < \delta \Rightarrow f(\Pi(t), p(t)) > \frac{\varepsilon}{4}$$

$$\|\Pi(t) - \Pi(t_n)\| = \left\| \int_{t_n}^{t} \frac{d\Pi}{d\tau} d\tau \right\| \leq \left| \int_{t_n}^{t} \left\|\frac{d\Pi}{d\tau}\right\| d\tau \right| \leq |t - t_n|$$

$$\|p(t)-p(t_n)\| = \left\| \int_{t_n}^{t} \frac{dp}{d\tau}\, d\tau \right\| \leq \left| \int_{t_n}^{t} \left\| \frac{dp}{d\tau} \right\| d\tau \right|$$

As ${}^t\frac{dp}{dt} = d\tilde{\eta}^{-1}(p(t))\, \partial\overline{\overline{X}}(\pi(t))\, {}^t\frac{d\Pi}{dt}$ and $\partial\tilde{\eta}^{-1}(p(t)).\ \partial\overline{\overline{X}}(\Pi(t))$ is continuous on K and hence bounded, there exists k such that :

$$\left\| \frac{dp}{d\tau} \right\| \leq k \qquad \forall \tau \in [t_n, t]$$

$$\Rightarrow \| p(t) - p(t_n) \| \leq k\, |t - t_n|$$

It follows that : $| t - t_n | < \frac{\delta}{\sqrt{1+k^2}} = \mu$ implies :

$$\| (\Pi(t), p(t)) - (\Pi(t_n), p(t_n)) \| < \delta$$

and hence $f(\Pi(t), p(t)) > \frac{\varepsilon}{4}$

Now the function $V_h^*(t) = u_h(X_h(\Pi(t))$ is continuous non decreasing.

$$\Rightarrow V_h^*(t) \underset{t \nearrow +\infty}{\rightarrow} V_h^* = u_h(X_h(\Pi^*)) \qquad \forall h\ (1 \ldots H)$$

$$\frac{dV_h^*}{dt}(t) = -\alpha_h(\Pi(t)).\ X_h(\Pi(t))\ .\ {}^t\frac{d\Pi}{dt}$$

$\alpha_h(\Pi(t))$ is bounded from below, uniformly in t (*) by α_h.

$$\Rightarrow \frac{dV_h^*}{dt}(t) \geq \alpha_h\, f(\Pi(t), p(t)).$$

(*) $\alpha_h(\Pi) = \frac{\partial u_h}{\partial x_k} / \Pi_k \qquad k\ (1 \ldots n)$

$\Rightarrow \alpha_h(\Pi) = \sqrt{\dfrac{\sum_{k=1}^{n} \left[\frac{\partial u_h}{\partial x_k}(X_h(\Pi))\right]^2}{\|\Pi\|^2}}$ which is bounded on K

by $\alpha_h > 0$ (H 2)

$$V_h^* = V_h^*(0) + \int_0^{+\infty} \frac{dV_h^*}{dt}(t)\,dt$$

It is easy to see that there exists a sequence u_n such that :

- u_n is a subsequence of $1 \ldots n \ldots$
- $u_1 = n_o$
- $\forall n : t_{u_n} + \mu \leq t_{u_{n+1}} - \mu$

Hence :

$$V_h^* \geq V_h(0) + \int_0^{t_{n_o}-\mu} \frac{dV_h}{dt}(t)\,dt + \sum_n \int_{t_{u_n}-\mu}^{t_{u_n}+\mu} \frac{dV_h}{dt}(t)\,dt$$

$$\geq V_h(t_{n_o}-\mu) + \sum_n \int_{t_{u_n}-\mu}^{t_{u_n}+\mu} \alpha_h\, f(\Pi(t), p(t))\,dt$$

$$\geq V_h^*(t_{n_o}-\mu) + \sum_n \alpha_h \frac{\epsilon}{4} 2\mu$$

This is impossible which proves proposition 2

A straight forward modification of the above argument proves theorem 2.

