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FINANCING PUBLIC GOODS WITH COMMODITY TAXES : THE TAX REFORM VIEW POINT.

BY

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This paper presents a study of the allocation and redistribution problems raised by the use of indirect commodity taxation for financing public goods. The formal description of the economic world which is adopted rests on a set of simplified behavioural and policy assumptions which conform to the prototype for theoretical analysis of commodity taxation proposed by Diamond-Mirrlees (1971) in their seminal article. However, the approach taken in this note, although inspired by normative considerations departs considerably from the traditional optimal taxation viewpoint illustrated by the Diamond-Mirlees article just cited.

The problems which are raised are of two types which correspond to the main subdivisions of the article.

The first part is an initial study which attempts to provide a more accurate and precise understanding of the structure of the set of tax equilibria which coincides with the basic feasible set of the second best model considered.

This study starts from the examination of existence questions : considering an exogenously given tax system —conveniently formally defined— does there exist (at least) one equilibrium relative to this tax system ?

It must be noted that despite the recent growing concern for the problem of existence of equilibria with taxes (Sontheimer (1971), Shoven-Whalley (1973), Mantel (1975)) there has been no apparent interaction between this stream of thought and reflections in the field of optimal taxation. The results of the former cannot be directly applied —to the best of our knowledge— to the main models developed by the latter. On the other hand, optimal tax theory puts the emphasis on characterizing optimal taxes i.e. on defining the relationship between elasticities, prices, levels of demand which are necessarily met when taxes are designed optimally, and sometimes on computing these optimal taxes : it is fully coherent to this approach to ignore existence problems for a given tax system since existence problems for an optimal set of taxes are automatically solved.

Section IA presents a statement of sufficient conditions assuring the existence of equilibrium relative to a given tax system. The proof rests on traditional fixed point techniques and its specific arguments have the same flavour as those used in the already mentioned previous contributions on existence of equilibria with taxes especially that of Mantel (1975).

According to Debreu (1972), the model builder has to face and solve two types of preliminary questions : has the model at least one outcome ? How is this outcome sensitive to changes in the exogeneous variables ?

The existence theorem of section IA provides an answer to the first question. An insight into the second question is gained from section IC devoted to the study of the continuity properties of equilibria in relation to the exogeneous characteristics of the tax system. The theorems of section IC follow from the elucidation of the mathematical structure of the set of tax equilibria, a study which is not a simple bridge, but derives conclusions which seem to be of independant interest. The tools used in this section are the central theorems of differential topology which have been widely applied for the study of the Walrasian model (Dierker (1975), Balasko (1976)).

The second part of the paper is directly concerned with normative problems.

Section IIA considers the problem of what can be termed the direction of tax reform : starting from a given tax equilibrium, what are the "small" changes of taxes and public good production which are first, feasible, and secondly satisfactory with respect to a given criterion. The chosen criterion is the Pareto improvement and the results are simply obtained by adapting previous arguments of the author (Guesnerie (1977)) to the public good case. The problem of temporary inefficiencies which appeared in this latter article in also briefly treated.

Section IIB is an attempt at evaluating the relative "size" of the set of tax equilibria and of the set of second best Pareto-optima. Obviously the first best model has accustomed us to the coïncidence of the set of Pareto optima and of the set of equilibria (without taxes) under standard assumptions. This fact no longer holds here. On the other hand, superficial familiarity with the optimal taxation approach which focuses the attention on specific second best optima —those corresponding to a particular social welfare function— might incline one to believe that there are "few" second best optima. A more careful analysis, although remaining at a very informal level, can be sketched as follows : since economic agents will have in general different ideas about the "good" tax system and the "good" level of public good , there will be conflicts between them, the basic features of which can be perceived through arguments similar for example to those of Zeckhauser-Weinstein (1974). Hence, there would be many tax systems and public good production levels where conflicts are unavoidable ; in other words there would be "many" tax equilibria which would be second best. This skeleton of argument captures a part of the truth and is partially validated in section IIB under conditions which make the role of the relative number of households and of "means of financement" for the public good clear.

Two remarks will close this introductory section.

The first one is aimed at justifying the title. In spite of its normative aspects, this contribution cannot be considered as belonging to the field of optimal taxation, since it is not aimed at defining the relationship between prices, levels of demand and elasticities which are necessarily met when taxes are designed optimally.

It could rather be tied up with the "tax reform" viewpoint advocated by Martin Fledstein (1975). Tax reform theory as defined in this latter article "takes as is starting point the initial tax system and considers the situation of each individual before as well as after any proposed change". It is opposed to "tax design", the purpose of usual optimal taxation analysis, which has been criticized on the grounds that it does not provide a fully satisfactory framework for a comprehensive tax theory either because of its exclusion of horizontal equity problems (Musgrave (1959)) or because the "knowledge of optimal taxes may be useless for practical purposes" (Dixit (1975)) since "actual changes are slow and piecemeal" (Feldstein (1975)). "Tax reform" would have attention focused on the improvement of the system through a succession of linked small changes which allow taking into account the evolution of individual welfare, rather than through a one step large change designed from an ideal social welfare function. Even if the topic considered in this note covers only a small part of the program proposed by Feldstein and if the opposition between "tax reform" and "tax design" must not be overestimated, the point of view adopted in this paper is indeed this of tax reform.

The second and last remark is that the paper attempts to cover a coherent —and rather wide— set of problems, with reference to a single model. An alternative solution would have been focusing the attention on the problems separately but in a more general framework. The consequences of the choice made here are two fold. On the one hand, most methods could be transposed into a more general framework (existence of income tax, etc...) and most results could probably be extended in several directions. On the other hand, some of the properties proved here are not presented under the most appropriate and elegant form. This is especially the case of the generic properties of section IIB for which a more satisfactory and general treatment would require the use of a heavy apparatus, and for which the approach taken here is an attempt at capturing the essential properties of the model, while conciling as much as possible rigor and brevity.

I - EXISTENCE AND CONTINUITY OF TAX EQUILIBRIA.

IA. MODEL AND DEFINITIONS.

We are considering an economy in which there are ℓ private commodities indexed by $h = 1,\dots,\ell$ and for the sake of simplicity one public good the quantity of which will be denoted q.

Consumers indexed by i, $(i = 1,\dots,m)$ have preferences represented by a utility function u_i defined on a subset $X_i \times \mathbb{R}_+$ of $\mathbb{R}^{\ell+1}$. When consumer i is given a consumption bundle x_i, and when the quantity q of public good is available, $(x_i,q) \in X_i \times \mathbb{R}_+$ the level of utility of i is $u_i(x_i,q)$.

One firm —numbered 0— can produce public goods from private commodities. The production possibilities of this firm are formalized through a production set $Y_o \subset \mathbb{R}^{\ell+1}$.

Other firms are indexed by $j = 1,\dots,v$ produce and use only private commodities. Their production sets are $Y_j \subset \mathbb{R}^{\ell}$, $j = 1,\dots,v$.

The vector of exogeneous endowments is zero (which means that endowments are only of labor type).

The set of attainable states "à la Debreu" for this economy can be defined as follows.

DEFINITION 1.

An attainable state "à la Debreu" consists is a sequence (x_i) $(i = 1,\dots,m)$, $y_j (j = 1,\dots,v)$ of vectors of $\mathbb{R}^{\ell}$ and a vector $\begin{pmatrix} y_o \\ q \end{pmatrix}$ of $\mathbb{R}^{\ell+1}$ such that

$$(x_i,q) \in X_i \times \mathbb{R}_+$$

$$y_j \in Y_j \quad , \quad (y_o,q) \in Y_o$$

$$\sum_{i=1}^{m} x_i \leq \sum_{j=0}^{v} y_j \qquad (1) \quad .$$

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(1) Throughtout the paper $x_i \geq 0 \iff x_{ih} \geq 0$, $x_i > 0 \iff x_{ih} \geq 0$, $x_i \neq 0$, $x_i \gg 0 \iff x_{ih} > 0$.

Now, the institutional assumptions are that the allocation of commodities are made through markets. On each market consumption prices and production prices may differ and consumers and firms 1,...,v have an autonomous behaviour :

Consumer i, faced with a price system π $(\pi \in \mathbb{R}^{\ell}_{+})$, where π_h is the price to be paid for buying commodity h, and consuming q units of public good, and having no other income than his labour income will choose a consumption bundle among the solutions of the following program :

$$\text{Max } u_i(x_i,q)$$

$$\pi \circ x_i \leq 0$$

$$x_i \in X_i$$

Let $\xi_i(\pi,q)$ denote the subset of solutions of this program.

Firms j = 1,...,v are said to be "uncontrolled" : faced with a price system p, firm j plans to implement a production plan belonging to the set $\eta_j(p)$, the set of competitive production plans :

$$\eta_j(p) = \{y_j \in Y_j \mid p \circ y_j = \text{Max } p.Y_j\}$$

$\xi_i : (\pi,q) \to \xi_i(\pi,q)$ is the demand correspondence of household i.

$\eta_j : p \to \eta_j(p)$ is the supply correspondence of uncontrolled firm j.

$\xi = \sum_{i=1}^{m} \xi_i$ is the total demand correspondence and

$\eta = \sum_{j=1}^{v} \eta_j$ is the total supply correspondence.

We are now in position to define the notion of feasible state "à la Diamond-Mirrlees" or of weak equilibrium.

DEFINITION 2.

A feasible state "à la Diamond-Mirrlees" or a weak equilibrium or a (p-π) *weak equilibrium* consists in a sequence of consumption bundles for consumers (x_i) i = 1,...,m, of production plans for uncontrolled firms (y_j), j = 1,...,v , of one production plan of the controlled firm (y_0,q) , and of price systems p and π such that :

$$x_i \in \xi_i(\pi,q) \qquad \forall\, i = 1,\dots,m$$

$$y_j \in \eta_j(p) \qquad \forall\, j = 1,\dots,v$$

$$(y_0,q) \in Y_0$$

$$\sum_{i=1}^{m} x_i \leq \sum_{j=0}^{v} y_j$$

So the set of weak equilibria describes the possible states of an economy in which no direct transfert can be made to the consumers —who consequently only have a labour income— and where the planner has three main policy tools :
He can, through a tax system, disconnect price systems p end π —since there is no a priori relationship between p and π— ; he can implement a 100 % tax on profits of uncontrolled firms —since no profits are distributed—, and finally he decides upon the level of public good to be produced and upon the production techniques to be chosen —since there are no other restriction on (y_0,q) than technical feasibility.
Models of this type, and assumptions underlying them, have been lengthily discussed otherwise, and for example in Diamond-Mirrlees (1971) : the reader interested in more comments, should refer to this contribution.

The definition of semi-market equilibrium introduces restrictions in the conditions of production of the public good.

DEFINITION 3.

A semi-market equilibrium or a $(p-\pi)$ *semi-market equilibrium* consists in a sequence of consumption bundles (x_i), production plans (y_j) , $j = 1,\dots,v$, of one production plan of the controlled firm (y_0,q) and of price systems p and π such that :

$$x_i \in \xi_i(\pi,q)$$

$$y_j \in \eta_j(p)$$

$$p\cdot y_0 \geq p\cdot y_0' \qquad \forall\, y_0' \text{ s.t. } (y_0',q) \in Y_0$$

$$\sum_i x_i \leq \sum_{j=0}^{v} y_j \quad .$$

If one calls $\mathcal{Y}_0(p,q) = \{y_0 \text{ s.t. } (y_0,q) \in Y_0, \text{ and } p\cdot y_0 \geq p\cdot y_0' \ \forall\, y_0' | (y_0',q) \in Y_0\}$ the set of cost minimizing inputs vectors for producing a level q of public good, an alternative definition of a SM equilibrium is :
A SM equilibrium is feasible state $(x_i)(y_j)(p)(\pi)\ q$ such that $y_0 \in \mathcal{Y}_0(p,q)$.

In a SM equilibria, the planner has not the freedom to choose the input combination for producing q units of public good, but is constrained by the cost minimization condition. In other words the production of the public good is compatible with a private management of the firm producing it : it is hence called a Semi-Market equilibrium.

In the set of feasible states the subset of semi-market equilibria is particularly interesting for the following two reasons :
It allows taking into account the case, frequent in real economic systems, where public goods are produced by firms with a private status which are consequently faced with the production price system.
Even when public goods are produced by public firms, these firms are not generally given special instructions concerning the inputs combination and make economic computations on the basis of the price system of private goods they actually face. Moreover in this model the private management rule is compatible with the attainment of second best Pareto optimal states. (cf. Diamond-Mirrlees (1971)).

A definition of tight equilibria is now given.

DEFINITION 4.

Weak and Semi-Market Equilibria are said to be tight when $\sum_{i}^{m} x_i = \sum_{o}^{v} y_j$.

We are now going to give a rather general definition of the notion of fixed tax system ; a definition whose abstraction is justified by the elementary homogeneity property.
If (x_i) (y_j) is a $(p-\pi)$ weak equilibrium or semi-market equilibrium, it is also a $(\lambda p-\mu\pi)$ weak equilibrium or semi-market equilibrium (for $\lambda > 0$, $\mu > 0$). This remark leads us to consider that the basic objects are not the price vectors, but the direction of price vectors, or the equivalence classes of the equivalence relation associated with proportionality : An equivalence class of π(resp. p) is the set of all price-systems π' (resp. p') such that $\pi' = \lambda\pi$ (resp. $p' = \lambda p$) for some $\lambda > 0$.

A general tax reform is then formally defined as follows.

DEFINITION 5.

A general *tax system ϕ is a single-valued application from the set of equivalence classes of vectors of $\mathbb{R}_+^{\ell}$ into itself.*

To every equivalence class of production price systems, ϕ associates one equivalence class of consumption price systems. So a tax system

does not define taxes in an usual meaning, since taxes depend upon the choice of one production price vector in the equivalence class of production prices and of one consumption price vector in the equivalence class of consumption prices. A tax system, in this general sense, is compatible with different tax schedules according to the normalization rule chosen.
On the other hand, it is obvious that a tax system is completely defined through a tax schedule, once a normalization rule has been chosen both for consumption and production prices.
For example a general tax system is completely determined by a single-valued mapping $\tilde{\phi}$ from S^{ℓ}, the simplex of $\mathbb{R}^{\ell}$ into itself.

In the following, one will generally reason with both p, π belonging to the simplex and will identify ϕ with the mapping $\tilde{\phi}$. The tax schedule associated with this normalization is $t(p) = \tilde{\phi}(p) - p$, and depends upon p, even if it describes "specific" taxes.

Two concepts of equilibria relative to a given general tax system, can then be presented.

DEFINITION 6.

Weak equilibrium relative to a tax system.

A weak equilibrium relatively to the general tax system ϕ, is a (p,π) weak equilibrium such that $\pi = \tilde{\phi}(p)$.

DEFINITION 7.

Semi market equilibrium relative to a tax sustem.

A semi-market equilibrium relatively to the general tax system ϕ is a $(p-\pi)$ semi-market equilibrium such that $\pi = \tilde{\phi}(p)$.

Equivalently a semi-market equilibrium $(x_i),(y_j),p,\pi,q$ relatively to tax system ϕ is a weak equilibrium relatively to ϕ such that $y_o \in Y_o(p,q)$.

Définition 8 concerns tax systems.

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DEFINITION 8.

A tax system ϕ is said to be an *effective financial source* if :

$(\alpha)\Delta(p,q) = - p\circ x$, $x \in \xi(\tilde{\phi}(p),q)$ is strictly positive, for every $p \in \mathring{\mathbb{R}}^{\ell}_{+}$, $q \geq 0$, where $\xi(p,q)$ is defined.

The rationale for this definition can be seen by looking at the amount of resources drawn from commodity taxation which is $[(\tilde{\phi}(p) - p)\circ\sum_i \xi_i (\tilde{\phi}(p),q)]$, an expression which reduces to $\Delta(p,q)$ (when $\pi\, \xi_i(\pi,q) = 0$).

Noting that $\Delta(p,q)$ is independant of the normalization of π and that $\Delta(\lambda p,q) = \lambda\Delta(p,q)$ (for $\lambda > 0$), one sees that the fact that ϕ is an effective financial source does not depend upon the normalization rule adopted for π and p, so that it is enough to check property (α) with $p \in \mathring{S}^{\ell}$ and $\tilde{\phi}$ as a specification of ϕ.

It must also be noted that the fact that ϕ is an effective financial source depends a priori both on ϕ itself and on the economy through its total demand function.

However, if one restricts oneself to considering economies in which commodities can be partitioned in two types, commodities H1 which can be only consumed in negative quantities (labor type), commodities H2 which can only be consumed in positive quantities by all households[(1)] (consumption goods), one can exhibit a class of tax system (the "ad valorem taxes" of public finance) which are effective financial sources independantly of the specific characteristics of demand ; namely those associated with $\tilde{\phi}$ such that [(2)] ;

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(1) Such constraints are taken through consumption sets X_i.

(2) In order to prove that ϕ is an effective financial source let us put

$$\mu = \sum_{h\in H1} p_h + \sum_{h\in H2} (1+\alpha_h)\, p_h$$ and let us consider

$$\frac{1}{\mu}(\Delta(p,q)) = \Delta(\frac{p}{\mu},q) = [\tilde{\phi}(p) - \frac{p}{\mu}] \circ \sum_i \xi_i(\tilde{\phi}(p),q)$$

$$= \frac{1}{\mu} \sum_{h\in H2} \alpha_h\, p_h \sum_i \xi_{ih}(\tilde{\phi}(p),q)$$

an expression which is positive, from the definition of commodities H2.

$$\tilde{\phi}_h(p) = \frac{p_h}{\sum_{h\in H1} p_h + \sum_{h\in H2} (1+\alpha_h)\, p_h} \qquad \forall\ h \in H1$$

$$\tilde{\phi}_h(p) = \frac{(1+\alpha_h)\, p_h}{\sum_{h\in H1} p_h + \sum_{h\in H2} (1+\alpha_h)\, p_h} \quad \alpha_h \geq 0 \ , \quad \forall\ h \in H2$$

On the other hand, the reader will convince himself that in a given economy it is always possible to define a tax system which is an effective financial source.

IB. EXISTENCE OF EQUILIBRTA RELATIVE TO A GIVEN TAX SYSTEM.

THEOREM 1.

Under the following assumptions :

H1) $X_i = a_i + \mathbb{R}^{\ell}_{+}$, $a_i < 0$, $\forall$ i .

H2) $\xi_i : (\pi,q) \to \xi_i(\pi,q)$ is unique continuous and satisfies $\pi \cdot \xi_i(\pi,q) = 0$ for every $(\pi,q) \in \mathbb{R}^{\ell+1}_{+}$ such that $\mathrm{Min}(\pi \cdot X_i) < 0$.

H3) Y_o and Y_j are closed and convex. $0 \in Y_o$.

H4) The set of attainable states (of definition 1) is compact.

H5) $(y_o,q) \in Y_o$ and $y_o' \ll y_o \implies \exists\ q' > q$ s.t. $(y_o',q') \in Y_o$.

And if the tax system satisfies

Hα) It is an effective financial source.

Hβ) $\tilde{\phi}$ is a continuous function on $\tilde{S}^{\ell}$.