Proof of Proposition 3

- From Proposition 1 a limit point $(\overline{\Pi}, \overline{p})$ of system (1') satisfies $\eta(\overline{p}) = X(\overline{\Pi})$
- Furthermore a limit point is such that the linear system

$$\begin{cases} X_h(\overline{\Pi}).{}^t a < 0 & \forall h \in [1 \ldots H] \\ \overline{p}\, \partial \overline{\overline{X}}(\overline{\Pi}).{}^t a = 0 \end{cases}$$

is inconsistent. It is equivalent to say that the system of inequalities

$$\begin{cases} X_h(\overline{\Pi}).{}^t a < 0 \\ \overline{p}\, \partial \overline{\overline{X}}(\overline{\Pi})\,{}^t a \leq 0 \\ -\overline{p}\, \partial \overline{\overline{X}}(\overline{\Pi})\,{}^t a \leq 0 \end{cases}$$

is inconsistent.

From Rockafellar [12] theorem 22.2, there exists :

$$\mu_h \geq 0 \qquad 1 \leq h \leq H$$

$$\nu \geq 0$$

$$\nu' \geq 0$$

such that . at least one of the numbers $\mu_1 \ldots \mu_H$ is non zero

and . $\sum_h \mu_h X_h(\bar{\Pi}) + (\nu - \nu')\,\bar{p}.\delta\overline{\overline{X}}(\bar{\Pi}) = 0$

It is impossible that $\nu - \nu' = 0$ because of H 4 and the (H 4) assumption of specialized commodities H 1 .

Hence either $\nu - \nu' > 0$ and $(\bar{\Pi}, \bar{p})$ is a local Pareto equilibrium.

Or $\nu - \nu' < 0$ and $\bar{p}.\delta X(\bar{\Pi})\,{}^t a \leq 0$ is then a consequence of

$X_h(\bar{\Pi})\,.\,{}^t a < 0 \quad \forall h \in [1, \ldots H]$

In other words $\mathring{K}(\bar{\Pi}) \cap Q(\bar{\Pi}, \bar{p}) \neq \emptyset$

Section III

Proof of proposition 4

First we verify that $U(p, \Pi, \lambda) = \delta\overline{\overline{X}}(\Pi)\,{}^t\frac{d\Pi}{dt} + \frac{d\lambda}{dt}\,{}^t\Pi + \lambda\,{}^t\frac{d\Pi}{dt}$

belongs to $V(p)$ so that $\frac{dp}{dt}$ is well defined .

$$p.U(p, \Pi, \lambda) = p.\delta\overline{\overline{X}}(\Pi)\,{}^t\frac{d\Pi}{dt} - p.{}^t\Pi\;\frac{p.\delta\overline{\overline{X}}(\Pi) + \lambda p}{p\,{}^t\Pi}\,.\,{}^t\frac{d\Pi}{dt} + \lambda p\,{}^t\frac{d\Pi}{dt}$$

$$= 0$$

Thus we have :

$$\partial \overline{\overline{\eta}}(p(t))\ {}^t\frac{dp}{dt} = \partial \overline{\overline{X}}(\Pi(t))\ {}^t\frac{d\Pi}{dt} + \frac{d}{dt}(\lambda(t)\ {}^t\Pi(t))$$

$$\Rightarrow \quad \eta(p(t)) = X(\Pi(t)) + \lambda(t)\ {}^t\Pi(t) + C$$

where C is a constant which is in fact zero because of the initial conditions

$$\Rightarrow \forall\ t \in [0,T] : \eta(p(t)) = X(\Pi(t)) + \lambda(t)\ {}^t\Pi(t) .$$

so we have proved the second assertion.

The third one is simply a consequence of : $\frac{d\Pi}{dt} \in \mathring{K}(\Pi(t))$

In order to prove the first one, let us suppose that $\exists\ t_1 : \lambda(t_1) < 0$.