Hγ) $p \gg 0 \implies \tilde{\phi}(p) \gg 0$.

Then, there exists p^*, $\pi^* = \tilde{\phi}(p^*)$, q^* , x_i^* , y_j^* , y_o^* such that

$x_i^* = \xi_i(\pi^*,q^*)$ or|and $\pi^* \cdot x_i^* = \mathrm{Min}\ \pi^* \cdot X_i = 0$

$y_j^* \in \eta_j(p^*)$, $y_o^* \in \mathcal{Y}_o(p^*,q^*)$

$$\sum_{i=1}^{m} x_i^* \leq \sum_{j=0}^{v} y_j^*$$

Such a state can be termed a quasi Semi-Market equilibrium with respect to the tax system ϕ.

Proof.

Let us consider an increasing sequence of compact convex subsets S^k of $\mathring{S}^{\ell}$ such that $\bigcup_k S^k \supset \mathring{S}^{\ell}$.

Let N be a compact disk of $\mathbb{R}^{\ell}$ containing all attainable x_i, y_j, y_o in its interior and $[0,\bar{q}]$ be a segment containing all attainable q in its interior.

Let $\tilde{\xi}_i(\tilde{\phi}(p),q) = k(p,q)\ \xi_i(\tilde{\phi}(p),q)$ be with $k(p,q)$

with $k(p,q) = \text{Min}\{1, \text{Max}\ t \mid t\ \xi_i(\tilde{\phi}(p),q) \in N,\}$

One can check that $k(p,q)$ is a continuous function[1]. Hence $\tilde{\xi}_i$ is a continuous function on $S^k \times \mathbb{R}_+$.

Let $\tilde{\eta}_j(p) = \{y_j \in Y_j \mid p\cdot y_j = \text{Max}\ p\cdot(Y_j \cap N)\}$ be. $\tilde{\eta}_j$ is a compact convex valued upper hemi-continuous correspondence on S^ℓ (hence on S^k) and $p\cdot\tilde{\eta}_j(p)$ is continuous and positive.

And let $\tilde{\delta}(p,q) = -\ p\cdot\sum_i \tilde{\xi}_i(\tilde{\phi}(p),q)\ +\ p\cdot\sum_j \tilde{\eta}_j(p)$.

$\tilde{\delta}$ is continuous and strictly positive on $S^k \times (0,\bar{q})$ (because of Hα),Hβ,Hγ,H1,H2)).

Let $\lambda(p,q) = Y_o \cap \{(y',q') \mid p.y' \geq -\tilde{\delta}(p,q)\} \cap \{N \times (0,\bar{q})\}$.

λ is a continuous and convex correspondence on $S^k \times (0,\bar{q})$ (as intersection of convex continuous correspondences, the intersection of which has a non empty interior).

Let $Q(p,q)$: $\text{Max}\ q' \mid \{(y',q') \in \lambda(p,q)\}$ be. It results from the maximum theorem that Q is a continuous function on $S^k \times (0,\bar{q})$ and that $\beta_o(p,q) = \{y \in \mathbb{R}^\ell \mid (y,Q(p,q) \in \lambda(p,q)\}$ which is compact and convex valued is also upper hemi continuous on $S^k \times (0,\bar{q})$.

Now let $\zeta(p,q) = \sum_i \tilde{\xi}_i(\tilde{\phi}(p),q) - \sum_j \tilde{\eta}_j(p) - \beta_o(p,q)$ be.

It comes from the definition of $\tilde{\delta}$ that $p.\zeta(p,q) \leq 0$. And ζ is a compact and convex valued upper hemi-continuous correspondence on $S^k \times (0,\bar{q})$.

Let us consider the compact set $T^k = \zeta(S^k \times (0,\bar{q}))$.

To each z in T^k let $\mu^k(z)$ be the set of prices of S^k which maximizes $p\circ z$.

To each $(p,z,q) \in S^k \times T^k \times (0,\bar{q})$ let us associate the set $\mu^k(z) \times \zeta(p,q)\times Q(p,q)$, a subset of $S^k \times T^k \times (0,\bar{q})$. This correspondence has a fixed point.

There exist p^{*k}, z^{*k}, q^{*k} and $x^{*k} = \tilde{\xi}_i(\tilde{\phi}(p^{*k}),q^{*k})$ $y_j^{*k} \in \tilde{\eta}_j(p^{*k})$, $y_o^{*k} \in \beta_o(p^{*k},q^{*k})$ such that :

$$z^{*k} = \sum_{i=1}^{m} x_i^{*k} - \sum_{j=0}^{v} y_j^{*k}, \quad z^{*k} \in \zeta(p^{*k},q^{*k})$$

$$0 \geq p^{*k}\circ z^{*k} \geq p\circ z^{*k}, \quad \forall\ p \in S^k.$$

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(1) The set $\{t \mid t\ \xi_i(\tilde{\phi}(p),q) \in N\}$ is convex and depends continuously upon (p,q). It follows from the maximum theorem that k is the minimum of two continuous functions.

Several remarks can be made :

1. Assumptions H1) to H5) do not call for long comments. H1)-H2)-H4) are standard assumptions. H2) can be derived from the strict convexity and monotonicity of preferences. H5) is rather inocuous : it tells that if all inputs are available in strictly greater quantity, strictly more public good can be produced.

2. The spirit of the proof of Theorem 1 is similar to this of previous contributions in the fiel. (cf. for example Sontheimer (1971), Mantel (1975)), where assumptions and the construction of the proof were intended to define a concept of excess demand meeting the Walras law. More precisely, the assumption Hα) for the tax system of being an effective financial source, is quite similar to G4) of Mantel (1975). The difference with Mantel's work are however twofold : Mantel whose reasonings apply to a more complex (unspecified general tax system) and more sophisticated (interdependant preferences) system, is only concerned by the existence of what we termed weak equilibria. Second, Mantel's C3) on continuity of demand correspondence does not hold here.

3. The proof of Theorem 1 could be straightforwardly adapted for existence of concepts different from those of semi-market equilibria. Let us suppose for example that the public firm be given special instructions for choosing its inputs, so that it minimizes its cost when the production price system is p, relatively to the shadow price system $\rho(p)$.
Let us call a ρ-equilibrium the corresponding equilibrium concept. A statement similar to Theorem 1 could be proved concerning ρ-equilibria through appropriate modification of the definition of $Q(p,q)$, $\beta_0(p,q)$.

4. Anticipating on section IC, when a concept of η-SM equilibrium will be introduced, the argument of Theorem 1 could also be adapted for proving that such as (quasi) η-SM equilibria do exist[1].

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(1) For that, the assumption should be made that the tax system is a η-effective financial source in the sense that the budgetary excess be large enough to finance the bundle $\eta(\Delta(p,q) > p\cdot\eta$ with the notation of section IC).
The reader is invited to see which modifications should be brought into the proof.

5. The existence of TSM equilibria with to respect to the tax system is based upon assumptions concerning the tax system which are not inocuous : In particular, Hβ) and Hγ) imply that the image of S^{ℓ} , $\tilde{\phi}(S^{\ell})$ is S^{ℓ} itself ($\tilde{\phi}$ is a surjection). However, they are not unreasonable on economic grounds : Hβ) means that a consumption price cannot be zero when the corresponding production price is not zero. Hγ) means that any commodity whose production price is zero cannot be neither taxed nor subsidized.

6. If H6(1) is an usual assumption, H6(2) which asserts that all commodities which are not desired are "productive" in a strong sense, is stronger than the assumptions made in similar contexts for the Walrasian model. Especially H6(2) is stronger than the resource relatedness of Arrow-Hahn (1971) or than HVI - VII of Arrow-Debreu (1954). However, the fact that those assumptions are insufficient for proving corollary II, can be checked by the reader (For example the Arrow-Hahn argument (1971) crucially rests upon the unicity of the price system).

IC. THE STRUCTURE OF THE SET OF TIGHT SEMI MARKET EQUILIBRIA.

For an attempt at exploring the mathematical structure of the sets of TSM equilibria and Pareto-equilibria, the following differentiability assumptions concerning supply and demand functions will always be made in this section.

Ha) For all $(\pi,q) \in \mathring{\mathbb{R}}_+^{\ell+1}$, ξ_i is a C^{∞} smooth function.
For all $p \in \mathring{\mathbb{R}}_+^{\ell}$, η_j is a C^{∞} smooth function.
For all $(p,q) \in \mathring{\mathbb{R}}_+^{\ell+1}$, Y_o is a C^{∞} smooth function.

Two normalisations conventions will be considered.

- the first one is that considered in section II : $p \in S^{\ell}$, $\pi \in S^{\ell}$. One will put $\mathring{S}^{\ell} \times \mathring{S}^{\ell} \times \mathring{\mathbb{R}}_+ \stackrel{def}{=} A$.

- The second one corresponds to $p_1 = \pi_1 = 1$ (commodity one is untaxed). In this case the set of possible p and π can be identified with $\mathbb{R}_+^{\ell-1}$. One will put $\mathring{\mathbb{R}}_+^{\ell-1} \times \mathring{\mathbb{R}}_+^{\ell-1} \times \mathring{\mathbb{R}}_+ = A'$.

Ha) Being given, TSM equilibria with strictly positive production and consumption prices and strictly positive production of public good, are complety defined by (p,π,q). One will restrict the attention to them :

$(p,\pi,q) \in A$(resp. A') define a tight semi-market equilibrium if and only if

$$(1) \quad \sum_i \xi_i(\pi,q) - \sum_j \eta_j(p) - y_0(p,q) = 0$$

<u>One will call E the subset of A(resp. E' the subset of A') of (p,π,q) meeting equations (1).</u>

This section is concerned with the elucidation of the mathematical structure of the set E(resp E'). In economic terms, we would like to understand "how many" tight semi-market equilibria there are and how they are related to each other. The preceding section has already given us the conclusion that there was at least one tight semi-market equilibrium as soon as one was able to exhibit tax systems being effective financial sources. It suggested at the same time, since a whole class of such tax systems can generally be generated, that there were actually many "tax equilibria".