Since $\lambda(0) \geq 0$, there exists $t_0 \geq 0$ such that $\lambda(t_0) = 0$ and for $t \in [t_0, t_1]$ $\lambda(t) \leq 0$.

From the definition of $Q(\Pi, p, \lambda)$ we have :

$$-\frac{p\,\partial\overline{\overline{X}}(\Pi)\ {}^t\frac{d\Pi}{dt}}{p.{}^t\Pi} \geq -\frac{\lambda\,\|p\|}{p.{}^t\Pi} \Leftrightarrow \frac{d\lambda}{dt} \geq \frac{-\lambda\,(\|p\| + p\ {}^t\frac{d\Pi}{dt})}{p.{}^t\Pi}$$

On $[t_0, t_1]$ we have $-\lambda \geq 0$. We always have $p.{}^t\Pi > 0$

The Cauchy Schartz inequality proves that :

$$\|p\| + p.{}^t\frac{d\Pi}{dt} \geq 0$$

Thus on $[t_0, t_1]$: $\frac{d\lambda}{dt} \geq 0$ which contradicts $\lambda(t_1) < 0$.

Proof of theorem 3

The sketch of the proof is as in theorem 1 but one has to deal with the additional variable $\lambda \in [\ 0,\ +\infty[$.

Step 1

System (4) can be written :

$$\left(\frac{d\Pi}{dt}, \frac{dp}{dt}, \frac{d\lambda}{dt}\right) \in G\,(\Pi, p, \lambda).$$

where :

$$G(\Pi,p,\lambda) = \{ (a, \partial\tilde{\eta}^{-1}(p).\partial\bar{\bar{X}}(\Pi).{}^t a - \frac{p\,\partial\bar{\bar{X}}(\Pi)+\lambda p}{p.{}^t\Pi}.{}^t a.\partial\tilde{\eta}^{-1}(p).{}^t\Pi + \lambda\partial\tilde{\eta}^{-1}(p).{}^t a,$$

$$-\frac{p\,\partial\bar{\bar{X}}(\Pi)+\lambda p}{p.{}^t\Pi}.{}^t a),\quad a \in M\,(\Pi, p, \lambda)\}$$

In order to apply Castaing's theorem we will prove that :

. for any open, relatively compact set $K \subset S$ such that $\bar{K} \subset S$, there exists ε such that φ is continuous on $K \times]-\varepsilon, +\infty[$, and hence M is u.h.c on this set.

. hence, all compact set in $S \times [0,\ +\infty[$ is contained in an open set where G is uhc and bounded.

1 - $\forall\, K \subset S$, $\exists\, \varepsilon > 0 : \forall\, (\Pi, p, \lambda) \in K \times]-\varepsilon, +\infty[\quad \varphi(\Pi, p, \lambda) \neq \emptyset$.

it is obvious for $\lambda \geq 0$.

when $\lambda < 0$:

$$\forall\, a \in Q\,(\Pi, p, \lambda)\quad \|a\| \geq \frac{|\lambda|\ \|p\|}{\|p.\partial\bar{\bar{X}}(\Pi)\|}$$

$$\Rightarrow \varphi\,(\Pi, p, \lambda) \neq \emptyset \Leftrightarrow \frac{|\lambda|\ \|p\|}{\|p.\partial\bar{\bar{X}}(\Pi)\|} \leq 1$$

$$\Leftrightarrow |\lambda| \leq \frac{\|p.\partial\bar{\bar{X}}(\Pi)\|}{\|p\|}$$

but on $\overline{K}$ the continuous function $(\Pi, p) \to \frac{\|p.\partial X(\Pi)\|}{\|p\|}$ has a minimum $\epsilon > 0$

And hence,

$\varphi(\Pi, p, \lambda) \neq \emptyset \quad \forall (\Pi, p, \lambda) \in K \times]-\epsilon, +\infty[.$

Note now that φ is continuous on $K \times]-\epsilon, +\infty[$: it is upper hemi continuous as intersection of upper hemi continuous correspondences and lower hemi continuous (see theorem 6 in appendix II).

The maximum theorem then implies that M is upper hemi continuous and compact valued on $K \times]-\epsilon, +\infty[$, and hence G is uhc , compact valued and clearly convex valued.

2 - For any compact set C in $S \times [0, +\infty[$, there exists an open, relatively compact neighbourhood $K \times]-\epsilon, c[$ of C such that $\overline{K} \subset S$ and G is upper hemi continuous and bounded on $K \times]-\epsilon, C[$

Hence by Castaing's theorem, there exists $T_C > 0$ such that for all $(\Pi°, p°, \lambda°) \in C$ there is a solution $(\Pi(t), p(t), \lambda(t))$ of system (4) defined on $[0, T_C]$ and starting from $(\Pi°, p°, \lambda°)$.

<u>Step 2</u>

Let be $(\Pi°, p°, \lambda°) \in S \times [0, +\infty[$

As in section II we consider a sequence of non decreasing compact sets C^k such that $S \times [0, +\infty[= \bigcup_k C^k$

We then build sequences $C^{k(n)}$, $T_{k(n)}$, $\Pi_n(t)$, $p_n(t)$, $\lambda_n(t)$ such that : $C^{k(n)}$ is the smallest compact set containing $(\Pi_{n-1}(T_{k(n-1)})$, $p_{n-1}(T_{k(n-1)})$, $\lambda_{n-1}(T_{k(n-1)}))$, $(\Pi_n(t)$, $p_n(t)$, $\lambda_n(t)$) is a solution path of system (4) defined on $[0, T_{k(n)}]$ and starting from $\Pi_{n-1}(T_{k(n-1)})$, $p_{n-1}(T_{k(n-1)})$, $\lambda_{n-1}(T_{k(n-1)})$.

Now, we have to prove that the sequence $\lambda_n(T_{k(n)})$ is bounded.

According to proposition 4. we have :

$$\forall\ t \in [0, T_{k(n)}] : \eta\ (p_n(t) = X\ (\Pi_n(t)) + \lambda_n(t).{}^t\Pi_n(t)$$

$$\lambda_n(t) \geq 0$$

and

$$\|\Pi_n(t)\| = \|\Pi°\|$$

$$\|p_n(t)\| = \|p°\|$$

$$\Rightarrow \quad \|\lambda_n(t).{}^t\Pi_n(t)\| = \|\eta(p_n(t)) - X(\Pi_n(t))\|$$

$$\Rightarrow \quad \lambda_n(t) = \frac{\|\eta(p_n(t)) - X(\Pi_n(t))\|}{\|\Pi_n(t)\|}$$

$$= \frac{\|\eta(p_n(t)) - X(\Pi_n(t))\|}{\|\Pi°\|}$$

but for all n,$(X(\Pi_n(t))$, $\eta(p_n(t))$ is a feasible state, hence is in a compact and thus bounded set.

$\Rightarrow \lambda_n(t)$ is bounded (uniformly).

Hence one can extract a convergent subsequence from $\lambda_n(T_{k(n)})$ and the end of the proof goes on as in step 2 of theorem 1 .

When one has a trajectory for system (4) it is possible to build a trajectory for system (3) in the same way as in section II .

<u>Proof of theorem 4</u>

The proof is a straight forward modification of the proof of proposition 2 using the uniform continuity of $f(\Pi, p, \lambda)$ on the compact set :

$$E(k_1, k_2, r_1, \ldots r_H) = \{ (\Pi, p, \lambda) \in P \times P \times [0, +\infty[\; : \; \|\Pi\| = k_1, \|p\| = k_2$$

$$\eta(p) = X(\Pi) + \lambda^t \Pi$$

$$u_h(X_h(\Pi)) \geq r_h \quad , \quad \forall \, h \; \}$$

where all r_h are strictly positive .