Such an intuition is deepened and made precise in theorem 2.

<u>THEOREM 2.</u>

If the following assumptions are made :

Ha)

Hb) $\forall\, p \gg 0$, let the $\ell \times \ell$ matrix $\overline{\overline{\partial\eta_j}} = \begin{bmatrix} & \vdots & \\ \cdots & \left(\dfrac{\partial \eta_{jh}}{\partial p_k}\right)_{(p)} & \cdots \\ & \vdots & \end{bmatrix}$ be

$\forall\, p \gg 0$, $\forall\, q > 0$, let the $\ell \times \ell$ matrix $\overline{\overline{\partial y_0}} = \begin{bmatrix} & \vdots & \\ \cdots & \left(\dfrac{\partial y_{oh}}{\partial p_k}\right)_{(p,q)} & \cdots \\ & \vdots & \end{bmatrix}$

$\forall\, p \gg 0,\ q > 0, \quad \{ \sum_j \overline{\overline{\partial\eta_j}} + \overline{\overline{\partial y_0}} \}$ is of rank $\ell-1$.

H7) The preference relation of each household can be represented by an utility function which is separable between private and public goods :

$$u_i(x_i,q) = u_{i1}(x_i) + u_{i2}(q)$$

H8) The marginal cost for producing the public good is always strictly positive $\forall\, p \gg 0$, $q > 0$.

If <u>E</u>(resp E') is non empty, it is <u>a smooth manifold of dimension $\ell-1$</u>.

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The proof can be tied up with lemma 1, which is of independant interest.

Lemma 1 : Let Ha), Hb) be

Let $\overline{p},\overline{\pi},\overline{q}$ define a TSM equilibrium such that either $\overline{p}\cdot(\overline{\overline{\partial\xi}})_{(\overline{\pi},\overline{q})} \neq 0$ or $\overline{p}\cdot\left(\frac{\partial\xi}{\partial q}\right)_{(\overline{\pi},\overline{q})} - \overline{p}\cdot\left(\frac{\partial \mathcal{Y}_o}{\partial q}\right)_{(\overline{p},\overline{q})} \neq 0$.

Then, there exists an open neighbourhood $\mathcal{U}$ of $(\overline{p},\overline{\pi},\overline{q})$ $(\mathcal{U} \subset A)$ such that $(\mathcal{U} \cap E)$ be a smooth manifold of dimension $\ell-1$.

Proof : Let $\mathcal{U}$ be an open neighbourhood of $\overline{p},\overline{\pi},\overline{q}$ in A such that everywhere in $\mathcal{U}$ either $p\cdot(\overline{\overline{\partial\xi}}) \neq 0$ or $p\cdot\left(\frac{\partial\xi}{\partial q} - \frac{\partial \mathcal{Y}_o}{\partial q}\right) \neq 0$. Let us consider the map $f : \mathcal{U} \to \mathbb{R}^n$ such that :

$$f(p,\pi,q) = \xi(\pi,q) - \sum \eta_j(p) - \mathcal{Y}_o(p,q).$$

The jacobian matrix df is

$$df(p,\pi,q) = \left(\left[\overline{\overline{\partial\xi}}_{(\pi,q)}\right] \vdots \left[-\overline{\overline{\partial\eta}}_{(p)} - \overline{\overline{\partial\mathcal{Y}}}_o\right] \left(\frac{\partial\xi}{\partial q} - \frac{\partial \mathcal{Y}_o}{\partial q}\right)_{(p,\pi,q)} \right)$$

where $\frac{\partial\xi}{\partial q} - \frac{\partial \mathcal{Y}_o}{\partial q}$ is the vector $\left(\frac{\partial\xi_h}{\partial q} \vdots - \vdots \frac{\partial\mathcal{Y}_{oh}}{\partial q}\right)$ and $\overline{\overline{\partial\eta}} = \sum_j \overline{\overline{\partial\eta_j}}$

$-(\overline{\overline{\partial\eta}} + \overline{\overline{\partial\mathcal{Y}}}_o)$ is of rank $\ell-1$ (cf Hb) and it is well known that $p\cdot(\overline{\overline{\partial\eta}}+\overline{\overline{\partial\mathcal{Y}}}_o) = 0$.

Hence, one can extract from $\overline{\overline{\partial\eta}} + \overline{\overline{\partial\mathcal{Y}}}_o$ $(\ell-1)$ column vectors $y_1,\ldots,y_{\ell-1}$ which are linearly independant and such that $p\cdot y_i = 0$; $i = 1,\ldots,\ell-1$.

If $p\cdot\overline{\overline{\partial\xi}} \neq 0$, one can extract from $\overline{\overline{\partial\xi}}$ one column vector y_ℓ such that $p\cdot y_\ell \neq 0$.

If $p\cdot\left(\frac{\partial\xi}{\partial q} - \frac{\partial \mathcal{Y}_o}{\partial q}\right) \neq 0$, one takes $y_\ell = \frac{\partial\xi}{\partial q} - \frac{\partial \mathcal{Y}_o}{\partial q}$.

In any case, (y_i) $i = 1,\ldots,\ell$ define a set of ℓ linearly independant vectors and rank $df = \ell, \forall\, p,\pi,q \in \mathcal{U}$.

(cf. Millnor (1965)p.11, or Guillemin et Pollack (1974)).

Hence by the preimage Theorem $\mathcal{U} \cap f^{-1}(0)$ is a smooth manifold of dimension $2\ell-1$ (dimension of the manifold $\mathcal{U}$) minus ℓ i.e. $\ell-1$.

Q.E.D.

Let us come back to Theorem 1 : H7) $\Longrightarrow$ $\left(\frac{\partial\xi}{\partial q}\right) = 0$ and H8) $\Longrightarrow$ $p\cdot\frac{\partial \mathcal{Y}_o}{\partial q} > 0$.

It follows that $p\cdot\left(\frac{\partial\xi}{\partial q} - \frac{\partial \mathcal{Y}_o}{\partial q}\right) \neq 0$ $\forall\, (p,\pi,q) \in A$. Hence conclusion follows.

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One can notice that for proving Theorem 2, one needed H8) which is rather inocuous and H7) which is strong. This does not mean that the fact that E is a $(\ell-1)$ manifold is a property which is rarely true, where one considers the "whole" set of economies. At contrary, this property which is always true in the small set of economies with separable utility functions, seems to be "generically" true : it may be wrong for some specific data defining the economy but is "nearly always" true.
Such an assertion can be made meaningful only if a notion of neighbourhood of economy has been introduced, and if a natural measure on the corresponding topological space of economies has been defined.
It is clear that a fully satisfactory treatment of this question —with all elements of the economy —endowments, preferences, production sets— as "parameters" of the space of economies— would require an effort which is out of the scope of this paper.
However, one can catch a part of the essence of the phenomenon and justify to some extent our above intuition, through the consideration of a crude notion of neighbourhood of economy.
Let us precisely define

$$E_\eta = \{(p,\pi,q) \in A | \sum_i \xi_i(\pi,q) - \sum_j \eta_j(p) - Y_0(p,q) = \eta\}.$$

η can be considered an exogeneous manna.
E_0 is nothing else than E and an η economy with η close to zero is close to the initial economy in the sense that all components of the exogeneous manna are "small". So the set of economies is identified with a subset of the Euclidean space $\mathbb{R}^\ell$ and a simple —although imperfect— concept of neighbourhood is deduced.

One has then Theorem 3.

THEOREM 3.

Let V be a neighbourhood in $\mathbb{R}^\ell$ of the economy considered in this section, neighbourhood such that $E_\eta \neq \emptyset$, $\forall \eta \in V$.(1)
Then, there exists a closed set, σ , of measure zero in $\mathbb{R}^\ell$ such that, $\forall \eta \in V | \sigma$, E_η is a smooth manifold of dimension $\ell-1$.

Proof : Let us consider $f : (p,\pi,q) \in A \to f(p,\pi,q) \in \mathbb{R}^\ell$ defined just above. According to SARD's theorem (cf Millnor (1965), the set of critical values

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(1) Cf. remark 4, p. 15.

of f in V is of measure zero. The inverse image Theorem already referred to leads immediately to the conclusion.

ID. THE CONTINUITY OF TAX EQUILIBRIA.

Let us consider now $E' \subset A'$ when E' is a smooth manifold of dimension $\ell-1$. Let then be $\tilde{v} : E' \to \mathbb{R}^{\ell-1}$ defined by $\tilde{v}(p,\pi,q) = \begin{pmatrix} \pi_2-p_2 \\ \vdots \\ \pi_\ell-p_\ell \end{pmatrix}$. $\tilde{v}$ is a (smooth) function associating every tight semi-market equilibrium with the tax vector corresponding to the second normalization rule adopted in section IC. (which is the more natural from an economic viewpoint). It is remarkable that the map "starts from" and "arrives in" manifolds of the same dimension.

Hence, a very informal reflection suggests that the function generally defines local diffeomorphisms between equilibria and taxes, which means in economic terms that the number of equilibria is locally constant and that these equilibria are continuous functions of taxes.

One will try in this subsection to make this argument precise while remaining reasonably simple. For that one will introduce assumption Hc) which provides a minor technical precision on the differentiability of ξ_i, Y_o, η_j and assumption H9) which mainly concerns the boundary properties of supply and demand functions :

Hc) : Ha) and Hb) above are true on $\mathring{\mathbb{R}}^{\ell}_{+} \times \mathring{\mathbb{R}}^{\ell}_{+} \times \mathbb{R}_{+}$.

Hc) extends the differentiability properties Ha) and the rank property Hb) to q = 0 , i.e from the boundaryless manifold $\mathring{\mathbb{R}}^{\ell}_{+} \times \mathring{\mathbb{R}}^{\ell}_{+} \times \mathring{\mathbb{R}}_{+}$ to the manifold with boundary $\mathring{\mathbb{R}}^{\ell}_{+} \times \mathring{\mathbb{R}}^{\ell}_{+} \times \mathbb{R}_{+}$.

H9) 1 - $\forall\, h = 2,\dots,\ell$, $\xi_h(\pi,q) > 0$ for all $(\pi,q) \in \mathring{\mathbb{R}}^{\ell}_{+} \times \mathbb{R}_{+}$

2 - $\forall\, h = 2,\dots,\ell$, if $\frac{\pi_h}{\pi_1} \to +0$ then $\|\xi(\pi)\| \to +\infty$

3 - $\forall\, h = 2,\dots,\ell$, $\forall\, q \geq 0$ if $\frac{p_1}{p_h} \to +0$, then $\|\eta(p) + Y_o(p,q)\| \to +\infty$

4 - $\forall h = 2,\dots,\ell$, $\forall q \geq 0$ if $\frac{p_h^k}{p_1} \to +0$, then $\forall \varepsilon \geq 0$, for k large enough, $\eta_h(p^k) + y_{oh}(p^k,q) < \varepsilon$.

H9) is economically justified if there is in the economy one type of labor —commodity 1—, if all others commodities are desired in the sense of H61) —which implies H92)—, if these commodities tend to be supplied in infinite amount when their prices tends to infinity relatively to labor price H93) —which is assured for example if labor is productive in the sense of H62)— and if their supply tends to be negative when their prices (relative to labor) tend to zero (H94)).

Then, one can state theorem 4.

<u>THEOREM 4.</u>

Let Hc), H4), H7), H8), H9) be.

Then, for all $T_o \in \mathbb{R}^{\ell-1}$, which are outside a closed subset of a measure zero in $\mathbb{R}^{\ell-1}$, the following holds :

. The number of tight semi-market equilibria associated with a given tax vector T_o is finite(1).

. $\exists$ a neighbourhood of T_o , $V(T_o)$ where

- $\forall$ T $\in V(T_o)$, the number of TSM equilibria associated with T is constant.
- In $V(T_o)$ characteristics of a given TSM equilibrium can be expressed as continuously differentiable functions of T.

<u>Proof</u> : We first note that $\mathring{\mathbb{R}}_+^{\ell} \times \mathring{\mathbb{R}}_+^{\ell} \times \mathbb{R}_+$ is a manifold with boundary denoted $\tilde{A}'$. Hence, with Hc), the argument of theorem 2 can be repeated and

$$\tilde{E}' = \{(p,\pi,q) \in \tilde{A}' \mid \xi_1(\pi,q) - \eta(p) - y_o(p,q) = 0\}$$

is a smooth manifold with boundary of dimension ℓ-1. Let then consider

$$\tilde{v} : (p,\pi,q) \in \tilde{E}' \to \begin{pmatrix} \pi_2 - p_2 \\ \vdots \\ \pi_\ell - p_\ell \end{pmatrix} \in \mathbb{R}^{\ell-1}$$ and let us prove that $\tilde{v}$ is proper .

Let C be a compact set in $\mathbb{R}^{\ell-1}$ and let an infinite sequence (p^k,π^k,q^k) be

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(1) Obviously this number may be zero. However, one may notice that for $T_o \gg 0$, the tax system is an effective financial source : hence, if H3) and H5) are added, theorem 1 assures that this number is positive for a such a T_o.

in $\tilde{v}^{-1}(C)$. This sequence is necessarily bounded : if not, there would exist at least one h such that both π_h^k <u>and</u> p_h^k be unbounded (since q^k (H4) and $|\pi_h^k - p_h^k|$ are bounded) ; but $\frac{p_1}{p_h^k}$ would tend to zero, which is excluded by H93) and H4). Hence the sequence has an accumulation point $p^\infty, \pi^\infty, q^\infty$.
By H92), π_h^∞ is different from zero, $\forall$ h. Hence $\lim \xi_h(\pi^k, q^k) = \xi_h(\pi^\infty, q^\infty) > 0$. But $p_h^\infty = 0$ would imply that for k large enough, $\eta_h(p^k) + y_{oh}(p^k, q^k) < \varepsilon$, $\forall \varepsilon > 0$ (H94)), which contradicts the fact that p^k, π^k, q^k is a TSM equilibrium, $\forall$ k. Hence $(p^\infty, \pi^\infty, q^\infty) \in \tilde{A}'$, which proves that $\tilde{v}$ is proper. Conclusion follows.

Let us briefly discuss the assumptions underlying Theorem 4.

- As we noticed that assumptions Hb), H7), H8) were not necessary for obtaining the statement of theorem 3 which could be proved in a suitable framework to be generic, we may remark that a corresponding generic version of theorem 4 could be given which would not refer neither to the rank assumption in Hc) nor to H7), H8).

- It also clear that Ha) could be relaxed, smoothness of η_j being only required on a set $\dot{T}$ ($\neq \mathring{\mathbb{R}}_+^\ell$), the interior of the polar of the asymptotic cone of Y_j($\neq \mathbb{R}_-^\ell$). H9) being modified accordingly, theorem 4 would remain true.

- It is more difficult to dispense with H9) which may look strong. However, it is only for the sake of simplicity that intermediary goods are excluded (cf H91) : they could be introduced without major difficulties. It must also be noted that H91) and H93) are standard. Even H94) is reasonable. Furthermore one was unable to see how it could seriously be relaxed without introducing considerable technical difficulties for proving the theorem. The most serious restriction is in the assumption of existence of one type of labor which plays apparently a decisive role for letting the argument simple.

Finally, let us briefly discuss the case, corresponding to the first normalization procedure, where the tax system is defined by a mapping $\tilde{\phi}_a$ depending upon a vector of parameters belonging to an open subset $\mathcal{O}$ of an euclidean space.

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If adequate differentiability ($\tilde{\phi}_a$ is C^∞ on $\overset{\circ}{S} \times \mathcal{O}$) and transversality conditions were assumed on $\tilde{\phi}$, and if H9.1) were reinforced in order to rule out cases were π_1 and p_1 tend to zero, a theorem similar to theorem 4 ($T_o \in \mathbb{R}^{\ell-1}$ being replaced by $a_o \in \mathcal{O}$) could be obtained.

II - THE PARETO RANKING OF TAX EQUILIBRIA.

Contrary to the first section, which focused the attention on the foundations of the model, its logical consistency and its basic properties, this section is devoted to the study of its normative properties and concentrates on the Pareto ranking of tax equilibria. This question is examined from two viewpoints :

Given a semi-market equilibrium, is it possible to find, a neighbour semi-market equilibrium which would be Pareto better ? This is the problem of the direction of tax reform treated in IIA.

Are there many tax equilibria which cannot be improved upon by tax and public good production manipulations ? This is the problem of the "size" of the set of second best Pareto optima relatively to the set of tax equilibria, treated in section IIB.

In the whole section, we will have to consider the marginal willingness to pay of agent i for the public good.

$C_i(\pi,q)$ will designate the marginal willingness to pay of household i when the consumption price system is π and the level of public good q.

Commodity 1 being the numeraire and taking a differentiable representation U_i of i's preordering —supposed to exist—, and assuming $\xi_i(\pi,q)$ unique one has :

$$C_i(\pi,q) \overset{\text{def}}{=} \frac{\frac{\partial U_i}{\partial q}(.)}{\frac{\partial U_i}{\partial x_1}(.)} \quad \text{where} \quad (.) = (\xi_i(\pi,q),q) .$$

We will need in section IIB smoothness of C_i^∞ , under the following form :

Hc) $C_i : (\pi,q) \to C_i(\pi,q)$ is a smooth C^∞ function on $\overset{\circ}{\mathbb{R}}{}^{\ell}_{+} \times \overset{\circ}{\mathbb{R}}_{+}$, $\forall i = 1,\ldots,m$.

IIA. THE DIRECTION OF TAX REFORM.

The line of argument for the analysis of the direction of tax reform proposed in this section does not depart very much from that proposed by the author in a preceding article, applying to a similar model without public good. Hence, the proofs will only be sketched, and the emphasis will be put on the conclusions and on their specificity.