APPENDIX II

The theorem on which our proof of trajectories' existence relies is the following :

Castaing's Theorem [2]

Let be the multivalued differential equation

(1) $\frac{dx}{dt} \in F(t,x) \quad t \in [0,a] \quad x \in \Omega$ non empty open set of $\mathbb{R}^n$

we suppose that :

1° - $F(t,x)$ is a non empty convex compact set of $\mathbb{R}^n$, $\forall t \in [0,a], \forall x \in \Omega$.

2° - $\forall t \in [0,a]$, $x \twoheadrightarrow F(t,x)$ is upper hemi continuous on Ω

3° - $\forall x \in \Omega$, $t \twoheadrightarrow F(t,x)$ is Lebesgue measurable on Ω.

4° - There exists a function g integrable on $[0,a]$ such that :

$\|u\| \leq g(t)$, $\quad \forall u \in F(t,x)$, $\forall t \in [0,a]$, $\forall x \in \Omega$.

A solution of the differential equation (1) is a function X from $[0,t_\circ]$ (with $t_\circ \leq a$) to Ω such that X is absolutely continuous and

$\frac{dX(t)}{dt} \in F(t, X(t))$ a.e on $[0,t_\circ]$.

Let be M any non empty convex compact set in Ω, and $t_\circ \in]0,a]$ such that $\int_0^{t_\circ} g(s)\,ds \leq d(M, C\Omega)$

Theorem

For any $\xi \in M$, there exists at least one solution X of the differential equation (1) on $[0, t_\circ]$ such that $X(0) = \xi$. The set S_ξ of all solutions X such that $X(0) = \xi$ is compact in the Banach space $C_{\mathbb{R}^n}[0,t_\circ]$. (*)

(*) Where $C_{\mathbb{R}^n}[0,t_\circ]$ is the set of all continuous functions from $[0,t_\circ]$ to $\mathbb{R}^n$, endowed with the topology of uniform convergence.

Furthermore, the correspondence $\xi \to S_\xi$ is upper hemi continuous on M.

Here, we use only the first part of the theorem. Our function F does not depend on t: so that we need not verify 3°), and we can take for g any constant such that :

$$\| u \| \leq k , \forall x \in \Omega$$

Theorem 5

Let φ and Ψ be (non empty) correspondences defined by :

$$\forall x \in \Omega \quad \varphi(x) = \{ y \in \mathbb{R}^n : a(x).y = b(x) \}$$

$$\Psi(x) = \{ y \in \mathbb{R}^n : c(x).y = d(x) \}$$

where Ω is a set in $\mathbb{R}^p$

a and c are continuous functions from Ω to $\mathbb{R}^n$
b and d are continuous functions from Ω to $\mathbb{R}$

and $a(x)$ and $c(x)$ are independent vectors of $\mathbb{R}^n$, for all $x \in \Omega$ then $\varphi \cap \Psi$ is a lower hemi continuous correspondence on Ω.

Proof of theorem 5

One has to show that :

For all sequence x^k in Ω converging to $\bar{x} \in \Omega$, for all $\bar{y} \in \varphi(\bar{x}) \cap \Psi(\bar{x})$, there exists a sequence $y^k \in \varphi(x^k) \cap \Psi(x^k)$ converging to $\bar{y}$.

$\bar{y}$ is a solution of the system :

$$(S) \quad \begin{cases} a_1(\bar{x}).y_1 + a_2(\bar{x}).y_2 + \ldots + a_n(\bar{x}).y_n = b(\bar{x}) \\ c_1(\bar{x}).y_1 + c_2(\bar{x}).y_2 + \ldots + c_n(\bar{x}).y_n = d(\bar{x}). \end{cases}$$

As $a(\bar{x})$ and $c(\bar{x})$ are independent there exists a matrix of order 2 in $\begin{pmatrix} a_1(\bar{x}) & \ldots\ldots & a_n(\bar{x}) \\ c_1(\bar{x}) & \ldots\ldots & c_n(\bar{x}) \end{pmatrix}$ which is of full rank 2