Let us first introduce some piece of notations :
The TSM equilibrium we start from is indexed by zero (as if it were the point of departure of a time process).
We put $x_i(o) = \xi_i(\pi(o))$, $y_j(o) = \eta_j(p(o))$, $y_o(o) = \mathcal{V}_o(p(o),q(o))$ and denote $\overline{\overline{\partial \xi_i}}(o)$, $\overline{\overline{\partial \eta_j}}(o)$, $\overline{\overline{\partial \mathcal{V}_o}}(o)$ the specification in $p(o)$, $\pi(o)$, $q(o)$ of the matrices precedently defined. $\frac{\partial \xi}{\partial q}(o)$ denotes the vector of partial derivatives of total demand with respect to q and $\frac{\partial \mathcal{V}_o}{\partial q}(o)$ is the vector of marginal inputs necessary for a marginal increase of public good.
$C_i(o) = C_i(\pi(o),q(o))$ is the marginal willingness to pay of i in state 0.
$\frac{\partial U_i}{\partial x}(o)$ is the vector of marginal utilities associated with a differentiable U_i in $x_i(o)$, $q(o)$ and $\frac{\partial U_i}{\partial q}(o)$ is the corresponding marginal utility of public good.
We are now in position of defining the following sets :

$$K(o) = \{(a,b) \in \mathbb{R}^n \times \mathbb{R} \mid a.x_i(o) - b\, C_i(o) \leq 0 \ , \ \forall i\}$$

$$\overset{\circ}{K}(o) = \{(a,b) \in \mathbb{R}^n \times \mathbb{R} \mid a.x_i(o) - b\, C_i(o) < 0 \ , \ \forall i\}$$

$$Q(o) = \{(a,b) \in \mathbb{R}^n \times \mathbb{R} \mid p(o).\overline{\overline{\partial \xi}}(o).a + b\, p(o) \cdot \left(\frac{\partial \xi}{\partial q}(o) - \frac{\partial \mathcal{V}_o}{\partial q}(o)\right) \leq 0\}$$

$$FrQ(o) = \{(a,b) \in \mathbb{R}^n \times \mathbb{R} \mid p(o).\overline{\overline{\partial \xi}}(o).a + b\, p(o). \left(\frac{\partial \xi}{\partial q}(o) - \frac{\partial \mathcal{V}_o}{\partial q}(o)\right) = 0\}.$$

One will give precise definition formalizing the intuitive ideas of feasible and advantageous directions of tax reform.

A direction of consumption prices changes —denoted $\frac{d\pi}{d\tau}$ as if it were related to an infinitesimal move of a time variable τ of production prices changes $\frac{dp}{d\tau}$, of public good production change denoted $\frac{dq}{d\tau}$, are equilibrium preserving if and only if :

$$\sum_i \frac{dx_i}{d\tau} \leq \frac{dy}{dt} \text{ with } \frac{dx_i}{d\tau} = \overline{\overline{\partial \xi_i}}(o) \cdot \frac{d\pi}{d\tau} + \frac{\partial \xi_i}{\partial q}(o) \cdot \frac{dq}{d\tau}$$

$$\frac{dy}{dt} = (\overline{\overline{\partial \eta}}(o) + \overline{\overline{\partial y_o}}(o)) \cdot \frac{dp}{d\tau} + \frac{\partial y_o}{\partial q}(o) \cdot \frac{dq}{d\tau}$$

$\left(\frac{dp}{d\tau}, \frac{d\pi}{d\tau}, \frac{dq}{d\tau}\right)$ will be tight equilibrium preserving if and only if $\sum_i \frac{dx_i}{d\tau} = \frac{dy}{d\tau}$.

$\left(\frac{dp}{d\tau}, \frac{d\pi}{d\tau}, \frac{dq}{d\tau}\right)$ will be strictly Pareto improving if and only if :

a) It is equilibrium preserving.

b) $\frac{dU_i}{d\tau} = \frac{\partial U_i}{\partial x}(o) . \overline{\overline{\partial \xi_i}}(o) \cdot \frac{d\pi}{d\tau} + \frac{\partial U_i}{\partial q}(o) \frac{dq}{d\tau} > 0 \quad , \quad \forall i$.

One can prove the following.

Proposition 1.

Let us suppose that $(p(o), \pi(o), q(o)) \in A'$ is a tight semi-market equilibrium such that :

- utility functions U_i are continuously differentiable in $x_i(o), q(o)$ and monotonic in the sense that $\frac{\partial U_i}{\partial x}(o) \gg 0$, $\frac{\partial U_i}{\partial q}(o) > 0$.
- ξ_i , η_j , y_o , C_i are continuously differentiable in state 0.
- $\overline{\overline{\partial \eta}}(o) + \overline{\overline{\partial y_o}}(o)$ is a matrix of rank $\ell-1$ (1).

Then, for any direction of consumption prices and public good production changes $\left(\frac{d\pi}{d\tau}, \frac{dq}{d\tau}\right)$ belonging to $Q(o)$, there exists at least one direction of production prices changes $\frac{dp}{d\tau}$ such that $\left(\frac{dp}{d\tau}, \frac{d\pi}{d\tau}, \frac{dq}{d\tau}\right)$ be equilibrium preserving.

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(1) Hence, in addition to the differentiability of U_i , we need local versions of Ha), Hb), Hc).

If, moreover $\left(\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right) \in FrQ(o)$, the direction of production prices changes is unique and $\left(\frac{dp}{d\tau},\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right)$ is tight equilibrium preserving.

The <u>proof</u> rests on the following preliminary result.

<u>Lemma</u> : Let $V(o) = \{u \in \mathbb{R}^n \mid p(o).u = 0\}$

Then $\overline{\overline{\partial\eta}}(o) + \overline{\overline{\partial Y}}(o)$ defines a one to one linear mapping from V(o) on to itself, denoted A(o).

The lemma rests on a simple argument of linear algebra, that the reader will find in Guesnerie (1977).

Let $\left(\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right) \in FrQ(o)$.

Let us put $\frac{dX}{d\tau} = \overline{\overline{\partial\xi}}(o)\ \frac{d\pi}{d\tau} + \left(\frac{\partial\xi}{\partial q}(o) - \frac{\partial Y_o}{\partial q}(o)\right)\ \frac{dq}{d\tau}$

As $p(o).\frac{dX}{d\tau} = 0$, the above lemma implics that one can define $\frac{dp}{d\tau} = A(o)^{-1}\ \frac{dX}{d\tau}$.

It follows that $\left(\frac{dp}{d\tau},\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right)$ is tight equilibrium preserving.

If $\left(\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right) \in \overset{\circ}{Q}(o)$, the argument will be slightly modified. In this case, the reader will notice that there are several ways of defining $\frac{dp}{d\tau}$ and that $\left(\frac{dp}{d\tau},\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right)$ is no longer tight equilibrium preserving.

<u>Proposition 2</u>.

Under, the same assumptions as in proposition 1, for any direction $\left(\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right) \in Q(o) \cap \overset{\circ}{K}(o)$, one can find at least one direction $\frac{dp}{d\tau}$ such that $\left(\frac{dp}{d\tau},\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right)$ be strictly Pareto-improving.

If, moreover, $\left(\frac{d\pi}{d\tau},\frac{dq}{d\tau}\right) \in FrQ(o) \cap \overset{\circ}{K}(o)$, $\frac{dp}{d\tau}$ is unique and is tight equilibrium preserving.

For proving it, it is necessary to consider $\frac{dU_i}{d\tau} = \left(\frac{\partial U_i}{\partial x}\right)\overline{\overline{\partial\xi_i}}\ \frac{d\pi}{d\tau} + \frac{\partial U_i}{\partial q}\ \frac{dq}{d\tau}$

It results from the assumptions that $\xi_i(\pi,q)$ is a solution of the program $\text{Max } U_i(x_i) | \{x_i | \pi(o).x_i = 0\}$. Hence, there exists $\mu_i(o) > 0$ s.t. $\frac{\partial U_i}{\partial x}(o) = \mu_i(o).\pi(o)$. It follows $\frac{\partial U_i}{\partial q}(o) = \mu_i(o)\ C_i(o)$.
Finally $\frac{\partial U_i}{\partial \tau}(o) = -\mu_i(o)x_i(o)\frac{d\pi}{dt} + \mu_i(o)\ C_i(o)\frac{dq}{dt}$.
Considering the definition of $Q(o) \cap K(o)$ and proposition 1 leads to the conclusion.

The directions of unanimously advantageous directions of tax reform have been proved to belong to the intersection of a certain number of hyperplanes. The various possible configurations of such an intersection are examined in theorem 6.

THEOREM 6.

Let $\Lambda(o)$ be the following cone in $\mathbb{R}^{\ell+1}$.

$$\Lambda(o) = \{(a,b) | a = \sum_i \lambda_i\ x_i(o)\ ,\ b = -\sum_i \lambda_i\ C_i(o)$$

for some $\lambda_i \geq 0\}$.

Then, the existence of strictly Pareto improving directions of tax reform depends upon the position of vector $\nu(0)$:

$$\nu(o) \stackrel{\text{def}}{=} (p(o).\overline{\overline{\partial\xi}}(o)\ ,\ p(o)\circ\left(\frac{\partial\xi}{\partial q}(o) - \frac{\partial Y_o}{\partial q}(o)\right))$$ relatively to $\Lambda(o)$.

a) If $\nu(o) \in -\Lambda(o)$, there does not exist strictly Pareto improving directions of tax reform.

b) If $\nu(o) \in +\Lambda(o)$, then there exist strictly Pareto improving directions of tax reform but none is tight equilibrium preserving

c) If $\nu(o) \in C(\Lambda(o) \cup -\Lambda(o))$, then there exist tight equilibrium preserving and strictly Pareto improving directions of tax reform.

The proof, which is only sketched, consists in establishing that cases a) and b) are respectively equivalent to $\overset{\circ}{K}(o) \cap Q(o) = \emptyset$, $\overset{\circ}{K}(o) \cap \overline{CQ(o)} = \emptyset$ (at least when $\overset{\circ}{K}(o)$ is non empty, which is clearly true here).

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The statement calls for three types of comments.

1. One must underline the intuitive content of proposition 1 et 2.
$\left(\frac{d\pi}{dt},\frac{dq}{dt}\right) \in Q(o)$ means that the change in total excess demand of consumers and of the public firm induced by the prices and public good production changes and measured with production prices is negative.
$\left(\frac{d\pi}{dt},\frac{dq}{dt}\right) \in \overset{\circ}{K}(o)$ means that for every consumer the variation of cost of his consumption bundles induced by the prices changes corrected by the willingness to pay for the public good production change is negative.
Proposition 1 also gives an intuitive understanding of the fact that the TSM equilibria define a smooth manifold of dimension $\ell-1$: $\ell-1$ "degrees of freedom" are given in choosing consumption prices, one corresponds to the public good production choice, and one is substracted since the preceding changes are related through the fact that the corresponding vectors belong to FrQ(o).

2. Part b) of the theorem points out a property which may look strange at first sight. It indicates that it tends to be impossible to obtain a small Pareto-improving movement of the system without keeping aside some of the produced goods. In others words, small Pareto improving movements may be impossible without inefficiencies in production. These inefficiencies will be temporary in the sense that second best Pareto optima of the model are known to imply —under minor restrictions— efficiency in production. This problem of the existence of "unavoidable temporary inefficiencies" has been lenghtily discussed in Guesnerie (1977), in a model where public goods are not taken into account. The specific insight which can be gained from the analysis of this section, where public goods are explicitly considered, is that the occurence of the phenomenon of unavoidable temporary inefficiencies is more unlikely, due to the fact that condition b) can be met only when $p(o)\,\frac{\partial \xi}{\partial q}(o)$ is negative and greater in absolute value that $p(o).\frac{\partial y_u}{\partial q}(o)$.
For example one can state, in the case where utility functions are separable between private and public goods that the phenomenon of unavoidable temporary inefficiencies disappears.

Proposition 3.

Under H 7 , case b) of Theorem 5 cannot occur.

IIB. THE SIZE OF THE SET OF SECOND BEST PARETO OPTIMA.

It is known from the literature that the second best Pareto optima relative to the feasible states "à la Diamond-Mirrlees" – i.e. the maximal elements of the Pareto preordering on the set of feasible states "à la Diamond-Mirrlees" – coïncide approximately (under minor restrictions) with the second best optima relative to the set of semi-market equilibria.
It is why one will directly focus the attention on these latter second best optima.
Let (p^*, π^*, q^*) be one of them supposed tight. Intuitively, in such a state there will not exist strictly Pareto improving directions of changes. Hence it will meet condition a) of Theorem 6. This is confirmed by Theorem 7.

THEOREM 7. (Diamond-Mirrlees)

Let (p^*, π^*, q^*) be a tight semi-market equilibrium second best Pareto optimal, and meeting the assumptions of proposition IIB.1
Then $\exists\ \lambda_i \geq 0$ s.t.

$$p^* \cdot (\overline{\overline{\partial \xi}})_{(*)} = - \sum_{i=1}^{m} \lambda_i\ \xi_i(*)$$

$$p^* \circ \left[\left(\frac{\partial \xi}{\partial q} \right)_{(*)} - \left(\frac{\partial y_o}{\partial q} \right)_{(*)} \right] = \sum_{i=1}^{m} \lambda_i\ C_i(*) .$$

A proof could be given from theorem 6, by proving that when condition a) is not satisfied and that $\left[p^* \circ (\overline{\overline{\partial \xi}})^* ,\ p^* \circ \left(\frac{\partial \xi}{\partial q} \right)_{(*)} - \left(\frac{\partial y_o}{\partial q} \right)_{(*)} \right] \neq 0$, there exists a small but finite strictly Pareto improving move of the system, by an ad hoc argument similar to this used in Guesnerie(1977). An alternative classical proof would use the Kuhn-Tucker theorem.

This preliminary characterization result being recalled, we will try to understand the basic features of our problem through a simple specific example.
Let us then look at an economy with 4 commodities. Commodity one is labour. Commodity 2, 3 are consumption commodities. Commodity 4 is a public good. Technologies are of Leontieff type so that competitive production prices equals "labour value" $\overline{p} = 1, \dots, \overline{p}_4$. Commodity one being supposed untaxed,

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a consumption price system is completely determined by the taxes t_2, t_3.

Taxes t_2, t_3 associated with a tight semi-market equilibrium define a subset of $\mathbb{R}^2$, which if one refers to theorem 1, may have roughly the shape indicated in figure 1.

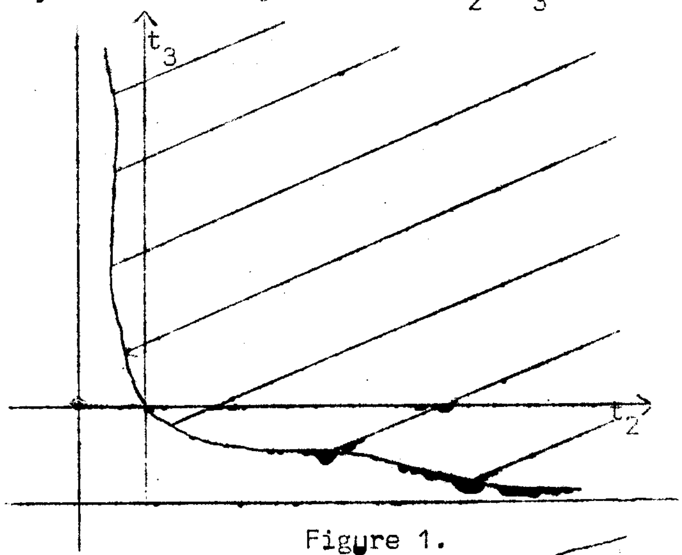

Figure 1.

Let us now suppose that t_2 being given the receipts of the Governement are an increasing function of t_3 (this is not a general property). Then, t_2 being given and production prices being constant, there is a one to one relationship between t_3 and the level of q which can be produced. The set of feasible (t_2, q) can be represented on figure 2, which visualizes the manifolds E' of section I[1].

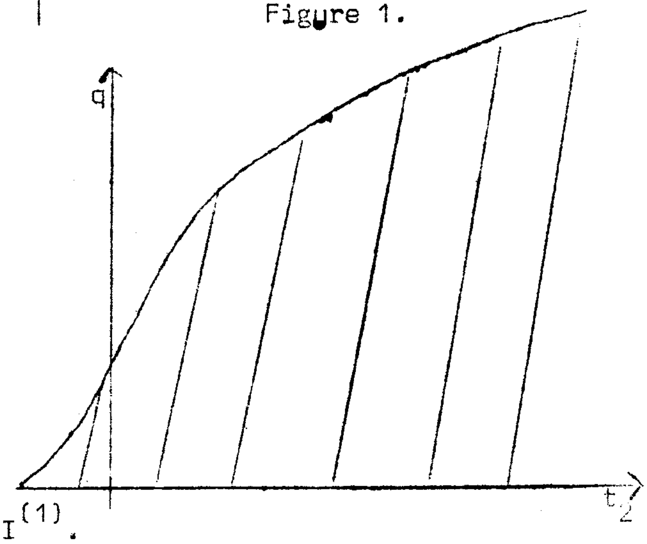

Figure 2.

Now one can represent preferences of each household as indirect preferences depending upon the tax system —implicitely determined by t_2 and q—, and upon the quantity of public good.

Generally these indirect preferences will determine a bliss point (the optimal tax system for the considered consumer) and indifferences curves surrounding it as shown in figure 3. Obviously these preferences have "no reason" to be convex .

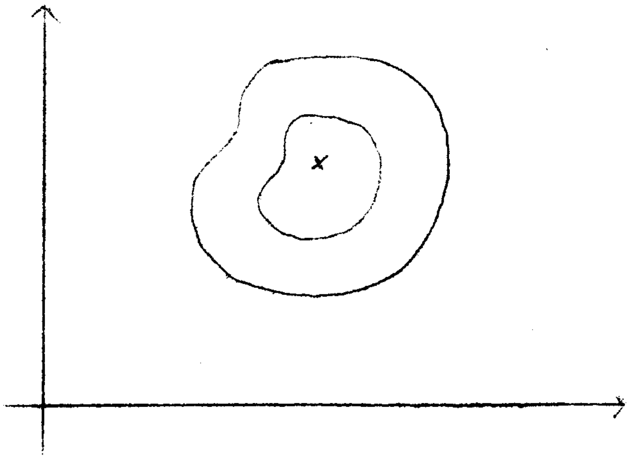

Figure 3.

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(1) Let us remark that this figure suggests that the manifold E' is arc wise connected, a property which does not seem to be general.

A synthetical view of the problem is given in figure 4 which considers the case of three consumers A B C. This diagram which bears some similarity with the diagram supporting the analysis of Zeckhauser-Weinstein (1974) in a related but different context, suggests the following remarks.

- All points outside the curviline triangle A B C are Pareto dominated by point inside A B C which are second best Pareto optima. If preferences where convex, the set A B C would be topologically close to a simplex as argued by Zeckhauser- Weinstein (1974). As they are not, the topological nature of the set of second best optima is generally more complicated than what is suggested here.

- If there were only two consumers A B, the set of second best Pareto optima would reduce to $\widehat{A\,B}$ and would generally be negligible in the set of tax equilibria. With three consumers, unless C is on A B, A B C has a positive measure and may be "big" with respect to the manifold E'.