For example, let us suppose that :

$$D(\bar{x}) = \begin{vmatrix} a_1(\bar{x}) & a_2(\bar{x}) \\ c_1(\bar{x}) & c_2(\bar{x}) \end{vmatrix} \neq 0$$

System (S) can then be written :

$$\begin{cases} a_1(\bar{x}).y_1 + a_2(\bar{x}).y_2 = b(\bar{x}) - a_3(\bar{x}).\ y_3 - \ldots\ldots - a_n(\bar{x}).y_n \\ c_1(\bar{x}).y_1 + c_2(\bar{x}).y_2 = d(\bar{x}) - c_3(\bar{x}).\ y_3 - \ldots\ldots - c_n(\bar{x}).y_n. \end{cases}$$

Let us denote $\tilde{b}(x, y_3 \ldots y_n) = b(x) - a_3(x).y_3 - \ldots - a_n(x).y_n$

$$\tilde{d}(x, y_3 \ldots y_n) = d(x) - c_3(x).y_3 - \ldots - c_n(x).y_n$$

$\tilde{b}$ et $\tilde{d}$ are clearly continuous.

And one knows that $\bar{y}$, as a solution of system (S), can be written :

$$\bar{y}_1 = \frac{\begin{vmatrix} \tilde{b}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) & a_2(\bar{x}) \\ \tilde{d}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) & c_2(\bar{x}) \end{vmatrix}}{D(\bar{x})}$$

$$\bar{y}_2 = \frac{\begin{vmatrix} a_1(\bar{x}) & \tilde{b}(\bar{x}, \bar{y}_3 \ldots \bar{y}_n) \\ c_1(\bar{x}) & \tilde{d}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) \end{vmatrix}}{D(\bar{x})}$$

As $D(\bar{x}) \neq 0$ and D is a continuous function, there exists a neighbourhood $\mathcal{V}$ of $\bar{x}$ such that :

$$\forall x^k \in \mathcal{V} \quad D(x^k) \neq 0$$

Then let $y^k = (y_1^k, y_2^k, \bar{y}_3, \ldots, \bar{y}_n)$ be defined by :

$$y_1^k = \frac{\begin{vmatrix} \tilde{b}(x^k, \bar{y}_3, \ldots, \bar{y}_n) & a_2(x^k) \\ \tilde{d}(x^k, \bar{y}_3, \ldots, \bar{y}_n) & c_2(x^k) \end{vmatrix}}{D(x^k)}$$

$$y_2^k = \frac{\begin{vmatrix} a_1(x^k) & \tilde{b}(x^k \; \bar{y}_3, \ldots, \bar{y}_n) \\ c_1(x^k) & \tilde{d}(x^k \; \bar{y}_3, \ldots, \bar{y}_n) \end{vmatrix}}{D(x^k)}$$

Then y^k is a solution of $\begin{cases} a(x^k).y = b(x^k) \\ c(x^k).y = d(x^k) \end{cases}$, i.e $y^k \in \varphi(x^k) \cap \Psi(x^k)$

and $y^k \to \bar{y}$, by the continuity of all functions $\tilde{b}$, $\tilde{d}$, a_1, a_2, c_1, c_2 and D.

<u>Theorem 6</u>

Let φ and Ψ be (non empty) correspondences defined by :

$\forall\, x \in \Omega : \varphi(x) = \{y \in \mathbb{R}^n : a(x).y = b(x)\}$

$\Psi(x) = \{y \in \mathbb{R}^n : c(x).y \leq d(x)\}$

where Ω, a, b, c and d are as in theorem 5 .

Then $\varphi \cap \Psi$ is a lower hemi continuous correspondence.

<u>Proof of theorem 6</u>

Let x^k be a sequence in Ω converging to some $\bar{x} \in \Omega$ and

$\bar{y} \in \varphi(\bar{x}) \cap \Psi(\bar{x})$