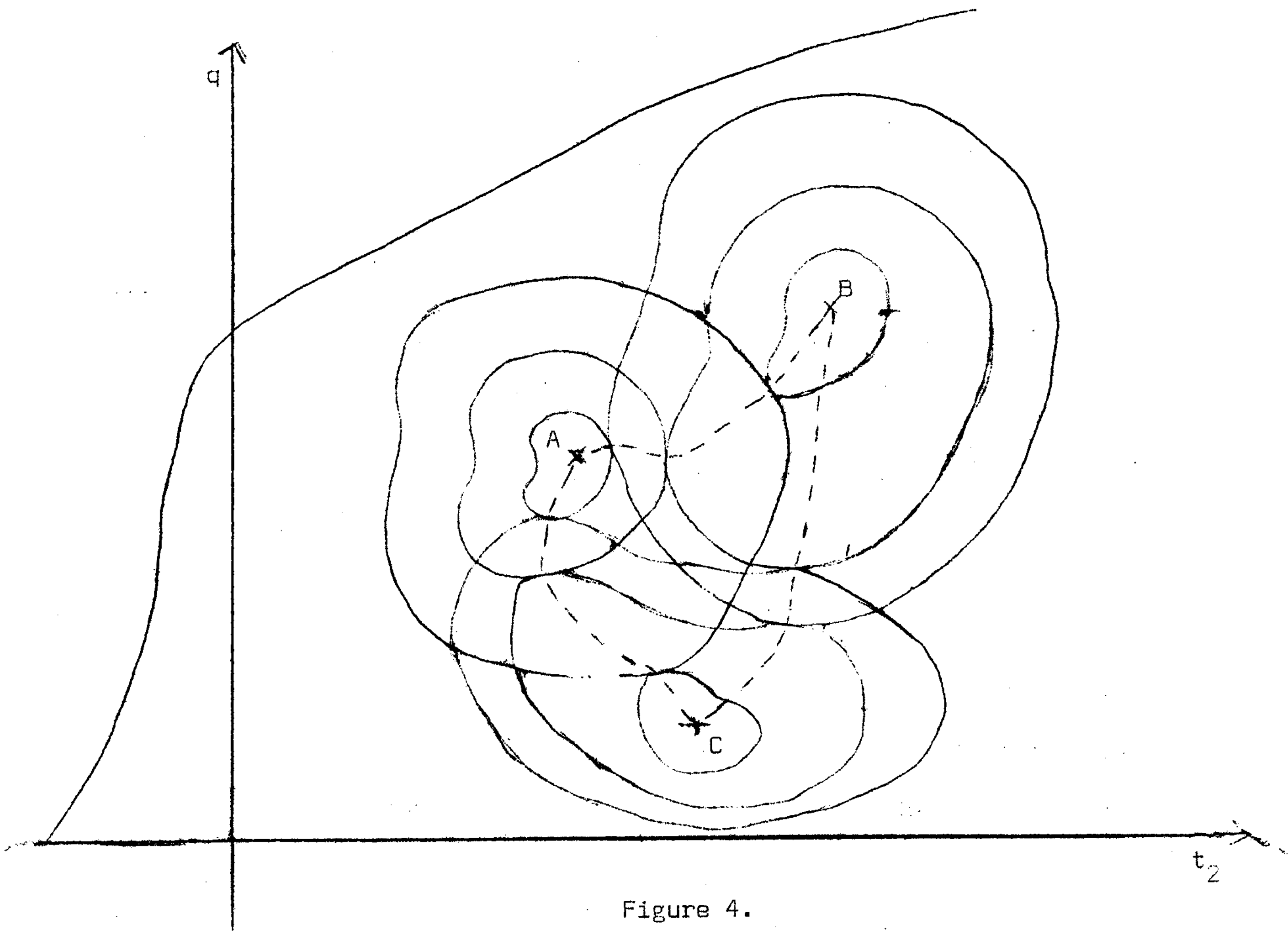

Figure 4.

This analysis shows that an important parameter of the size of the set of second best Pareto optima is the relative number of goods and consumers. This point will now be explored.

As following will make it clear, the property we want to focus on is not always true, but only "generic" in the sense of differential topology. So we are led to consider as in section IB. a set of economies associated with a notion of distance of economies which realize a compromise between simplicity and generality.

To that extent, one will suppose that an economy is associated with a vector of $\mathbb{R}^{2\ell}$ $\alpha = (\nu,\varepsilon,\mu)$, $\nu \in \mathbb{R}^{\ell}$, $\varepsilon \in \mathbb{R}^{\ell-1}$, $\mu \in \mathbb{R}$.

The reference economy we consider in the paper is characterized by its total supply function $\eta(p)$, the public supply function $Y_o(p,q)$, its demand functions $\xi_i(\pi,q)$ and marginal willingness to pay function $C_i(\pi,q)$.

The α-economy is characterized as follows :

$$\eta(p,\alpha) = \eta(p) + \nu$$
$$Y_{oh}(p,q,\alpha) = Y_{oh}(p,q) - m\,\varepsilon_h \quad , \quad h = 1,\dots,\ell-1$$
$$Y_{o\ell}(p,q,\alpha) = Y_{o\ell}(p,q)$$
$$\xi_{ih}(\pi,q,\alpha) = \xi_{ih}(\pi,q) + \varepsilon_h \quad , \quad h = 1,\dots,\ell-1 \quad , \quad i = 1,\dots,m$$
$$\xi_{i\ell}(\pi,q,\alpha) = \xi_{i\ell}(\pi,q)$$
$$C_i(\pi,q,\alpha) = C_i(\pi,q) + \mu \qquad\qquad i = 1,\dots,m.$$

One defines naturally :

$$E(\alpha) = \{(p,\pi,q) \in A \mid \sum \xi_i(\pi,q,\alpha) - \sum_j \eta_j(p,\alpha) - Y_o(p,q,\alpha) = 0\}$$

the set of TSM equilibria of the economy α .

And

$$P(\alpha) = \{(p,\pi,q) \in E(\alpha) \mid \exists\, \lambda_i > 0 \text{ such that}$$
$$p\cdot\overline{\overline{\partial\xi}}(\pi,q,\alpha) + \sum_i \lambda_i\, \xi_i(\pi,q,\alpha) = 0 \quad , \quad p\cdot\left(-\frac{\partial Y_o}{\partial q}(p,q,\alpha) + \frac{\partial\xi}{\partial q}(\pi,q,\alpha)\right)$$
$$- \sum_i \lambda_i\, C_i(\pi,q,\alpha) = 0\}$$

the set of Pareto equilibria in economy α , i.e. the subset of TSM equilibria which satisfy the necessary conditions of second best Pareto optimality established in Theorem 7.

The main conclusion suggested by the above specific example can now be confirmed by Theorem 8.

THEOREM 8.

Let us suppose that the assumptions Ha) to Hc) hold.

Let Θ be an open neighbourhood of 0 in $\mathbb{R}^{2\ell}$ such that $\forall\ \alpha \in \Theta$, $P(\alpha) \neq \emptyset$.

Then, for all $\alpha \in \Theta\backslash\tau$ where τ is a closed subset of measure zero the following holds :

If $m < \ell$ $P(\alpha)$ is of measure zero in $E(\alpha)$.

Proof. Let us consider $\psi : A \times \overset{\circ}{\mathbb{R}}{}^m_+ \rightarrow \mathbb{R}^{2\ell}$ (1) defined by :

$$\psi_h(p,\pi,q,\lambda) = \sum_i \xi_{ih}(\pi,q) - \eta_h(p) - y_{oh}(p,q) \quad , \quad h = 1,\ldots,\ell$$

$$\psi_h(p,\pi,q,\lambda) = \frac{1}{\sum_i \lambda_i}\left[\sum_k p_k\left(\frac{\partial\xi_k}{\partial\pi_h}\right)(\pi,q) - \sum_i \lambda_i \xi_{ik}(\pi,q)\right] \quad h = \ell+1,\ldots,2\ell-1$$

$$\psi_{2\ell}(p,\pi,q,\lambda) = \frac{1}{\sum_i \lambda_i}\left[p.\left[\left(\frac{\partial\xi}{\partial q}\right)(\pi,q) - \frac{\partial y_c}{\partial q}(p,q)\right] - \sum_i \lambda_i C_i(\pi,q)\right]$$

Let $\mathcal{P}(\alpha) = \psi^{-1}(\alpha)$. It is not difficult to check that

$P(\alpha) = \mathrm{Proj}_A\ \mathcal{P}(\alpha)$.

But Sard's theorem assures that the set τ' of critical values of ψ in Θ is of measure zero.

For any $\alpha \in \Theta \setminus \tau'$, $\mathcal{P}(\alpha)$ is a smooth manifold of dimension $2\ell-1 + m - 2\ell = m-1$ (inverse image theorem).

Let now $\phi : A \rightarrow \mathbb{R}^\ell$ be.

$$\phi(p,\pi,q) = \sum_i \xi_i(\pi,q) - \eta(p) - y_o(p,q).$$

One can check that $E(\alpha) = \phi^{-1}(\alpha)$ and for all $\alpha \in \Theta / \tau''$, where τ'' is a subset of measure zero in Θ , $E(\alpha)$ is a smooth manifold of dimension $\ell-1$.

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(1) It is important to notice that in the optimal tax formula, λ_i and p cannot both normalized. This implies that one has to consider $\overset{\circ}{\mathbb{R}}{}^m_+$ as the set of possible λ in this proof.

Let then v be the projection function : $v(p,\pi,q,\lambda) = (p,\pi,q)$ and $\tilde{v}$ its restriction to $\mathcal{P}(\alpha)$: $\tilde{v} : \mathcal{P}(\alpha) \rightarrow E(\alpha)$.
It is clear that Image $\tilde{v} = \mathcal{P}(\alpha)$. Then, according to Sard's theorem $m-1 < \ell-1$ implies the searched property, for $\alpha \in \Theta / \tau' \cup \tau''$.

The result calls for one final remark.

Remark : It is clear that our notion of α economy is not fully satisfactory so far as one did not prove that the supply and demand functions can be derived from preferences and production sets, a point which is not rigorously true (cf the boundary problems).
One can justify the approach presented here, by arguing again that a fully satisfactory treatement would require a heavy apparatus and a longer and more technical analysis and by noting that a less ambitious interpretation of the property can be given, $E(\alpha)$ being considered the set of states which are "nearly" equilibria "à ν près" and $P(\alpha)$ being a set of states which are nearly Pareto equilibria "à ε près".

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